Blurred complex exponentiation

Jonathan Kirby

version 1.4, 18^{th} January 2007

Abstract

I describe a way of blurring an exponential field in a way which preserves the natural closure operator. The results of blurring the complex exponential field and of blurring Zilber's exponential field are isomorphic.

1 Introduction

Zilber constructed an exponential field $K_{\exp} = \langle K; +, \cdot, \exp \rangle$ which is defined up to isomorphism by a list of axioms which is essentially as short and simple as possible [Zil05]. He called this field *pseudoexponentiation*. However, from an algebraic perspective it is a perfectly genuine exponential field so in this paper I will call it *Zilber's exponential field*. The purpose of the construction was to make the following conjecture:

Conjecture 1.1 (Zilber). *The complex exponential field* $\mathbb{C}_{\exp}$ *is isomorphic to* $K_{\exp}$.

This conjecture contains Schanuel's Conjecture:

Conjecture 1.2 (Schanuel). *If* $a_1, \ldots, a_n$ *are complex numbers then*

$$\operatorname{td}_{\mathbb{Q}}(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) - \operatorname{ldim}_{\mathbb{Q}}(a_1, \ldots, a_n) \geqslant 0$$

where $\operatorname{ldim}_{\mathbb{Q}}$ *is* $\mathbb{Q}$*-linear dimension.*

This conjecture answers essentially all number theoretic questions relating to exponentiation. As a simple example, taking $a_1 = 1$ and $a_2 = 2i\pi$, it implies that e and π are algebraically independent. Even this very special case has been an open question for over a century. Faced with this, Zilber's conjecture seems out of reach.

However, Schanuel's conjecture is not the main idea in Zilber's conjecture. If, for example, it were discovered that e and π were algebraically dependent then it should be possible to adapt Zilber's conjecture accordingly. The main idea is rather that the "large scale" behaviour of $K_{\exp}$ is controlled by a pregeometry, that is, a closure operator satisfying the Steinitz exchange property. This pregeometry gives rise to a dimension theory in the same way that algebraic closure gives a dimension theory on a pure field.

There is a notion of *exponential algebraic closure*, ecl, on $\mathbb{C}_{\exp}$ which also defines a pregeometry. In this paper I show that ecl can be studied without answering any of the difficult number theoretic questions. I replace $K_{\exp}$ and $\mathbb{C}_{\exp}$ by the *blurred exponential fields* $K_{\mathcal{B}} = \langle K; +, \cdot, \mathcal{B} \rangle$ and $\mathbb{C}_{\mathcal{B}} = \langle \mathbb{C}; +, \cdot, \mathcal{B} \rangle$. In each case $\mathcal{B}$ is a "blurring" of the graph of exponentiation defined by $(x, y) \in \mathcal{B}$ iff $\exp(x)/y$ is exponentially algebraic. The main theorem of this paper can be viewed as a blurred version of Zilber's conjecture.

Theorem 1.3 (Main theorem). *The blurred exponential fields $K_{\mathcal{B}}$ and $\mathbb{C}_{\mathcal{B}}$ are isomorphic.*

I also show that the blurring process preserves the exponential algebraic closure operator, ecl. That is, ecl is definable from the blurred structure and controls that structure in the same way the it controls Zilber's exponential field. Hence it follows that the large scale behaviour of complex exponentiation is controlled by a pregeometry, and furthermore that this pregeometry is isomorphic to the pregeometry controlling Zilber's field. This can be viewed as establishing the "geometric part" of Zilber's conjecture. Unfortunately, although ecl controls $\mathbb{C}_{\mathcal{B}}$ very well (for example every permutation of a basis extends to an automorphism of the structure) one cannot immediately deduce that this also holds for $\mathbb{C}_{\exp}$ itself. Indeed, Mycielski's question of whether $\mathbb{C}_{\exp}$ has more than the two obvious automorphisms remains open.

In section 2, I define exponential algebraic closure and give some basic properties. In section 3, I describe the Schanuel predimension and the Schanuel property, not just in their usual absolute form but also relative to a subfield. This relative Schanuel predimension gives rise to a pregeometry which I compare with the exponential algebraic closure. Section 4 describes Zilber's exponential field and its theory. In section 5, I describe the process of blurring an exponential field and give an axiomatization of $K_{\mathcal{B}}$. Section 6 shows that $\mathbb{C}_{\mathcal{B}}$ also satisfies these axioms.

The deduction that $K_{\exp}$ and $\mathbb{C}_{\exp}$ are isomorphic relies on the axiomatization being uncountably categorical. This is proved in a separate paper [Kir] using the same method of quasiminimal excellence that Zilber used to show that his description of $K_{\exp}$ is categorical.

This paper could be viewed as an exploration of the consequences for the complex exponential field itself of James Ax's work [Ax71] on power series rings over $\mathbb{C}$.

Some of this work was done as part of my DPhil thesis in Oxford, under the supervision of Boris Zilber. I would like to thank him for many useful conversations. Theorem 6.5 was conceived and proved at the Isaac Newton Institute. The remainder of the work was carried out at the University of Illinois at Chicago.

2 Exponential algebraic closure

The definition of exponential algebraic closure is taken from the Angus Macintyre's survey paper [Mac96].

Definition 2.1. An *exponential ring* is a structure $\langle R; +, \cdot, \exp\rangle$ which is a field together with a homomorphism from the additive group $\langle R; +\rangle$ to the multiplicative group $\langle R^\times, \cdot\rangle$. If R is a field this is an *exponential field*.

