

# LIPSCHITZ CELL DECOMPOSITION IN O-MINIMAL STRUCTURES. I

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ABSTRACT. A main tool in studying topological properties of sets definable in o-minimal structures is the Cell Decomposition Theorem. The present paper proposes its metric counterpart based on the idea of a Lipschitz cell. In contrast to earlier results, we give an algorithm of a Lipschitz cell decomposition involving only permutations of variables as changes of coordinates.

## 1. Introduction.

A main tool in studying topological properties of sets definable in o-minimal structures is the Cell Decomposition Theorem (cf. [vdD]). The present paper proposes its metric counterpart based on the idea of a Lipschitz cell, called here an  $M$ -cell. Of course, in general a decomposition into such cells requires linear changes of coordinate systems (cf. [K,P]). We will give an algorithm showing that in fact permutations of coordinates suffice as changes of coordinate systems.

The present article deals only with Lipschitz cell decomposition of open sets. The case of general o-minimal sets, easily reducible to the previous one, with some additional properties and applications will be treated in a separate paper.

Fix any o-minimal structure on a real closed field  $R$  (for the definition and fundamental properties of o-minimal structures the reader is referred to [vdD]). Let  $n$  be a positive integer.

A subset  $S$  of  $R^n$  will be called an (*open*) *cell* in  $R^n$  iff

$$(1.1) \quad S = \{(x', x_n) \in R^n : x' \in \Delta, \varphi_1(x') < x_n < \varphi_2(x')\},$$

where  $x' = (x_1, \dots, x_{n-1})$ ,  $\Delta$  is an open definable subset of  $R^{n-1}$ , every  $\varphi_i$  ( $i \in \{1, 2\}$ ) is either a definable continuous function  $\varphi_i : \Delta \rightarrow R$  or  $\varphi_i \equiv -\infty$  or  $\varphi_i \equiv +\infty$  and, for each  $x' \in \Delta$ ,  $\varphi_1(x') < \varphi_2(x')$ .

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For any positive  $M \in R$ , a definable continuous function  $\varphi : \Delta \longrightarrow R$  defined on an open subset  $\Delta$  of  $R^{n-1}$  will be called an  $M$ -function iff

$$(1.2) \quad \left| \frac{\partial \varphi}{\partial x_j}(a) \right| \leq M \quad (j \in \{1, \dots, n-1\}),$$

at each point  $a \in \Delta$  in a neighborhood of which  $\varphi$  is of class  $\mathcal{C}^1$ .

An cell  $S$  in  $R^n$  will be called an  $M$ -cell (a *semi- $M$ -cell*) iff, for each  $i \in \{1, 2\}$  (for at least one  $i \in \{1, 2\}$ ), if  $\varphi_i$  is finite, it is an  $M$ -function. A cell  $S$  in  $R^n$  will be called a *regular  $M$ -cell* iff it is any open interval in the case  $n = 1$  and, in the case  $n > 1$ , for each  $i \in \{1, 2\}$ , if  $\varphi_i$  is finite it is an  $M$ -function of class  $\mathcal{C}^1$  on  $\Delta$  and the projection  $\Delta$  of  $S$  into  $R^{n-1}$  is a regular  $M$ -cell in  $R^{n-1}$ .

An  $M$ -cell will be called an  $M$ -disc iff it is any open interval in the case  $n = 1$  and, in the case  $n > 1$ , the both  $\varphi_i$  ( $i \in \{1, 2\}$ ) are finite and admit continuous extensions

$$(1.3) \quad \varphi_i : \overline{\Delta} \longrightarrow R$$

onto the closure of  $\Delta$  in  $R^{n-1}$ , and

$$(1.4) \quad \varphi_1 = \varphi_2 \quad \text{on} \quad \partial\Delta.$$

**Proposition 1.** *Let  $S$  be a regular  $M$ -cell in  $R^n$  and let  $\varphi : S \longrightarrow R$  be an  $L$ -function ( $L > 0$ ) of class  $\mathcal{C}^1$ .*

