

DEFINABLE NON-DIVISIBLE HENSELIAN VALUATIONS

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ABSTRACT. On a Henselian valued field (K, V) where V is the valuation ring, if the value group contains a convex p -regular subgroup which is not p -divisible, then V is definable in the language of rings. A Henselian valuation ring with a regular non-divisible value group is always 0-definable. In particular, some results of Ax's and Koenigsmann's are generalized.

In the model theoretic study of valued fields, it is natural to ask whether the valuation rings are definable in terms of addition and multiplication. Among the first and simplest observations about definable valuations is that \mathbf{Z}_p is definable over \mathbf{Q}_p in the language of rings. It is this very observation that inspired our investigation about definable Henselian valuations with non-divisible value groups in this article. While there are many examples where Henselian valuation rings with non-divisible value groups are not definable (see e.g. Corollary 4.3 and the proof of Theorem 4.4 in [3].), we provide some sufficient conditions for them to be definable. This enables us to generalize Ax's [1] (definability and decidability) and Koenigsmann's Lemma 3.6 in [8] to a broader class of non-divisible Henselian valuations.

We use the following convention: K always denotes a field; if V is a valuation ring on K , then the corresponding valuation map is denoted by $v: K \rightarrow vK = vK^\times \cup \{\infty\}$, where the value group is denoted by vK^\times ; the residue field and the maximal ideal of V are denoted by Kv and M_v respectively. The language of rings is denoted by \mathcal{L}_r . Unless mentioned otherwise, 'definable' means with parameters. Finally, $p > 1$ denotes a fixed prime number and ' $\zeta_p \in K$ ' means that K contains a primitive p -th root of unity.

Recall that an ordered abelian group is n -regular if every infinite convex subset has at least one n -divisible element, or equivalently, if every quotient over a non-zero convex subgroup is n -divisible. See e.g. [10] and [2]. A valued field is p -Henselian (our major reference would be [7]) if it has exactly one extension of the valuation to its maximal Galois p -extension.

Definition 1. *An ordered abelian group would be called p -rendible if it is p -regular but not p -divisible; it would be called rendible if it is regular (i.e. p -regular for all p) but not divisible.*

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Proposition 2. *Suppose that the valued field (K, V) satisfies one of the following conditions (in which case we call (K, V) p -pleasant):*

- V is Henselian.
- V is p -Henselian, $\zeta_p \in K$, $\text{char}(Kv) \neq p$;
- V is p -Henselian, $\text{char}(K) = p$.

If $\gamma \in vK^\times$ is not p -divisible, then the following set is definable in K in \mathcal{L}_r ,

$$(1) \quad \Phi = \{x \in K \mid pv(x) + \gamma > 0\}.$$

PROOF. We first show it for the Henselian case.

Note that $\Phi \neq \emptyset$. Pick $\epsilon \in K$ with $v(\epsilon) = \gamma$. The following formula defines Φ :

$$(2) \quad \phi(x) := \text{"}\exists y (y^p - y^{p-1} = \epsilon x^p)\text{"}.$$

For the other two cases, replace $y^p - y^{p-1}$ in $\phi(x)$ above, by $y^p - 1$ and $y^p - y$ respectively. Note that these polynomials split in K in each case and then one can use the " p -Hensel's Lemma" (see e.g. [7]). \square

Corollary 3. *Suppose that the valued field (K, V) is p -pleasant. Then vK^\times being discrete implies that V is definable in \mathcal{L}_r .*

PROOF. Let $v(\epsilon)$ be the smallest positive element in vK^\times , where $\epsilon \in K$. Then $v(\epsilon)$ is not p -divisible. So $\Phi = \{x \in K \mid v(\epsilon x^p) > 0\}$ is definable in \mathcal{L}_r . Clearly, $V \subseteq \Phi$. On the other hand if $v(x) < 0$ then $v(\epsilon x^p) < 0$ as well since $v(\epsilon)$ is the smallest positive element of vK^\times . \square

Theorem 4. *Suppose that (K, V) is p -pleasant. If vK^\times contains a p -rendible convex subgroup C , then V is definable in \mathcal{L}_r .*

PROOF. We assume that vK^\times is dense (Corollary 3 proves the discrete case).

