K-ANALYTIC VERSUS CCM-ANALYTIC SETS IN NONSTANDARD COMPACT COMPLEX MANIFOLDS

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Abstract. It is shown that in an elementary extension of a compact complex manifold M, the K-analytic sets (where K is the algebraic closure of the underlying real closed field) agree with the ccm-analytic sets if and only if M is essentially saturated. In particular, this is the case for compact Kähler manifolds.

1. Introduction and Preliminaries

Let $\mathcal{R} = (\mathbb{R}, \dots)$ be a fixed o-minimal expansion of $\mathbb{R}_{\mathrm{an}}$ and M a compact complex manifold. We will view the underlying real-analytic manifold of M as living definably in $\mathcal{R}$ and equipped with all the structure induced from $\mathcal{R}$. One way to do so would be to consider M as a *definable space in* $\mathcal{R}$ in the sense of [16]. However we proceed more concretely as follows: the Morrey-Grauert imbedding theorem allows us to identify M with a real-analytic submanifold of $\mathbb{R}^m$ for some $m \geq 0$. Moreover, by compactness, $M \subseteq \mathbb{R}^m$ is globally subanalytic and hence definable in $\mathbb{R}_{\mathrm{an}}$. We therefore can, and do, view M as equipped with the full induced structure from $\mathcal{R}$: a subset of M^n is *definable* if as a subset of $\mathbb{R}^{mn}$ it is definable in $\mathcal{R}$. For example, every real-analytic subset of M^n is definable.

Now consider the complex manifold structure on M. Every complex-analytic subset of M^n is real-analytic and hence definable. The complex-analytic structure on M is thus viewed as a reduct of its real-analytic structure. Formally, let us denote by M_{ccm} this reduct where the underlying set is M and where the language consists of a predicate for each complex-analytic subset of each cartesian power of M. A definable subset of M (or its cartesian powers) will be called ccm-*definable* if it is definable in the reduct M_{ccm}. Recall that $\mathrm{Th}(M_{\mathrm{ccm}})$ admits quantifier elimination and is of finite Morley rank ([18]). A more detailed survey of the model theory of $\mathrm{Th}(M_{\mathrm{ccm}})$ can be found in [6].

Next, we pass to an elementary extension of the entire situation: let $\widehat{\mathcal{R}} = (R, \dots)$ be a sufficiently saturated elementary extension of $\mathcal{R}$, and let $\widehat{M} \subseteq R^m$ be the interpretation of M in $\widehat{\mathcal{R}}$. We obtain a corresponding elementary extension of M_{ccm}, denoted by $\widehat{M}_{\mathrm{ccm}}$, which is itself a reduct of the induced structure on $\widehat{M}$ from $\widehat{\mathcal{R}}$. So on the "nonstandard" manifold $\widehat{M}$ we have the *definable* sets, namely those that are definable in $\widehat{\mathcal{R}}$, and the ccm-*definable* sets, those that are definable in the reduct $\widehat{M}_{\mathrm{ccm}}$.

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Notation 1.1. If $\mathcal{N} \preceq \widehat{\mathcal{N}}$ is an elementary extension of first-order structures, and

$$\mathcal{F} = \{F_a = \phi(c,a)^{\mathcal{N}} : a \in A = \psi(x)^{\mathcal{N}}\}$$

is a definable family of sets in $\mathcal{N}$, then we let

$$\widehat{\mathcal{F}} := \{\phi(c,a)^{\widehat{\mathcal{N}}} : a \in \psi(x)^{\widehat{\mathcal{N}}}\}$$

be the corresponding definable family in $\widehat{\mathcal{N}}$. A definable set in $\widehat{\mathcal{N}}$ is then just a member of some $\widehat{\mathcal{F}}$ where $\mathcal{F}$ is a definable family of definable sets in $\mathcal{N}$. The study of definable sets in $\widehat{\mathcal{N}}$ thus amounts to the study of definable families in $\mathcal{N}$.

Among the definable sets in $\widehat{M}$ there are two natural candidates for the "non-standard complex-analytic" ones. The first of these comes from working entirely in $\mathrm{Th}(M_{\mathrm{ccm}})$:

Definition 1.2. A subset $S \subseteq \widehat{M}^n$ is ccm-*analytic* if there exists a ccm-definable family $\mathcal{F}$ of complex-analytic subsets of M^n such that $S \in \widehat{\mathcal{F}}$.