*Then*

- (1) *for any two different points  $a, b \in S$ , there is a definable continuous mapping*

$$\lambda = (\lambda_1, \dots, \lambda_n) : [0, |a - b|] \longrightarrow S$$

*such that  $\lambda(0) = a, \lambda(|a - b|) = b$  and  $|\lambda'_j(t)| \leq (j - 1)!M^{j-1}$ , for any  $j \in \{1, \dots, n\}$  and any  $t$  such that  $\lambda'_j(t)$  exists<sup>1</sup>;*

- (2) *for any two points  $a, b \in S$ ,*

$$|\varphi(a) - \varphi(b)| \leq n!M^{n-1}L|a - b|.$$

*Proof.* (1) Let  $S$  be as in (1.1). Arguing by induction and assuming that  $a' \neq b'$ , one can find a mapping

$$\omega = (\omega_1, \dots, \omega_{n-1}) : [0, |a' - b'|] \longrightarrow \Delta$$

such that  $\omega(0) = a', \omega(|a' - b'|) = b'$  and  $|\omega'_j(\tau)| \leq (j - 1)!M^{j-1}$ , for any  $j \in \{1, \dots, n-1\}$  and any  $\tau$  such that  $\omega'_j(\tau)$  exists. Let  $\varepsilon > 0$  be such that

$$\varphi_1(\omega(\tau)) + \varepsilon < \varphi_2(\omega(\tau)) - \varepsilon, \quad \text{for any} \quad \tau \in [0, |a' - b'|],$$

and

$$\varphi_1(a') + \varepsilon < a_n < \varphi_2(a') - \varepsilon \quad \text{and} \quad \varphi_1(b') + \varepsilon < b_n < \varphi_2(b') - \varepsilon.$$

Now, it suffices to put

$$\lambda_j(t) = \omega_j\left(t \frac{|a' - b'|}{|a - b|}\right), \quad \text{for} \quad j \in \{1, \dots, n-1\},$$

and

$$\lambda_n(t) = \max\left\{\varphi_1\left(\omega\left(t \frac{|a' - b'|}{|a - b|}\right)\right) + \varepsilon, \min\left\{\varphi_2\left(\omega\left(t \frac{|a' - b'|}{|a - b|}\right)\right) - \varepsilon, a_n + t \frac{b_n - a_n}{|a - b|}\right\}\right\}.$$

- (2) follows from (1), by the Mean Value Theorem (see [vdD, Chapter 7, (2.3)]).

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<sup>1</sup> $|a - b| = \sqrt{\sum_{j=1}^n (a_j - b_j)^2}$

**Kurdyka-Parusiński Theorem ([K, P]).** *Any open definable subset  $G$  of  $R^n$  has a finite decomposition*

$$G = S_1 \cup \cdots \cup S_k \cup \Sigma,$$

where every  $S_\nu$  is a regular  $M_n$ -cell in some linear coordinate system in  $R^n$  and  $\Sigma$  is nowhere dense,  $M_n$  being a constant depending only on  $n$ .

The aim of the present article is to give an algorithm showing that in fact permutations of coordinates are sufficient in the above theorem. We will prove simultaneously by induction on  $n$  the following three theorems.

**Theorem  $1_n(2_n, 3_n)$ .** *Any open definable subset  $G$  of  $R^n$  has a finite decomposition*

$$(1.5) \quad G = S_1 \cup \cdots \cup S_k \cup \Sigma,$$

where every  $S_\nu$  is an  $M_{1n}$ -cell ( $M_{2n}$ -disc, a regular  $M_{3n}$ -cell) in  $R^n$  after a permutation of coordinates and  $\Sigma$  is nowhere dense,  $M_{1n}$  ( $M_{2n}$ ,  $M_{3n}$ ) being a constant  $\geq 1$  depending only on  $n$ .

