# GEOMETRIC STRUCTURES WITH A DENSE INDEPENDENT SUBSET

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Abstract. We generalize the work of [13] on expansions of o-minimal structures with dense independent subsets, to the setting of geometric structures. We introduce the notion of an $H$-structure of a geometric theory $T$, show that $H$-structures exist and are elementarily equivalent, and establish some basic properties of the resulting complete theory $T^{ind}$, including quantifier elimination down to "$H$-bounded" formulas, and a description of definable sets and algebraic closure. We show that if $T$ is strongly minimal, supersimple of SU-rank 1, or superrosy of thorn rank 1, then $T^{ind}$ is $\omega$-stable, supersimple, and superrosy, respectively, and its U-/SU-/thorn rank is either 1 (if $T$ is trivial) or $\omega$ (if $T$ is non-trivial). In the supersimple SU-rank 1 case, we obtain a description of forking and canonical bases in $T^{ind}$. We also show that if $T$ is (strongly) dependent, then so is $T^{ind}$, and if $T$ is non-trivial of finite dp-rank, then $T^{ind}$ has dp-rank greater than $n$ for every $n < \omega$, but bounded by $\omega$. In the stable case, we also partially solve the question of whether any group definable in $T^{ind}$ comes from a group definable in $T$.

## 1. Introduction

We say a theory $T$ is *geometric* if for any model $M \models T$ the algebraic closure satisfies the exchange property and $T$ eliminates the quantifier $\exists^\infty$ (see [20, Def. 2.1], [17]). There are many examples of geometric theories, among them dense o-minimal theories, strongly minimal theories, SU-rank 1 theories, the p-adics in a single sort, etc.

Expansions of geometric theories with a unary predicate have been studied extensively. There are expansions where the underlying model $M$ is an algebraically closed or a real closed field and the predicate is interpreted as a multiplicative subgroup, for example to study groups with the Mann property [15]. This expansion created a nice framework for studying groups satisfying the Mordell-Lang property inside a fixed field. In the same way, the work on rational points of eliptic curves from [18] gives connections with number theory.

Another such expansion corresponds to lovely pairs [6, 3]. Let $\mathcal{L}$ be the language of $T$, let $M \models T$ and let $H$ be a new unary predicate that does not belong to $\mathcal{L}$. For $M \models T$, we say that $(M, H(M))$ is a lovely pair if $H(M)$ is an elementary substructure of $M$, the predicate satisfies the *density property* (for any infinite $\mathcal{L}$-formula $\varphi(x, \vec{b})$ with parameters in $M$, $\varphi(H(M)) \neq \emptyset$) and $M$ satisfies the *extension property* over $H(M)$ (for any infinite $\mathcal{L}$-formula $\varphi(x)$ with parameters in $M$, $(\varphi(M) \setminus \mathrm{acl}(H(M)\vec{b})) \neq \emptyset$). Lovely pairs are a tool for understanding the properties of the

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underlying geometry such as linearity. The structure imposed on the predicate, i.e. being an elementary substructure, a subgroup, allows one to use the expansion to get an insight into different properties of $T$ or structures living inside a model of $T$.

Generic (random) predicate expansions have also been studied extensively, e.g. by Chatzidakis and Pillay in [11]. In the strongly minimal case they provide "natural" examples of unstable supersimple structures of SU-rank 1. Baldwin and Benedikt [1] have also considered expansions by indiscernible sequences.

In this paper we will explore an expansion, introduced in the o-minimal case in [13], which in some sense is dual to the lovely pairs expansion. We will assume that $H(M)$ is a collection of algebraically independent elements satisfying the density and extension properties. The construction is a dual in the sense that instead of assuming the predicate to be an elementary substructure, we assume it is a collection of "geometrically unrelated" (algebraically independent) elements. We call such an expansion $(M, H)$ an *$H$-structure*, and we write $T^{ind}$ for its theory.

Examples of $T^{ind}$ include the theory of a vector space with a distinguished basis and the theory of a real closed field with a distinguished dense transcendence basis.

Some properties of this expansion are very similar to those of lovely pairs. For example, we show in Section 2 that saturated models of $T^{ind}$ are again $H$-structures. We show in Section 3 that the definable subsets of $H(M)$ are just intersections of $\mathcal{L}$-definable formulas with $H$. We also show that the definable subsets of $(M, H(M))$ come as boolean combinations of $\mathcal{L}$-formulas enlarged by existential quantifiers over $H$. While the question of elimination of $\exists^\infty$ in $T^{ind}$ remains open, we show that it holds for formulas where parameters are assumed to be in $H(M)$. As in the pairs setting, one of the central notions in the study of $H$-structures is that of the large and small set. What is different from the case of pairs is that in $H$-structures we also have the notion of "$H$-basis" of a tuple (over $\emptyset$ or another set). On one hand, this notion allows one to "coordinatize" the structure by elements of $H$ and elements orthogonal to $H$, while on the other hand it generates a variety of new definable functions from definable sets in $(M, H)$ to $H(M)$.

In Section 4 we explore some additional topics related to $H$-structures motivated by the analogies with the pair expansions. In subsection 4.1 we compare $H$-structures with lovely pairs and show how to build a lovely pair out of an $H$-structure. Subsection 4.2 iterates the construction of $H$-structures to tuples, following similar ideas of Poizat on beautiful pairs [24] and of Fornasiero for closure relations [16]. In subsection 4.3, we show elimination of $\exists y \in H$ for the expansion of $H(M)$ by externally definable sets.

In Section 5 we show that if $T$ is strongly minimal (respectively $T$ has SU-rank 1, thorn-rank 1), then $MR(T^{ind}) \leq \omega$ (respectively SU-rank$(T^{ind}) \leq \omega$, thorn-rank$(T^{ind}) \leq \omega$). We obtain a description of forking and canonical bases in $T^{ind}$ when $T$ is supersimple of SU-rank 1. We also observe a (somewhat surprising) fact that that one-basedness is not preserved when passing to $T^{ind}$.

In the lovely pair case, the rank of the expansion captured the geometric complexity of the base theory (along the lines of the trivial/linear/non-linear trichotomy). Similar, but much less refined, connection takes place in the case of $H$-structures: non-triviality of the base theory guarantees that the expansion will have the maximal rank $\omega$.

Finally in this section we show that if $T$ is (strongly) dependent, $T^{ind}$ is also (strongly) dependent. As before there is a connection between the triviality of $T$

and the dp-rank. When $T$ is trivial and $T$ has dp-rank $n$, then so does $T^{ind}$, if $T$ is not trivial and $T$ has dp-rank $n$, then $T^{ind}$ has dp-rank greater than or equal to $k$ for all $k$.

In Section 6 we study groups definable in a $H$-structure $(M,H)$. Since the geometry on $H$ is trivial, we expected that adding $H$ should not introduce new definable groups. We managed to show this claim only partially. When $T$ is stable we show that every connected group definable in $(M,H)$ is definably isomorphic to a group interpretable in $M$.

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## 2. $H$-structures: definition and first properties

Let $T$ be a complete geometric theory in a language $\mathcal{L}$. Thus, in any model $M \models T$, the algebraic closure satisfies the Exchange Property and $T$ eliminates the quantifier $\exists^\infty$. Let $H$ be a new unary predicate and let $\mathcal{L}_H = \mathcal{L} \cup \{H\}$. Let $T'$ be the $\mathcal{L}_H$-theory of all structures $(M,H)$, where $M \models T$ and $H(M)$ is an $\mathcal{L}$-algebraically independent subset of $M$. Note that saying that $H(M)$ is independent is a first order property, it is simply the conjunctions of formulas of the form $\neg\varphi(x_1,\ldots,x_n)$, where $dim(\varphi(x_1,\ldots,x_n)) < n$.

**Notation 2.1.** *Let $(M,H(M)) \models T'$ and let $A \subset M$. We write $H(A)$ for $H(M) \cap A$.*

**Notation 2.2.** *Throughout this paper* independence *(and the corresponding notation $\downarrow$) means* acl-*independence, where* acl *stands for the algebraic closure in the sense of $\mathcal{L}$. We write* $\mathrm{tp}(\vec{a})$ *for the $\mathcal{L}$-type of $a$ and* dcl, acl *for the definable closure and the algebraic closure in the language $\mathcal{L}$. Similarly we write* $\mathrm{dcl}_H, \mathrm{acl}_H, \mathrm{tp}_H$ *for the definable closure, the algebraic closure and the type in the language $\mathcal{L}_H$. For $A \subset B$ sets and $q \in S_n(B)$, we say that $q$ is* free *over $A$ or that $q$ is a free extension of $q \restriction_A$ if for any (all) $\vec{c} \models q$, $\vec{c}$ is independent from $B$ over $A$.*

**Definition 2.3.** We say that $(M,H(M))$ is an *$H$-structure* if

1. $(M,H(M)) \models T'$
2. (Density/coheir property) If $A \subset M$ is finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in H(M)$ such that $a \models q$.
3. (Extension property) If $A \subset M$ is finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in M$, $a \models q$ and $a \notin \mathrm{acl}(A \cup H(M))$.

**Lemma 2.4.** *Let $(M,H(M)) \models T'$. Then $(M,H(M))$ is an $H$-structure if and only if:*

- (2') *(Generalized density/coheir property) If $A \subset M$ is finite dimensional and $q \in S_n(A)$ has dimension $n$, then there is $\vec{a} \in H(M)^n$ such that $\vec{a} \models q$.*
- (3') *(Generalized extension property) If $A \subset M$ is finite dimensional and $q \in S_n(A)$, then there is $\vec{a} \in M^n$ realizing $q$ such that $\mathrm{tp}(\vec{a}/A \cup H(M))$ is free over $A$.*

*Proof.* We prove (2') and leave (3') to the reader. Let $\vec{b} \models q$, we may write $\vec{b} = (b_1,\ldots,b_n)$. Since $(M,H(M))$ is an $H$-structure, applying $n$ times the density property we can find $a_1,\ldots,a_n \in H(M)$ such that

$$\mathrm{tp}(a_1,\ldots,a_n/\,\mathrm{acl}(A)) = \mathrm{tp}(b_1,\ldots,b_n/\,\mathrm{acl}(A)).$$

□

Note that if $(M, H(M))$ is an $H$-structure, the extension property implies that $M$ is $\aleph_0$-saturated.

**Remark 2.5.** *Assume now that $T$ is a geometric theory expanding DLO and that $(M, H(M))$ is an $H$-structure. Let $a, b \in M$ be such that $a < b$; then the partial type $a < x < b$ is non-algebraic and by the density property it is realized in $H(M)$. Thus, the density property implies that $H(M)$ is dense in $M$. The density property that we use in this paper can be traced back to Macintyre* [21]*, it also appears under the name of coheir property in* [2].

**Definition 2.6.** Let $A$ be a subset of an $H$-structure $(M, H(M))$. We say that $A$ is *$H$-independent* if $A$ is independent from $H(M)$ over $H(A)$.

**Lemma 2.7.** *Any model $M$ of $T$ with a distinguished independent subset $H(M)$ can be embedded in a model of $T'$ in an $H$-independent way.*

*Proof.* Given any model $M$ with a distinguished independent subset $H(M)$, we can always find an elementary extension $N$ of $M$ and a set $H(N)$ extending $H(M)$ such that for every non-algebraic 1-type $p(x, \operatorname{acl}(\vec{m}))$, where $\vec{m} \in M$, there are $d \in N$ and $b \in H(N)$ such that both $b$ and $d$ realize $p(x, \operatorname{acl}(\vec{m}))$ and $d \notin \operatorname{acl}(M, H(N))$. Now apply a chain argument. □

In particular, for a geometric theory $T$, $H$-structures exist.

**Lemma 2.8.** *Let $(M, H)$ and $(N, H)$ be sufficiently saturated models of $T'$, $\vec{a} \in M$ and $\vec{a}' \in N$ $H$-independent tuples such that $\operatorname{tp}(\vec{a}, H(\vec{a})) = \operatorname{tp}(\vec{a}', H(\vec{a}'))$. Then $tp_H(\vec{a}) = \operatorname{tp}_H(\vec{a}')$.*

*Proof.* Let $\vec{a} = \vec{a}_0\vec{a}_1\vec{h}$, where $\vec{a}_0$ is independent over $H$, $\vec{h} \in H$ and $\vec{a}_1 \in \operatorname{acl}(\vec{a}_0\vec{h})$. Similarly write $\vec{a}' = \vec{a}_0'\vec{a}_1'\vec{h}'$.

To prove the Lemma we show that the partial isomorphism that sends $\vec{a}$ to $\vec{a}'$ can be extended, so it suffices to show that for any $b \in M$ there are $\vec{h}_1 \in H(M)$, $\vec{h}_1' \in H(N)$ and $b' \in N$ such that $\vec{a}\vec{h}_1 b$ and $\vec{a}'\vec{h}_1' b'$ are each $H$-independent, $\operatorname{tp}(\vec{a}_0\vec{a}_1\vec{h}\vec{h}_1 b) = \operatorname{tp}(\vec{a}_0'\vec{a}_1'\vec{h}'\vec{h}_1' b')$, and $b \in H(M)$ iff $b' \in H(N)$.

Case 1: $b \in \operatorname{acl}(\vec{a})$. By $H$-independence, either $b \in \vec{h}$ or $b \notin H(M)$. Let $b' \in \operatorname{acl}(\vec{a}')$ be such that $\operatorname{tp}(b'\vec{a}') = \operatorname{tp}(b\vec{a})$. Clearly, $b \in H(M)$ iff $b' \in H(N)$. Here we can take $\vec{h}_1$ and $\vec{h}_1'$ to be empty.

Case 2: $b \in H$ and is non algebraic over $\vec{a}$. By the density property, we can find $b' \in H(N)$ such that $\operatorname{tp}(b'\vec{a}') = \operatorname{tp}(b\vec{a})$. Here again we can take $\vec{h}_1$ and $\vec{h}_1'$ to be empty.

Case 3: $b \in \operatorname{acl}(H\vec{a})$. Add a tuple $\vec{h}_1 \in H$ such that $\vec{a}b\vec{h}_1$ is $H$-independent, and use Cases 1 and/or 2.

Case 4: $b \notin \operatorname{acl}(H\vec{a})$. By the extension property, there is $b' \in N$ such that $b' \notin \operatorname{acl}(H\vec{a}')$ and $\operatorname{tp}(b'\vec{a}') = \operatorname{tp}(b\vec{a})$. The tuples stay $H$-independent, so again we can take $\vec{h}_1$ and $\vec{h}_1'$ to be empty.

□

The previous result has the following consequence:

**Corollary 2.9.** *All $H$-structures are elementarily equivalent.*

We write $T^{ind}$ for the common complete theory of all $H$-structures of models of $T$.

To axiomatize $T^{ind}$ we follow the ideas of [27, Prop 2.15]. Here we use for the first time the fact that $T$ eliminates $\exists^\infty$. Recall that whenever $T$ eliminates $\exists^\infty$ the expression *the formula* $\varphi(x,\vec{b})$ *is nonalgebraic* is first order.

**Proposition 2.10.** *The theory $T^{dim}$ is axiomatized by:*

(1) $T'$.
(2) *For all $\mathcal{L}$-formulas $\varphi(x,\vec{y})$*
$\forall \vec{y}(\varphi(x,\vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x,\vec{y}) \wedge x \in H))$.
(3) *For all $\mathcal{L}$-formulas $\varphi(x,\vec{y})$, $m \in \omega$, and all $\mathcal{L}$-formulas $\psi(x,z_1,\dots,z_m,\vec{y})$ such that for some $n \in \omega$ $\forall \vec{z}\forall \vec{y}\exists^{\leq n} x\psi(x,\vec{z},\vec{y})$ (so $\psi(x,\vec{y},\vec{z})$ is always algebraic in $x$)*
$\forall \vec{y}(\varphi(x,\vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x,\vec{y}) \wedge$
$\forall w_1 \dots \forall w_m \in H \neg\psi(x,w_1,\dots,w_m,\vec{y}))$

*Furthermore, if $(M,H(M)) \models T^{ind}$ is $|T|^+$-saturated, then $(M,H(M))$ is an $H$-structure.*

The second scheme of axioms corresponds to the density property and the third scheme to the extension property. The first axiom says that $H$ is a collection of independent elements. The proof is the same on as the one in [3, Thm 2.10].