By quantifier elimination for $\mathrm{Th}(M_{\mathrm{ccm}})$, every ccm-definable set in $\widehat{M}$ is a finite boolean comination of ccm-analytic sets.

Remark 1.3. We will eventually use an equivalent characterisation of ccm-analytic subsets that is more along the lines of [13, 6], and it is worth describing this now. Let $\mathcal{A}$ be the many-sorted structure where there is a sort for each irreducible compact complex-analytic space (not necessarily smooth) and the language consists of a predicate for each complex-analytic subset of each cartesian product of sorts. The $\mathrm{Th}(\mathcal{A})$ admits quantifier elimination and is of finite Morley rank (sort by sort). If M is a compact complex manifold then M is a sort of $\mathcal{A}$ and M_{ccm} is just the structure induced on M by $\mathcal{A}$. The whole many-sorted structure $\mathcal{A}$ is interpretable in $\mathcal{R}$. Hence there is a corresponding elementary extension $\widehat{\mathcal{A}}$ of $\mathcal{A}$ interpretable in $\widehat{\mathcal{R}}$. Now, a subset $S \subseteq \widehat{M}^n$ is ccm-analytic if and only if there exist:

- a sort B of $\mathcal{A}$,
- a complex-analytic subset $Z \subseteq B \times M^n$, and
- a generic point e of B in $\widehat{\mathcal{A}}$,

such that S is the fibre of Z above e in $\widehat{\mathcal{A}}$ (see Section 6 of [6]). This equivalence follows from quantifier elimination as well as definability of types in $\mathrm{Th}(\mathcal{A})$. The point is that by definability of types the family of fibres of Z over B, $\mathcal{F} := \{Z_b : b \in B\}$, forms a family of complex-analytic subsets of M^n that is definable *in* M_{ccm}.

The second notion of "complex-analyticity" in $\widehat{M}$ is informed by the work of Y. Peterzil and the second author on complex analysis over algebraically closed fields in o-minimal structures. Let R be a real closed field and K its algebraic closure. In [11, 12, 10] Y. Peterzil and the second author investigate differentiability with respect to the field K for functions of several variables definable in o-minimal expansions of R. Their analysis leads to a notion of K-*analytic* set. We will not give their definition here as we will be concerned with only a special case. Suffice it to say that K-analyticity is definable in parameters (see Corollary 4.13 of [10]), and that if $K = \mathbb{C}$ and one is working in an o-minimal expansion of $\mathbb{R}_{\mathrm{an}}$ then K-analytic is just complex-analytic. It follows that if R is the underlying real closed field of an elementary extension of an o-minimal expansion of $\mathbb{R}_{\mathrm{an}}$, then the notion of K-analytic subset of $\widehat{M}$ introduced in [12] has the following more direct characterisation – which will serve as a definition for us.

Definition 1.4. Let R be the underlying real closed field of $\widehat{\mathcal{R}}$ and K its algebraic closure. A subset $S \subseteq \widehat{M}^n$ is *K-analytic* if there exists a definable family $\mathcal{F}$ of complex-analytic subsets of M^n such that $S \in \widehat{\mathcal{F}}$.

In particular, every ccm-analytic set is K-analytic. Our purpose in this note is to characterise the class of compact complex manifolds M for which the converse holds.

Example 1.5. Consider the case when M is a projective algebraic variety V (or rather the set of $\mathbb{C}$-rational points of V). Then $\widehat{M}$ can be identified with the K-rational points of V. Moreover, by quantifier elimination and Chow's theorem, every ccm-definable family of complex-analytic subsets of M^n is a Zariski-constructible family of algebraic subsets of M^n. It follows that the ccm-analytic subsets of $\widehat{M}^n$ in this case are exactly the algebraic subsets over K. Now, in [10] Y. Peterzil and the second author prove a nonstandard version of Chow's theorem so that the K-analytic subsets of $\widehat{M}^n$ are also just the algebraic subsets over K. Hence, for M a projective variety, K-analyticity and ccm-analyticity coincide in $\widehat{M}$.

