EQUIVARIANT DEFINALE MORSE FUNCTIONS IN DEFINABLY COMPLETE STRUCTURES

TOMOHIRO KAWAKAMI

Abstract. Let G be a compact definable C^r group, X a compact affine definable C^rG manifold and $2 \leq r < \infty$. We prove that the set of equivariant definable Morse functions on X whose loci are finite unions of nondegenerate critical orbits is open and dense in the set of G invariant definable C^r functions with respect to the definable C^r topology.

1. Introduction

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an expansion of a real closed field R. We say that $\mathcal{N}$ is *definably complete* if every nonempty definable subset A of R, $\sup A, \inf A \in R \cup \{\infty, -\infty\}$. Every o-minimal expansion of R is definably complete. Definably complete structures are studied in [1], [2]. A weakly o-minimal structure is not always definably complete. For example $(\mathbb{R}_{alg}, +, \cdot, <, (-\pi, \pi) \cap \mathbb{R}_{alg})$ is weakly o-minimal but not definably complete.

If R is the field $\mathbb{R}$ of real numbers, then an expansion $\mathcal{M}$ of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ is definably complete.

In this paper we consider its equivariant definable C^r version of Morse theory on $\mathcal{M}$. It is a generalization of [8], [12]. Everything is considered in $\mathcal{M}$, $r \geq 2$ and every definable map is continuous unless otherwise stated. Remark that the condition that $r \geq 2$ is necessary to define Morse functions. Definable C^rG manifolds in o-minimal structures are studied in [10], [9]. Their definitions work in $\mathcal{M}$.

Let X be an n-dimensional definable C^r manifold and $f : X \to \mathbb{R}$ a definable C^r function. We say that a point $p \in X$ is a *critical point* of f if the differential of f at p is zero. We say that $f(p)$ is called a *critical value* of f if p is a critical point of f. Let p be a critical point of f and (U, u) a definable C^r coordinate system on X at p. The critical point p is *nondegenerate* if the Hessian matrix of $f \circ u^{-1}$ at 0 is nonsingular. Direct computations show this definition is well-defined.

In the non-equivariant setting, Y. Peterzil and S. Starchenko [15] introduced definable C^r Morse functions in an o-minimal expansion of the standard structure of a real closed field.

Let G be a definable C^r group, X a definable C^rG manifold and $f : X \to \mathbb{R}$ a G invariant definable C^r function on X. A closed definable C^rG submanifold Y of X is a *critical manifold* (resp. a *nondegenerate critical manifold*) of f if each $p \in Y$ is a critical point (resp. a nondegenerate critical point) of f. We say that f is an *equivariant*

2010 *Mathematics Subject Classification*. 14P10, 14P20, 57R35, 58A05, 03C64.
Keywords and Phrases. Definably complete structures, Morse theory, equivariant definable Morse functions.

definable Morse function if the critical locus of f is a finite union of nondegenerate critical manifolds of f without interior.

Theorem 1.1. *Let G be a compact definable C^r group, f an equivariant definable Morse function on a compact affine definable C^rG manifold X and $2 \leq r < \infty$. If f has no critical value in $[a, b]$, then $f^a := f^{-1}((-\infty, a])$ is definably C^rG diffeomorphic to $f^b := f^{-1}((-\infty, b])$. If $\mathcal{M}$ is exponential, then we can take $r = \infty$.*

Theorem 1.1 is an equivariant definable C^r version of Theorem 4.3 [17]. An O-minimal version of Theorem 1.1 is considered in [8], [12].

Note that the method of the proof Theorem 4.3 [17] is the integration of a G invariant C^∞ vector field. This method does not work in the definable category because the integration of a G invariant definable C^r vector field is not always definable.

In the non-equivariant o-minimal case, T.L. Loi [13] proved density and openness of definable Morse functions.