**Example 2.11.** *Let $T$ be the theory of infinite dimensional vector spaces over a fixed finite field, say $F_2$. Note that $T$ is strongly minimal so $T$ is geometric. Let $V \models T$ be countable and let $H = \{v_j : j \in \omega\}$ be its basis. Then $(V,H) \models T^{ind}$ but it is NOT an $H$-structure since it does not satisfy the extension property.*

**Example 2.12.** *Let $T = Th(\mathbb{R},+,\times,0,1,<)$, $T$ is o-minimal extending DLO so $T$ is geometric. Let $H = \{e_i : i \in I\}$ be a dense trascendence basis, then $(\mathbb{R},H) \models T^{ind}$. Note that $(\mathbb{R},H)$ is not an $H$-structure, since it does not satisfy the extension property.*

## 3. Definable sets in $H$-structures

Fix $T$ a geometric theory and let $(M,H(M)) \models T^{ind}$. Our next goal is to obtain a description of definable subsets of $M$ and $H(M)$ in the language $\mathcal{L}_H$. We will also address the question of the elimination of $\exists^\infty$ in $T^{ind}$.

**Notation 3.1.** *Let $(M,H(M))$ be an $H$-structures. Let $\vec{a}$ be a tuple in $M$. We denote by $\mathrm{etp}_H(\vec{a})$ the collection of formulas of the form $\exists x_1 \in H \dots \exists x_m \in H\varphi(\vec{x},\vec{y})$, where $\varphi(\vec{x},\vec{y})$ is an $\mathcal{L}$ formula such that there exists $\vec{h} \in H$ with $M \models \varphi(\vec{h},\vec{a})$.*

**Lemma 3.2.** *Let $(M,H(M))$, $(N,H(N))$ be $H$-structures. Let $\vec{a}$, $\vec{b}$ be tuples of the same arity from $M$, $N$ respectively. Then the following are equivalent:*

(1) $\mathrm{etp}_H(\vec{a}) = \mathrm{etp}_H(\vec{b})$.
(2) *$\vec{a}$, $\vec{b}$ have the same $\mathcal{L}_H$-type.*

*Proof.* Clearly (2) implies (1). Assume (1), then $\mathrm{tp}(\vec{a}) = \mathrm{tp}(\vec{b})$.

**Claim** $\dim(\vec{b}/H) = \dim(\vec{a}/H)$.

Let $\vec{h} = (h_1, \ldots, h_l) \in H(M)$ be such $k := \dim(\vec{a}/\vec{h}) = \dim(\vec{a}/H(M))$. We may assume that $\vec{a}^1 = (a_1, \ldots, a_k)$ are independent over $H$ and $\vec{a}^2 = (a_{k+1}, \ldots, a_n) \in acl(a_1, \ldots, a_k, h_1, \ldots, h_l)$. Choose $\psi(\vec{x}, \vec{y}, \vec{z})$ such that for any $\vec{b} \in M$, $\vec{c} \in M$ $\psi(\vec{b}, \vec{c}, \vec{z})$ is always algebraic in $\vec{z}$ and $M \models \psi(\vec{h}, \vec{a}^1, \vec{a}^2)$. Since $\mathrm{etp}_H(\vec{a}) = \mathrm{etp}_H(\vec{b})$ we get that $dim(\vec{b}/H) \leq k$. A similar argument shows that $\dim(\vec{a}/H(M)) \leq \dim(\vec{b}/H(N))$.

**Claim** $\mathrm{tp}_H(\vec{b}) = \mathrm{tp}_H(\vec{a})$.

As before, let $\vec{h} = (h_1, \ldots, h_l) \in H(M)$ be such that $k := \dim(\vec{a}/\vec{h}) = dim(\vec{a}/H(M))$. Then $\vec{a}\vec{h}$ is $H$-independent. Since $N$ is saturated as an $\mathcal{L}$-structure there are $\vec{h}' = (h'_1, \ldots, h'_l) \in H$ such that $\mathrm{tp}(\vec{a}, \vec{h}) = \mathrm{tp}(\vec{b}, \vec{h}')$. By the claim above $\vec{b}\vec{h}'$ is $H$-independent, so the result follows from Lemma 2.8. □

**Corollary 3.3.** *Let $(M, H(M))$ be a sufficiently saturated $H$-structure, assume that $T = Th(M)$ is trivial and that* dcl = acl *in $T$. Then every $\mathcal{L}_H$ formula $\varphi(\vec{x})$ in $(M, H(M))$ is equivalent to a boolean combination of $\mathcal{L}$ formulas and formulas of the form $H(f(\vec{x}))$, where $f$ is an $\mathcal{L}$-definable function.*

*Proof.* It suffices to check that types of tuples in $(M, H(M))$ are isolated by the the $\mathcal{L}$-formulas that they satisfy and the values of expressions of the form $H(f(\vec{x}))$, where $f$ is an $\mathcal{L}$-definable function..

Let $\vec{a}$, $\vec{b}$ be tuples of the same arity from $M$ and assume that they satisfy the same $\mathcal{L}$-type and that for every $\mathcal{L}$-definable function $f(\vec{x})$ we have that $H(f(\vec{a}))$ holds if and only if $H(f(\vec{b}))$ holds. We will prove that $\mathrm{tp}_H(\vec{a}) = \mathrm{tp}_H(\vec{b})$.

**Claim** $\dim(\vec{b}/H) = \dim(\vec{a}/H)$.

Let $\vec{h} = (h_1, \ldots, h_l) \in H(M)$ be such $k := \dim(\vec{a}/\vec{h}) = \dim(\vec{a}/H(M))$ and assume that $\vec{h}$ is a minimal such tuple. Then since $T$ is trivial, for each $i \leq k$ we have that $h_i = f_i(a_{j_i})$ for some $j_i$ and some $\mathcal{L}$-definable function $f_i$. Let $h'_i = f_i(b_{j_i})$ and let $\vec{h}' = (h'_1, \ldots, h'_l)$. Then $H(h'_i)$ holds for each $i \leq l$ and $\dim(\vec{b}/H) \leq l = \dim(\vec{a}/H)$. The other inequality follows in the same way.

Note that for $\vec{h}$ and $\vec{h}'$ defined as above, we have that $\vec{a}\vec{h}$ and $\vec{b}\vec{h}'$ are $H$-independent and thus by Lemma 2.8 we have that $\mathrm{tp}(\vec{a}) = \mathrm{tp}(\vec{b})$ as desired. □

Now we are interested in the $\mathcal{L}_H$-definable subsets of $H(M)$. This material is very similar to the results presented in [14, Theorem 2].

**Lemma 3.4.** *Let $(M_0, H(M_0)) \preceq (M_1, H(M_1))$ and assume that $(M_1, H(M_1))$ is $|M_0|$-saturated. Then $M_0$ (seen as a subset of $M_1$) is a $H$-independent set.*

*Proof.* Assume not. Then there are $a_1, \ldots, a_n \in M_0 \setminus H(M_0)$ such that $a_n \in \mathrm{acl}(a_1, \ldots, a_{n-1}, H(M_1))$ and $a_n \notin \mathrm{acl}(a_1, \ldots, a_{n-1}, H(M_0))$. Let $\varphi(x, \vec{y}, \vec{z})$ be a formula and $\vec{b} \in H(M_1)_{\vec{z}}$ be a tuple such that

$$\varphi(a_n, a_1, \ldots, a_{n-1}, \vec{b}) \wedge \exists^{\leq n} x \varphi(x, a_1, \ldots, a_{n-1}, \vec{b})$$

holds. Since $(M_0, H(M_0)) \preceq (M_1, H(M_1))$ there is $\vec{b}' \in H(M_0)_{\vec{y}}$ such that

$$\varphi(a_n, a_1, \ldots, a_{n-1}, \vec{b}') \wedge \exists^{\leq n} x \varphi(x, a_1, \ldots, a_{n-1}, \vec{b}')$$

holds, so $a_n \in \mathrm{acl}(a_1, \ldots, a_{n-1}, H(M_0))$, a contradiction. □

**Proposition 3.5.** *Let $(M, H(M))$ be an $H$-structure and let $Y \subset H(M)^n$ be $\mathcal{L}_H$-definable. Then there is $X \subset M^n$ $\mathcal{L}$-definable such that $Y = X \cap H(M)^n$.*

*Proof.* Let $(M_1, H(M_1)) \succeq (M, H(M))$ be $\kappa$-saturated where $\kappa > |M| + |L|$ and let $\vec{a}, \vec{b} \in H(M_1)^n$ be such that $\mathrm{tp}(\vec{a}/M) = \mathrm{tp}(\vec{b}/M)$. We will prove that $\mathrm{tp}_H(\vec{a}/M) = \mathrm{tp}_H(\vec{b}/M)$ and the result will follow by compactness. Since $\vec{a}, \vec{b} \in H(M_1)^n$, we get by Lemma 3.4 that $M\vec{a}$, $M\vec{b}$ are $H$-independent sets and thus by Lemma 2.8 we get $\mathrm{tp}_H(\vec{a}/M) = \mathrm{tp}_H(\vec{b}/M)$. $\square$

**Remark 3.6.** *A small warning is due here. In the previous proof, we may need extra parameters in the small model to define an $\mathcal{L}$-formula equivalent to the original $\mathcal{L}_H$-formula.*

**Definition 3.7.** Let $(M, H) \models T^{ind}$ be saturated. We say that an $\mathcal{L}_H$ formula $\psi(x, \vec{c})$ defines a *large subset* of $M$ is there is $b \models \psi(x, \vec{c})$ such that $b \notin \mathrm{scl}(\vec{c})$. This is equivalent as requiring that there are infinitely many realizations of $\psi(x, \vec{c})$ that are algebraically independent over $H(M)\vec{c}$.

**Definition 3.8.** Let $(M, H) \models T^{ind}$ be $\kappa$-saturated and let $A \subset M$ be smaller than $\kappa$. Let $\vec{b} \in M$ be a tuple. We say that $\vec{b}$ is in the *small closure* of $A$ if $\vec{b} \in \mathrm{acl}(AH(M))$ and write $\vec{b} \in \mathrm{scl}(A)$. Let $X \subset M^n$ be $A$-definable. We say that $X$ is *small* if $X \subset \mathrm{scl}(A)$.

Since $T$ is geometric, scl satisfies the exchange property and thus it is a closure operator.

Next, we introduce the notion of the *H-basis*, which first appeared in [13] in the o-minimal context.

**Proposition 3.9.** *Let $(M, H(M))$ be an $H$-structure. Let $\vec{a} = (a_1, \ldots, a_n) \in M$. Then there is a unique smallest tuple $\vec{h} \in H(M)$ such that $\vec{a} \perp_{\vec{h}} H$.*

*Proof.* Clearly there is a tuple $\vec{h} \in H$ such that $\vec{a} \perp_{\vec{h}} H$. Choose such a tuple so that $|\vec{h}|$ (the length of the tuple) is minimal. We will now show such a tuple $\vec{h}$ is unique (up to permutation).

We can write $\vec{a} = (\vec{a}_1, \vec{a}_2)$ so that $\vec{a}_1$ is independent over $H(M)$ and $\vec{a}_2 \in \mathrm{scl}(\vec{a}_1)$. If $\vec{a}_2 = \emptyset$, then $\vec{h} = \emptyset$ and the result follows. So we may assume that $\vec{a}_2 \neq \emptyset$.

Then $\vec{a}_2 \in \mathrm{acl}(\vec{a}_1, \vec{h})$. Let $\vec{h}'$ be another such tuple. Let $\vec{h}_1$ be the list of common elements in both $\vec{h}$ and $\vec{h}'$, so we can write $\vec{h} = (\vec{h}_1, \vec{h}_2)$ and $\vec{h}' = (\vec{h}_1, \vec{h}'_2)$.

**Claim** $\vec{h}_2 = \vec{h}'_2 = \emptyset$.

Assume otherwise. Since $\vec{a}_2 \in \mathrm{acl}(\vec{a}_1, \vec{h}_1, \vec{h}_2) \setminus \mathrm{acl}(\vec{a}_1, \vec{h}_1)$ and $\vec{a}_2 \in \mathrm{acl}(\vec{a}_1, \vec{h}_1, \vec{h}'_2) \setminus \mathrm{acl}(\vec{a}_1, \vec{h}_1)$ then by the exchange property $\dim(\vec{h}'_2/\vec{a}_1\vec{h}_1\vec{h}_2) < \dim(\vec{h}'_2/\vec{a}_1\vec{h}_1)$. Since $\vec{a}_1$ is independent over $H$ we get that $\dim(\vec{h}'_2/\vec{h}_1\vec{h}_2) < \dim(\vec{h}'_2/\vec{h}_1)$ and thus since $H$ is independent, $\vec{h}_2$ has a common element with $\vec{h}'_2$, a contradiction. $\square$

**Remark 3.10.** *Let $(M, H(M))$ be an $H$-structure. Let $\vec{a} = (a_1, \ldots, a_n) \in M$ and let $C \subset M$ be such that $C = \mathrm{acl}(C)$ and $C$ is $H$-independent. As before, there is a unique smallest tuple $\vec{h} \in H(M)$ such that $\vec{a} \perp_{\vec{h}C} H$.*

**Definition 3.11.** Let $(M, H(M))$ be an $H$-structure. Let $\vec{a} = (a_1, \ldots, a_n) \in M$. Let $\vec{h} \in H(M)$ be the smallest tuple such that $\vec{a} \perp_{\vec{h}} H$. We call $\vec{h}$ the *H-basis* of $\vec{a}$ and we denote it as $HB(\vec{a})$. Given $C \subset M$ such that $C = \mathrm{acl}(C)$ and $C$ is $H$-independent, let $\vec{h} \in H(M)$ the smallest tuple such that $\vec{a} \perp_{C\vec{h}} H$. We call $\vec{h}$ the *H-basis of $\vec{a}$ over $C$* and we denote it as $HB(\vec{a}/C)$. Note that $H$-basis is unique up to permutation, therefore we will view the $H$-basis $\vec{h} = (h_1, \ldots, h_k)$ either as a finite set $\{h_1, \ldots, h_k\}$ or as the imaginary representing this finite set. If we view it as a tuple, we will explicitly say so.

We will first apply the $H$-basis to characterize definable sets in terms of $\mathcal{L}$-definable sets.

**Proposition 3.12.** *Let $(M, H(M))$ be an $H$-structure and let $Y \subset M$ be $\mathcal{L}_H$-definable. Then there is $X \subset M$ $\mathcal{L}$-definable such that $Y \triangle X$ is small, where $\triangle$ stands for a boolean connective for the symmetric difference.*

*Proof.* If $Y$ is small or cosmall, the result is clear, so we may assume that both $Y$ and $M \setminus Y$ are large. Assume that $Y$ is definable over $\vec{a}$ and that $\vec{a} = \vec{a}HB(\vec{a})$. Let $b \in Y$ be such that $b \notin scl(\vec{a})$ and let $c \in M \setminus Y$ be such that $c \notin scl(\vec{a})$. Then $b\vec{a}$, $c\vec{a}$ are $H$-independent and thus there is $X_{bc}$ an $\mathcal{L}$-definable set such that $b \in X_{bc}$ and $c \notin X_{bc}$. By compactness, we may first assume that $X_{bc}$ only depends on $\text{tp}(b/\vec{a})$ and applying compactness again we may assume that $X_{bc}$ does no depend on $\text{tp}(b/\vec{a})$ and we will call it simply $X$. Thus for $b' \in Y$ and $c' \in M \setminus Y$ not in the small closure of $\vec{a}$, we have $b' \in X$ and $c' \in M \setminus X$. This shows that $Y \triangle X$ is small. □

Our next goal is to characterize the algebraic closure in $H$-structures. The key tool is the following result:

**Lemma 3.13.** *Let $T$ be a geometric theory, $M \models T$, $(M, H(M))$ an $H$-structure, and let $A \subset M$ be* acl*-closed and $H$-independent. Then $A$ is* $\text{acl}_H$*-closed.*

*Proof.* Suppose $a \in M$, $a \notin A$. If $a \notin \text{scl}(A)$, then $A \cup \{a\}$ is $H$-independent, and using the extension property, we can find $a_i$, $i \in \omega$, acl-independent over $A \cup H(M)$, realizing $\text{tp}(a/A)$. By Lemma 2.8, each $a_i$ realizes $\text{tp}_H(a/A)$, and thus $a \notin \text{acl}_H(A)$.