Example 1.6 (See Section 3.3 of [12]). Fix real numbers a and b lying strictly between 0 and 1, and consider the Hopf surface M obtained as the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the action of the group of automorphisms generated by $(z,w) \mapsto (az, bw)$. It is explained in [14] that there are no infinite ccm-definable families of automorphisms of M. Hence $\widehat{M}$ has no ccm-analytic automorphisms (i.e., no automorphims whose graph is a ccm-analytic subset of $\widehat{M}^2$) other than the interpretations of the complex-analytic automorphisms of M itself. On the other hand, as pointed out in [12], it is not hard to construct infinite definable (in $\mathcal{R}$) families of complex-analytic automorphisms of M. This gives rise to K-analytic subsets of $\widehat{M}^2$ that are not ccm-analytic.

The characterisation we obtain will be in terms of the following notion introduced by the first author in [7]: a compact complex manifold M is *essentially saturated* if there exists a countable collection of complex-analytic subsets of M and its cartesian powers, $\mathcal{L}_0$, such that every ccm-definable set is definable (with paramaters) in the language where there is a predicate for each complex-analytic set in $\mathcal{L}_0$.[1] The terminology comes from the fact (Proposition 2.3 of [7]) that if M is essentially saturated and $\mathcal{L}_0$ is as above, then the structure $(M, \mathcal{L}_0)$ is saturated, and definability in $(M, \mathcal{L}_0)$ and M_{ccm} agree.

Our main observation is the following result which we prove in Section 3.

Theorem 1.7. *The following are equivalent:*
(a) *M is essentially saturated.*
(b) *The K-analytic and* ccm*-analytic subsets of $\widehat{M}^n$, for all $n > 0$, coincide.*

Since compact Kähler manifolds are essentially saturated (see Section 4 of [7]), we obtain:

Corollary 1.8. *If M is a compact Kähler manifold then every K-analytic subset of $\widehat{M}^n$ is* ccm*-analytic, for all $n > 0$.* □

[1] In fact, one can ask for less: that $\mathcal{L}_0$ is a countable sublanguage of the language of the many-sorted structure $\mathcal{A}$ of Remark 1.3. That these are equivalent is definability of types again, see Proposition 2.3 of [7].

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2. The Barlet Space, Essential Saturation, and Volume

In [7] essential saturation is given a geometric characterisation in terms of certain universal families of analytic subspaces. In this section we will recall Barlet's space of compact cycles of a complex manifold, recast essential saturation as a condition on the Barlet spaces, and discuss a theorem of Lieberman's which will be used in the proof of Theorem 1.7. A more detailed exposition of this material can be found in Section 3 of [8].

For M any complex manifold, a *k-cycle of M* is a finite linear combination $Z = \sum_i n_i Z_i$ where the Z_i's are distinct k-dimensional irreducible compact complex-analytic subsets of M, and each n_i is a positive integer called the *multiplicity* of Z_i in Z. By $|Z|$ we mean the underlying set or *support* of Z, namely $\bigcup_i Z_i$. We denote the set of all k-cycles of M by $\mathcal{B}_k(M)$, and the set of all cycles of M by $\mathcal{B}(M) := \bigcup_n \mathcal{B}_k(M)$. In [1] Barlet endowed $\mathcal{B}_k(M)$ with a natural structure of a complex-analytic space whereby if for $s \in \mathcal{B}_k(M)$ we let Z_s denote the cycle respresented by s, then the set $\{(s,x) : s \in \mathcal{B}_k(M), x \in |Z_s|\}$ is a complex-analytic subset of $\mathcal{B}_k(M) \times M$. Equipped with this complex structure, $\mathcal{B}(M)$ is called the *Barlet space of M*. When M is a projective variety the Barlet space coincides with the Chow scheme. In [3] it is shown that

$$\mathcal{B}^*(M) := \{s \in \mathcal{B}(M) : Z_s \text{ is irreducible with multiplicity } 1\}$$

is a *Zariski open* subset of $\mathcal{B}(M)$: its complement in $\mathcal{B}(M)$ is a proper complex-analytic subset. An irreducible component of $\mathcal{B}(M)$ is *prime* if it has nonempty intersection with $\mathcal{B}^*(M)$.

The characterisation of essential saturation referred to above can be expressed in terms of the Barlet space as follows.