Given a set X, the *exponential polynomial ring* over X is $\mathbb{Z}[X]^E$, the free exponential ring generated by X. If R is an exponential ring we can also consider the exponential ring $R[X]^E$ of exponential polynomials with coefficients from R. As with polynomial rings, when $X = \{x_1, \ldots, x_n\}$ these exponential polynomial rings act as rings of functions from R^n to R.

Definition 2.2. Let R be an exponential ring, let B be a subset of R and let $\alpha \in R$. Then α is exponentially algebraic over B iff there are $n, m \in \mathbb{N}$, tuples $a = (a_1, \ldots, a_n) \in R^n$, $b = (b_1, \ldots, b_m) \in B^m$, and functions $f_1, \ldots, f_n \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_m]^E$ such that:

i) $a_1 = \alpha$,

ii) For each $i = 1, \ldots, n$, $f_i(a, b) = 0$, and

iii) $\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}(a,b) \neq 0$

Define $\operatorname{ecl}(B)$, the *exponential algebraic closure* of B in R, to be the set of elements of R which are exponentially algebraic over B.

Most of the following basic properties of exponential algebraic closure were given in [Mac96].

Proposition 2.3. *Suppose that R has no zero divisors. Then* ecl *is a closure operator, and, for any subset B of R,* $\operatorname{ecl}(B)$ *is an exponential subring of R which is relatively algebraically closed and contains the kernel of* exp. *Furthermore,* ecl *has finite character, that is* $\operatorname{ecl} B = \bigcup \{\operatorname{ecl}(B_0) \mid B_0 \subseteq_{\text{fin}} B\}$.

Proof. All simple calculations from the definition. □

Macintyre also notes the following.

Proposition 2.4. *If R is $\mathbb{C}_{\exp}$ and B is an exponential subfield then $\alpha \in$* $\operatorname{ecl}(B)$ *iff there is an n-tuple a containing α which is an isolated zero of an exponential polynomial map f, whose n components lie in $B[x_1, \ldots, x_n]^E$.*

Proof. Take a and f witnessing that $\alpha \in \operatorname{ecl} B$. By the implicit function theorem for holomorphic maps, f is locally invertible at a, and so a is an isolated zero. □

Alex Wilkie [Wil05] has shown that ecl in $\mathbb{C}_{\exp}$ is equal to the definable closure operator in the real field with restricted real exponentiation and restricted sine, when $\mathbb{C}$ is interpreted as $\mathbb{R}^2$. In particular, this shows that ecl is a pregeometry on $\mathbb{C}_{\exp}$. It appears to be an open question whether ecl is a pregeometry on any exponential field. We will not use Wilkie's result as our methods will be those of complex analytic geometry. Indeed, we will later give an independent proof that ecl is a pregeometry on $\mathbb{C}_{\exp}$. However, all the results proven here could presumably also be obtained using Wilkie's methods and real analytic geometry.

3 The Schanuel property

From now on, we assume that R is an exponential ring with no zero divisors and, furthermore, that it is a $\mathbb{Q}$-algebra.

Definition 3.1. Let R be an exponential ring as above, and let $a = (a_1, \ldots, a_n)$ be an n-tuple from R. Define the *absolute Schanuel predimension*

$$\mathrm{S}\delta(a) = \mathrm{td}_{\mathbb{Q}}(a, \exp(a)) - \mathrm{ldim}_{\mathbb{Q}}(a),$$

and, for a subset $C \subseteq R$, the *relative Schanuel predimension*

$$\mathrm{S}\delta(a/C) = \mathrm{td}_C(a, \exp(a)) - \mathrm{ldim}_{\mathbb{Q}}(a/C)$$

where $\mathrm{td}_C(x)$ is the transcendence degree of the extension $\mathbb{Q}(C, x)/\mathbb{Q}(C)$ and $\mathrm{ldim}_{\mathbb{Q}}(x/B)$ is the $\mathbb{Q}$-linear dimension of the quotient $\mathbb{Q}$-vector space $\langle C, x\rangle / \langle C\rangle$.

Then R has the *absolute Schanuel property*, SP, iff for each $n \in \mathbb{N}$ and every $a \in R^n$,

$$\mathrm{S}\delta(a) \geqslant 0,$$

and it has the *Schanuel property relative to C* iff for each such n and a,

$$\mathrm{S}\delta(a/C) \geqslant 0.$$

Schanuel's conjecture is the assertion that $\mathbb{C}_{\exp}$ has the Schanuel property.

The quantity $\mathrm{S}\delta$ gives rise to a pregeometry on any exponential ring. This is the pregeometry used in the construction of Zilber's exponential field. The method of construction is due to Hrushovski and is documented in many places, for example [Kir06]. We fix a subset $C \subseteq R$ and first assume that $\mathrm{S}\delta(x/C)$ is bounded below on R, that is, there is $N \in \mathbb{Z}$ such that for any tuple a, $\mathrm{S}\delta(a/C) \geqslant N$. In this case, for a finite tuple a and a subset B of R we define the *Schanuel dimension (with respect to C)* of a to be

$$\mathrm{Sd}_C(a) = \min\left\{\mathrm{S}\delta(ab/C) \mid b \text{ is a finite tuple from } R\right\},$$

the *relative Schanuel dimension (with respect to C)* of a over B to be

$$\mathrm{Sd}_C(a/B) = \min\left\{\mathrm{Sd}_C(ab) - \mathrm{Sd}_C(b) \mid b \text{ is a finite tuple from } B\right\},$$

and we define the *Schanuel closure (with respect to C)* of B to be

$$\mathrm{Scl}_C(B) = \left\{x \in R \mid \mathrm{Sd}_C(x/B) = 0\right\}.$$

If $\mathrm{S}\delta(x/C)$ is not bounded below then we declare $\mathrm{Scl}_C(B) = R$ for every subset B of R.