If $a \in \text{scl}(A)$, take a minimal tuple $\vec{b} \in H(M)$ such that $a \in \text{acl}(A\vec{b})$. Using the coheir property of $H$-structures, we can find $\vec{b}_i \in H(M)$, $i \in \omega$, such that $\vec{b}_i$ are acl-independent over $A$ and realize $\text{tp}(\vec{b}/A)$. Take $a_i \in \text{acl}(A\vec{b}_i)$ such that $\text{tp}(a_i\vec{b}_i/A) = \text{tp}(a\vec{b}/A)$. Then $\{a_i : i \in \omega\}$ are acl-independent over $A$. On the other hand, for any $i \in \omega$, $A\vec{b}_i a_i$ is a $H$-independent set and thus by Lemma 2.8 $\text{tp}_H(a_i\vec{b}_i/A) = \text{tp}_H(a\vec{b}/A)$ and in particular $\text{tp}_H(a_i/A) = \text{tp}_H(a/A)$. □

**Corollary 3.14.** *Let $T$ be a geometric theory, $M \models T$, $(M, H(M))$ an $H$-structure, and let $A \subset M$. Then $\text{acl}_H(A) = \text{acl}(A, HB(A))$.*

*Proof.* By Proposition 3.9, it is clear that $HB(A) \in \text{acl}(A)$, so $\text{acl}_H(A) \supset \text{acl}(A, HB(A))$. On the other hand, $A, HB(A)$ is $H$-closed, so by the previous Proposition, $\text{acl}(A, HB(A)) = \text{acl}_H(A, HB(A))$ and thus $\text{acl}_H(A) \subset \text{acl}(A, HB(A))$ □

It is interesting to check which properties of $T$ are preserved in $T^{ind}$.

**Question 3.15.** *Does $T^{ind}$ eliminate the quantifier $\exists^\infty$?*

We give a partial answer. Namely, we will show the elimination of the $\exists^\infty x$ for $L_H$-formulas $\phi(x, \vec{z})$ implying $H(\vec{z})$. Note that if $a \in H(M)$ and $\vec{h} \in H(M)$ then $a \in \text{acl}_H(\vec{h})$ exactly when $a$ is a part of $\vec{h}$. Thus we may assume that $\phi(x, \vec{z})$ implies $\neg H(x) \wedge H(\vec{z})$. We will be working in a sufficiently saturated $H$-structure $(M, H)$ of a geometric theory $T$.

First, note that if $\vec{h} \in H(M)$ and $|\phi(M, \vec{h})| = n < \omega$, then for any $a \in M$ with $\models \phi(a, \vec{h})$, we have $a \in \mathrm{acl}(\vec{h})$, and there is an $L$-formula $\phi_n(x, \vec{z})$ such that $\phi_n(M, \vec{h}) = \phi(M, \vec{h})$. By compactness, $\phi_n(x, \vec{z})$ does not depend on the choice of $\vec{h} \in H(M)$, but it may still depend on $n$ (unless $T$ is $\omega$-categorical). Thus, this approach does not seem to work. Instead, we will take a closer look at the $L_H$-formula $\phi(x, \vec{z})$.

We say that an $L$-formula $\psi(x, \vec{y})$ *has a bounded finite number of realizations in* $x$, if there exists $n < \omega$ such that for any $\vec{b}$, $|\psi(M, \vec{b})| < n$. Thus, $\phi(x, \vec{b})$ is either inconsistent or witnesses $x \in \mathrm{acl}(\vec{b})$.

**Lemma 3.16.** *For any $\vec{h} \in H(M)$ and $a \in M$ with $a \notin H(M)$, $\mathrm{tp}_H(a, \vec{h})$ is axiomatized by $\neg H(x)$, $H(\vec{z})$, $L$-formulas, and the formulas of the form*

$$\exists \vec{y} \in H \ \theta(x, \vec{y}, \vec{z})$$

*or*

$$\neg \exists \vec{y} \in H \ \theta(x, \vec{y}, \vec{z}),$$

*where $\theta(x, \vec{y}, \vec{z})$ is an $L$-formula having a bounded finite number of realizations in $x$.*

*Proof.* Assuming $a \notin H(M)$ and $\vec{h} \in M$, the $L_H$-type of the tuple $a\vec{h}$ is determined by its $L$-type, and either the fact that $a \notin \mathrm{scl}(\vec{h})$ or the $L$-type of some $\vec{k} \in H(M)$ over $\vec{h}$, such that $a \in \mathrm{acl}(\vec{k}, \vec{h})$. All these properties can be expressed with the given types of formulas. □

Note that $\phi(x, \vec{z})$ is a conjunction of $H(\vec{z})$, $\neg H(x)$, Boolean combination of $L$-formulas and formulas of the form $\exists \vec{y} \in H \ \theta(x, \vec{y}, \vec{z})$, where $\theta(x, \vec{y}, \vec{z})$ is an $L$-formula having a bounded finite number of realizations in $x$. Note that elimination of $\exists^\infty x$ is preserved under disjunction. Thus, we may assume that $\phi(x, \vec{z})$ is a conjunction of $H(\vec{z})$, $\neg H(x)$, an $L$-formula $\Gamma(x, \vec{z})$, and/ or formulas of the form $\exists \vec{y} \in H \ \theta(x, \vec{y}, \vec{z})$ or $\neg \exists \vec{y} \in H \ \theta(x, \vec{y}, \vec{z})$, where $\theta(x, \vec{y}, \vec{z})$ is an $L$-formula having a bounded finite number of realizations in $x$. Note that the class of $L$-formulas $\theta(x, \vec{y}, \vec{z})$ having a bounded finite number of realization in $x$ is closed under conjunction and disjunction. Thus, we may assume that $\phi(x, \vec{z})$ has one of the four forms:

(1) $\neg H(x) \wedge H(\vec{z}) \wedge \Gamma(x, \vec{z})$,
(2) $\neg H(x) \wedge H(\vec{z}) \wedge \Gamma(x, \vec{z}) \wedge \exists \vec{y} \in H \ \theta(x, \vec{y}, \vec{z})$,
(3) $\neg H(x) \wedge H(\vec{z}) \wedge \Gamma(x, \vec{z}) \wedge \neg \exists \vec{y} \in H \ \theta(x, \vec{y}, \vec{z})$,
(4) $\neg H(x) \wedge H(\vec{z}) \wedge \Gamma(x, \vec{z}) \wedge \exists \vec{y} \in H \ \theta_1(x, \vec{y}, \vec{z}) \wedge \neg \exists \vec{y}' \in H \ \theta_2(x, \vec{y}', \vec{z})$,

where $\Gamma(x, \vec{z})$ is an $L$-formula, and $\theta(x, \vec{y}, \vec{z})$, $\theta_1(x, \vec{y}, \vec{z})$ and $\theta_2(x, \vec{y}', \vec{z})$ are $L$-formulas having a bounded finite number of realizations in $x$.

Clearly, in cases (1) and (3), the algebraicity of $\phi(x, \vec{h})$ is determined by algebraicity of $\Gamma(x, \vec{z})$. Indeed, in (3), if $\Gamma(x, \vec{h})$ is infinite, it defines a large set, and clearly has an infinite number of realizations that do not satisfy the small formula $\exists \vec{y} \in H \ \theta(x, \vec{y}, \vec{h})$. Also in cases (2) and (4) we can absorb $\Gamma(x, \vec{z})$ in $\theta(x, \vec{y}, \vec{z})$ or $\theta_1(x, \vec{y}, \vec{z})$.

We will now reduce case (4) to case (2). Note that we can assume that $\theta_1(x, \vec{y}, \vec{z})$ implies that $\vec{y}$ is a tuple of distinct elements and is disjoint from $\vec{z}$, and $\theta_2(x, \vec{y}', \vec{z})$ implies the same about $\vec{y}'$.

The idea of the proof is the following. If we assume that $\theta(a, \vec{k}, \vec{h})$ holds for some $\vec{k} \in H(M)$, then to analyze $\exists \vec{y}' \in H \ \ \theta_2(a, \vec{y}', \vec{h})$, we will look at the relationship between $\vec{k}$ and $\vec{y}'$, namely, how much of an overlap do we have between $\vec{y}'$ and $\vec{k}$. For each subtuple $\vec{k}^*$ of $\vec{k}$, we consider all the tuples $\vec{y}' \in H(M)$ such that $\vec{y}' \cap \vec{k} = \vec{k}^*$. It will turn out that the existence of such $\vec{y}'$ in $H(M)$ with $\models \theta_2(a, \vec{y}', \vec{h})$ is (uniformly) $L$-definable in $x\vec{y}\vec{z}$. Then we take the disjunction over all the subtuples of $\vec{y}$.

**Claim 1:** Let $\vec{h} \in H(M)$, $a \notin H(M)$. Suppose $\vec{k} \in H(M)$ is such that $\models \theta_1(a, \vec{k}, \vec{h})$. Let $\vec{k}'$ be a subtuple of $\vec{k}$. Suppose $\vec{b} \in H(M)$ is such that $\vec{b} \cap \vec{k} = \vec{k}'$ and $\models \theta_2(a, \vec{b}, \vec{h})$. Then $a \in \operatorname{acl}(\vec{k}', \vec{h})$.

Proof of Claim 1: Since $\models \theta_2(a, \vec{b}, \vec{h})$, we have $a \in \operatorname{acl}(\vec{b}, \vec{h})$. On the other hand, $a \in \operatorname{acl}(\vec{k}, \vec{h})$. Since $\vec{b}\vec{h}$ is independent from $\vec{k}\vec{h}$ over $\vec{k}'\vec{h}$, we have $a \in \operatorname{acl}(\vec{k}', \vec{h})$.

**Claim 2:** Suppose $\vec{y}^*$ is a subtuple of $\vec{y}$. Then the formula

$$\exists \vec{y}' \in H\ (\vec{y}' \cap \vec{y} = \vec{y}^* \wedge \theta_2(x, \vec{y}', \vec{z}))$$

is equivalent to an $L$-formula $\Delta(x, \vec{y}, \vec{z})$ modulo

$$H(\vec{z}) \wedge \neg H(x) \wedge H(\vec{y}) \wedge \theta_1(x, \vec{y}, \vec{z}).$$

Proof of Claim 2: By Claim 1 and compactness, there exists an $L$-formula $\psi(x, \vec{y}^*, \vec{z})$ having a bounded finite number of realizations in $x$, such that

$$\models (H(\vec{z}) \wedge \neg H(x) \wedge H(\vec{y}) \wedge \theta_1(x, \vec{y}, \vec{z}) \wedge \vec{y}' \cap \vec{y} = \vec{y}^* \wedge \theta_2(x, \vec{y}', \vec{z})) \rightarrow \psi(x, \vec{y}^*, \vec{z}).$$

Let $\vec{y}^{**}$ be such that $\vec{y}' = \vec{y}^*\vec{y}^{**}$ (permute the variables if needed). Then modulo $H(\vec{z}) \wedge \neg H(x) \wedge H(\vec{y}) \wedge \theta_1(x, \vec{y}, \vec{z})$,

$$\exists \vec{y}' \in H\ (\vec{y}' \cap \vec{y} = \vec{y}^* \wedge \theta_2(x, \vec{y}', \vec{z}))$$

is equivalent to

$$\exists \vec{y}^{**} \in H\ (\theta_2(x, \vec{y}^*\vec{y}^{**}, \vec{z}) \wedge \psi(x, \vec{y}^*, \vec{z})).$$

The latter is equivalent, modulo $H(\vec{z}) \wedge \neg H(x) \wedge H(\vec{y}^*) \wedge \vec{y}^{**} \cap \vec{y}^*\vec{z} = \emptyset$ and the statement that $\vec{y}^{**}$ is a tuple of distinct elements, to the existence of a tuple $\vec{y}^{**}$, acl-independent over $x\vec{y}\vec{z}$, such that $\theta_2(x, \vec{y}^*\vec{y}^{**}, \vec{z}) \wedge \psi(x, \vec{y}^*, \vec{z})$ holds true. Indeed, suppose $\vec{h}, \vec{k}^* \in H(M)$, $a \notin H(M)$. Note that

$$\theta_2(a, \vec{k}^*\vec{y}^{**}, \vec{h}) \wedge \psi(a, \vec{k}^*, \vec{h})$$

implies that $a \in \operatorname{acl}(\vec{k}^*, \vec{h})$. Then there exists a tuple of distinct elements $\vec{k}^{**} \in H(M)$, disjoint from $\vec{k}^*\vec{h}$, such that

$$\models \theta_2(a, \vec{k}^*\vec{k}^{**}, \vec{h}) \wedge \psi(a, \vec{k}^*, \vec{h})$$

exactly when there exists a tuple $\vec{k}^{**} \in M$, acl-independent over $a\vec{k}^*\vec{h}$, such that

$$\models \theta_2(a, \vec{k}^*\vec{k}^{**}, \vec{h}) \wedge \psi(a, \vec{k}^*, \vec{h}).$$

This condition is $L$-definable in $x\vec{y}\vec{z}$, as needed, which proves Claim 2.

Next, rewrite

$$\neg H(x) \wedge H(\vec{z}) \wedge \exists \vec{y} \in H\ \theta_1(x, \vec{y}, \vec{z}) \wedge \neg \exists \vec{y}' \in H\ \theta_2(x, \vec{y}', \vec{z})$$

as

$$\neg H(x) \wedge H(\vec{z}) \wedge \exists \vec{y} \in H\ (\theta_1(x, \vec{y}, \vec{z}) \wedge \neg \exists \vec{y}' \in H\ \theta_2(x, \vec{y}', \vec{z})),$$

and note that it is equivalent to

$$\neg H(x) \wedge H(\vec{z}) \wedge \exists \vec{y} \in H\ (\theta_1(x,\vec{y},\vec{z}) \wedge \neg \bigvee_{\vec{y}^* \subset \vec{y}} \exists \vec{y}' \in H\ (\vec{y}' \cap \vec{y} = \vec{y}^* \wedge \theta_2(x,\vec{y}',\vec{z}))).$$

By Claim 2, the disjunction above can be replaced with an $L$-formula in $x\vec{y}\vec{z}$, and therefore can be absorbed in $\theta_1(x,\vec{y},\vec{z})$. This reduces case (4) to case (2). Thus, we may assume that $\phi(x,\vec{z})$ has form

$$\neg H(x) \wedge H(\vec{z}) \wedge \exists \vec{y} \in H\ \theta(x,\vec{y},\vec{z}),$$

where $\theta(x,\vec{y},\vec{z})$ is an $L$-formula having a bounded finite number of realizations in $x$.

**Lemma 3.17.** *Suppose $\theta(x,y,\vec{z})$ is an $L$-formula having a bounded finite number of realizations in $x$. Then for any $\vec{h} \in H(M)$, $\exists y \in H\ \theta(x,y,\vec{h})$ is infinite if and only if $M \models \exists^\infty x \exists y\ \theta(x,y,\vec{h})$.*

*Proof.* Left to right is clear. For the other direction, suppose $M \models \exists^\infty x \exists y\ \theta(x,y,\vec{h})$. Then we can find a sequence $(a_i : i \in \omega)$ of realizations of $\exists y\ \theta(x,y,\vec{h})$, acl-independent over $\vec{h}$. For each $a_i$ there exists $b_i \in M$ such that $M \models \theta(a_i,b_i,\vec{h})$. We have $a_i \in \operatorname{acl}(b_i\vec{h})$. Thus, $b_i \notin \operatorname{acl}(\vec{h})$, and $\operatorname{acl}(a_i\vec{h}) = \operatorname{acl}(b_i\vec{h})$. Then the sequence $(b_i : i \in \omega)$ is also acl-independent over $\vec{h}$, and thus we may assume that $b_i \in H(M)$. Then $a_i$ all realize $\exists y \in H\ \theta(x,y,\vec{h})$, as needed.