Let $\epsilon \in K$ be such that $v(\epsilon) \in C$ is not p -divisible. Now by Proposition 2, $\Psi := \{\epsilon x^p \mid v(\epsilon x^p) > 0\}$ is definable in \mathcal{L}_r .

We show it for the case where (K, V) is Henselian, as proofs for the other two cases are similar. We define

$$(3) \quad \Omega := \{x^p - x^{p-1} \in K \mid (\exists y \in K) (\exists z \in \Psi) [z(y^p - y^{p-1}) = x^p - x^{p-1}]\},$$

Then $\Omega \subseteq M_v$ because $v(z)$ is not p -divisible and is positive for all $z \in \Psi$; $M_v \subseteq \Omega$ because of Hensel's Lemma and the fact that Ψ contains arbitrarily small positively valued elements of K thanks to the p -regularity of C . \square

The above results about definable valuations all involve one parameter whose value is not p -divisible. We can say more in the case where the value group is rendible; in particular, Koenigsmann's Lemma 3.6 in [8] follows from the following.

Theorem 5. *Suppose that (K, V) is a Henselian valued field with vK^\times rendible. Then V is 0-definable in \mathcal{L}_r .*

PROOF. If vK^\times is discrete, then the minimal element of vK^\times is not l -divisible for all prime l . Choose one l not equal to $\text{char}(Kv)$, then one applies Corollary 3 and Ax's trick in [1], to show that if vK^\times is l -rendible with v being Henselian and $l \neq \text{char}(Kv)$, then V is 0-definable. The proof is essentially the same as Ax's.

If vK^\times is dense, then for $\epsilon \in K^\times$, let $\Psi_\epsilon = \{\epsilon x^p \mid \exists y(y^p - y^{p-1}) = \epsilon x^p\}$; define $\Omega_\epsilon := \{x^p - x^{p-1} \in K \mid (\exists y \in K) (\exists z \in \Psi_\epsilon) [z(y^p - y^{p-1}) = x^p - x^{p-1}]\}$.

Then the proof of Theorem 4 implies that $M_v = \bigcap_{\epsilon \neq 0} \Omega_\epsilon$, because for all $\epsilon \neq 0$, Ψ_ϵ contains arbitrarily small positively valued elements. \square

The conditions in Theorem 4 are not necessary for a non-divisible Henselian valuation ring to be definable in \mathcal{L}_r . To see this, we recall that for a field K which is not p -closed, there is a so-called "canonical p -Henselian valuation" on it, denoted by \mathcal{O}_p , whose valuation ring is the smallest (with respect to " \subseteq ") p -Henselian valuation ring if there is no p -Henselian valuation on K with a p -closed residue field, or the biggest p -Henselian valuation on K with a p -closed residue field. See e.g. [7] for details.

We have the following fact about the canonical p -Henselian valuation \mathcal{O}_p .

Fact 6 ([7]). *Assume that K , not p -closed, satisfies that $\zeta_p \in K$ or $\text{char}(K) = p$. Then \mathcal{O}_p is 0-definable in \mathcal{L}_r , unless $p = 2$ and the residue field of \mathcal{O}_p is euclidean.*

Remark 7. *Given any ordered abelian group G , consider the ordered valued field $\mathbf{Q}((t^G))$ with the natural valuation V . By Lemma 4.3.6 of [4], every 2-Henselian valuation ring on K is convex, hence comparable to V . But V must be the \mathcal{O}_2 which is 0-definable by Fact 6. So the non-necessity of the conditions of Theorem 4 follows; e.g. when G is the lexicographical product of \mathbf{Q} and \mathbf{Z} with elements in \mathbf{Z} bigger.*