Let $Def^r(\mathbb{R}^n)$ denote the set of definable C^r functions on $\mathbb{R}^n$. For each $f \in Def^r(\mathbb{R}^n)$ and for each positive definable function $\epsilon : \mathbb{R}^n \to \mathbb{R}$, the ϵ-neighborhood $N(f; \epsilon)$ of f in $Def^r(\mathbb{R}^n)$ is defined by $\{h \in Def^r(\mathbb{R}^n) | |\partial^\alpha(h - f)| < \epsilon, \forall \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \leq r\}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n, |\alpha| = \alpha_1 + \cdots + \alpha_n, \partial^\alpha F = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. We call the topology defined by these ϵ-neighborhoods the *definable C^r topology*.

Theorem 1.2 ([13])**.** *Let $\mathcal{L}$ be an o-minimal expansion of $\mathcal{R}$ and X a definable C^r submanifold of $\mathbb{R}^n$. Then the set of definable C^r functions on $\mathbb{R}^n$ which are Morse functions on X and have distinct critical values are open and dense in $Def^r(\mathbb{R}^n)$ with respect to the definable C^r topology.*

Theorem 1.2 is generalized in o-minimal expansions of real closed fields ([6]).

Remark that the definable C^r topology and the C^r Whitney topology do not coincide in general. If X is compact, then these topologies of the set $Def^r(X)$ of definable C^r functions on X are the same (P156 [16]).

A nondegenerate critical manifold of an equivariant Morse function on a definable C^rG manifold is called a *nondegenerate critical orbit* if it is an orbit.

Theorem 1.3. *Let G be a compact definable C^r group, X a compact affine definable C^rG manifold and $2 \leq r < \infty$.*

(1) The set $Def_{equi-Morse,o}(X)$ of equivariant definable Morse functions on X whose critical loci are finite unions of nondegenerate critical orbits is dense in the set $C^r_{inv}(X)$ of G invariant C^r functions on X with respect to the C^r Whitney topology. Moreover $Def_{equi-Morse,o}(X)$ is open and dense in the set $Def^r_{inv}(X)$ of G invariant definable C^r functions with respect to the definable C^r topology.

(2) If $\mathcal{M}$ is exponential, then the set $Def_{equi-Morse,o}(X)$ of equivariant definable Morse functions on X whose critical loci are finite unions of nondegenerate critical orbits is dense in the set $C^\infty_{inv}(X)$ of G invariant C^∞ functions on X with respect to the C^r Whitney topology. Moreover $Def_{equi-Morse,o}(X)$ is open and dense in the set $Def^\infty_{inv}(X)$ of G invariant definable C^∞ functions with respect to the definable C^r topology.

Definable G CW complexes are introduced in [5]. They are generalized in o-minimal expansions of real closed fields ([4]). In the o-minimal setting $\mathcal{L}$, the following result holds.

Theorem 1.4 ([8])**.** *Let $\mathcal{L}$ be an o-minimal expansion of $\mathcal{R}$, G a compact definable group and X a definable G manifold.*

1. *X is definably G homeomorphic to a finite union of open G cells of a definable G CW complex.*
2. *If X is compact, then X is definably G homeomorphic to a definable G CW complex. In particular, X is G homeomorphic to a finite G CW complex.*

However if $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \mathbb{Z})$, then Theorem 1.4 does not hold even when G is the trivial group because a definable set $\mathbb{Z}$ is not homeomorhic to a finite union of open cells.

By a way similar to the proof of 1.6 [8], we have the following result. It is a definable version of a well-known topological result (e.g. 6.2.4 [3]).

Theorem 1.5. *Let X be an n-dimensional compact definable C^r manifold having a definable Morse function $f : X \to \mathbb{R}$ with only two critical points and $2 \le r < \infty$. Then X is definably homeomorphic to the n-dimensional unit sphere S^n. If $n \le 6$, then X is definably C^r diffeomorphic to S^n. If $\mathcal{M}$ is exponential, then we can take $r = \infty$.*

Remark that if $n = 7$, then there exsits a C^∞ manifold which is homeomorphic to S^7, but not C^∞ diffeomorphic to S^7 ([14]).