Fact 2.1 (Moosa [7])**.** Suppose M is a compact complex manifold. Then M is essentially saturated if and only if every prime component of $\mathcal{B}(M^n)$ is compact, for all $n > 0$.

Actually, in Theorem 3.3 of [7] this is done with the universal family of analytic subspaces – the *Douady space* – rather than cycles. However, the same arguments work for Barlet spaces as well.[2] Or alternatively, since there is a holomorphic map from the Douady space to the Barlet space whose image contains $\mathcal{B}^*(M)$ (see Chapitre V of [1]), compactness of the components of the Douady space implies compactness of the prime components of the Barlet space.

The main technique for determining whether a prime component of the Barlet space is compact comes from a theorem of Lieberman's which involves some hermitian geometry: Recall that if M is a complex manifold then there is a one-to-one correspondence between hermitian metrics and positive real $(1,1)$-forms on M, given by $h \longmapsto \omega := -\operatorname{Im}(h)$. Moreover, $\operatorname{Re}(h)$ is a reimannian metric on

[2] Indeed it is easier as one can replace the use of Hironaka's flattening theorem by the much simpler geometric flattening theorem of [2].

M. Wirtinger's theorem allows us to compute the volume of a compact complex submanifold of M with respect to this riemannian metric (denoted by vol_h) by integrating the appropriate exterior power of the assocated $(1,1)$-form over the submanifold: if $Z \subseteq M$ is a compact complex submanifold of complex-dimension k, then the volume of Z is given by

$$\mathrm{vol}_h(Z) = \frac{1}{k!}\int_Z \omega^k \tag{2.1}$$

where ω^k is the kth exterior power of ω (see Section 3.1 of [17]).

If Z is a possibly singular complex-analytic subset (irreducible, compact, dimension k), then equation (2.1) can serve as the *definition* of volume; it agrees with the volume of the regular locus of Z. More generally, if $Z = \sum_i n_i Z_i$ is a k-cycle of M, then the volume of Z with respect to h is $\mathrm{vol}_h(Z) := \sum_i n_i \mathrm{vol}_h(Z_i)$.

Note that taking volumes of cycles induces a function $\mathrm{vol}_h : \mathcal{B}(M) \to \mathbb{R}$ given by $\mathrm{vol}_h(s) := \mathrm{vol}_h(Z_s)$.

Fact 2.2 (Lieberman [5])**.** Suppose M is a compact complex manifold equipped with a hermitian metric h, and $X \subseteq \mathcal{B}(M)$. Then X is relatively compact in $\mathcal{B}(M)$ if and only if vol_h is bounded on X.

One application of this fact is to show that compact Kähler manifolds are essentially saturated (see Section 3 of [8]). We will use Lieberman's theorem together with the following consequence of a result of van den Dries' to prove Theorem 1.7.

Proposition 2.3. *Suppose M is a compact complex manifold (viewed as a definable space in an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}_{\mathrm{an}}$) and h is a hermitian metric on M. If $\mathcal{F}$ is a definable family of complex-analytic subsets of M then $\{\mathrm{vol}_h(F) : F \in \mathcal{F}\}$ is bounded.*

Proof. In the e-print [9] there is an argument for how this follows from the fact (Proposition 4.1 of [15]) that Hausdorff measure in definable families is bounded. For the sake of completeness we include it here.

Let $\mathcal{F} = \{F_a : a \in A\}$. Since complex-dimension is twice the real-dimension and the latter is definable in parameters by o-minimality, we may assume that there is a $k \geq 0$ such that $\dim_{\mathbb{C}}(F_a) = k$ for all $a \in A$. By compactness of M it suffices to show that for each $p \in M$ there is an open neighbourhood V of p such that $\mathrm{vol}_h(F_a \cap V)$ is bounded as a varies in A.

