FORKING GEOMETRY ON THEORIES WITH AN INDEPENDENT PREDICATE

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ABSTRACT. We prove that a simple geometric theory of SU-rank 1 is n-ample if and only if the associated theory equipped with an predicate for an independent dense subset is n-ample for n at least 2.

1. INTRODUCTION

The notion of n-ampleness, introduced by Pillay in [5], roughly measures the complexity of the forking geometry: the Hrushovski construction *ab-initio* exhibits that, in strongly minimal theories, forking geometry can be more complicated than the geometry of vector spaces (1-based theories) and yet less than algebraic geometry (theories interpreting a field). The property that holds in Hrushovski's example is called CM-triviality and it fits in this hierarchy of n-amples. In fact, (see [5] and [6]) a theory T is 1-based if and only if is not 1-ample and is CM-trivial if and only if is not 2-ample. Furthermore if T interprets a field, then T is n-ample for all n. However, the converse is not true: Evans [4] constructs a 1-based theory with a reduct which is n-ample for every n but does not interpret an infinite group.

It is a major problem to find theories of finite rank that are not CM-trivial but do not interpret a field. Baudisch and Pillay ([1]) obtained a 2-ample theory which is of infinite rank not interpreting any infinite group.

On the other hand, Berenstein and Vassiliev in [2] exhibit a 1-based (not 1-ample) theory T such that T^{ind} is not 1-based, where T^{ind} stands for the theory of the pair (M, H), where $M \models T$ and H is an independent dense subset of M. We prove in this paper that in this case T^{ind} is CM-trivial. Moreover,we prove that for $n \geq 2$, T is not n-ample if and only if T^{ind} is not n-ample.

2. INDEPENDENT PREDICATES IN GEOMETRIC THEORIES

In this section we write down the principal definitions and results on geometric theories with an independent predicate that we will use in this paper. All proofs can be found in [2] and [3].

Definition 2.1. A complete theory T is geometric if eliminates $\exists^\infty$ and algebraic closure satisfies the exchange property in every model of T.

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Let T be a complete geometric theory in a language L and let $L_H = L \cup \{H\}$ where H is a new unary predicate. T^{ind} is the L_H theory extending T together with the axioms:

(1) for all L-formulas $\varphi(x, \bar{y})$

$$\forall \bar{y}(\varphi(x, \bar{y})\text{nonalgebraic} \rightarrow \exists x \in H\varphi(x, \bar{y})) \text{ (Density property)}$$

(2) for all L-formulas $\varphi(x, \bar{y})$, for all $n \in \omega$ and for all $\psi(x, \bar{y}, \bar{z})$

$$\forall \bar{y}(\varphi(x, \bar{y})\text{nonalgebraic} \wedge \forall \bar{y}\bar{z}\exists^{\leq n} x\psi(x, \bar{y}, \bar{z})$$
$$\rightarrow (\exists x \notin H \forall \bar{z} \in H(\varphi(x, \bar{y}) \wedge \neg\psi(x, \bar{y}, \bar{z}))) \text{ (Extension property)}$$

In these axioms, "non algebraicity" can be expressed in a first order way due to the elimination of $\exists^\infty$.

From now on by $acl()$ and $\mathop{\smile\!\!\!\!\!\!|}$ we mean algebraic closure and algebraic independence in the sense of T.

Proposition 2.1. *(Berenstein, Vassiliev [2]) If T is a geometric theory and*

$$(M, H) \models T^{ind}$$

is $\aleph_0$-saturated, then:

(1) If $A \subset M$ is finite dimensional and $q \in S_n(A)$ has dimension n, then there is $\bar{a} \in H(M)^n$ such that $\bar{a} \models q$ (Generalized density property).

(2) If $A \subset M$ is finite dimensional and $q \in S_n(A)$ then there is $\bar{a} \models q$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A H$ (Generalized extension property).

Definition 2.2. An *H-structure* is an $\aleph_0$-saturated model of T^{ind} .

Definition 2.3. Let (M, H) be an H-structure and c a tuple in M. We denote by $\mathrm{HB}(c)$, the *H-basis* of c, the smallest tuple $h \subseteq H$ such that $c \mathop{\smile\!\!\!\!\!\!|}_h H$.