2. Preliminaries and proof of Theorem 1.1.

A group G is a *definable C^r group* if G is a definable C^r manifold such that the group operations $G \times G \to G$ and $G \to G$ are definable C^r maps. Let G be a definable C^r group. A *definable C^rG manifold* is a pair (X, ϕ) consisting of a definable C^r manifold X and a group action $\phi : G \times X \to X$ such that ϕ is a definable C^r map. For simplicity, we write X instead of (X, ϕ).

Let G be a definable C^r group. A *representation map* of G means a group homomorphism from G to some $O_n(\mathbb{R})$ which is of class definable C^r and the *representation* of this representation map is $\mathbb{R}^n$ with the orthogonal action induced by the representation map. In this paper, we always assume that every representation is orthogonal. A *definable C^rG submanifold* of a representation Ω of G is a G invariant definable C^r submanifold of Ω. We say that a definable C^rG manifold is *affine* if it is definably C^rG diffeomorphic to a definable C^rG submanifold of some representation of G.

Theorem 2.1. *Let X and Y be compact affine definable C^rG manifolds possibly with boundary and $2 \le r < \infty$. Then X and Y are C^1G diffeomorphic if and only if they are definably C^rG diffeomorphic. If $\mathcal{M}$ is exponential, then we can take $r = \infty$.*

Let G be a compact group, f a map from a C^rG manifold X to a representation Ω of G and $0 \le r \le \infty$. Denote the Haar measure of G by dg, and let x be a point in X. Recall the *averaging operator* A defined by

$$A(f)(x) = \int_G g^{-1} f(gx) dg.$$

Proposition 2.2 (e.g. 2.11 [7])**.** *Let G be a compact group and $0 \le r \le \infty$. Suppose that $C^r(X, \Omega)$ denotes the set of C^r maps from a C^rG submanifold X of a representation of G to a representation Ω of G.*

1. *The averaged map $A(f)$ of f is equivariant, and $A(f) = f$ if f is equivariant.*

(2) *If* $f \in C^r(X, \Omega)$*, then* $A(f) \in C^r(X, \Omega)$*.*
(3) *If* f *is a polynomial map, then so is* $A(f)$*.*
(4) *If* X *is compact and* $r < \infty$*, then* $A : C^r(X, \Omega) \to C^r(X, \Omega)$ *is continuous in the* C^r *Whitney topology.*

By a way similar to the proofs of 4.5, 4.6 [7], we have the following two propositions.

Proposition 2.3. *Let* X *be a compact definable* C^rG *submanifold possibly with boundary of a representation* Ω *of* G *and* $1 \leq r < \infty$*. Then there exists a definable* C^rG *tubular neighborhood* (U, θ) *of* X *in* Ω*. If* $\mathcal{M}$ *is exponential, then we can take* $r = \infty$*.*

Proposition 2.4. *Let* X *be a compact affine definable* C^rG *manifold with boundary and* $2 \leq r < \infty$*. Then* X *admits a definable* C^rG *collar, namely there exists a definable* C^rG *imbedding* $\phi : \partial X \times [0, 1] \to X$ *such that* $\phi|(\partial X \times \{0\})$ *is the inclusion* $\partial X \to X$*, where the action on* $[0, 1]$ *is trivial. If* $\mathcal{M}$ *is exponential, then we can take* $r = \infty$*.*

Theorem 2.5 (P 38 [3])**.** *(1) Let* X, Y *be* C^1 *manifolds. Then the set of* C^1 *diffeomorphisms from* X *onto* Y *is open in the set* $C^1(X, Y)$ *of* C^1 *maps from* X *to* Y *with respect to the* C^1 *Whitney topology.*
(2) Let X, Y *be* C^1 *manifolds with boundary* $\partial X, \partial Y$*, respectively. Then the set of* C^1 *diffeomorphisms from* X *onto* Y *is open in* $\{f \in C^1(X, Y) | f(\partial X) \subset f(\partial Y)\}$ *with respect to the* C^1 *Whitney topology.*

By a way similar to the proof of 2.5 [11], we have the following theorem.