We define the absolute Schanuel dimension and closure as above with $C = \emptyset$. Write Scl for the absolute Schanuel closure.

Proposition 3.2. *Let $C \subseteq R$. Then Scl_C is a pregeometry on R.*

Proof. If $\mathrm{S}\delta(x/C)$ is not bounded below this is trivial. Otherwise, it is immediate from the definition that Scl_C is increasing and monotone. It has finite character because td and ldim both do. It is standard to show that $\mathrm{S}\delta(x/C)$ is *submodular*: for any a, b,

$$\mathrm{S}\delta(a \cap b/C) + \mathrm{S}\delta(ab/C) \leqslant \mathrm{S}\delta(a/C) + \mathrm{S}\delta(b/C),$$

and it follows quickly from this that Scl_C is idempotent. If a is a singleton then $\mathrm{Sd}_C(a/B) = 0$ or 1, and the exchange property follows from this and submodularity. □

However, even if $\mathrm{S}\delta(x/C)$ is bounded below, the pregeometry is only non-trivial if $\mathrm{S}\delta(x/C)$ is bounded below by 0, that is, if R has the Schanuel property relative to C.

Lemma 3.3. *If R is an exponential ring which does not satisfy the Schanuel property relative to C then $\mathrm{Scl}_C(\emptyset) = R$.*

Proof. Let a be such that $\mathrm{S}\delta(a/C) < 0$, and let $b \in R$. Then $\mathrm{Sd}_C(b) \leqslant \mathrm{S}\delta(ab/C) \leqslant \mathrm{S}\delta(a/C) + 1 \leqslant 0$. Hence $b \in \mathrm{Scl}_C(\emptyset)$. □

Suppose that C is a subring of R and $C' = \mathrm{Scl}_C(C)$. Then the predimensions Scl_C and $\mathrm{Scl}_{C'}$ coincide. Furthermore, C' contains all of the zero dimensional tuples over itself. That is, if a is a tuple from R then $\mathrm{Sd}_{C'}(a) \geqslant 1$ or a lies in C'. In this case we say that R has the *strong Schanuel property relative to C'*. The proofs of the following two properties are standard predimension calculations such as those in [Kir06].

Lemma 3.4. *If R has the Schanuel property relative to C and relative to C', and $C \subseteq C' \subseteq R$, then $\mathrm{Scl}_{C'}(A) = \mathrm{Scl}_C(A \cup C')$.*

If R has the strong Schanuel property relative to C and $C \subseteq C' \subseteq R$ then R has the strong Schanuel property relative to C' iff C' is Scl_C-closed. □

Now suppose R has the Schanuel property relative to a given subring C, and $C' = \mathrm{Scl}_C(C)$. Then for any intermediate ring $C \subseteq S \subseteq R$, the information in Scl_S is captured in Scl_C, and thus also in $\mathrm{Scl}_{C'}$. So it is enough to look at Scl_C for subrings C relative to which R has the strong Schanuel property. However, there may be a proper subfield C of R such that R has the strong Schanuel property relative to C but nonetheless R does not have the absolute Schanuel property. The introduction of this relative Schanuel closure allows us to extend methods previously used only for exponential rings with SP to a more general setting.

We now have two closure operators on exponential rings: the exponential algebraic closure operator ecl and the Schanuel closure operator Scl. It is natural to ask how they are related. We can show inclusion in one direction. For the complex numbers, we will later give an analytic argument to show the reverse inclusion. As far as I know, the reverse inclusion is an open problem in general.

Proposition 3.5. *If* $B \subseteq R$ *then* $\mathrm{ecl}(B) \subseteq \mathrm{Scl}_B(B)$.

We first prove two lemmas. If $a \in \mathrm{ecl}(B)$ this is witnessed by a tuple of n exponential polynomials in n variables. If $a \in \mathrm{Scl}_C(B)$ this is witnessed by a tuple of polynomials in $2m$ variables, together with the fixed exponential dependencies $h_j = Y_{m+j} = \exp(Y_j) = 0$ for $j = 1, \ldots, m$. We must show that the exponential polynomials can be decomposed into this special form.

Lemma 3.6. *Let* B *be an exponential subring of* R *containing* $\ker\exp$. *Suppose* $\bar{a} = (a_1, \ldots, a_n) \in R^n$ *and* $f_1, \ldots, f_n \in B[\bar{X}]^E$ *are such that* $\bar{f}(\bar{a}) = 0$. *Then there is a tuple* $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{2m})$ *extending* $\bar{a}$, *say with* $\bar{\alpha} = \bar{a}\bar{b}$, *such that*

- *each* b_i *lies in the exponential subring of* R *generated by* $B(\bar{a})$,
- *there are* $g_j \in B[Z_1, \ldots, Z_m, Y_1, \ldots, Y_m]$ *for* $j = 1, \ldots, m$ *such that* $\bar{g}(\bar{\alpha}) = 0$, $\bar{h}(\bar{\alpha}) = 0$ *and*

$$\left|\frac{\partial \bar{f}}{\partial \bar{X}}\right|(\bar{a}) = \begin{vmatrix} \frac{\partial \bar{g}}{\partial \bar{Z}} & \frac{\partial \bar{g}}{\partial \bar{Y}} \\ \frac{\partial \bar{h}}{\partial \bar{Z}} & \frac{\partial \bar{h}}{\partial \bar{Y}} \end{vmatrix}(\bar{\alpha}).$$

Proof. Each exponential polynomial f_i is constructed from variables and elements of B by finitely many additions, multiplications and applications of exponentiation. By induction, it suffices to reduce the number of applications of exponentiation by one.