Consider a chart $\alpha : V \to D$ at p such that D is a definable bounded open ball in $\mathbb{C}^m$ (where $m = \dim_{\mathbb{C}}(M)$) and $\alpha(p) = 0$. Then $\mathrm{Re}(h)|_V$ induces via α a riemannian metric g on D. For each $d \in D$, let r_d be the maximum of $g_d(v,v)$ for all v in the closed unit sphere in the tangent space T_dD of D at d. By the continuity of the map $d \mapsto r_d$, for a smaller open ball D' compactly contained in D, there is an $r > 0$ such that $r_d \leq r$ for all $d \in D'$. Shrinking the chart if necessary we may assume $D' = D$. Replacing α with $\dfrac{\alpha}{\sqrt{r}}$, we can now assume that for all $d \in D$ and all $v \in T_dD$, $g_d(v,v) \leq \|v\|_d^2$ where $\|\ \|$ denotes the standard norm. In other words, the length of tangent vectors with respect to the riemannian metric induced on D from $h|_V$ is not greater than the length with respect to the standard metric. It follows that the volume of $\alpha(F_a \cap V)$ with respect to the standard riemannian metric on D is not less than $\mathrm{vol}_h(F_a \cap V)$, for any $a \in A$. But the volume of $\alpha(F_a \cap V)$ with respect to the standard Riemannian metric is equal to the $2k$-dimensional Hausdorff measure of $\alpha(F_a \cap V)$. Since $\{F_a \cap V : a \in A\}$ is a definable family, the $2k$-dimensional Hausdorff measure of $\alpha(F_a \cap V)$ is bounded as a varies in A by Proposition 4.1 of [15]. Hence, $\mathrm{vol}_h(F_a \cap V)$ is bounded as a varies in A, as desired. □

3. Proof of Theorem 1.7

Let us recall the set-up of the theorem:

- M is a compact complex manifold viewed as a definable set in a fixed o-minimal expansion $\mathcal{R}$ of $\mathbb{R}_{\mathrm{an}}$.
- $\widehat{\mathcal{R}}$ is a sufficiently saturated elementary extension of $\mathcal{R}$ with underlying real closed field R whose algebraic closure is K.
- $\widehat{M}$ is the corresponding elementary extension of M viewed as a definable set in $\widehat{\mathcal{R}}$.

We first show (a) implies (b): that if M is essentially saturated then every K-analytic subset of $\widehat{M}^n$ is ccm-analytic, for all $n > 0$. So assume that M is essentially saturated and let $S \subseteq \widehat{M}^n$ be K-analytic. Let $\mathcal{F} = \{F_a : a \in A\}$ be a definable family of complex-analyitc susbets of M^n such that $S \in \widehat{\mathcal{F}}$.

First of all, by the definable trivialisation theorem in o-minimal structures (see Chapter 9 of [16]) the connected components of $\mathrm{reg}(F_a)$, and hence the irreducible components of F_a, are uniformly definable. By taking unions it therefore suffices to consider the case when each F_a is irreducible.

Hence, for each $a \in A$ there is a unique $b \in \mathcal{B}^*(M^n)$ such that $F_a = Z_b$. Let $X \subseteq \mathcal{B}^*(M^n)$ be the set of such b's. Now fix a hermitian metric h on M^n. By Proposition 2.3 $\{\mathrm{vol}_h(F_a) : a \in A\}$ is bounded, and hence vol_h is bounded on X. It follows that X is relatively compact in $\mathcal{B}(M^n)$ (this is Fact 2.2). So X is contained in the union of finitely many prime components of $\mathcal{B}(M^n)$, say $B := \bigcup_{i=1}^{\ell} B_i$. By essential saturation, each B_i is compact (this is Fact 2.1). Hence $\mathcal{G} := \{Z_b : b \in B \cap \mathcal{B}^*(M^n)\}$ is a ccm-definable family of complex-analytic subsets of M^n (see Remark 1.3). Since $X \subseteq B \cap \mathcal{B}^*(M^n)$, every member of $\mathcal{F}$ is in $\mathcal{G}$. It follows that every member of $\widehat{\mathcal{F}}$ is in $\widehat{\mathcal{G}}$. In particular $S \in \widehat{\mathcal{G}}$, and so S is ccm-analytic as desired.