□

**Lemma 3.18.** *Suppose $\vec{h} \in H(M)$, $\theta(x,\vec{y},\vec{z})$ an $L$-formula having a bounded finite number of realizations in $x$, and implying that $\vec{y} = y_1 \dots y_n$ is a tuple of distinct elements and $\vec{y} \cap \vec{z} = \emptyset$. Then the formula $\exists \vec{y} \in H\ \theta(x,\vec{y},\vec{h})$ is infinite if and only if there exists $1 \leq i \leq n$ and $b_1, \dots b_{i-1}, b_{i+1}, \dots, b_n \in H(M)$ such that*

$$\exists y_i \in H\ \theta(x, b_1, \dots, b_{i-1}, y_i, b_{i+1}, \dots, b_n, \vec{h})$$

*is infinite.*

*Proof.* Right to left is clear. For the other direction, suppose $\exists \vec{y} \in H\ \theta(x,\vec{y},\vec{h})$ is infinite. Then it has infinitely many realizations $a \notin \operatorname{acl}(\vec{h})$. For each such $a$ there exists $\vec{b} \in H$ such that

$$\models \theta(a,\vec{b},\vec{h}).$$

Then $a \in \operatorname{acl}(\vec{b},\vec{h})$. Note that the tuple $\vec{b}$ acl-independent over $\vec{h}$. There is a nonempty minimal subtuple $\vec{b}'$ of $\vec{b}$ such that $a \in \operatorname{acl}(\vec{b}',\vec{h})$. Take any $b_i$ contained in $\vec{b}'$. Then clearly $a$ is interalgebraic (in terms of acl) with $b_i$ over $b_1 \dots b_{i-1} b_{i+1} \dots b_n \vec{h}$. Taking infinitely many acl-independent $L$-conjugates $a'b_i'$ of $ab_i$ over $b_1 \dots b_{i-1} b_{i+1} \dots b_n \vec{h}$, with $b_i \in H(M)$, we get infinitely many realizations of

$$\exists y_i \in H\ \theta(x, b_1, \dots, b_{i-1}, y_i, b_{i+1}, \dots, b_n, \vec{h}).$$

□

**Proposition 3.19.** *Suppose $\phi(x,\vec{z})$ is an $L_H$-formula implying $H(\vec{z})$. Then $\exists^\infty x\ \phi(x,\vec{z})$ is first order.*

*Proof.* We may assume that $\phi(x,\vec{z})$ has form

$$\neg H(x)\wedge H(\vec{z})\wedge \exists \vec{y}\in H\ \theta(x,\vec{y},\vec{z}),$$

where $\theta(x,\vec{y},\vec{z})$ is an $L$-formula having a bounded finite number of realizations in $x$, and $\theta(x,\vec{y},\vec{z})$ implies that $\vec{y}=y_1\dots y_n$ is a tuple of distinct elements, disjoint form $\vec{z}$.

Then by Lemma 3.18, $\exists^\infty x\ \phi(x,\vec{z})$ is equivalent (modulo $\neg H(x)\wedge H(\vec{a})$) to

$$\bigvee_{1\le i\le n} \exists y_1\in H\ \dots y_{i-1}\in H\ y_{i+1}\in H\ \dots y_n\in H\ \exists^\infty x\exists y_i\in H\ \theta(x,\vec{y},\vec{z}).$$

By Lemma 3.17, $\exists^\infty x\exists y\in H\ \theta(x,\vec{y},\vec{z})$ is a first order formula. □

We finish this section with a property of non-trivial geometric theories that we will use in the next sections.

**Definition 3.20.** Let $T$ be a geometric theory, let $M\models T$ and let $\vec{a}=(a_1,\dots,a_{n-1},a_n)\in M^n$ be such that $\dim(\vec{a})=n-1$ but any $n-1$ subset of $\{a_1,\dots,a_{n-1},a_n)\}$ is independent. We call such a tuple an *algebraic $n$-gon.*

**Proposition 3.21.** *Let $T$ be a non-trivial geometric theory and let $M\models T$ be saturated. Then for every $n$ there is $m\ge n$ and an algebraic $m$-gon.*

*Proof.* Working over a finite independent tuple, if necessary, me may assume that $T$ has an algebraic triangle, i.e. an algebraic 3-gon (triangle) $abc$. Let $a'\models \mathrm{tp}(a/b)$ be independent from $ac$ over $b$. Note that then $aca'$ is an independent tuple. Let $c'$ be such that $\mathrm{tp}(a'c'/b)=\mathrm{tp}(ac/b)$. Then $aca'c'$ is an algebraic 4-gon (quadrangle). Then take $a''$ such that $\mathrm{tp}(a''c')=\mathrm{tp}(ac)$ and $a''$ is independent from $abca'c'$ over $c'$. Then $aca'a''$ is an independent tuple. Let $b''$ be such that $\mathrm{tp}(a''b''c')=\mathrm{tp}(abc)$. Then $aca'a''b''$ is an algebraic 5-gon. Continuing in this way, we can generate algebraic $n$-gons for an arbitrarily large $n$. □

## 4. Lovely pairs, iterated $H$-structures, and externally definable sets

In this section we will explore topics motivated by analogies between $H$-structures and lovely pairs. First, we take a closer look at the connections between the two constructions. Then we look at the iterated version of the $H$-structures (similar to "tuples" of structures and "double pairs"). Finally, we look at the expansion of $H(M)$ with traces of externally definable sets.

4.1. **Independent subsets and lovely pairs.** In this subsection we study the connections between $H$-structures and lovely pairs. Let $T$ be a geometric theory in a language $\mathcal{L}$ and let $N\preceq M\models T$. We say that the pair $(M,N)$ is a *lovely pair* of models of $T$ if

1. (Density/coheir property) If $A\subset M$ is finite dimensional and $q\in S_1(A)$ is non-algebraic, there is $a\in N$ such that $a\models q$.
2. (Extension property) If $A\subset M$ is finite dimensional and $q\in S_1(A)$ is non-algebraic, there is $a\in M$, $a\models q$ and $a\notin \mathrm{acl}(A\cup N)$.

Note that the properties characterizing lovely pairs are very similar to the ones of $H$-structures, the role of the independent set $H$ is played by the elementary substructure $N$. In this section we will only use the definition of lovely pairs. More information on lovely pairs of geometric structures can be found in [3].

**Proposition 4.1.** *Let $T$ be a geometric theory and let $(M,H)$ be an $H$-structure. Let $N = \operatorname{acl}(H)$. Then $(M,N)$ is a lovely pair of models of $T$.*

*Proof.* Let $T$, $(M,H)$ and $N$ be as above.

**Claim** $N \preceq M$.

We apply the Tarski-Vaught test. Let $\vec{a} \in N$, let $b \in M$ and assume that $M \models \varphi(b,\vec{a})$. If $b \in \operatorname{acl}(\vec{a})$ then $b \in N$ and $N \models \varphi(b,\vec{a})$. If $b \notin \operatorname{acl}(\vec{a})$ let $p(x) = \operatorname{tp}(b/\vec{a})$. By the coheir property for $H$-structures there is $b' \in H$ such that $\operatorname{tp}(b'/\vec{a}) = \operatorname{tp}(b/\vec{a})$

Now we check that $(M,N)$ satisfies the coheir property. Let $A \subset M$ be finite dimensional and let $q \in S_1(A)$ be non-algebraic. By the coheir property for $H$-structures, there is $b \in H$ such that $b \models q$. Since $N = \operatorname{acl}(H)$ we have $b \in N$.

Now we check that $(M,N)$ satisfies the extension property. Let $A \subset M$ be finite dimensional and let $q \in S_1(A)$ be non-algebraic. By the extension property for $H$-structures, there is $b \in M$ such that $b \models q$ and $b \notin \operatorname{acl}(A \cup H)$. Since $N = \operatorname{acl}(H)$ then $b \notin \operatorname{acl}(A \cup N)$ as desired. □

Let $P$ be a new predicate that does not appear in $\mathcal{L}$ and let $L_P = L \cup \{P\}$ be the old language extended with a new predicate symbol. If $(M,N)$ is a lovely pair of models of $T$, we can study $(M,N)$ as an $L_P$ structure by interpreting $P$ as $N$. In [3] it is shown that if $(M,N)$ and $(M',N')$ are lovely pairs of models of $T$, then $Th((M,N)) = Th((M',N'))$ (seen as $L_P$ structures). Note that Corollary 2.9 is the analogous result for $H$-structures. We denote by $T_P$ this common theory in the language $L_P$.

It is shown in [3] that when $T$ is geometric, weakly 1-based, and $\omega$-categorical, then the associated theory $T_P$ of lovely pairs is also $\omega$-categorical. This is not the case for the associated theory $T^{ind}$:

**Example 4.2.** *Let $T$ be the theory of infinite dimensional vector spaces over a fixed finite field, say $F_2$. Note that $T$ is strongly minimal, $\omega$-categorical and 1-based. Let $V \models T$ be countable and let $H = \{v_j : j \in \omega\}$ be an enumeration of a basis. Let $i < \omega$ and let $H_i = \{v_j : j \in \omega, j > i\}$. Then $(V,H_i) \models T^{ind}$ for every $i$ and the models $(V,H_i)$, $(V,H_j)$ are not isomorphic for $i < j$. Thus the theory $T^{ind}$ is not $\omega$-categorical.*

*Now let $H_{even} = \{v_{2j} : j \in \omega\}$, then as before $(V,H_{even}) \models T^{ind}$ and it is not isomorphic to any of the pairs $(V,H_i)$. Also note that $(V,H_{even})$ is an $H$-structure, but for every $i \in \omega$ the pair $(V,H_i)$ is NOT an $H$-structure.*

*If we take algebraic closures, then we see that for every $i < \omega$, $(V,\operatorname{acl}(H_i))$ is not a model of $T_p$, since it does not satisfy the axiom corresponding to the extension property (see the third scheme of axioms in 2.10). On the other hand, $(V,\operatorname{acl}(H_{even}))$ is a model of $T_p$ and it is the unique model up to isomorphism.*

The previous example shows:

**Remark 4.3.** *Let $T$ be geometric and let $(M,H) \models T^{ind}$. Then $(M,\operatorname{acl}(H))$ may not be a model of $T_P$. The pair $(M,\operatorname{acl}(H))$ will satisfy the scheme of axioms corresponding to the density property, but it may fail to satisfy the scheme of axioms corresponding to the extension property.*

4.2. **Iterating the construction: $H$-tuples.** In this subsection we show how to iterate the process of expanding by $H$-structures. We will do the details for an expansion with two extra predicates but this procedure can be easily generalized to $n$ tuples of predicates. As before, we start with $T$ a geometric theory in a language $\mathcal{L}$ and we consider $H_1, H_2$ two new predicate symbols. Let $\mathcal{L}_{2H} = \mathcal{L} \cup \{H_1\} \cup \{H_2\}$. Let $T'$ be the $\mathcal{L}_H$-theory of all structures $(M, H_1, H_2)$, where $M \models T$ and $H_1(M) \cup H_2(M)$ is an $\mathcal{L}$-algebraically independent subset of $M$ and $H_1(M) \cap H_2(M) = \emptyset$.

**Definition 4.4.** We say that $(M, H_1(M), H_2(M))$ is an $H$-*triple* if

(1) $(M, H(M)) \models T'$
(2) (Density/coheir property for $H_1$) If $A \subset M$ is finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in H_1(M)$ such that $a \models q$.
(3) (Density/coheir property for $H_2/H_1$) If $A \subset M$ is finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in H_2(M)$ such that $a \models q$ and $a \notin \text{acl}(A \cup H_1(M))$.
(4) (Extension property) If $A \subset M$ is finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in M$, $a \models q$ and $a \notin \text{acl}(A \cup H_1(M) \cup H_2(M))$.

As before, if $(M, H_1(M), H_2(M))$, $(N, H_1(N), H_2(N))$ are $H$-triples, then $Th(M, H_1(M), H_2(M)) = Th(N, H_1(N), H_2(N))$, we denote the common theory by $T^{tri}$.

We will now follow the approach from Fornasiero [16] and consider an $H$-structure associated to the small closure in $(M, H_1)$. Fornasiero [16] considers lovely pairs in a general framework of a closure operator associated to an existential matroid. In this paper we will only consider the special case of the small closure.

Let $T_2$ be the $\mathcal{L}_H$-theory of all structures $(M, H_1, H_2)$, where $(M, H_1(M))$ is an $H$-structure and $H_2(M)$ is an algebraically independent subset of $M$ over $H_1(M) = \text{scl}(\emptyset)$.

**Definition 4.5.** We say that $(M, H(M), H_2(M))$ is an scl-structure if

(1) $(M, H_1(M), H_2(M)) \models T_2$
(2) (Density/coheir property for scl) If $A \subset M$ is finite dimensional and $q \in S_1^{ind}(A)$ is non-small, there is $a \in H_2(M)$ such that $a \models q$.
(3) (Extension property) If $A \subset M$ is finite dimensional and $q \in S_1^{ind}(A)$ is non-small, there is $a \in M$, $a \models q$ and $a \notin \text{scl}(A \cup H_2(M))$.

Now we will show that considering $H$-triples is equivalent as considering scl-structures

**Proposition 4.6.** *Let $T$ be a geometric structure, let $M \models T$ and let $H_1(M) \subset M$, $H_2(M) \subset M$ be distinguished subsets. Then $(M, H_1(M), H_2(M))$ is a scl-structure if and only if $(M, H_1(M), H_2(M))$ is an $H$-triple.*

*Proof.* Assume first that $(M, H_1(M), H_2(M))$ is a *scl*-structure. Then the pair $(M, H_1(M))$ is an $H$-structure and thus $(M, H_1(M), H_2(M))$ satisfies the density/coheir axiom for $H_1$. Now let $A \subset M$ be finite dimensional and let $q \in S_1(A)$ be non-algebraic. Let $\hat{q} \in S_1^{ind}(A)$ be an extension of $q$ that contains no small formula with parameters in $A$. Then by the Density/coheir property for scl it follows that there is $a \in H_2(M)$ such that $a \models \hat{q}$. In particular, $a \models q$ and $a \notin \text{acl}(A \cup H_1(M))$ and it follows the density/coheir property for $H_2/H_1$. Finally, since the same $\hat{q}$ is not small, there is $c \in M$, $c \models \hat{q}$ and $c \notin \text{scl}(A \cup H_2(M)) = \text{acl}(A \cup H_1(M) \cup H_2(M))$. Thus the extension property $H$-triples holds.

Now assume that $(M, H_1(M), H_2(M))$ is an $H$-triple. By the density property for $H_1$ and the extension property it follows that $(M, H_1(M))$ is an $H$-structure and that $(M, H_1(M), H_2(M)) \models T_2$. Now let $A \subset M$ be finite dimensional and let $\hat{q} \in S_1^{ind}(A)$ be non-small. We may enlarge $A$ and assume that $A = A \cup HB(A)$, so that $A$ is $H_1$-independent. Let $q$ be the restriction of $\hat{q}$ to the language $\mathcal{L}$. Note that $\hat{q}$ is the unique extension of $q$ to a non-small type. By the density/coheir property for $H_2/H_1$ there is $a \in H_2(M)$ such that $a \models q$, $a \notin \operatorname{acl}(H_1 A)$ and thus $a \models \hat{q}$. Finally the extension property follows from the extension property for $H$-triples. $\square$

We will now show that the class of scl-structures is "first order", that is, that there is a set of axioms whose $|T|^+$-saturated models are the scl-structures. For this we consider $H$-triples.

**Proposition 4.7.** *Assume $T$ eliminates $\exists^\infty$. Then the theory $T^{tri}$ is axiomatized by:*

1. $T'$.
2. *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$*
   $\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \in H_1))$.
3. *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$, $m \in \omega$, and all $\mathcal{L}$-formulas $\psi(x, z_1, \ldots, z_m, \vec{y})$ such that for some $n \in \omega$ $\forall \vec{z} \forall \vec{y} \exists^{\leq n} x \psi(x, \vec{z}, \vec{y})$ (so $\psi(x, \vec{y}, \vec{z})$ is always algebraic in $x$)*
   $\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \in H_2) \wedge$
   $\forall w_1 \ldots \forall w_m \in H_1 \neg \psi(x, w_1, \ldots, w_m, \vec{y}))$
4. *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$, $m \in \omega$, and all $\mathcal{L}$-formulas $\psi(x, z_1, \ldots, z_m, \vec{y})$ such that for some $n \in \omega$ $\forall \vec{z} \forall \vec{y} \exists^{\leq n} x \psi(x, \vec{z}, \vec{y})$ (so $\psi(x, \vec{y}, \vec{z})$ is always algebraic in $x$)*
   $\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \notin H_1 \wedge x \notin H_2) \wedge$
   $\forall w_1 \ldots \forall w_m \in H_1 \cup H_2 \neg \psi(x, w_1, \ldots, w_m, \vec{y}))$

   *Furthermore, if $(M, H_1(M), H_2(M)) \models T^{tri}$ is $|T|^+$-saturated, then $(M, H_1(M), H_2(M))$ is an $H$-triple.*

The proof is the same one as for $H$-structures and we leave it for the reader.

Note that since $T$ eliminates the quantifier $\exists^\infty$, then $T^{ind}$ eliminates the quantifier $\exists^{large}$. This is the main reason why the theory of $H$-triples is axiomatizable.

4.3. **Elimination of $\exists y \in H$.** In this subsection we will look at elimination of quantifiers in the structure obtained by naming all the externally definable relations on $H(M)$ in an $H$-structure (note we have already shown any $\mathcal{L}_H$-definable relation on $H(M)$ is $\mathcal{L}$-definable). This problem is known as elimination of $\exists y \in P$, where $P$ is a unary predicate symbol. Such an elimination is known to hold in the case when $P$ is an elementary submodel of a model of a stable theory (by definability of types), or an elementary submodel of a sufficiently saturated model of an NIP theory (established by Shelah [25]). The case when $P$ is the smaller model in a lovely pair of models of a simple theory has been considered in [22], where the elimination of $\exists y \in P$ has been shown to be equivalent to the property called weak lowness. In the case of lovely pairs of geometric structures, the elimination of $\exists y \in P$ was shown in [3]. Here we will show that any $H$-structure of a geometric theory eliminates $\exists y \in H$.