If all the f_i are polynomials, set $g_i = f_i(\bar{Z})$ (replacing the variable X_i by Z_i). Otherwise, without loss of generality, f_1 has the form $p(\bar{X}, \exp(q(\bar{X})))$ where p is an exponential polynomial and q is a polynomial. Replace f_1 by the exponential polynomial $g_1 = p(\bar{Z}, Y_{n+1})$, and set $g_i = f_i(\bar{Z})$ for $i = 2, \ldots, n$. Then, for $i = 1, \ldots, n+1$, let $h_i = Y_i - \exp(Z_i)$. Let

$$\alpha_i = \begin{cases} a_i & \text{for } i = 1, \ldots, n \\ q(\bar{a}) & \text{for } i = n+1 \\ \exp(\alpha_i) & \text{for } i = 1, \ldots, n \end{cases} .$$

Then $\bar{g}(\alpha) = \bar{h}(\alpha) = 0$ and a standard Jacobian calculation with the chain rule shows that

$$\left| \frac{\partial \bar{f}}{\partial \bar{X}} \right| (\bar{a}) = \begin{vmatrix} \frac{\partial \bar{g}}{\partial \bar{Z}} & \frac{\partial \bar{g}}{\partial \bar{Y}} \\ \frac{\partial \bar{h}}{\partial \bar{Z}} & \frac{\partial \bar{h}}{\partial \bar{Y}} \end{vmatrix} (\bar{\alpha})$$

as required. □

Lemma 3.7. *Let B be as above, and suppose $a \in \operatorname{ecl} B$. Then there is $m \in \mathbb{N}$, an m-tuple $\bar{s}$ from $\operatorname{ecl}(B)$ and $g_1, \ldots, g_m \in B[\bar{X}, \bar{Y}]$ (with $\bar{X}$, $\bar{Y}$ both m-tuples of variables) such that*

- *a lies in the exponential subring of R generated by B and $\bar{s}$,*
- *the s_i are $\mathbb{Q}$-linearly independent over B,*
- *taking $t_i = \exp(s_i)$, we have $\bar{g}(\bar{s}, \bar{t}) = 0$ and*

$$\begin{vmatrix} \frac{\partial \bar{g}}{\partial \bar{X}} & \frac{\partial \bar{g}}{\partial \bar{Y}} \\ \frac{\partial \bar{h}}{\partial \bar{X}} & \frac{\partial \bar{h}}{\partial \bar{Y}} \end{vmatrix} (\bar{s}, \bar{t}) \neq 0$$

Proof. Take a tuple $\bar{a}$ containing a and exponential polynomials $f_i \in B[\bar{X}]^E$ witnessing that $a \in \operatorname{ecl} B$. Then apply lemma 3.6 to produce a $2m'$ tuple $\bar{\alpha}$. Let $\bar{\sigma}$ be the first m' elements of $\bar{\alpha}$ and let $\bar{\tau}$ be the latter m' elements. Let $\bar{s}$ be a maximal subset of $\bar{\sigma}$ which is $\mathbb{Q}$-linearly independent over B, and let $\bar{t}$ be the corresponding tuple from $\bar{\tau}$. Replace each g_i by a polynomial in the variables corresponding to $\bar{s}$ and $\bar{t}$ by substituting in for the other variables according to the $\mathbb{Q}$-linear dependencies amongst the σ_i and the multiplicative dependencies amonst the τ_i. The result will be $m' + m$ equations in $2m$ variables for some $m \leqslant m'$. The rank of the Jacobian matrix as above will be $2m$ by lemma 3.6 and the fact that $\left| \frac{\partial \bar{f}}{\partial \bar{X}} \right| (\bar{a}) \neq 0$,since the original f_i were witnesses to $a \in \operatorname{ecl} B$. Thus we may throw away some of the equations g_i to leave $2m$ equations in $2m$ variables, with nonzero Jacobian as desired. □

Proof of proposition 3.5. Suppose $a \in \operatorname{ecl}(B)$. We may replace B by the exponential subring of R generated by B and $\ker \exp$, since $\operatorname{ecl}(B)$ and $\operatorname{Scl}_C(B)$

both contain this subring. Apply lemma 3.7 to produce $\bar{s}$ and $\bar{g}$, with

$$\begin{vmatrix} \frac{\partial \bar{g}}{\partial \bar{X}} & \frac{\partial \bar{g}}{\partial \bar{Y}} \\ \frac{\partial h}{\partial \bar{X}} & \frac{\partial h}{\partial \bar{Y}} \end{vmatrix} (\bar{s}, \exp(\bar{s})) \neq 0.$$

Then the matrix $\begin{pmatrix} \frac{\partial \bar{g}}{\partial \bar{X}} & \frac{\partial \bar{g}}{\partial \bar{Y}} \end{pmatrix} (\bar{s}, \exp(\bar{s}))$ has rank m. Hence $\operatorname{td}(\bar{s}, \exp(\bar{s}/B) \leqslant m$. The s_i are $\mathbb{Q}$-linearly independent over B, and so $\mathrm{S}\delta(\bar{s}/B) \leqslant 0$. Hence $a \in \mathrm{Scl}_B(B)$. □

4 Zilber's exponential field

Zilber considered exponential fields with SP and three further properties: *standard kernel*, the *countable closure property* (CCP), and *strong exponential closedness* (SEC).