Also for $A \subseteq M$, A algebraically closed (in the sense of T^{ind}), the *H-basis of c relative to A*, denoted by $\mathrm{HB}(c/A)$, will stand for the smallest tuple $h_A \in H$ such that $c \mathop{\smile\!\!\!\!\!\!|}_{h_A A} H$.

Proposition 2.2. *For every c, the basis $HB(c)$ exists.*

Proof. Let h and h' be tuples of H such that $c \mathop{\smile\!\!\!\!\!\!|}_h H$ and $c \mathop{\smile\!\!\!\!\!\!|}_{h'} H$. It suffices to prove that if $h'' = h \cap h'$ then $c \mathop{\smile\!\!\!\!\!\!|}_{h''} H$.

We can write c as $c_1 c_2$ where c_1 is independent over H and $c_2 \subseteq \mathrm{acl}(c_1 H)$. So by definition of h and h' we know that $c_2 \subseteq \mathrm{acl}(c_1 h)$ and $c_2 \subseteq \mathrm{acl}(c_1 h')$. If $c_2 \nsubseteq \mathrm{acl}(c_1 h'')$ then, by exchange property, there is an element in g in $h \setminus h'$ (or in $h' \setminus h$), such that $g \in \mathrm{acl}(c_1 h')$. But c_1 was chosen to be independent from H so actually $g \in \mathrm{acl}(h')$. This yields a contradiction as H is an independent subset. $\square$

Proposition 2.3. *Let (M, H) an H-structure, let c and A be subsets of M and assume that $A = \mathrm{acl}(A)$ and $\mathrm{HB}(A) \subseteq A$, then $\mathrm{HB}(c/A)$ exists.*

Proof. Again, let h and h' be minimal such that $c \underset{hA}{\perp} H$ and $c \underset{h'A}{\perp} H$. In particular we have that $hh' \cap A = \emptyset$.

Write c as c_1c_2 where c_1 is independent over AH and $c_2 \subseteq \operatorname{acl}(c_1AH)$. Then $c_2 \subseteq \operatorname{acl}(c_1Ah)$ and $c_2 \subseteq \operatorname{acl}(c_1Ah')$. Let $h'' = h \cap h'$, if $c_2 \nsubseteq \operatorname{acl}(c_1Ah'')$ then by exchange there is an element $g \in h \setminus h'$ (or viceversa) such that $g \in \operatorname{acl}(c_1Ah')$.

Claim: we have that $g \notin \operatorname{acl}(Ah')$.

If not, as $g \notin h'$ then by exchange there is an element a' and a subset A' of A such that $a \notin \operatorname{acl}(A')$ and $a' \in \operatorname{acl}(A'gh')$, then some (non empty) subset of gh' must be contained in $\mathrm{HB}(A)$, $\mathrm{HB}(A) \subseteq A$ and $h'g \cap A \subseteq h'h \cap A = \emptyset$. Contradiction.

Therefore, as $g \in \operatorname{acl}(c_1Ah') \setminus \operatorname{acl}(Ah')$, c_1 is not independent over AH.

□

We defined the relative H-basis over algebraic closed sets A with $HB(A) \subseteq A$. The next theorem shows that actually these hypothesis impose that the set over which the H-base is defined must be algebraically closed in T^{ind}.

Theorem 2.1. *(Berenstein, Vassiliev [2]) If $(M, H(M))$ is an H-structure and A is a subset of M then the algebraic closure of A in the sense of L_H (that we will denote by $\operatorname{acl}_H(A)$) is the algebraic closure in the sense of L of $A \cup \mathrm{HB}(A)$.*

From now on, by $HB(A/B)$ we mean $HB(A/\operatorname{acl}_H(B))$.

The next theorem provides a characterization of the canonical bases in T^{ind} in terms of H-basis and algebraic closure.

Theorem 2.2. *(Berenstein, Vassiliev [2]) Let T an SU-rank 1 geometric theory and (M, H) be an H-structure (sufficiently saturated), a a tuple of M and $B \subset M$ acl_H-closed. Then the canonical base $\operatorname{cb}_H(a/B)$ of $stp_H(a/B)$, is interalgebraic (in the sense of L_H) with $\operatorname{cb}(a\,\mathrm{HB}(a/B)/B)$.*

Example 2.1. *Let V a vector space over $\mathbb{Q}$ such that $|V| > \aleph_0$ and let $H = \{h_0, h_1, ...\}$ be a countable independent subset of V. Then (V, H) is an H-structure.*

Moreover, if t is a vector independent of H and $t_0 = t + v_0$ then $\operatorname{cb}_H(t/t_0)$ is interalgebraic with $\operatorname{cb}(tv_0/t_0) = t_0$. So $t \not\perp t_0$, but $\operatorname{acl}_H(t) \cap \operatorname{acl}_H(t_0) = \emptyset$ hence $Th(V, H)$ is not 1-based.