We will follow the Definition 1.1 from [22].

**Definition 4.8.** Let $T$ be a first order theory in a language $\mathcal{L}$, and let $(M,H)$ be an expansion of $M$ with a new unary predicate. We say that $(M,H)$ eliminates the quantifier $\exists y \in H$, if for any $\mathcal{L}$-formula $\phi(\vec{x},y,\vec{z})$ and $\vec{a} \in M$, there exists an $\mathcal{L}$-formula $\psi(\vec{x},\vec{w})$ and $\vec{b} \in M$, such that for any $\vec{c} \in H(M)$,

$$(M,H) \models \exists y \in H\ \phi(\vec{c},y,\vec{a}) \iff M \models \psi(\vec{c},\vec{b}).$$

If the choice of $\psi(\vec{x},\vec{w})$ does not depend on the choice of $\vec{a} \in M$ (i.e. depends only of the formula $\phi(\vec{x},y,\vec{y})$), we say that the elimination is uniform.

**Proposition 4.9.** *Let $T$ be a geometric theory, and let $(M,H)$ be an $H$-structure of $T$. Then $(M,H)$ eliminates the quantifier $\exists y \in H$.*

*Proof.* Let $\phi(\vec{x},y,\vec{z})$ be an $\mathcal{L}$-formula, and let $\vec{a} \in M$. Let $\vec{c} \in H(M)$.

If the formula $\phi(\vec{c},y,\vec{a})$ is non-algebraic, then clearly , it is realized in $H(M)$.

Now, suppose the formula $\phi(\vec{c},y,\vec{a})$ is algebraic and is realized by $e \in H(M)$, where $e$ is not a part of the tuple $\vec{c}$. Let $\vec{d} = HB(\vec{a})$, viewed as a tuple. If $e$ is not a part of $\vec{d}$, then $e$ is not algebraic over $\vec{c}\vec{d}$, and thus $\vec{c}e \not\perp_{\vec{d}} \vec{a}$. This contradicts the definition of $HB(\vec{a})$. Thus $e$ is a part of $\vec{d}$.

Thus for any $c \in H(M)$, we have $(M,H) \models \exists y \in H\ \phi(c,y,\vec{a})$ if and only if
either $\phi(\vec{c},y,\vec{a})$ is non-algebraic,
or $M \models \phi(\vec{c},e,\vec{a})$ where $e$ is a part of $\vec{c}HB(\vec{a})$.
Both conditions on $\vec{c}$ are $\mathcal{L}$-definable over the elements of $\vec{a}HB(\vec{a})$.

□

**Question 4.10.** *Is this elimination uniform? Note that the $\mathcal{L}$-definition of $\exists y \in H\ \phi(\vec{c},y,\vec{a})$ involves $HB(\vec{a})$, and this tuple could be arbitrarily long.*

## 5. Strongly minimal, $SU$-rank 1 and thorn rank 1 cases

In this section we study four special cases of geometric theories, when the underlying theory $T$ is strongly minimal, $SU$-rank 1, thorn rank 1 or strongly dependent of finite dp-rank. In these cases, we show that the theory $T^{ind}$ becomes $\omega$-stable, supersimple of $SU$-rank less than or equal to $\omega$, super-rosy of thorn-rank less than or equal to $\omega$ or strongly dependent respectively. We also characterize in each of these cases when $T$ is trivial in terms of the rank of $T^{ind}$.

### 5.1. Strongly minimal case.

Let $T$ be a strongly minimal theory (in particular it is a geometric theory). In this section we prove that $T^{ind}$ is $\omega$-stable and has Morley rank less than or equal to $\omega$.

**Proposition 5.1.** *Let $T$ be strongly minimal. Then $T^{ind}$ is $\omega$-stable.*

*Proof.* Suppose $(M,H(M))$ is a sufficiently saturated model of $T^{ind}$, and $A \subset M$ is a countable set. We may assume that $A$ is $H$-independent. We will count the number of types of the form $\mathrm{tp}_H(b/A)$ where $b \in M$.

Case 1: $b \in H(M)$. Then $bA$ is $H$-independent, and $\mathrm{tp}_H(b/A)$ is determined by $\mathrm{tp}(b/A)$ and the fact that $b \in H(M)$. By strong minimality of $T$, there are at most countably many such types.

Case 2: $b \in \mathrm{scl}(A)$. Then there are $h_1,\dots,h_l \in H(M)$ such that $b \in \mathrm{acl}(h_1 \dots h_l A)$. By Case 1, there are at most countably many types of the form $\mathrm{tp}_H(h_1,\dots,h_l/A)$ where $h_i \in H(M)$, and thus at most countably many types of the form $\mathrm{tp}_H(b/A)$ for $b$ as above.

Case 3: $b \notin \operatorname{scl}(A)$. Note that $bA$ is $H$-independent, and thus $\operatorname{tp}_H(b/A)$ is determined by $\operatorname{tp}_H(b/A)$ and the fact that $b \notin \operatorname{scl}(A)$. There is a unique such type. □

In the setting of lovely pairs of strongly minimal theories, there is a strong connection between the underlying geometry of the theory $T$ and the Morley rank of the associated theory of lovely pairs $T_P$. Buechler [9] showed that $T$ is trivial iff $MR(T_P) = 1$, $T$ is locally modular non-trivial iff $MR(T_P) = 2$ and $T$ is not locally modular iff $MR(T_P) = \omega$. He used this result to prove that pseudolinearity implies linearity. We will now show that the Morley rank of the theory $T^{ind}$ "detects" trivial theories, in the sense that $T$ is trivial iff $MR(T^{ind}) \leq 2$ and non-trivial iff $MR(T^{ind}) = \omega$.

**Proposition 5.2.** *Let $T$ be strongly minimal and trivial (i.e $\operatorname{acl}(A) = \bigcup_{a \in A} \operatorname{acl}(a)$). Let $(M, H) \models T^{ind}$. Then $T^{ind}$ has Morley rank 1 iff for all $a \in M$, $a \notin \operatorname{acl}(\emptyset)$, $\operatorname{acl}(a) \setminus \operatorname{acl}(\emptyset)$ is finite. Otherwise, $T$ has Morley rank 2.*

*Proof.* Let $(M, H)$ be a sufficiently saturated model of $T^{ind}$. Note that because of triviality, $\operatorname{acl}_H = \operatorname{acl}$ in $(M, H)$.

Let $b \in H(M)$. Note for any (small) $A \subset M$, there is a unique non-algebraic $L_H$-type of an element of $H(M)$ over $A$. Thus $MR(b) = 1$. This shows that $MR(H(x)) = 1$.

Now, let $b \notin H(M)$. Let $A$ be a small subset of $M$. If $b \in \operatorname{scl}(A)$, by triviality of $T$, either $b \in \operatorname{acl}(A)$, in which case, $MR(b/A) = 0$, or $b \in \operatorname{acl}(h) \setminus \operatorname{acl}(A)$ for some $h \in H(M)$. Note also that $h \in \operatorname{acl}(b)$. Then $MR(b/A) = MR(h/A) = MR(h) = 1$. This shows that Morley rank of any small definable set in $(M, H)$ is $\leq 1$ (=1 if the set is infinite).

Note that any two large definable sets in $(M, H)$ have a large intersection, so there is a unique large type. It follows that $T^{ind}$ has Morley rank $\leq 2$.

Suppose $\operatorname{acl}(a) \setminus \operatorname{acl}(\emptyset)$ is finite for all non-algebraic $a \in M$, say of size $n$. Let $\theta(x)$ be the first order formula expressing "$x \in \operatorname{acl}(h) \setminus \operatorname{acl}(\emptyset)$ for some $h \in H(M)$". Then $\theta(x)$ has $n$ non-algebraic extensions over any small $A \subset M$. Since there is a unique large type over $A$, there are only finitely many non-algebraic types over $A$. Thus, in this case $T^{ind}$ has Morley rank 1.

Suppose $\operatorname{acl}(a) \setminus \operatorname{acl}(\emptyset)$ is infinite for all non-algebraic $a \in M$. Then we can find $L$-formulas $\phi_n(x, y)$, $n \in \omega$, such that for $a \in M \setminus \operatorname{acl}(\emptyset)$, we have $\phi_n(M, a) \subset \operatorname{acl}(a) \setminus \operatorname{acl}(\emptyset)$, and $\phi_n(M, a)$ are finite, disjoint and non-empty. Let $\psi_n(x) = \exists y \in H\ \phi_n(x, y)$. Then $\psi_n(M)$ are infinite and small. From the disjointness of $\phi_n(M, a)$ for a fixed $a$ and independence of $H(M)$ it follows that $\psi_n(M)$ are disjoint. Thus, in this case, $T^{ind}$ has Morley rank 2.

□

**Proposition 5.3.** *Let $T$ be strongly minimal and non-trivial. Then $T^{ind}$ has Morley rank $\omega$.*

*Proof.* Suppose $(M, H(M))$ is sufficiently saturated model of $T^{ind}$, and $A \subset M$ is a countable set. We may assume that $A$ is $H$-independent. Let $b \in M$.

Case 1: $b \in H(M)$. Then $bA$ is $H$-independent, and $\operatorname{tp}_H(b/A)$ is determined by $\operatorname{tp}(b/A)$ and the fact that $b \in H(M)$. In this case $MR(b/A) \leq 1$.

Case 2: $b \in \operatorname{scl}(A)$. Then there are $h_1, \ldots, h_l \in H(M)$ such that $b \in \operatorname{acl}(h_1 \ldots h_l A)$. We may assume that $l$ is minimal. Then $b$ is $\mathcal{L}_H$-interalgebraic with $h_1 \ldots h_l$ over

$A$. Thus $MR(b/B) = MR(h_1 \dots h_l/A) = l$. Since $T$ is not trivial, by Proposition 3.21 for every $n$ there exists an algebraic $n$-gon $a_1, \dots, a_{n-1}, a_n$, and we can assume that $a_1, \dots, a_{n-1} \in H(M)$ (and thus $a_n \notin H$). We may also assume that $a_1 \dots a_n$ is independent from $A$ over $\emptyset$. Thus for any $b \in \mathrm{scl}(A)$, $MR(b/A) < \omega$ but can have arbitrarily large finite values.

Case 3: $b \notin \mathrm{scl}(A)$. As noted in the proof of Proposition 5.1, there a unique such 1-type over $A$. Then $MR(b/A) \leq \omega$. Since $T$ is not trivial, for every $n$ there exists an algebraic $n{+}2$-gon $a_1, \dots, a_{n+2}$, where $a_{n+2} = b$, $a_{n+1} \notin H(M)$, $a_1, \dots, a_n \in H(M)$ and $a_1 \dots a_{n+2}$ is independent from $A$ over $\emptyset$. Then $\mathrm{tp}_H(b/a_1 \dots a_{n+1}A)$ has Morley rank $n$. Therefore $MR(b/A) \geq \omega$. Thus $MR(b/A) = \omega$.

□

Next we will take a look at the geometric properties of $T^{ind}$. It is well-known that in case of lovely pairs (or belles paires, in the stable case), if $T$ is one-based, then so is the pair theory $T_P$. This is no longer the case for $T^{ind}$, as the following example illustrates.

**Example 5.4.** *Let $(V, +, 0, H)$ be a vector space over $\mathbb{Q}$, where $H(V) = \{v_i : i \in \omega\}$ consists of linearly independent vectors over $\mathbb{Q}$. Furthermore assume that $V \neq span(\{v_i : i \in \omega\})$. Then $(V, +, 0, \{v_i : i \in \omega\}) \models T^{ind}$ where $T$ is the theory of vector spaces over $\mathbb{Q}$. Let $u \in V \setminus span(\{v_i : i \in \omega\})$. Note that $u$ being generic is $H$-independent and that $\mathrm{acl}_H(u) = span(u)$. Let $t = u + v_1$.*

***Claim** $T^{ind}$ is not 1-based.*

*Note that $t$ is small over $u$, $t$ is interdefinable with $v_1$ over $u$ and that $MR(\mathrm{tp}(t/u)) = 1$. Let $t' = u + v_2$, then $\mathrm{tp}_H(v_1/u) = \mathrm{tp}_H(v_2/u)$ (since they are $H$-independent) and thus $\mathrm{tp}_H(t/u) = \mathrm{tp}_H(t'/u)$. Note that $t \mathop{\smile\!\!\!\!\!\!|}^{ind}_{u} t'$. Note also that $t - t' = v_1 - v_2$ so $t$ is interdefinable with $\{v_1, v_2\}$ over $t'$. Thus $MR(\mathrm{tp}_H(t/t')) = 2$ and thus $t \not\mathop{\smile\!\!\!\!\!\!|}^{ind}_{t'} u$. Hence $T^{ind}$ is not 1-based.*

Carmona showed in [10] that when $T$ is linear, $T^{ind}$ is CM-trivial.

5.2. **SU-rank one theories.** Let $T$ be an $SU$-rank one theory.

**Theorem 5.5.** *The theory $T^{ind}$ is supersimple.*

*Proof.* We will prove that non-dividing has local character.

Let $(M, H(M)) \models T^{ind}$ be saturated. Let $C \subset D \subset M$ be small sets and assume that $C = \mathrm{acl}_H(C)$ and $D = \mathrm{acl}_H(D)$. Note that both $C$ and $D$ are $H$-independent. Let $\vec{a} \in M$. We will find "geometric conditions" for the type of $\vec{a}$ over $C$ and $D$ that guarantee that $\mathrm{tp}_H(\vec{a}/D)$ does not divide over $C$.

We may write $\vec{a} = (\vec{a}_1, \vec{a}_2, \vec{a}_3) \in M$ so that $\vec{a}_1$ is an independent tuple over $CH$, $\vec{a}_2$ is an independent tuple over $C\vec{a}_1$, $\vec{a}_2 \in \mathrm{acl}(HC\vec{a}_1)$ and $\vec{a}_3 \in \mathrm{acl}(C\vec{a}_1\vec{a}_2)$. Assume that $\vec{a}_1$ is an independent tuple over $DH$ and that $HB(\vec{a}/D) = HB(\vec{a}/C)$.

**Claim** $\mathrm{tp}_H(\vec{a}/D)$ does not divide over $C$.

Let $p(\vec{x}, D) = tp(\vec{a}_1, D)$. Let $\{D_i : i \in \omega\}$ be an $\mathcal{L}_H$-indiscernible sequence over $C$. Since $\vec{a}_1$ is independent over $D$, $tp(\vec{a}_1, D)$ does not divide over $C$ and $\cup_{i\in\omega} p(\vec{x}, D_i)$ is consistent. We can find $\vec{a}_1' \models \cup_{i\in\omega} p(\vec{x}, D_i)$ such that $\{\vec{a}_1' D_i : i \in \omega\}$ is indiscernible and $\vec{a}_1'$ is independent over $\cup_{i\in\omega} D_i$. By the generalized extension property, we may assume that $\vec{a}_1'$ is independent over $\cup_{i\in\omega} D_i H$. Note that $\vec{a}_1 D$ is $H$-independent, $\vec{a}_1 D_i$ is also $H$-independent for any $i \in \omega$. So by Lemma 2.8 $\mathrm{tp}_H(\vec{a}_1 D) = \mathrm{tp}_H(\vec{a}_1' D_i)$ for any $i \in \omega$.

Now let $\vec{h} = HB(\vec{a}/C)$ (viewed as a tuple) and let $q(\vec{y}, \vec{a}_1, D) = tp(\vec{h}, \vec{a}_1, D)$. Note that $\vec{h}$ is an independent tuple over $\vec{a}_1 D$ (as well as an independent tuple over $\vec{a}_1 C$). Since $\{D_i a_1' : i \in \omega\}$ is an $\mathcal{L}$-indiscernible sequence, there is $\vec{h}' \models \cup_{i\in\omega} q(\vec{y}, \vec{a}_1', D_i)$. We may assume that $\vec{h}'$ is independent from $\cup_{i\in\omega} D_i a_1'$. By the generalized coheir/density property, we may assume that $\vec{h}' \in H$. Note that since each $\vec{a}_1' D_i$ is $H$-independent, then $\vec{h}'\vec{a}_1' D_i$ is also $H$-independent. On the other hand, $\mathrm{tp}(\vec{h}, \vec{a}_1, D) = \mathrm{tp}(\vec{h}', \vec{a}_1', D_i)$ for each $i$, so by Lemma 2.8 we have $\mathrm{tp}_H(\vec{h}, \vec{a}_1, D) = \mathrm{tp}_H(\vec{h}', \vec{a}_1', D_i)$. This shows that $\mathrm{tp}(\vec{a}_1, \vec{h}/D)$ does not divide over $C$ and since $\vec{a} \in \mathrm{acl}(\vec{a}_1, \vec{h}C)$ we get that $\mathrm{tp}(\vec{a}/D)$ does not divide over $C$.