Theorem 2.6. *Let* G *be a compact definable* C^r *group and* X *a compact affine definable* C^rG *manifold and* $1 \leq r < \infty$*. Suppose that* A, B *are* G *invariant definable disjoint closed subsets of* X*. Then there exists a* G *invariant definable* C^r *function* $f : X \to \mathbb{R}$ *such that* $f|A = 1$ *and* $f|B = 0$*. If* $\mathcal{M}$ *is exponential, we can take* $r = \infty$*.*

Proof of Theorem 2.1. Let Ω (resp. Ξ) be a representation of G containing X (resp. Y) as a definable C^rG submanifold of Ω (resp. Ξ). We first assume that $\partial X = \partial Y = \emptyset$. By Polynomial Approximation Theorem, Proposition 2.2, Proposition 2.3 and Theorem 2.5, X and Y are C^1G diffeomorphic if and only if they are definably C^rG diffeomorphic. If $\mathcal{M}$ is exponential, then we can take $r = \infty$.

We assume that $\partial X \neq \emptyset$ and $\partial Y \neq \emptyset$. Let $f : X \to Y$ be a C^1G diffeomorphism. Since $f|\partial X : \partial X \to \partial Y$ is a C^1G diffeomorphism and ∂X is compact, one can find a definable C^rG diffeomorphism $f' : \partial X \to \partial Y$ as an approximation of $f|\partial X : \partial X \to \partial Y$ in the C^1 Whitney topology. Using definable C^rG collars of ∂X and ∂Y in X and Y, respectively, we have a G invariant definable open neighborhoods U and V of ∂X and ∂Y in X and Y, respectively, and a definable C^rG diffeomorphism $f_1 : U \to V$ with $f_1|\partial X = f'$.

Take a G invariant definable open neighborhood U' of ∂X in X with $U' \subsetneq U$. By Theorem 2.6, there exists a G invariant definable C^r function $\lambda : X \to \mathbb{R}$ such that $\lambda = 1$ on U' and the support lies in U. By Proposition 2.3 and since Y is compact, there exists a definable C^rG tubular neighborhood (V, θ) of Y in Ξ. By Polynomial Approximation Theorem, Proposition 2.2 and since X is compact, there exists a polynomial G map $f_2 : X \to \Xi$ which is an approximation of $i \circ f$ in the C^1 Whitney topology, where $i : Y \to \Xi$ denotes the inclusion. If our approximation is sufficiently close, then $H : X \to Y, H(x) = \theta(\lambda(x) f_1(x) + (1 - \lambda(x)) f_2(x))$ is a definable C^rG map such that it is an approximation of f in the C^1 Whitney topology and $H(\partial X) \subset \partial Y$. Therefore by Theorem 2.5 and the inverse function theorem, H is the required definable $C^r G$ diffeomorphism. If $\mathcal{M}$ is exponential, then we can take $r = \infty$ in the general case. □

Proof of Theorem 1.1. By the proof of Theorem 4.3 [17], $f^a = f^{-1}((-\infty, a])$ is $C^{r-1}G$ diffeomorphic to $f^b = f^{-1}((-\infty, b])$. Since X is compact and affine, these two manifolds are compact affine definable $C^r G$ manifolds with boundary. Thus Theorem 1.1 follows from Theorem 2.1. □

3. Proof of Theorem 1.3.

By the proof of Lemma 4.8 [17] proves the following.

Theorem 3.1 ([17]). *Let G be a compact C^r group, X a compact $C^r G$ manifold and $2 \le r \le \infty$. Then the set $C^r_{equi-Morse,o}(X)$ of equivariant Morse functions on X whose critical loci are finite unions of nondegenerate critical orbits is open and dense in the set $C^r_{inv}(X)$ of G invariant C^r functions on X with respect to the C^r Whitney topology.*