This example shows that 1-basedness is not preserved in T^{ind}.

We will see in the next section that if T is a SU-rank 1 geometric theory, then T^{ind} is 1-based iff T is trivial.

3. Ampleness

Definition 3.1. A simple theory T is *not n-ample* if for every sets $a_0, ..., a_n$ of M^{eq} which satisfy the next conditions:

For all $1 \leq i \leq n - 1$.

(1) $a_{i+1} \underset{a_i}{\perp} a_{i-1}...a_0$,

(2) $\mathrm{acl}^{eq}(a_0...a_{i-1}a_{i+1}) \cap \mathrm{acl}^{eq}(a_0...a_{i-1}a_i) = \mathrm{acl}^{eq}(a_0...a_{i-1})$.

We have $a_n \underset{\mathrm{acl}^{eq}(a_1)\cap\mathrm{acl}^{eq}(a_0)}{\downarrow} a_0$.

Here we use the definition given by Evans in [4]. This definition seems more natural that the one given by Pillay in [5]. Nevertheless all the results that we present here work for both definitions.

From now on we will assume that T is a SU-rank 1 geometric theory eliminating imaginaries. By Theorem 1.2. canonical basis are interalgebraic with a tuple of elements, so T^{ind} has geometric elimination of imaginaries. Then, for the definition of n-ampleness, it suffices to work with real elements in an H-structure (M, H).

Proposition 3.1. *The H-basis are transitive in the sense that*

$$\mathrm{HB}(c/B) \cup \mathrm{HB}(B) = \mathrm{HB}(cB).$$

In particular, if $A \subseteq B$ and $\mathrm{acl}_H(cA) \cap \mathrm{acl}_H(B) = \mathrm{acl}_H(A)$ then

$$\mathrm{HB}(c/A) \subseteq \mathrm{HB}(c/B).$$

Proof. It's clear that $\mathrm{HB}(c/B) \cup \mathrm{HB}(B) \subseteq \mathrm{HB}(cB)$. On the other hand,

$$c \underset{B\cup\mathrm{HB}(c/B)}{\downarrow} H$$

and

$$B \underset{\mathrm{HB}(c/B)\,\mathrm{HB}(B)}{\downarrow} H$$

then

$$cB \underset{\mathrm{HB}(c/B)\,\mathrm{HB}(B)}{\downarrow} H.$$

□

Lemma 3.1. *If T is trivial, then for every set A, $\mathrm{acl}(A) = \mathrm{acl}_H(A)$.*

Proof. Let $h = \mathrm{acl}(A) \cap H$. If $x \in \mathrm{acl}(A) \cap \mathrm{acl}(H)$, then by triviality $x \in \mathrm{acl}(h')$ for some $h' \in H \cap \mathrm{acl}(A) = h$. Hence $\mathrm{HB}(A) \subseteq h \subseteq acl(A)$. □

The previous lemma and Proposition 2.1 implies that, in a trivial theory, for every set A and every $B = acl_H(B)$ we have $\mathrm{HB}(A/B) \subseteq \mathrm{acl}(A)$. Because

$$\mathrm{HB}(A/B) \subseteq \mathrm{acl}_H(AB) \setminus B = \mathrm{acl}(AB) \setminus B \subseteq \mathrm{acl}(A).$$

Proposition 3.2. *T is trivial iff T^{ind} is 1-based.*

Proof. If T is trivial then for every a and b with $b = acl_H(b)$ we have

$$h = \mathrm{HB}(a/b) \subseteq \mathrm{acl}(a),$$

so $\mathrm{acl}_H(\mathrm{cb}_H(a/b)) = \mathrm{acl}_H(\mathrm{cb}(ah/b)) \subseteq \mathrm{acl}_H(a)$.