Since for any $D$ and $\vec{a}$ we can always choose a countable set $C$ with the properties described above, $T$ is simple.

Given any $D = \mathrm{acl}_H(D)$ and a tuple $\vec{a} \in M$, write $\vec{a} = (\vec{a}_1, \vec{a}_2, \vec{a}_3) \in M$ so that $\vec{a}_1$ is an independent tuple over $DH$, $\vec{a}_2$ is an independent tuple over $D\vec{a}_1$, $\vec{a}_2 \in \mathrm{acl}(HD\vec{a}_1)$ and $\vec{a}_3 \in \mathrm{acl}(D\vec{a}_1\vec{a}_2)$. We can always choose a finite $B \subset D$ such that for $C = \mathrm{acl}_H(B)$ we have $HB(\vec{a}/C) = HB(\vec{a}/D)$ and $\vec{a}_3 \in \mathrm{acl}(C, \vec{a}_1, \vec{a}_2)$. Then $\mathrm{tp}(\vec{a}/D)$ does not divide over $B$ and $T$ is supersimple. □

**Proposition 5.6.** *Let $(M, H) \models T^{ind}$ be saturated, let $C \subset D \subset M$ be small and such that $C = \mathrm{acl}_H(C)$, $D = \mathrm{acl}_H(D)$ and let $a \in M$. Then $\mathrm{tp}(a/D)$ forks over $C$ iff $a \in D \setminus C$ or $a \in \mathrm{scl}(D) \setminus \mathrm{scl}(C)$ or $HB(a/C) \neq HB(a/D)$.*

*Proof.* In the proof of Theorem 5.5 we showed that if $a \in C$ or if $HB(a/C) = HB(a/D)$ then $\mathrm{tp}(a/D)$ does not fork over $C$. So it remains to show the other direction, which we do case by case.

**Case 1:** Assume that $a \in D \setminus C$, then $a$ became algebraic over $D$ and $\mathrm{tp}(a/D)$ forks over $C$.

**Case 2:** Assume that $a \in \mathrm{scl}(D) \setminus \mathrm{scl}(C)$. Let $\vec{d} \in D$ and let $\vec{c} \in C$ be such that $a \in acl(\vec{c}\vec{d}H)$. We can choose $\vec{d}$ independent over $HC$. Let $\vec{h} \in H$ be such that $a \in acl(\vec{c}\vec{d}\vec{h})$. Let $p(x, \vec{y}) = tp_H(a, \vec{d}/C)$.

Let $\{\vec{d}_i : i \in \omega\}$ be an $\mathcal{L}$ -indiscernible sequence in $\mathrm{tp}(\vec{d}/C)$ over $C$ such that $\{\vec{d}_i : i \in \omega\}$ is independent over $C$. By the generalized extension property, we may assume that $\{\vec{d}_i : i \in \omega\}$ is independent over $HC$. Note that by Lemma 2.8 $\{\vec{d}_i : i \in \omega\}$ is an $\mathcal{L}_H$-indiscernible sequence over $C$. Assume, in order to get a contradiction, that there is $a' \models \cup_{i\in\omega} p(x, \vec{d}_i)$. Then there are $\{\vec{h}_i : i \in \omega\}$ such that $a' \in \mathrm{acl}(\vec{d}_i, \vec{c}, \vec{h}_i)$ for every $i$. But $a' \notin \mathrm{acl}(CH)$, so $\vec{d}_0 \not\downarrow_{CH} \vec{d}_1$, a contradiction.

**Case 3:** Assume that $HB(a/D) \neq HB(a/C)$. Then $HB(a/D)$ is a proper subset of $HB(a/C)$. Write $\vec{h}_C = HB(a/C)$, $\vec{h}_D = HB(a/D)$ and let $\vec{h}_E \in H$ be such that $\vec{h}_C = \vec{h}_D\vec{h}_E$. Note that $\vec{h}_E \neq \emptyset$ and that $\vec{h}_E \in D$ is an independent tuple over $C$.

Let $p(x, \vec{y}) = tp_H(a, \vec{h}_E/C)$. Let $\{\vec{h}_E^i : i \in \omega\}$ be an indiscernible sequence in $\mathrm{tp}(\vec{h}_E/C)$ such that $\{\vec{h}_E^i : i \in \omega\}$ is independent over $C$. Then by the generalized density property, we may assume that the sequence $\{\vec{h}_E^i : i \in \omega\}$ belongs to $H$. Note that by Lemma 2.8, the sequence $\{\vec{h}_E^i : i \in \omega\}$ is indiscernible over $C$. We will show that $\cup_{i\in\omega} p(x, \vec{h}_E^i)$ is inconsistent.

Assume, not, so there is $a' \models \cup_{i\in\omega} p(x, \vec{h}_E^i)$. Then we can find $\vec{h}_{D_i}$ in $H$ such that $HB(a'/C) = \vec{h}_{D_i}\vec{h}_E^i$. Since the $h_E^i$ are independent, we get that the $H$-basis of $a'$ over $C$ is not unique, a contradiction. □

**Corollary 5.7.** *Let $(M, H) \models T^{ind}$ be saturated, let $C \subset M$ be small and such that $C = \operatorname{acl}_H(C)$ and let $a \in H(M)$. Then $SU(a/C) \leq 1$.*

*Proof.* Clearly $SU(a/C) = 0$ iff $a \in C$. If $a \notin C$, then $HB(a/C) = \{a\}$ so if it forks over some superset $D$ of $C$ by Proposition 5.6 we must have that $HB(a/\operatorname{acl}_H(D)) = \emptyset$ and that means that $a \in \operatorname{acl}_H(D)$. □

**Corollary 5.8.** *Let $T$ be non-trivial and let $(M, H) \models T^{ind}$ be saturated, let $C \subset M$ be small such that $C = \operatorname{acl}_H(C)$ and let $a \in M$. Then*

1. *$a \in C$ iff $SU(\operatorname{tp}_H(a/C)) = 0$*
2. *$a \in scl(C)$ iff $SU(\operatorname{tp}_H(a/C)) < \omega$ and $SU(\operatorname{tp}_H(a/C)) = |HB(a/C)|$.*
3. *$a \notin scl(C)$ iff $SU(\operatorname{tp}_H(a/C)) = \omega$.*

*Proof.* If $a \in scl(C)$, then $a$ is interalgebraic with $HB(a/C)$ over $C$. By Corollary 5.7, $SU(\operatorname{tp}_H(HB(a/C)/C)) = |HB(a/C)|$ and thus $SU(\operatorname{tp}_H(a/C)) = |HB(a/C)|$. Since $T$ is not trivial, there are algebraic $n$-gons for $n$ large enough and thus we can get arbitrarily large values for $SU(a/C)$. If $a \notin scl(C)$, then by Proposition 5.6 and the existence of algebraic $n$-gons for $n$ large enough shows that $SU(\operatorname{tp}(a/C)) = \omega$. The other statements are clear. □

**Corollary 5.9.** *Let $T$ be trivial and let $(M, H) \models T^{ind}$ be saturated, let $C \subset M$ be small such that $C = \operatorname{acl}_H(C)$ and let $a \in M$. Then*

1. *$a \in C$ iff $SU(a/C) = 0$*
2. *If $a \in scl(C) \setminus C$ then $SU(\operatorname{tp}_H(a/C)) = 1$.*
3. *If $a \notin scl(C)$ then $SU(\operatorname{tp}_H(a/C)) = 1$.*

*Proof.* The first statement is clear. If $a \in scl(C) \setminus C$ then by triviality of $T$ there is a single $h \in H$ such that $a \in \operatorname{acl}(h)$ and by Corollary 5.7 $SU(a/C) = SU(h/C) = 1$. If $a \notin scl(C)$ and $D$ is a superset of $C$ such that $\operatorname{tp}_H(a/D)$ forks over $C$, then by Proposition 5.6 and triviality we must have that $a \in \operatorname{acl}_H(D)$. □

**Remark 5.10.** *Note that in the case when $T$ is strongly minimal, the behavior of Morley rank maybe different form that of the SU-rank (U-rank). Namely, as we showed in Proposition 5.3, for a trivial strongly minimal theory $T$ where $\operatorname{acl}(a) \setminus \operatorname{acl}(\emptyset)$ is infinite for $a \notin \operatorname{acl}(\emptyset)$, the theory $T^{ind}$ has Morley rank 2 (while its U-rank is 1).*

**Corollary 5.11.** *(Coordinatization) Let $(M, H) \models T^{ind}$ be $\kappa$-saturated, let $C \subset M$ be such that $C = \operatorname{acl}_H(C)$, $|C| < \kappa$ and let $\vec{a} \in M^n$. Write $\vec{a} = \vec{a}_1\vec{a}_2\vec{a}_3$ where $\vec{a}_1$ is algebraically independent over $HC$, $\vec{a}_2$ is algebraically independent over $C\vec{a}_1$ and $\vec{a}_2 \in \operatorname{scl}(C\vec{a}_1)$ and $\vec{a}_3 \in \operatorname{acl}(\vec{a}_1\vec{a}_2C)$. Then for every $e \in \vec{a}_1$, $\operatorname{tp}_H(e/C)$ is regular, $\vec{a}_2$ is interalgebraic with $HB(\vec{a}/C)$ over $C\vec{a}_1$ and for each $h \in HB(\vec{a}/C)$, $\operatorname{tp}_H(h/C\vec{a}_1)$ is regular. So there is an explicit coordinatization in $T^{ind}$ in terms of types of real elements.*

Our next goal is to describe canonical bases in $T^{ind}$, for any SU-rank 1 theory $T$. Note that since $T^{ind}$ is supersimple, it eliminates hyperimaginaries, so canonical bases exist as imaginaries, both in $T$ and $T^{ind}$. Let $Cb(\vec{a}/B)$ denote $Cb(\operatorname{stp}(\vec{a}/B))$, and $Cb_H(\vec{a}/B)$ denote $Cb(\operatorname{stp}_H(\vec{a}/B))$.

**Lemma 5.12.** *Let $(M, H)$ be a sufficiently saturated $H$-structure of $T$, $B \subset M$ an $H$-independent set, and $\vec{a} \in M$, $h = HB(\vec{a}/B)$ (viewed as an imaginary representing a finite set). Suppose $e \in \operatorname{acl}^{eq}(B)$ (in the original theory) is such that $\vec{a}h \mathop{\smile\!\!\!\!\!\!\mid}_e B$. Then $\vec{a} \mathop{\smile\!\!\!\!\!\!\mid}_e^{ind} B$.*

*Proof.* We may assume that $\vec{a} = \vec{a}_1\vec{a}_2\vec{a}_3$, where $\vec{a}_1$ acl-independent over $B \cup H(M)$, $\vec{a}_2 \in \operatorname{acl}(H(M)B\vec{a}_1)\setminus \operatorname{acl}(B\vec{a}_1)$, $\vec{a}_3 \in \operatorname{acl}(B\vec{a}_1\vec{a}_2)$. Note that $\vec{a}_2 \in \operatorname{acl}(\vec{a}_1 Bh)$, so $\vec{a}h \mathop{\smile\!\!\!\!\!\!\,|}_e B$ implies that $\vec{a}_2 \in \operatorname{acl}(\vec{a}_1 eh)$ and thus $HB(\vec{a}/B) = HB(\vec{a}/e)$. Since $\vec{a}h \mathop{\smile\!\!\!\!\!\!\,|}_e B$, we also have $\vec{a}_3 \in \operatorname{acl}(e\vec{a}_1\vec{a}_2)$. Since $HB(\vec{a}/B) = HB(\vec{a}/e)$ and $\vec{a}h \mathop{\smile\!\!\!\!\!\!\,|}_e B$ by our characterization of forking in $T^{ind}$ we get $\vec{a} \mathop{\smile\!\!\!\!\!\!\,|}_e^{ind} B$. □

**Proposition 5.13.** *Let $(M,H)$ be a sufficiently saturated $H$-structure of $T$, $B \subset M$ an $H$-independent set, and $\vec{a} \in M$. Then $Cb_H(\vec{a}/B)$ and $Cb(\vec{a}HB(\vec{a}/B)/B))$ are interalgebraic.*

*Proof.* Let $e = Cb(\vec{a}HB(\vec{a}/B)/B))$. We saw in the previous lemma that $\vec{a} \mathop{\smile\!\!\!\!\!\!\,|}_e^{ind} B$ and thus $Cb_H(\vec{a}/B) \in \operatorname{acl}^{eq}(e)$. Now let $\{\vec{a}_i : i < \omega\}$ be an $\mathcal{L}_H$-Morley sequence in $\operatorname{tp}_H(\vec{a}/\operatorname{acl}^{eq}_H(B))$. Let $h_j = HB(\vec{a}_j/B)$ (viewed as an imaginary representing a finite set). Note that $h_j \in \operatorname{dcl}_H(\vec{a}_j B)$. Thus $\{\vec{a}_i h_i : i < \omega\}$ is also an $\mathcal{L}_H$-Morley sequence over $B$. This implies $h_j = HB(\vec{a}_j/B\vec{a}_{<j}h_{<j})$, and hence $\operatorname{tp}(\vec{a}_j h_j/B\vec{a}_{<j}h_{<j})$ does not fork (in the sense of $\mathcal{L}$) over $B$. Thus, $\{\vec{a}_i h_i : i < \omega\}$ is also an $\mathcal{L}$-Morley sequence over $B$ in $\operatorname{tp}(\vec{a}h/B)$. Since $\operatorname{tp}(\vec{a}_0 h_0/\{\vec{a}_i h_i : 0 < i < \omega\}B)$ is a free extension of $\operatorname{tp}(\vec{a}_0 h_0/\{\vec{a}_i h_i : 0 < i < \omega\})$ we also get that $e = Cb(\vec{a}_0 h_0/\{\vec{a}_i h_i : 0 < i < \omega\})$. It follows that $e \in \operatorname{acl}^{eq}(\{\vec{a}_i h_i : i < \omega\})$.

Since $T^{ind}$ is supersimple there is $N \in \omega$ such that for all $n \geq N$, $\vec{a}_n \mathop{\smile\!\!\!\!\!\!\,|}_{\vec{a}_{<N}}^{ind} B$. In particular $HB(\vec{a}_n/B) = HB(\vec{a}_n/\vec{a}_{<N})$ and thus $h_n \in \operatorname{dcl}_H(\vec{a}_i : i < \omega)$ for every $n$. We then get $e \in \operatorname{acl}^{eq}_H(\{\vec{a}_i : i < \omega\})$. Now, since $\{\vec{a}_i : i < \omega\}$ is a Morley sequence in $\operatorname{tp}_H(\vec{a}/\operatorname{acl}^{eq}_H(B))$, we have

$$\{\vec{a}_i : i < \omega\} \mathop{\smile\!\!\!\!\!\!\,|}_{Cb_H(\vec{a}/B)}^{ind} B,$$

and thus also

$$\{\vec{a}_i : i < \omega\} \mathop{\smile\!\!\!\!\!\!\,|}_{Cb_H(\vec{a}/B)}^{ind} e.$$

It follows that $e \in \operatorname{acl}^{eq}_H(Cb_H(\vec{a}/B))$, as needed.

□

**Remark 5.14.** *Note that Proposition 5.13 implies geometric elimination of imaginaries in $T^{ind}$ down to imaginaries of $T$, when $T$ is a supersimple SU-rank 1 structure.*

**Question 5.15.** *If $T$ is a geometric theory, does $T^{ind}$ have (geometric) elimination of imaginaries down to imaginaries of $T$?*

**Example 5.16.** Let $(V,+,0,H) = (V,+,0,\{v_i : i \in \omega\})$ be the structure from Example 5.4. We will give another proof of non-1-basedness of $T^{ind} = Th(V,+,0,H)$, using Lemma 5.13. Take $t,u,v_1$ as in Example 5.4, so $u,t$ are generic and $t = u+v_1$.