Proof of Theorem 1.3. Let $f \in C^r_{inv}(X)$ and $\mathcal{N} \subset C^r_{inv}(X)$ an open neighborhood of f in $C^r_{inv}(X)$. By Theorem 3.1, there exists an open subset $\mathcal{N}' \subset \mathcal{N}$ such that each $h \in \mathcal{N}'$ is an equivariant Morse function whose critical locus is a finite union of nondegenerate critical orbits. Let $C^r(X)$ denote the set of C^r functions on X. Since $A : C^r(X) \to C^r(X)$ is continuous and $A(C^r(X)) = C^r_{inv}(X)$, $A : C^r(X) \to C^r_{inv}(X)$ is continuous. Fix $h \in \mathcal{N}'$. Since $A(h) = h$, $A^{-1}(\mathcal{N}')$ is an open neighborhood of h in $C^r(X)$. Applying Polynomial Approximation Theorem, we have a polynomial function h' lies in $A^{-1}(\mathcal{N}')$. Applying the averaging function, we have a G invariant polynomial function $F := A(h')$ lies in $\mathcal{N}'$. Since F is a G invariant polynomial function, it is a G invariant definable C^r function. Thus F is an equivariant definable Morse function lies in $\mathcal{N}$.

We now prove the second part. By the first part, $Def_{equi-Morse,o}(X)$ is dense in C^r_{inv} (X). Thus it is dense in $Def^r_{inv}(X)$. Let $h \in Def_{equi-Morse,o}(X)$. By Theorem 3.1, there exists an open neighborhood $\mathcal{V}$ of h in $C^r_{inv}(X)$ such that each $h \in \mathcal{V}$ is an equivariant Morse function whose critical locus is a finite union of nondegenerate critical orbits. Thus $\mathcal{V} \cap Def^r_{inv}(X)$ is the required open neighborhood of h in $Def^r_{inv}(X)$.

If $\mathcal{M}$ is exponential, then the above argument works when $r = \infty$. □

References

[1] M. Aschenbrenner and A. Fischer, *Definable versions of theorems by Kirszbraun and Helly*, Proc. Lond. Math. Soc. **102** (2011), 468–502.

[2] A. Fornasiero and T. Servi, *Definably complete Baire structures*, Fund. Math. **209** (2010), 215–241.

[3] M.W. Hirsch, *Differential topology*, Graduate Texts in Mathematics **33**, Springer-Verlag, (1976).

[4] T. Kawakami, *A definable strong G retract of a definable G set in a real closed field*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **61** (2011), 7–11.

[5] T. Kawakami, *Definable G CW complex structures of definable G sets and their applications*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **54**, (2004), 1–15.

[6] T. Kawakami, *Definable Morse functions in a real closed field*, Bull. Fac. Edu. Wakayama Univ. **62** (2012), 35–38.

[7] T. Kawakami, *Equivariant definable C^r approximation theorem, definable C^rG triviality of G invariant definable C^r functions and compactifications*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **55** (2005), 23–36.
[8] T. Kawakami, *Equivariant definable Morse functions on definable C^rG manifolds*, Far East J. Math. Sci. (FJMS) **28** (2008), 175–188.
[9] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology Appl. **123** (2002), 323-349.
[10] T. Kawakami, *Imbeddings of manifolds defined on an o-minimal structures on* $(\mathbb{R}, +, \cdot, <)$, Bull. Korean Math. Soc. **36** (1999), 183–201.
[11] T. Kawakami, *Relative properties of definable C^∞ manifolds with finite abelian group actions in an o-minimal expansion of $R_{\exp}$*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **59** (2009), 21–27.
[12] T. Kawakami and H. Tanaka, *Equivariant definable Morse functions on definable $C^\infty G$ manifolds*, Surikaisekikenkyusho Kokyuroku **1718** (2010), 58-63.
[13] T.L. Loi, *Density of Morse functions on sets definable in o-minimal structures*, Ann. Polon. Math. **89**, (2006), 289–299.
[14] J. Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Ann. of Math. (2) **64** (1956), 399–405.
[15] Y. Peterzil and S. Starchenko, *Computing o-minimal topological invariants using differential topology*, Trans. Amer. Math. Soc. **359**, (2006), 1375-1401.
[16] M. Shiota, *Geometry of subanalytic and semialgebraic sets*, Progress in Mathematics **150**, Birkhäuser, Boston, 1997.
[17] G. Wasserman, *Equivariant differential topology*, Topology **8**, (1969), 127-150.

Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani Wakayama 640-8510, Japan
E-mail address: kawa@center.wakayama-u.ac.jp