Standard kernel An exponential ring R is said to have *standard kernel* iff the kernel of exponentiation is $\eta\mathbb{Z}$ for a transcendental element η.

Note that $\mathbb{C}_{\exp}$ has standard kernel, since $2i\pi$ is transcendental.

CCP If $B \subseteq R$ is finite then $\mathrm{Scl}(B)$ is countable.

Note that CCP makes sense for any closure operator.

The Schanuel property gives a restriction on what systems of equations can have solutions, and the SEC property effectively says that this is the only restriction. It uses the notion of *rotundity* of a subvariety, which we must define. G is an abelian group, hence a $\mathbb{Z}$-module. Thus G^n is naturally a module over the ring $\mathrm{Mat}_{n \times n}(\mathbb{Z})$ of $n \times n$ matrices with integer coefficients. The action of $\mathrm{Mat}_{n \times n}(\mathbb{Z})$ is by linear combinations in the additive part of G and by multiplicative combinations in the multiplicative part. Given a subset A of G^n, and a matrix $M \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$, write $M \cdot A$ for the image of A under the action. If V is an algebraic subvariety of G^n then $M \cdot V$ is also an algebraic subvariety.

Definition 4.1. A subvariety V of G^n is said to be *rotund* iff for every $M \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$, $\dim M \cdot V \geqslant \operatorname{rk} M$.

Each $M \cdot V$ is a projection of V in some direction that makes sense in the group, and the idea is that a rotund subvariety is "large in every direction". (Zilber's terminology is *normal*, but this is not so descriptive.)

SEC If V is a rotund subvariety of G^n then there is a generic point of V in the intersection $V \cap \mathcal{G}^n$, where $\mathcal{G}$ is the graph of the exponential map.

Theorem 4.2 (Zilber). *For each uncountable cardinal κ, there is a unique exponential field of cardinality κ with standard kernel satisfying SP + SEC + CCP.*

In some sense, these are the "simplest" nontrivial exponential fields, or at least those admitting the simplest possible description. Zilber's conjecture is that the unique model $K_{\exp}$ of cardinality $2^{\aleph_0}$ is isomorphic to $\mathbb{C}_{\exp}$.

5 Blurred exponentiation

Let F be an exponential field and let G be the group $F \times F^{\times}$. Let θ be the group homomorphism $G \xrightarrow{\theta} F^{\times}$ given by $\theta(x, y) = y/\exp(x)$. The graph of exp is of course the kernel of this map: $\mathcal{G}_{\exp} = \theta^{-1}(1)$. Let $\mathcal{H}$ be a subgroup of $F^{\times}$. We define the *blurred graph* of exponentiation, with respect to $\mathcal{H}$, to be

$$\mathcal{B}_{\mathcal{H}} = \theta^{-1}(\mathcal{H})$$

or, in other words, $\mathcal{B}_{\mathcal{H}} = \{(x, y) \in G \mid \exists h \in \mathcal{H}[y = h \cdot \exp(x)]\}$. We call the structure $\langle F; +, \cdot, \mathcal{B}_{\mathcal{H}}\rangle$ a *blurred exponential field.*

Now consider $K_0 = \mathrm{Scl}(\emptyset)$ in Zilber's exponential field K. Let $K_{\mathcal{B}} = \langle K; +, \cdot, \mathcal{B}\rangle$ be the result of blurring Zilber's exponential field with respect to $K_0^{\times}$. Observe that K_0 is recoverable from $\mathcal{B}$ as $\{x \mid (x, 1) \in \mathcal{B}\}$.

Theorem 5.1. *This blurred exponential field $K_{\mathcal{B}}$ satisfies the following axioms.*

Field $\langle K; +, \cdot\rangle$ is an algebraically closed field of characteristic zero, and K_0 is a proper algebraically closed subfield.

Group $\mathcal{B}$ is a subgroup of the algebraic group $G = K \times K^{\times}$.

Fibres The fibres of the projection of $\mathcal{B}$ to K are cosets of $K_0^{\times}$. The fibres of the projection to $K^{\times}$ are cosets of K_0.

Blurred Schanuel Property (BSP) Let $n \in \mathbb{N}$ and let V be an algebraic subvariety of G^n, defined over K_0. Then if $(\bar{a}, \bar{b}) \in V \cap \mathcal{B}^n$ and $\dim V < n + 1$ then there are $m_1, \ldots, m_n \in \mathbb{Z}$ such that $\sum_{i=1}^{n} m_i a_i \in K_0$.

Existential Closedness (BEC) For each $n \in \mathbb{N}$, and each rotund algebraic subvariety V of G^n, the intersection $V \cap \mathcal{B}^n$ is nonempty.

Proof. The Field, Group, and Fibres axioms are immediate from the definition.

Let $(a,b) \in V \cap \mathcal{B}^n$ with V a subvariety of G^n defined over K_0 with $\dim V < N+1$. Then there is $c \in (K_0^\times)^n$ such that $b = c \cdot \exp(a)$, so $\mathrm{td}_C(a, \exp(a)) < n+1$. Thus either $\mathrm{S}\delta(a/C) \leqslant 0$ or a is $\mathbb{Q}$-linearly dependent over K_0. But K_0 is Scl-closed, so in the first case $a \in K_0^n$. Thus, either way, there are $m_i \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^n m_i a_i \in K_0$. So BSP holds.