First note that $HB(t/u) = \{v_1\}$. Now, by Lemma 5.13, $Cb_H(t/u)$ is interalgebraic (in $(T^{ind})^{eq}$) with $Cb(tv_1/u)$. Note that $Cb(tv_1/u)$ is interdefinable with $u$. On the other hand, $u \notin \operatorname{acl}_H(t) = \operatorname{acl}(t) = span(t)$. Thus, $Cb_H(t/u)$ is not algebraic over $t$, and therefore $T^{ind}$ is not 1-based.

Let $t' = u + v_2$, then $t,t'$ are the first two elements in a Morley sequence in $\operatorname{tp}_H(t/u)$. Note that $t - t' = v_1 - v_2$, so $v_1, v_2 \in \operatorname{acl}_H(t,t')$ and thus $u \in \operatorname{acl}_H(t,t')$. We will show below that when $T$ is 1-based, $T^{ind}$ is *2-based*: we need two elements in a Morley sequence in $T^{ind}$ to recover the canonical base.

Carmona [10] proved that when $T$ is linear $SU$-rank one theory, $T^{ind}$ is CM-trivial. We will show below that if $T$ is 1-based, then $T^{ind}$ is 2-based:

**Proposition 5.17.** *Let $T$ be a simple theory of $SU$-rank 1 and assume that $T$ is 1-based. Let $(M,H) \models T^{ind}$ be saturated, let $A \subset M$ be small and let $p \in S_k^H(A)$. Then whenever $\{\vec{a}^i : i \in \omega\}$ is an $\mathcal{L}_H$-Morley sequence in $p$ over $A$ we have that $\vec{a}^2 \downarrow^{ind}_{\vec{a}^0\vec{a}^1} A$.*

*Proof.* Let $\{\vec{a}^i : i \in \omega\}$ be an $\mathcal{L}_H$-Morley sequence in $p$ over $A$. We can write $\vec{a}^i = \vec{a}^i_1\vec{a}^i_2\vec{a}^i_3$ where $\vec{a}^i_1$ is an independent tuple over $AH(M)$, $\vec{a}^i_2$ is an independent tuple over $A\vec{a}^i_1$ and $\vec{a}^i_2 \in \mathrm{scl}(A\vec{a}^i_1)$ and $\vec{a}^i_3 \in \mathrm{acl}(A\vec{a}^i_1 a^i_2)$. Let $\vec{h}^i = HB(\vec{a}^i/A)$ seen as a tuple. We may choose the ordering of $\vec{h}^i$ so that $\{\vec{a}^i\vec{h}^i : i \in \omega\}$ is an $\mathcal{L}_H$-Morley sequence in $\mathrm{tp}(\vec{a}^0\vec{h}^0/A)$. Note that both $\{\vec{a}^i : i \in \omega\}$ and $\{\vec{a}^i\vec{h}^i : i \in \omega\}$ are $\mathcal{L}$-Morley sequences over $A$. Indeed, since $\vec{a}^i\vec{h}^i \downarrow^{ind}_A \vec{a}^0\vec{h}^0 \ldots \vec{a}^{i-1}\vec{h}^{i-1}$, we have that $\vec{a}^i_1\vec{h}^i$ is an acl-independent tuple over $A\vec{a}^0\vec{h}^0 \ldots \vec{a}^{i-1}\vec{h}^{i-1}$. Since $\vec{a}^i\vec{h}^i \in \mathrm{acl}(A\vec{a}^i_1\vec{h}^i)$, it follows that $\vec{a}^i\vec{h}^i \downarrow_A \vec{a}^0\vec{h}^0 \ldots \vec{a}^{i-1}\vec{h}^{i-1}$, and thus, $\{\vec{a}^i\vec{h}^i : i \in \omega\}$ is an $\mathcal{L}$-Morley sequence over $A$. Then clearly $\{\vec{a}^i : i \in \omega\}$ is also $\mathcal{L}$-Morley over $A$.

Since $T$ is 1-based and $\{\vec{a}^i\vec{h}^i : i \in \omega\}$ is an $\mathcal{L}$-Morley sequence, we have that $\vec{a}^1\vec{h}^1 \downarrow_{\vec{a}^0\vec{h}^0} A$, in particular $\vec{a}^1_2 \in \mathrm{acl}(\vec{a}^1_1\vec{h}^1\vec{a}^0_1\vec{a}^0_2\vec{h}^0)$. Since $T$ is 1-based and $\{\vec{a}^i : i \in \omega\}$ is a $\mathcal{L}$-Morley sequence, we also get $\vec{a}^1_3 \in \mathrm{acl}(\vec{a}^1_1\vec{a}^1_2\vec{a}^0_1\vec{a}^0_2\vec{a}^0_3)$. Thus $HB(\vec{a}^1/\vec{a}^0) \subset \{\vec{h}^0\vec{h}^1\}$. Similarly, $HB(\vec{a}^2/\vec{a}^0) \subset \{\vec{h}^0\vec{h}^2\}$ and $HB(\vec{a}^2/\vec{a}^1) \subset \{\vec{h}^1\vec{h}^2\}$.

Note that $\vec{h}_2 = HB(\vec{a}^2/A) = HB(\vec{a}^2/A\vec{a}^0\vec{a}^1) \subset HB(\vec{a}^2/\vec{a}^0\vec{a}^1)$. We want to show that $\vec{a}^2 \downarrow^{ind}_{\vec{a}^0\vec{a}^1} A$, so it suffices to show that $\vec{h}_2 = HB(\vec{a}^2/\vec{a}^0\vec{a}^1)$ and to show this it suffices to prove that $HB(\vec{a}^2/\vec{a}^0\vec{a}^1) \subset \vec{h}_2$. Note that $HB(\vec{a}^2/\vec{a}^0\vec{a}^1) \subset HB(\vec{a}^2/a^0) \cap HB(\vec{a}^2/\vec{a}^1) = \{\vec{h}^0\vec{h}^2\} \cap \{\vec{h}^1\vec{h}^2\}$. Since $\vec{h}^i = HB(\vec{a}^i/A)$ are disjoint from $\mathrm{acl}(A)$, and $\{\vec{a}^i\vec{h}^i : i \in \omega\}$ is an $\mathcal{L}$-Morley sequence, the tuples $\vec{h}^i$ are disjoint. Thus, $\{\vec{h}^0\vec{h}^2\} \cap \{\vec{h}^1\vec{h}^2\} = \vec{h}^2$. Hence $HB(\vec{a}^2/\vec{a}^0\vec{a}^1) \subset \vec{h}_2$, as needed.

□

**Remark 5.18.** *Note that a 2-based SU-rank 1 theory is 4-pseudolinear, meaning that canonical bases of plane curves have SU-rank $\leq 4$. Indeed, suppose $SU(ab/A) = 1$, and let $\{a_ib_i : i \in \omega\}$ be a Morley sequence in $\mathrm{tp}(ab/A)$. Then 2-basedness implies $Cb(ab/A) \subset \mathrm{acl}^{eq}(a_0b_0, a_1b_1)$, and therefore $SU(Cb(ab/A)) \leq 4$. In [9], it is shown that pseudolinear strongly minimal theories are locally modular (1-based). In [26], it is shown that a pseudolinear $\omega$-categorical SU-rank 1 theory is 1-based. In [19], Hrushovski gives an example of an $\omega$-categorical SU-rank 1 theory which is not 1-based. By the above, this theory is not 2-based (or even finitely based), but it is known to be CM-trivial. Thus, CM-triviality does not imply 2-basedness.*

5.3. **Thorn rank one.** Assume that $T$ is a thorn rank one theory. The goal of this subsection is to show that $T^{ind}$ is super-rosy of thorn-rank $\leq \omega$. Our proof follows the proof of super-rosyness given by Boxall for lovely pairs of thorn rank one theories.

**Theorem 5.19.** *The theory $T^{ind}$ is super-rosy of thorn rank less that or equal to $\omega$.*

*Proof.* Let $(M,H) \models T^{ind}$ be highly saturated. In order to show that $T^{ind}$ is super-rosy, we need to understand two steps:

**Claim 1** Let $\varphi(x,\vec{c})$ define an infinite subset of $H(M)$. Then $\varphi(x,\vec{c})$ does not thorn divide over $\emptyset$.

The proof is word by word the same one as the one presented in [6].

**Claim 2** Let $\theta(x,\vec{a})$ be an $\mathcal{L}_H$ formula defining a large subset of $M$. Then $\theta(x,\vec{a})$ does not thorn divide over $\emptyset$.

The proof is again very similar to the one presented by Boxall in [6] for lovely pairs of thorn rank one theories, but we will do some small changes to see how the arguments adapt to the new setting.

Suppose $\theta(x,\vec{a})$ thorn divides. Let $\hat{a}$ be the canonical parameter of $\theta(x,\vec{a})$, we will also write the definable set as $\theta(x,\hat{a})$. Let $D$ be a finite set such that $\hat{a}\notin \mathrm{acl}_H^{eq}(D)$ and such that $\{\theta(x,\hat{a}') : \hat{a}'\models \mathrm{tp}(\hat{a}/D)\}$ is $k$ -inconsistent. We may assume that $D\subset M$, that is, it contains only real elements. By noticing that $HB(D)\in acl_H(D)$ and exchanging $D$ for $D\cup HB(D)$ we may also assume that $D$ is $H$-independent.

Let $b\in\theta(x,\hat{a})$, since the family $\{\theta(x,\hat{a}') : \hat{a}'\models \mathrm{tp}(\hat{a}/D)\}$ is $k$ -inconsistent, there are at most $k-1$ conjugates of $\hat{a}$ over $bD$, so $\hat{a}\in\mathrm{acl}_H(bD)$. Since $\theta(x,\hat{a})$ defines an infinite large set, we may assume that $b\notin\mathrm{scl}_H(\hat{a}D)$. Let $\varphi(\hat{y},x)$ be an algebraic formula in the variable $\hat{y}$ such that $(M,H)\models\varphi(\hat{a},b)$. Let $\hat{a}^*$ be the canonical parameter of $\varphi(\hat{a},x)$. Note that $\hat{a}^*\in\mathrm{dcl}(\hat{a})$.

**Claim 3** $\hat{a}\in\mathrm{acl}(\hat{a}^*D)$.

Let $n$ be the multiplicity of $\varphi(\hat{y},x)$ (in the variable $\hat{y}$). Let $\hat{a}_1,\dots,\hat{a}_{n+1}$ be realizations of $\mathrm{tp}(\hat{a}/\hat{a}^*D)$. Then for any $b'$ with $\varphi(\hat{a}^*,b')$, we also have $\varphi(\hat{a}_1,b'),\dots,\varphi(\hat{a}_{n+1},b')$ and thus there are $i<j\leq n+1$ such that $\hat{a}_i=\hat{a}_j$.

Thus, $\hat{a},\hat{a}^*$ be interalgebraic over $D$. By Proposition 3.12 there is an $\mathcal{L}$-definable set $\psi(x,\vec{c})$, where $\vec{c}$ is a real tuple, such that $\psi(x,\vec{c})\triangle\varphi(\hat{a}^*,x)$ is small, where $\triangle$ is a boolean connective for the symmetric difference. Note that we can choose $\vec{c}$ to be a real base of $\hat{a}^*$.

Let $E(\vec{u},\vec{v})$ be the equivalence relation $\psi(x,\vec{u})\triangle\psi(x,\vec{v})$ is finite. Since $T$ eliminates $\exists^\infty$, $E(\vec{z},\vec{w})$ is a definable equivalence relation. Let $e=\vec{c}/E$.

Let $\psi(x,\vec{c}')$ be such that $\psi(x,\vec{c}')\triangle\psi(x,\vec{c})$ is small. It is were infinite, since $\psi(x,\vec{c}')\triangle\psi(x,\vec{c})$ is an $\mathcal{L}$ definable set it would be large. Thus if $\psi(x,\vec{c}')\triangle\psi(x,\vec{c})$ is small, then $\psi(x,\vec{c}')\triangle\psi(x,\vec{c})$ is finite and $E(\vec{c},\vec{c}')$. Thus $e=\vec{c}/E\in\mathrm{acl}_H(\hat{a}D)$.

**Claim 4** $\hat{a}\in\mathrm{acl}(eD)$.

Recall that $n$ is the multiplicity of $\varphi(\hat{y},x)$ (in the variable $\hat{y}$). Let $\hat{a}_1,\dots,\hat{a}_{n+1}$ be realizations of $\mathrm{tp}(\hat{a}/eD)$. Then there are $c_1,\dots,c_{n+1}$ such that $\psi(x,c_i)\triangle\varphi(\hat{a}_i,x)$ is small for $i\leq n+1$. Since $e=c_i/E$ for $i\leq n+1$, we have that $\psi(x,c_i)\triangle\psi(x,c_i)$ is finite for $i\leq n+1$. Let $b''\in\bigwedge_{i\leq n+1}\psi(x,c_i)\wedge\bigwedge_{i\leq n+1}\varphi(\hat{a}_i,x)$. Then we have $\varphi(\hat{a}_1,b''),\dots,\varphi(\hat{a}_{n+1},b'')$ and thus there are $i<j\leq n+1$ such that $\hat{a}_i=\hat{a}_j$.

Thus $e$ and $\hat{a}$ are interalgebraic over $D$. Note that $e\in acl_H(bD)\setminus acl_H(D)$. Since $b\notin\mathrm{scl}(D)$ the set $bD$ is $H$-independent and thus $e\in\mathrm{acl}(bD)\setminus acl(D)$, but $b\notin\mathrm{acl}(eD)$, a contradiction since $T$ has thorn rank one. □

As with the supersimple case, when $T$ is trivial, the thorn rank of $T^{ind}$ is one and when $T$ is not trivial, the thorn rank of $T^{ind}$ is $\omega$. The proof follows easily from the previous theorem and we leave the details to the reader.

**Question 5.20.** *In* [4] *the authors developed a theory of weakly one-based geometric theories. A generalization of this notion appears in* [7] *in the setting of structures with a robust independence notion (for example rosy theories), where it is proved that when $T$ is rosy of thorn rank one, weakly one-basedness coincides with linearity.*

*Find a reasonable notion of weak 2-basedness in the setting of rosy theories and explore its properties, in particular does Proposition 5.17 hold in this setting?*

5.4. **NIP theories.** We finish this section by addressing the question of preservation of NIP.

**Proposition 5.21.** *Let $T$ be a geometric theory, $(M,H)$ a sufficiently saturated $H$-structure of $T$, and suppose $T$ has NIP. Then $Th(M,H)$ also has NIP.*

*Proof.* We apply the criterion from [12, Thm 2.4]. We begin by showing that every formula $\phi(\vec{x},\vec{y})$ has NIP over $H(\vec{x})$. Assume otherwise, so there is an $\mathcal{L}_H$-formula $\phi(\vec{x},\vec{y})$, $I=(\vec{b}_i : i\in\omega)$ an indiscernible sequence of elements in $H(M)$ and $\vec{a}\in M$ such that $\phi(\vec{b}_i,\vec{a})$ holds iff $i$ is even. Then by Proposition 3.5 we have that there is an $\mathcal{L}$-formula $\psi(\vec{x},\vec{z})$ and an element $\vec{d}$ such that $\psi(\vec{x},\vec{d})\wedge H(\vec{x})$ holds if and only if $\phi(\vec{x},\vec{a})\wedge H(\vec{x})$ holds. Thus the $\mathcal{L}$-formula $\psi(\vec{x},\vec{y})$ has the IP, a contradiction. By Proposition 3.2 every formula in $(M,H)$ is equivalent to a boolean combination of existential formulas over $H$. This fact together with Theorem 2.4 [12] shows that $Th(M,H)$ also has NIP. □

**Remark 5.22.** *The above result could also have been proved doing very small modifications on Theorem 2.8* [5]. *Also, Theorem 2.11* [5] *can be easily modified to show that if $T$ is strongly dependent then $Th(M,H)$ is strongly dependent.*

Now we will study the special case when $T$ is geometric and has finite dp-rank

**Proposition 5.23.** *Let $T$ be a geometric theory of dp-rank $k<\omega$ and let $(M,H)$ be a sufficiently saturated $H$-structure of $T$. If $T$ is trivial and* $\mathrm{dcl}=\mathrm{acl}$, *then $Th(M,H)$ has dp-rank $k$.*

*Proof.* Since $T$ is trivial, every formula $\psi(x,\vec{y})$ in $T^{ind}$ is a boolean combination of $\mathcal{L}$-formulas and formulas of the form $H(f(x,\vec{y}))$ where $f(x,\vec{y})$ is an $\mathcal{L}$-definable function over $\emptyset$.