Suppose now T^{ind} is 1-based and assume that T is not trivial, then there exists a tuple a and elements b and h such that $b \in \mathrm{acl}(ah)$ and $b \notin \mathrm{acl}(a) \cup \mathrm{acl}(h)$. We can assume that a is an independent tuple minimal with this property and therefore that $a \downarrow H$ (by the Generalized Extension Property). Moreover, as $tp(h/a)$ is not algebraic, we can assume that h belongs to H by density. It is clear that $h = \mathrm{HB}(b/a)$ so $\mathrm{cb}_H(b/a)$ is interalgebraic (in T^H) with $\mathrm{cb}(bh/a)$. Now, the theory

T^{ind} is 1-based, hence $\mathrm{acl}_H(\mathrm{cb}_H(b/a)) = \mathrm{acl}_H(b) \cap \mathrm{acl}_H(a)$. On the other hand $\mathrm{acl}_H(a) = \mathrm{acl}(a)$ as $a \mathop{\smile\!\!\!\!\!\!|}\, H$ and also $\mathrm{acl}_H(b) = acl(b)$ because $a \mathop{\smile\!\!\!\!\!\!|}_h H$ and $b \in \mathrm{acl}(ah)$, then $b \mathop{\smile\!\!\!\!\!\!|}_h H$. But $b \notin \mathrm{acl}(h)$ so $\mathrm{HB}(b) = \emptyset$.

However, minimality of a yields $\mathrm{acl}(\mathrm{cb}(bh/a)) = \mathrm{acl}(a)$, hence $\mathrm{acl}(a) \subseteq \mathrm{acl}(b)$ and $h \in \mathrm{acl}(ab) \subseteq \mathrm{acl}(b)$. This is a contradiction. □

Theorem 3.1. *For $n \geq 2$ T is n-ample iff T^{ind} is n-ample.*

Proof. ($\Rightarrow$) Assume T is n-ample, then there are sets $a_0, ..., a_n$ such that:
(1) $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_{i-1}...a_0$,
(2) $\mathrm{acl}(a_0...a_{i-1}a_{i+1}) \cap \mathrm{acl}(a_0...a_{i-1}a_i) = \mathrm{acl}(a_0...a_{i-1})$.
(3) $a_n \quad \not\mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(a_1)\cap\mathrm{acl}(a_0)} \quad a_0$

By the generalized extension property, there are $a'_0...a'_n$ such that $tp(a'_0...a'_n) = tp(a_0...a_n)$ and $a'_0...a'_n \mathop{\smile\!\!\!\!\!\!|}\, H$.

As the H-bases of any subset of $\{a'_0..., a'_n\}$ are empty, algebraic closure in T^{ind} is the same as in T. So condition (2) holds in T^{ind}.

By the characterization of the canonical bases (since H-bases are empty), condition (1) holds also in T^{ind}. But if

$$a'_n \quad \mathop{\smile\!\!\!\!\!\!|}^{H}_{\mathrm{acl}_H(a'_1)\cap\mathrm{acl}_H(a'_0)} \quad a'_0$$

then

$$a'_n \quad \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(a'_1)\cap\mathrm{acl}(a'_0)} \quad a'_0$$

This is a contradiction.

($\Leftarrow$)Assume T is not n-ample. Let $a_0, ..., a_n$ be such that for all $1 \leq i \leq n-1$
(1) $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}^{H}_{a_i} a_{i-1}...a_0$
(2) $\mathrm{acl}_H(a_0...a_{i-1}a_{i+1}) \cap \mathrm{acl}_H(a_0...a_{i-1}a_i) = \mathrm{acl}_H(a_0...a_{i-1})$.
We may assume that $a_i = acl_H(a_i)$ for every $i \leq n$.

Claim 1. In these conditions we have the following chain:

$$\mathrm{HB}(a_n/a_0) \subseteq \mathrm{HB}(a_n/a_0a_1) \subseteq ... \subseteq \mathrm{HB}(a_n/a_0...a_{n-1}).$$

Because

$$a_n \mathop{\smile\!\!\!\!\!\!|}^{H}_{a_{i+1}} a_i...a_0$$

therefore

$$\mathrm{acl}_H(a_na_{i-1}...a_0) \cap \mathrm{acl}_H(a_i, ..., a_0) \subseteq \mathrm{acl}_H(a_{i+1}a_{i-1}...a_0),$$

hence, by (2),

$$\mathrm{acl}_H(a_na_{i-1}...a_0) \cap \mathrm{acl}_H(a_ia_{i-1}...a_0) = \mathrm{acl}_H(a_{i-1}...a_0).$$

The conclusion follows from Proposition[2.1]. Note that this only make sense if $n \geq 2$.