If V is rotund then SEC says $V \cap \mathcal{G}_{\exp}^n \neq \emptyset$. But $\mathcal{G}_{\exp} \subseteq \mathcal{B}$, so $V \cap \mathcal{B}^n \neq \emptyset$. Thus BEC holds. $\square$

In my thesis [Kir06] I proved:

Theorem 5.2. *The above axioms are expressible as first order axiom schemes in the language $\mathcal{L} = \langle +, \cdot, \mathcal{B} \rangle$ and the $\mathcal{L}$-theory T which they define is complete.*

We now construct a pregeometry on $K_{\mathcal{B}}$ along the same lines as the Schanuel pregeometry. The construction works on any model of the axioms Field, Group, Fibres and BSP. The last axiom, BEC, is not used in the construction.

For a tuple $(a,b) \in \mathcal{B}^n$, let $\mathrm{B}\delta'(a,b) = \mathrm{td}_C(a,b) - \mathrm{ldim}_{\mathbb{Q}}(a/C)$. Then for any finite subset $A \subseteq K$, define $\mathrm{B}\delta(A) = \max \{\mathrm{B}\delta'(a,b) \mid \text{each } a_i, b_i \in A\}$.

For a finite subset A of R and a subset B of R we define the *blurred Schanuel dimension* of A to be

$$\mathrm{Bd}(A) = \min \{\mathrm{B}\delta(AX) \mid X \text{ is a finite subset of } R\},$$

the blurred Schanuel dimension of A over B to be

$$\mathrm{Bd}(A/B) = \inf \{\mathrm{Bd}(AX) - \mathrm{Bd}(X) \mid X \text{ is a finite subset of } B\},$$

and we define the *blurred Schanuel closure* of B to be

$$\mathrm{Bcl}(B) = \{x \in R \mid \mathrm{Bd}(x/B) = 0\}.$$

The same proof as for proposition 3.2 shows:

Proposition 5.3. Bcl *is a pregeometry on K.* $\square$

Theorem 5.4. *The pregeometries* Scl *and* Bcl *on Zilber's field K are equal.*

Proof. Suppose $\alpha \in \mathrm{Scl}(B)$. Since $K_0 = \mathrm{Scl}(\emptyset)$, we may assume $K_0 \subseteq B$. Then there is a finite tuple a from K containing α such that $\mathrm{td}(a, \exp(a)/B) - \mathrm{ldim}_{\mathbb{Q}}(a/B) \leqslant 0$. Since $(a, \exp(a)) \in \mathcal{B}^n$, this shows that $\alpha \in \mathrm{Bcl}(B)$.

Conversely, suppose $\alpha \in \mathrm{Bcl}(B)$. By BSP, $K_0 = \mathrm{Bcl}(\emptyset)$, so we may assume $K_0 \subseteq B$. Then there are n-tuples a and b of K, one of them containing α, such that $(a, b) \in \mathcal{B}^n$ and $\mathrm{td}(a, b/B) - \mathrm{ldim}_{\mathbb{Q}}(a/B) \leqslant 0$. Let $c = \exp(a)/b$. Then $c \in K_0^n$ and, since $K_0 \subseteq B$, we get that $\mathrm{td}(a, \exp(a)/B) - \mathrm{ldim}_{\mathbb{Q}}(a/B) \leqslant 0$. Hence $a \in \mathrm{Scl}(B)$. But $\mathrm{Scl}(B)$ is an exponential subfield of K containing K_0, so $b \in \mathrm{Scl}(B)$ as well. Hence $\alpha \in \mathrm{Scl}(B)$. □

As a corollary, we see that Bcl on $K_{\mathcal{B}}$ has the countable closure property.

The same arguments can be applied more generally, to prove the following.

Proposition 5.5. *Let F be any exponential field and let C be an exponential subfield. Then the result $F_{\mathcal{B}_C} = \langle F; +, \cdot, \mathcal{B}_C \rangle$ of blurring F by $C^\times$ is a model of the axioms Field, Group and Fibres, except that F may not be algebraically closed and some fibres of $\mathcal{B}_C$ may be empty. Furthermore, $\langle F; +, \cdot, \mathcal{B}_C \rangle$ satisfies BSP iff F has the strong Schanuel property relative to C. In this case,* Bcl *is a well-defined pregeometry on F and is equal to* Scl_C. □

We will also make use of the following theorem, proven in [Kir].

Theorem 5.6. *For each uncountable cardinal κ, there is exactly one model of T of cardinality κ such that* Bcl *has the countable closure property and* $\mathrm{td}(C/\mathbb{Q}) = \aleph_0$*. In other words, the abstract elementary class defined by the $L_{\omega_1\omega}(Q)$-theory T + CCP +* $\mathrm{td}(C/\mathbb{Q}) = \aleph_0$ *is uncountably categorical.*

Thus to prove the main theorem it remains to show that we can blur complex exponentiation to produce a model of this theory.

6 Blurred complex exponentiation

We now turn to the complex exponential field.

Theorem 6.1. *If C is a proper* ecl*-closed subfield of $\mathbb{C}_{\exp}$ then $\mathbb{C}_{\mathcal{B}_C}$ is a model of the theory T. If in addition C is countable then $\mathbb{C}_{\mathcal{B}_C}$ satisfies CCP.*

Let $C_0 = \operatorname{ecl}\emptyset$, and define $\mathbb{C}_{\mathcal{B}}$ to be $\mathbb{C}_{\mathcal{B}_{C_0}}$. Note that $\operatorname{td}(C_0/\mathbb{Q}) = \aleph_0$, and so the main theorem of this paper, Theorem 1.3, follows from theorems 6.1 and 5.6.