For an ordered abelian group G , recall that in Schmitt's [9], for $g \in G$, $B(g)$ is the smallest convex subgroup containing g ; for $n > 1$ an integer, $A_n(g)$ is the smallest convex subgroup C not containing g in G such that $B(g)/C$ is n -regular; $A_n(0) := \emptyset$. Then $A_n(g)$ is definable using g in G in the language of ordered abelian groups, for all n . Some cases of Theorem 4 follow from Fact 6:

Corollary 8. *Suppose that K satisfies the hypothesis of Fact 6. If V is a p -Henselian valuation on K with vK^\times containing a p -rendible convex subgroup C , then $\mathcal{O}_p \subseteq V$ and V is definable in \mathcal{L}_r .*

PROOF. If a field K' is p -closed, and K' contains ζ_p or $\text{char}(K') = p$, then vK'^\times is p -divisible for all valuations w on K' . If $V \not\subseteq \mathcal{O}_p$, then because C is p -rendible (so any convex subgroup of vK^\times is not p -divisible), the residue field $K\mathcal{O}_p$ (by our convention) of \mathcal{O}_p is not p -closed, because $\zeta_p \in K\mathcal{O}_p$ or $\text{char}(K\mathcal{O}_p) = p$. This contradicts the definition of \mathcal{O}_p .

Therefore, $\mathcal{O}_p \subseteq V$. Let $\gamma \in o_p K^\times$ be such that $\gamma + \Delta \in C$ is not p -divisible, where Δ is the convex subgroup of $o_p K^\times$ corresponding to V . Then $A_p(g) = \Delta$ in $o_p K^\times$. So V is definable using \mathcal{O}_p . \square

One can combine Corollary 8 and Theorem 4 to get that if K , being not p -closed, with V p -Henselian and either $\zeta_p \in K$ or $\text{char}(K) = p$, then V is definable if vK^\times contains a p -rendible convex subgroup.

Proposition 9. *Given an ordered abelian group G , not p -divisible, with a non-dense order type (henceforth called the principal p -regular rank) of the collection of $A_p(g)$ in G for $g \geq 0$, there is some convex subgroup C of G such that G/C contains a convex p -rendible subgroup.*

PROOF. It is enough to show that $B(g)/A_p(g)$ is not p -divisible for some g . By assumption, there are some $g_1 < g_2$ such that $A_p(g_1) \subsetneq A_p(g_2)$ and for all $g_1 < g < A_p(g_2)$, $A_p(g) = A_p(g_1)$. We may assume that $g_1 \neq 0$. Because $B(g_2)/A_p(g_1)$ is not p -regular,

$$B(g_2)/A_p(g_2) = \frac{B(g_2)/A_p(g_1)}{A_p(g_2)/A_p(g_1)}$$

implies that either $B(g_2)/A_p(g_2)$ is not p -divisible, or $B(g)/A_p(g) = B(g)/A_p(g_1)$ is not p -divisible for some $g_1 < g \in A_p(g_2)$. \square

Without the assumption of the non-denseness of the principal p -regular rank of G , the conclusion of Proposition 9 could be false. For example, let $I = \mathbf{Q}$ be the index set, and let $H := \prod'_{i \in I} G_i$ be the subgroup of $\prod_{i \in I} G_i$ consisting of those elements with well-ordered support, where $G_i = \mathbf{Q}$. H is linearly ordered lexicographically. For $a, b \in I$ with $a < b$, let $s(a, b) = \{b - (b - a)/n\}_{n=1}^\infty$. Then the subgroup of H generated by $\bigoplus_{i \in I} G_i$ and the characteristic functions of $s(a, b)$ (for all $a < b$), does not have that property.

It is then clear that if V is p -pleasant on K with vK^\times not p -divisible with a non-dense principal p -regular rank, then some valuation ring containing V is definable. This is a result similar to that about defining the valuation up to equivalence of topologies in Koenigsmann [6].

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