Let's call $h = \mathrm{HB}(a_n/a_0)$ and $h' = \mathrm{HB}(a_n/a_0,...,a_{n-1})$. Hence $h \subseteq h'$ by the previous claim. As the canonical basis $\mathrm{cb}_H(a_n/acl_H(a_0...a_{n-1}))$ is interalgebraic (in T^H) with $\mathrm{cb}(a_nh'/acl_H(a_0...a_{n-1}))$, then

$$a_nh \underset{a_{n-1}}{\downarrow} \mathrm{acl}_H(a_{n-1}a_{n-2}...a_0).$$

Define recursively tuples a_i', b_i for $0 \leq i \leq n-1$ in the following way:
For the case $i = 0$ let $a_0' = \emptyset$ and $b_0 = a_0$.
For $i > 0$ let $a_i' \subseteq \mathrm{acl}_H(a_i, b_{i-1},..,b_0)$ be a maximal tuple independent over $\mathrm{acl}(a_ib_{i-1},...,b_0)$ (in the sense of T), and $b_i = \mathrm{acl}(a_ia_i')$. In particular we have that

$$\mathrm{acl}(b_i,...,b_0) = \mathrm{acl}_H(a_i,...,a_0).$$

Note that we can take $a_i' = \mathrm{HB}(a_i, b_{i-1},...,b_0)$.
Define also b_n as a_nh'.

Claim 2. For $i \leq n-1$ we have $b_i \underset{b_{i-1}}{\downarrow} b_{i-2}...b_0$:

By definition $a_i' \downarrow a_ib_{i-1}..b_0$, hence $a_i' \underset{a_i}{\downarrow} b_{i-1}...b_0$. On the other hand, as $a_i \underset{a_{i-1}}{\downarrow^H} a_{i-1}...a_0$ (by (1)) and $a_{i-1} \subset b_{i-1} \subseteq acl_H(a_{i-1},...,a_0)$, then $a_i \underset{b_{i-1}}{\downarrow^H} b_{i-1}...b_0$. This implies by transitivity that $b_i \underset{b_{i-1}}{\downarrow} b_{i-2}...b_0$ for $i \leq n-1$.

Note also that $b_n \underset{b_{n-1}}{\downarrow} b_{n-2}...b_0$ by definition of h' and the characterization of canonical bases in T^{ind}.

Claim 3. For $i \leq n-1$ we have $\mathrm{acl}(b_{i+1}b_{i-1}...b_0) \cap \mathrm{acl}(b_ib_{i-1}...b_0) = \mathrm{acl}(b_{i-1}...b_0)$.
Because, for every $i \leq n$,

$$\mathrm{acl}(b_i...b_0) = \mathrm{acl}_H(a_i...a_0),$$

then by (2)

$$\mathrm{acl}(a_{i+1}b_{i-1}...b_0) \cap \mathrm{acl}(b_ib_{i-1}...b_0) = \mathrm{acl}(b_{i-1}...b_0).$$

So, if $\mathrm{acl}(a_{i+1}b_{i-1}...b_0) \cap \mathrm{acl}(b_ib_{i-1}...b_0) \subsetneq \mathrm{acl}(b_{i+1}b_{i-1}...b_0) \cap (b_ib_{i-1}...b_0)$, then by exchange there exists $a \in \mathrm{acl}(a_{i+1}') \cap \mathrm{acl}(a_{i+1}b_i...b_0)$. Contradiction.

Therefore, Claims 2, 3 and non n-ampleness of T imply that $b_n \underset{b_1 \cap b_0}{\downarrow} b_0$, moreover we get that $a_nh \underset{a_1 \cap a_0}{\downarrow} a_0$ because $h \subseteq h'$ and $b_1 \cap b_0 = a_1 \cap b_0$.

Hence, again by definition of h and characterization of canonical bases, $a_n \underset{a_1 \cap a_0}{\downarrow^H} a_0$.

□

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