The remainder of the paper is devoted to proving theorem 6.1. An ecl-closed subfield is an exponential subfield, so axioms Field, Group, and Fibres hold. We prove BSP, BEC and CCP below.

Before continuing with the proof of theorem 6.1, we give an alternative characterization of ecl-closed subfields of $\mathbb{C}$ in the style of a nullstellensatz.

Definition 6.2. An (affine complex) exponential-algebraic variety is the zero set of a finite set of exponential polynomials in $\mathbb{C}^n$. It is defined over an exponential field C iff the exponential polynomials may be taken to have coefficients in C.

Lemma 6.3. *A subfield C of $\mathbb{C}$ is* ecl*-closed iff for every exponential-algebraic subvariety A of $\mathbb{C}^n$, defined over C, every connected component X of A has a C-point.*

Proof. First note that if X is a component of dimension zero, that is, a singleton $\{x\}$, then x is an isolated zero of the system of exponential polynomial equations. We may choose just n of the equations whose loci intersect transversally at x. Then, by the implicit function theorem, we see that this is the same thing as saying that $x \in \operatorname{ecl}(C)$. Thus C has this property for components of dimension 0 iff C is ecl-closed.

We now reduce the question to components of dimension 0. Suppose X is a connected component of dimension strictly greater than 0, and let $x \in X$ be a regular point. X is an analytic subset of $\mathbb{C}^n$, so for small enough ε it contains a good approximation to a disc about x of radius ε. Let r, s be Gaussian rational points in $\mathbb{C}^n$ which are very close to the disc, one on each side. If either of $r, s \in X$ then we are done, since $\mathbb{Q}(i) \subseteq C$. Suppose not. Let L be the (real) line joining r and s and let L' be its Zariski closure. Then $L' \cap A$ is an exponential-algebraic subvariety of dimension strictly less than the dimension of A, with nonempty intersection with X. By induction on dimension, each component has a C-point. In particular, $L' \cap X$ has a C-point. □

We give a consequence of Ax's theorem for analytic sets.

Theorem 6.4. *Let V be an algebraic subvariety of $G^n(\mathbb{C})$.*

If W is a connected component of the analytic variety $V \cap \mathcal{G}_{\exp}^n$ with analytic dimension $\dim W$ *satisfying* $\dim W = (\dim V - n) + t$ *for some $t > 0$, then there are t linearly independent n-tuples of integers (m_{ij}) and $c_j \in \mathbb{C}$ such that W is contained in the algebraic variety defined by*

$$\bigwedge_{j=1}^{t} \sum_{i=1}^{n} m_{ij} x_i = c_j \qquad \textit{and} \qquad \bigwedge_{j=1}^{t} \prod_{i=1}^{n} y_i^{m_{ij}} = \exp(c_j)$$

Proof. Let w be a regular point of W and let F be the differential field of germs at w of meromorphic functions on $G^n(\mathbb{C})$, the differential operators being the usual $2n$ partial differentiation operators (with respect to a chosen basis $z_1, \ldots, z_{2n}$). The field of constants is $\mathbb{C}$.

Let (x, y) be coordinate functions on W at w, with $x = (x_1, \ldots, x_n)$ being a basis of coordinate functions on $\mathbb{G}_{\mathrm{a}}^n$ and $y = (y_1, \ldots, y_n)$ being a basis of coordinate functions on $\mathbb{G}_{\mathrm{m}}^n$. Then $y_i = \exp(x_i)$, and so $\frac{\partial y_i}{\partial z_j} = y_i \frac{\partial x_i}{\partial z_j}$, for each i and j. Also $(x, y) \in V$ and the rank of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial x_i}{\partial z_j} & \frac{\partial y_i}{\partial z_j} \end{pmatrix}_{i=1,\ldots,n; j=1,\ldots,2n}$$

is equal to $\dim W$, by the definition of the analytic dimension of W. By Ax's theorem [Ax71, Theorem 3], there are m_{ij} and c_j as described. These algebraic relations hold on the coordinate functions, and so they hold on the whole component W as required. $\square$

Proposition 6.5. $\mathbb{C}_{\mathcal{B}_C}$ *satisfies BSP.*

Proof. Let V be an irreducible subvariety of G^n, defined over C. Suppose that $(a, b) \in V \cap \mathcal{B}^n$ and $\dim V < n + 1$.

Let $c = \exp(a)/b$ and define $V' = \{(x, y) \in G^n \mid (x, y/c) \in V\}$. Then V' and V are isomorphic and $\dim V' = \dim V < n+1$. Let W be the connected component of $V' \cap \mathcal{G}_{\exp}^n$ containing $(a, \exp(a))$.

If $\dim W = 0$ then W is the singleton $\{(a, \exp(a))\}$. Now $V' \cap \mathcal{G}_{\exp}^n$ is an exponential-algebraic variety definable over C, and C is ecl-closed, so $(a, \exp(a)) \in G^n(C)$. Also $c \in C^n$ by definition of $\mathcal{B}$, and so $(a, b) \in G^n(C)$.