For the converse assume that X is not essentially saturated. By Fact 2.1, for some $n > 0$, there exists a non-compact prime component B_0 of $\mathcal{B}(M^n)$. Let $B_1 := \mathrm{reg}(B_0) \cap \mathcal{B}^*(M^n)$. Then B_1 is a nonempty Zariski open subset of B_0 and is a complex-analytic manifold. Using the Morrey-Grauert imbedding theorem we may assume that $B_1 \subseteq \mathbb{R}^\ell$ for some $\ell \geq 0$, and that as a real-analytic manifold it is a submanifold of $\mathbb{R}^\ell$. Since $G_0 := \{(s,x) : s \in B_0, x \in |Z_s|\}$ is a complex-analytic subset of $B_0 \times M^n$, $G_1 := \{(s,x) : s \in B_1, x \in |Z_s|\}$ is real-analytic as a subset of $\mathbb{R}^\ell \times \mathbb{R}^{mn}$. Now fix an arbitrary $a \in B_1$, choose $V \subseteq \mathbb{R}^\ell$ an open ball of radius one centred at a, and set $D := V \cap B_1$. Then $G_1 \cap (D \times \mathbb{R}^{mn})$ is a real-analytic subset of $\mathbb{R}^\ell \times \mathbb{R}^{mn}$ which is relatively compact (being contained in $D \times M^n$). Hence $G_1 \cap (D \times \mathbb{R}^{mn})$ is definable in $\mathbb{R}_{\mathrm{an}}$. It follows that $\mathcal{F} := \{Z_d : d \in D\}$ is definable in $\mathcal{R}$. The point here is that this definable family of complex-analytic subsets of M^n has as its parameter space a nonempty open subset of the non-compact prime component B_0. We will show that this forces some member of $\widehat{\mathcal{F}}$ to not be ccm-analytic. (Note that $\mathcal{F}$ itself fails to be ccm-definable because D is not ccm-definable.)

Claim 3.1. If $S \in \widehat{\mathcal{F}}$ is ccm-analytic then there exists a *compact* complex-analytic subset $C \subseteq \mathcal{B}(M^n)$ such that $S \in \widehat{\mathcal{G}}$ where $\mathcal{G} := \{Z_c : c \in C \cap \mathcal{B}^*(M^n)\}$.

Proof of claim. As mentioned in Remark 1.3, there exists an irreducible compact complex-analytic set A and a complex-analytic subset $F \subseteq A \times M^n$, such that S is a generic fibre of F over A in $\widehat{\mathcal{A}}$. Note that since $S \in \widehat{\mathcal{F}}$, every non-empty Zariski open subset of A contains an element a such that $F_a \in \mathcal{F}$. As the members of $\mathcal{F}$ are irreducible, it follows that over a nonempty Zariski open subset of A the fibres of F are irreducible. In particular, there is a unique irrreducible component of F that projects onto A and which contains the general fibres of F, and hence we may replace F by this irreducible component. Note also that after possibly shrinking the Zariski open set, these general fibres are of constant dimension. It follows by the universal property of the Barlet space (together with the geometric flattening theorem of [2]) that there exists a meromorphic map $g : A \to \mathcal{B}(M^n)$ and a nonempty Zariski open subset $U \subseteq A$ such that for all $a \in U$, $g(a) \in \mathcal{B}^*(M^n)$ and $F_a = Z_{g(a)}$ (see for example Proposition 2.20 of [4]). Let C be the closure of the image of U under g – which will be an irreducible compact complex-analytic subset of $\mathcal{B}(M^n)$ since A is compact and g is meromorphic – and let $\mathcal{G} := \{Z_c : c \in C \cap \mathcal{B}^*(M^n)\}$. Since S is a generic fibre of F over A, $S \in \widehat{\mathcal{G}}$. □

Suppose for a contradiction that every member of $\widehat{\mathcal{F}}$ is ccm-analytic. By saturation and the claim $\mathcal{F}$ must be covered by finitely many families of the form given by the claim. Taking unions this means that there exists a compact complex-analytic subset $C \subseteq \mathcal{B}(M^n)$ such that every member of $\mathcal{F}$ is contained in $\mathcal{G} := \{Z_c : c \in C \cap \mathcal{B}^*(M^n)\}$. But this means that $D \subseteq C \cap B_0$. By construction, D is a nonempty open subset of B_0 while $C \cap B_0$ is a complex-analytic subset of the irreducible complex-analytic space B_0. Hence $B_0 = C \cap B_0$, and so B_0 is an irreducible component of C. But this contradicts the fact that C is compact while B_0 is not. There must therefore exist some member of $\widehat{\mathcal{F}}$ that is not ccm-analytic. We have shown that not every K-analytic set is ccm-analytic, as desired.

This completes the proof of Theorem 1.7. □

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