**Claim** Let $(\vec{a}_i : i\in\omega)$ be an $\mathcal{L}_H$-indiscernible sequence and let $b\in M$ and let $f(x,\vec{y})$ be an definable function over $\emptyset$ in the language $\mathcal{L}$. Then for all $i\in\omega$ either $H(f(b,\vec{a}_i))$ or $\neg H(f(b,\vec{a}_i))$.

Since $T$ is trivial and $f(x,\vec{y})$ is an $\mathcal{L}$-definable function, either there is an $\mathcal{L}$-definable function $h(x)$ such that for all $i$ $t(x,\vec{a}_i)=h(x)$ or there is $\mathcal{L}$-definable function $g(\vec{y})$ such that for all $i$ $t(x,\vec{a}_i)=g(\vec{a}_i)$. The existence of $h$ or $g$ only depends on the type of $\vec{a}_i$. If $t(x,\vec{a}_i)=g(\vec{a}_i)$ then since $(\vec{a}_i : i\in\omega)$ is $\mathcal{L}_H$-indiscernible we have that the value of $H(f(b,\vec{a}_i))$ agrees with the value of $H(g(\vec{a}_0))$. If $t(x,\vec{a}_i)=h(x)$ then the value of $H(f(b,\vec{a}_i))$ agrees with the value of $H(h(b))$. In any case, the value of $H(f(b,\vec{a}_i))$ does not depend on $i$.

Assume there is an ICT pattern of depth $n$ in $(M,H)$. Then there are $\mathcal{L}_H$ formulas $\psi_1(x,\vec{y}_1)$, …, $\psi_n(x,\vec{y}_n)$ and there are $\mathcal{L}_H$-indiscernible sequences sequences $\{(\vec{a}^j_i : i<\omega) : j\leq n\}$ that form a ICT pattern of depth $n$. Let $i_1,i_2,\dots,i_n<\omega$ and let $b$ realize $\psi_1(x,\vec{a}^{i_1}_1)\wedge\psi_2(x,\vec{a}^{i_2}_2)\wedge\cdots\wedge\psi_n(x,\vec{a}^{i_n}_n)$ and the negation of all other formulas. Each formula $\psi_1(x,\vec{y}_1)$ is a boolean combination of $\mathcal{L}$-formulas and formulas of the form $H(f(x,\vec{y}_1))$ where $f(x,\vec{y}_1)$ is an $\mathcal{L}$-definable function. Since the value of $H(f(b,\vec{a_1}^i))$ does not depend on $i$ we may replace each $\psi_i(x,\vec{y}_1)$ just by the $\mathcal{L}$-formulas inside it and obtain an ICT pattern of depth $n$ in $(M,H)$. □

**Question 5.24.** *Does the result of the previous Proposition remain true if we remove the assumption* $\mathrm{acl}=\mathrm{dcl}$*?*

**Proposition 5.25.** *Let $T$ be a geometric theory of dp-rank $k < \omega$. Then $T^{ind}$ has dp-rank greater than $n$ for every $n \in \mathbb{N}$ but bounded by $\aleph_0$.*

*Proof.* Since $T^{ind}$ is strongly dependent, then the dp-rank is bounded by $\aleph_0$. Let $(M, H)$ be a sufficiently saturated $H$-structure. Let $a_1, \dots, a_n, a_{n+1} \in M$ be an algebraic $n+1$-gon. We may assume that $a_1, \dots, a_n \in H(M)$, thus $HB(a_{n+1}) = \{a_1, \dots, a_n\}$ and $a_1, \dots, a_n \in \operatorname{acl}(a_{n+1})$. Since $dp - rk(H(x_1) \wedge \dots \wedge H(x_n)) \geq n$, we have dp-rank$(x = x) \geq n$. $\square$

**Remark 5.26.** *Let $T$ be a geometric theory which is dp-minimal and let $(M, H)$ be a sufficiently saturated $H$-structure of $T$. Let $C \subset M$ and let $\vec{a} \in M^n$ be such that $\vec{a} \in \operatorname{scl}(C)$. Let $\vec{h} = HB(\vec{a}/C)$. Then $\vec{a}$ is interalgebraic with $\vec{h}$ over $C$. Each $d \in \vec{h}$ has a type of dp-rank one over $C$, so small tuples can be coordinatized in terms of dp-rank one types. In the stable setting, for $a \notin \operatorname{scl}(C)$ we had that $\operatorname{tp}(a/C)$ was regular. What is the corresponding notion in the setting of strongly dependent theories? Does it have burden one?*

## 6. Groups

In this section we study the special case where $M$ is rosy. As before we consider the $H$-structure $(M, H)$ and our aim in this section is to study definable groups in $(M, H)$. We will show that there are no small definable groups. Then we consider the special case where $M = G$ is a group with $RM(G) = 1$. We will show that the $\mathcal{L}_H$-definable subgroups of $G^n$ are $\mathcal{L}$-definable. Finally we show that if $M$ is stable of $U$-rank one, the connected component of any $\mathcal{L}_H$-definable group is isomorphic to an $\mathcal{L}$-type definable group.

We will use the following tool in the rosy setting:

**Fact 6.1.** *Let $T$ be rosy, let $M \models T$ and let $G \subset M^n$ be a definable group. Then $G$ has generics in the sense that there is $g \in G$ such that for $h \in G$ with $g \mathop{\smile\!\!\!\!\!\!\!\perp} h$ we have $gh \mathop{\smile\!\!\!\!\!\!\!\perp} h$.*

We start by showing that the generic elements in definable groups are independent from $H$:

**Proposition 6.2.** *Let $M$ be a model of a thorn rank one theory and assume that $(M, H)$ is a $\aleph_0$-saturated $H$-structure. Let $A \subset M$ be finite and let $T \leq M^n$ be a $\mathcal{L}_H$-definable group over $A$. Let $\vec{b} \in T$ be a generic element of the group, then $HB(\vec{b}/A) = \emptyset$.*

*Proof.* We may assume that $A = A \cup HB(A)$ and thus that $A$ is $H$-independent. Let $\vec{b}, \vec{c} \in T$ be independent generics (over $A$) and let $\vec{b} \cdot \vec{c}$ stand for the product of $\vec{b}, \vec{c}$ in the group $T$. Since $\vec{c} \mathop{\smile\!\!\!\!\!\!\!\perp}^H_A \vec{b}$, $HB(\vec{c}/A) = HB(\vec{c}/A\vec{b})$ and $HB(\vec{c}\vec{b}/A) = HB(\vec{c}/A) \cup HB(\vec{b}/A)$ and thus $HB(\vec{c} \cdot \vec{b}/A) \subset HB(\vec{c}/A) \cup HB(\vec{b}/A)$.

Since $\vec{b}, \vec{c}$ are independent generics, $\vec{b} \cdot \vec{c} \mathop{\smile\!\!\!\!\!\!\!\perp}^H_A \vec{b}$ and thus $HB(\vec{b} \cdot \vec{c}/A) \mathop{\smile\!\!\!\!\!\!\!\perp}^H_A HB(\vec{b}/A)$ and so $HB(\vec{b} \cdot \vec{c}/A) \cap HB(\vec{b}/A) = \emptyset$. Similarly one has $HB(\vec{b} \cdot \vec{c}/A) \cap HB(\vec{c}/A) = \emptyset$. This together with $HB(\vec{c} \cdot \vec{b}/A) \subset HB(\vec{c}/A) \cup HB(\vec{b}/A)$ proves that $HB(\vec{c} \cdot \vec{b}/A) = \emptyset$. Since $\vec{c} \cdot \vec{b}$ is generic, and $\vec{c}^{-1}$ is also generic and independent from $\vec{c} \cdot \vec{b}$ over $A$, we get that $HB(\vec{b}/A) = HB(\vec{b} \cdot \vec{c} \cdot \vec{c}^{-1}/A) = \emptyset$. $\square$

**Corollary 6.3.** *Let $M$ be a model of a thorn rank one theory and assume that $(M,H)$ is a $\aleph_0$-saturated $H$-structure. Let $A \subset M$ be finite and let $T \leq M^n$ be a $\mathcal{L}_H$-definable group over $A$ and assume that $T$ is small. Then $T$ is finite.*

We will now consider two settings, first the stable one and then topological one. Our goal is to see how close are $\mathcal{L}_H$-definable groups from being $\mathcal{L}$-definable.

6.1. **Stable groups.**

**Notation 6.4.** *Let $M$ be a structure of Morley rank one and assume that $(M,H)$ is an $H$-structure. For an $\mathcal{L}$-formula $\psi$, we denote by $RM(\psi)$ the Morley rank and $dM(\psi)$ the Morley degree calculated within $Th(M)$. Similarly, for an $\mathcal{L}_H$-formula $\psi$ we denote by $RM_H(\psi)$, $dM_H(\psi)$ the Morley rank and the Morley degree inside the theory $Th(M,H)$.*

*Assume now that $M$ is a group. For a complete $\mathcal{L}$-type $p$, we denote by $Stab(p)$ the stabilizer of $p$ in the language $\mathcal{L}$ and for a complete $\mathcal{L}_H$-type $q$ we write $Stab_H(p)$ for its stabilizer.*

**Proposition 6.5.** *Let $G$ be group with $RM(G) = 1$ and assume that $(G,H)$ is a $\aleph_0$-saturated $H$-structure. Let $A \subset G$ be finite and let $T \leq G^n$ be a $\mathcal{L}_H$-definable subgroup over $A$. Then $T$ is $\mathcal{L}$-definable.*

*Proof.* Let $A$ and $T$ be as above. By exchanging $A$ for $A \cup HB(A)$ we may assume that $A$ is $H$-independent. Let $\vec{b} \in T$ be generic, so $HB(\vec{b}/A) = \emptyset$.

Assume first that $T = T^0$, that is, $T$ is connected. Since $HB(\vec{b}/A) = \emptyset$, we may assume that we can write $\vec{b} = (b_1, \dots, b_l, b_{l+1}, \dots, b_n)$, where $(b_1, \dots, b_l)$ are $H$-independent and $b_{l+1}, \dots, b_n \in \operatorname{acl}(b_1, \dots, b_l)$. Let $p = \operatorname{tp}_H(\vec{b}/A)$ and note that $RM_H(\vec{b}/A) = \omega l$. Since $\vec{b}$ is generic, $Stab_H(p) = T$. Let $q = \operatorname{tp}(\vec{b}/A)$, so $q$ is the restriction of $p$ to the old language.

**Claim** Let $\vec{c} \models q$ be such that $\vec{b} \mathop{\smile\!\!\!\!\!\!|}_A \vec{c}$, then $\vec{b} \cdot \vec{c} \models q$.

We may assume that $\vec{b}, \vec{c} \mathop{\smile\!\!\!\!\!\!|}_A H$, so $\vec{b}, \vec{c} \models p$ and $\vec{b} \mathop{\smile\!\!\!\!\!\!|}_A^H \vec{c}$. Since $p$ is the unique generic type of $T$, we must have that $\vec{b} \cdot \vec{c} \models p$ and thus $\vec{b} \cdot \vec{c} \models q$ as we wanted.

Let $D = Stab(q)$, where we now take the stabilizer in the $\mathcal{L}$ language. Since $q$ is closed under generic multiplication and inverses, every member of $D$ is a product of two realizations of $q$, we have that $D = q \cdot q$. Since $q$ is a generic for $D$, we obtain then that $RM(D) = l$, $dM(D) = 1$, $RM_H(D) = \omega l$, $dM_H(D) = 1$. Also $p \cdot p \subset q \cdot q = D$, so $T = p \cdot p \leq D$. Since $MR_H(T) = \omega l$ and $dM_H(D) = 1$ we must have that $D = T$.

Now assume that $T$ is not necessarily connected. Then $T = T^0 \cup \vec{b}_1 T^0 \cup \dots \cup \vec{b}_k T^0$ for some finite $\vec{b}_1, \dots, \vec{b}_k \in T$. Since $T^0$ is $\mathcal{L}$-definable so is $T$. □

**Proposition 6.6.** *Let $M$ be a stable structure of $U$-rank one and let $H$ be a subset of $M$ such that $(M,H)$ is a $\aleph_1$-saturated $H$-structure. Let $A \subset M$ be countable and let $T \subset M^n$ be a $\mathcal{L}_H$-definable group over $A$. Let $T^0$ be the connected component of $T$. Then $T^0$ is definably isomorphic to an $\mathcal{L}$-definable group.*

*Proof.* We may assume by enlarging $A$ that $(A, H \cap A) \preceq (M,H)$ and that $(M,H)$ is saturated and strongly homogeneous over $A$. Thus every complete $\mathcal{L}$-type and every complete $\mathcal{L}_H$-type over $A$ is stationary. Also note that $T^0$ is $A$-definable. Let $b_1, b_2, a_2$ be independent generics in $T^0$. Let $a_3 = b_1 \cdot a_2$, let $a_1 = b_2^{-1} \cdot a_3$, $b_3 = b_2 \cdot b_1^{-1}$. Note that since $b_1$ and $a_2$ are independent generics, $HB(b_1, a_2/A) = \emptyset$

and thus by Lemma 2.8 $a_3 \in \operatorname{acl}(b_1, a_2, A)$. To simplify the argument we will assume that $a_3 \in \operatorname{dcl}(b_1, a_2, A)$. Similarly $a_1 \in \operatorname{dcl}(b_2, a_3, A)$, $b_3 \in \operatorname{dcl}(b_1, b_2, A)$. Then $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ forms an algebraic quadrangle in the language $\mathcal{L}$. Applying the group configuration construction to $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ (see for example [8, 23]) we obtain a connected $\mathcal{L}$-type-definable group $G$. We will now follow the proof in [8] to understand how $G$ is related to $T^0$. Let $p = \operatorname{tp}(b_2/A)$, note that $p = \operatorname{tp}(a_1/A)$, and that $b_2$ defines a unique germ given by $h_{b_2}(x) = b_2 \cdot x$, and this $\mathcal{L}$-function is defined for elements satisfying $p$ that are independent from $b_2$. The group $G$ is given by $p \times p/E$, where for $d_1, d_2 \in p$, $c_1, c_2 \in p$, $(d_1, d_2)E(c_1, c_2)$ if for $a \models p$ independent from $\{d_1, d_2, c_1, c_2\}$, $d_1 \cdot (d_2 \cdot a) = c_1 \cdot (c_2 \cdot a)$. In few words, every element in $G$ is formed as the product of two generics (realizations of $p$) and we identify the product $d_1 \cdot d_2$ with $c_1 \cdot c_2$ if they agree generically. We will now build a definable isomorphism between $T^0$ and $G$. For $t \in T^0$, let $b \models p$ be independent from $t$ and define $\varphi(t) = (b^{-1}, b \cdot t)/E$. Note that $\varphi$ is $A$-definable and does not depend on the choice of $b$.

**Claim** The map $\varphi$ is $1-1$.

Let $t_1, t_2 \in T$ and let $b \models p$ be independent from $t_1, t_2$. If $(b^{-1}, b \cdot t_1)E(b^{-1}, b \cdot t_2)$ then for $a \models p$ generic, $b^{-1} \cdot b \cdot t_1 \cdot a = b^{-1} \cdot b \cdot t_2 \cdot a$ and thus $t_1 = t_2$.

**Claim** The map $\varphi$ is onto.

Let $d_1, d_2 \models p$ be generics. Let $a \models p$ be generic such that $a \mathop{\smile\!\!\!\!\!\mid}_A H d_1 d_2$. Note that by stationarity $a$ realizes the unique extension of $p$ which is $H$-independent and thus $a$ is a generic of the group $T^0$. Again by stationarity, $d_1 \cdot a$ is generic in $T^0$, $a^{-1} \cdot d_2$ is also generic in $T^0$ and $(d_1, d_2)E(d_1 \cdot a, a^{-1} \cdot d_2)$. Thus $t = (d_1 \cdot a) \cdot (a^{-1} \cdot d_2)$ being a product of generics in $T^0$ belongs to $T^0$ and $\varphi(t) = (d_1, d_2)/E$.

It is easy to see that $\varphi$ is a homomorphism of groups and thus $\varphi$ is an isomorphism between $T^0$ and $G$. □

**Remark 6.7.** *Let $(M, H)$, $T$ and $A$ be as in the previous proposition. Note that the connected component $T^0$ of $T$ is isomorphic to an $\mathcal{L}$-definable group and thus $T$ is $\mathcal{L}$-definable-by-bounded. In the case where $M$ is $\omega$-stable, the index of $T^0$ in $T$ is finite and thus $T$ is $\mathcal{L}$-definable-by-finite. In the later case, $T$ can be written as a semidirect product of $T^0$ and $T/T^0$, but the action of $T^0$ is $\mathcal{L}_H$-definable and it is not clear we can recover a $\mathcal{L}$-definable copy of $T$.*

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