Otherwise, $\dim W > 0 \geqslant \dim V' - n$. Hence, by theorem 6.4, there are $m_i \in \mathbb{Z}$, not all zero, and $\gamma \in \mathbb{C}$ such that $\sum_{i=1}^n m_i x_i = \gamma$. Since V' is irreducible, and (x, y) is a generic point of V' over $\mathbb{C}$, this equation holds for all points in V'. In particular, $\sum_{i=1}^n m_i a_i = \gamma$. Now W is a connected component of an exponential-algebraic variety defined over C, so contains a C-point, say (α, β). Then $\sum_{i=1}^n m_i \alpha_i = \gamma$, but C is a field and so $\gamma \in C$. $\square$

Proposition 6.6. $\mathbb{C}_{\mathcal{B}_C}$ *satisfies BEC.*

Proof. Let V be an irreducible, rotund subvariety of $G^n(\mathbb{C})$. By intersecting with generic hyperplanes (which preserves rotundity) we may assume that $\dim V = n$. We may also assume that V is free, that is there are no $\mathbb{Q}$-linear dependences on its x-coordinates and no multiplicative dependences on its y-coordinates, over a field of definition.

Let v be a regular point of $V(\mathbb{C})$, and let U be a small neighbourhood of v in V. We may take U to be analytically diffeomorphic to an open ball in $\mathbb{C}^n$. We have the map $G(\mathbb{C}) \xrightarrow{\theta} \mathbb{C}^\times$ given by $\theta(x,y) = y/\exp(x)$. Let ψ be its restriction to $V(\mathbb{C})$.

Let $t = \psi(v) \in S(\mathbb{C})$ and let $A = \psi^{-1}(t) \cap U$. Then A is an analytic subset of U, so, by taking U sufficiently small, we may assume that A is connected.

Now let V' be the translation of V given by

$$V' = \{(x, y/t) \mid (x,y) \in V\},$$

let $V \xrightarrow{\tau} V'$ be the translation map and let $A' = \tau A$. By construction, $A' \subseteq \mathcal{G}_{\exp} \cap V'$. Now V is rotund and free, so V' is also rotund and free. Thus by theorem 6.4 the intersection is typical, that is,

$$\dim A' \leqslant 2n - \dim V' - \dim \mathcal{G}_{\exp} = 0.$$

Thus $\dim A = \dim A' = 0$, and A is connected, so A is the singleton $\{v\}$.

By the inverse function theorem, ψ is locally invertible at v. Thus, choosing the neighbourhood U even smaller if necessary, $\psi \restriction_U$ is a topological embedding. So $\psi(U)$ is open in $\mathbb{C}^{\times n}$. Now C is dense in $\mathbb{C}$ and so there is $c \in \psi(U) \cap C^{\times n}$. Then $\theta^{-1}(c) \in U \cap \mathcal{B}^n \subseteq V \cap \mathcal{B}^n$, which is nonempty as required. □

This completes the proof that $\mathbb{C}_{\mathcal{B}_C}$ is a model of T. Recall that $C_0 = \mathrm{ecl}(\emptyset)$, and $\mathbb{C}_{\mathcal{B}} = \mathbb{C}_{\mathcal{B}_{C_0}}$. Let Bcl be the blurred Schanuel pregeometry on $\mathbb{C}_{\mathcal{B}}$.

Proposition 6.7. *The blurred Schanuel pregeometry* Bcl *is equal to the exponential algebraic closure operator* ecl *on* $\mathbb{C}_{\exp}$*. In particular,* ecl *is a pregeometry on* $\mathbb{C}$*.*

Proof. By proposition 5.5, $\mathrm{Bcl} = \mathrm{Scl}_{C_0}$. Thus by proposition 3.5, for every $A \subseteq \mathbb{C}$, $\mathrm{ecl}(A) \subseteq \mathrm{Bcl}(A)$. Thus every Bcl-closed subfield is ecl-closed. By lemma 3.4 and proposition 6.5, every ecl-closed subfield is Bcl-closed. Hence ecl = Bcl. □

This provides the promised converse to proposition 3.5, in the complex case.

Proposition 6.8. Bcl *satisfies CCP. More generally, if* C *is any countable* ecl-*closed subfield of* $\mathbb{C}$ *then* Bcl_C *satisfies CCP.*

Proof. It is enough to show that ecl has the CCP on $\mathbb{C}$, since $\mathrm{Bcl}_C(A) = \mathrm{Bcl}(A \cup C)$.

If A is a finite subset of $\mathbb{C}$ then $x \in \mathrm{ecl}(A)$ iff x is a coordinate of an isolated zero of an exponential polynomial map defined over $\mathbb{Q}(A)$. There are only countably many such maps and each can have only countably many isolated zeros because $\mathbb{C}^n$ is a countable union of compact subspaces. Hence $\mathrm{ecl}(A)$ is countable. □

This completes the proof of theorem 6.1, and thus also of theorem 1.3.

References

[Ax71] James Ax. On Schanuel's conjectures. *Ann. of Math. (2)*, **93**:pp. 252–268, 1971.

[Kir] Jonathan Kirby. On exponential differential equations. Forthcoming.

[Kir06] Jonathan Kirby. *The theory of exponential differential equations.* DPhil thesis, University of Oxford, 2006. `http://eprints.maths.ox.ac.uk/275/`.

[Mac96] Angus J. Macintyre. Exponential algebra. In *Logic and algebra (Pontignano, 1994)*, volume 180 of *Lecture Notes in Pure and Appl. Math.*, pp. 191–210. Dekker, New York, 1996.

[Wil05] A. J. Wilkie. Some local definability theory for holomorphic functions, October 2005. Available on the MODNET preprint server.

[Zil05] Boris Zilber. Pseudo-exponentiation on algebraically closed fields of characteristic zero. *Ann. Pure Appl. Logic*, **132**(1):pp. 67–95, 2005.