For simplicity we will often skip the adjective *definable*, when considering subsets of spaces  $R^n$  and mappings between such subsets. Also, we adopt the following conventions. A local property  $(w)$  of a mapping  $f : A \rightarrow R^m$ , where  $A \subset R^n$ , is said to be satisfied *almost everywhere* iff there is a closed subset  $E$  of  $A$  such that  $\dim E < \dim A$  and  $(w)$  is satisfied at each point of  $A \setminus E$ . A finite sequence  $B_1, \dots, B_k$  of subsets of a set  $A \subset R^n$  is said to be an *almost decomposition* of  $A$  iff  $B_\nu$  ( $\nu = 1, \dots, k$ ) are pairwise disjoint and  $\dim(A \setminus (B_1 \cup \cdots \cup B_k)) < \dim A$ . This will be denoted by writing

$$A \simeq B_1 \cup \cdots \cup B_k.$$

Since Theorem  $2_n$  together with  $3_{n-1}$  easily imply both Theorems  $1_n$  and  $3_n$ , it suffices to derive first Theorem  $1_n$  from Theorem  $2_{n-1}$  and then Theorem  $2_n$  from Theorems  $1_n$ ,  $2_{n-1}$  and  $3_{n-1}$ . From now on, we will assume that  $n \geq 2$  is fixed.

## 2. A preparation.

**Lemma 1.** *If  $G \subset R^{n-1}$  is open and  $E \subset \partial G$  is closed of dimension  $< n - 2$  and Theorem  $2_{n-1}$  is true, then  $G$  has an almost decomposition*

$$G \simeq \Delta_1 \cup \cdots \cup \Delta_p,$$

where every  $\Delta_\nu$ , after a permutation of coordinates in  $R^{n-1}$ , is an  $M_{2n-1}$ -disc:

$$\Delta_\nu = \{(x'', x_{n-1}) : x'' \in \Omega_\nu, \sigma_\nu(x'') < x_{n-1} < \rho_\nu(x'')\}^2,$$

such that the both (graphs of)<sup>3</sup>  $\sigma_\nu$  and  $\rho_\nu$  are disjoint with  $E$ .

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<sup>2</sup> $x'' = (x_1, \dots, x_{n-2})$

<sup>3</sup>We will identify functions with their graphs.

*Proof.* Take the projections

$$\pi_j : R^{n-1} \ni (x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}) \in R^{n-2},$$

for  $j \in \{1, \dots, n-1\}$ , and set

$$Z = \text{the closure of } \bigcup_j \pi_j^{-1}(\pi_j(E)).$$

Then  $\dim Z \leq n-2$  and it suffices to use Theorem 2<sub>n-1</sub> to  $G \setminus Z$ .

As a corollary one easily gets (see [vdD]) the following

**Lemma 2.** *If  $G \subset R^{n-1}$  is open and  $\varphi : G \rightarrow R$  is continuous, then  $G$  has an almost decomposition*

$$G \simeq \Delta_1 \cup \dots \cup \Delta_p,$$

where every  $\Delta_\nu$ , after a permutation of coordinates in  $R^{n-1}$ , is an  $M_{2n-1}$ -disc

$$\Delta_\nu = \{(x'', x_{n-1}) : x'' \in \Omega_\nu, \sigma_\nu(x'') < x_{n-1} < \rho_\nu(x'')\}$$

such that  $\varphi|_{\Delta_\nu}$  has a continuous extension

$$\varphi_\nu : \Delta_\nu \cup \sigma_\nu \cup \rho_\nu \rightarrow \bar{R} = R \cup \{-\infty, +\infty\}$$

such that  $\varphi_\nu(\sigma_\nu) \subset R$  or  $\varphi_\nu(\sigma_\nu) = \{-\infty\}$ , or  $\varphi_\nu(\sigma_\nu) = \{+\infty\}$  and the same for  $\rho_\nu$ .

**Proposition 2.** *Let  $f : S \rightarrow R$  be a definable  $\mathcal{C}^1$ -function defined on a cell*

$$S = \{(x', x_n) \in R^n : x' \in \Delta, \varphi(x') < x_n < \psi(x')\}$$

in  $R^n$  such that  $\varphi : \Delta \rightarrow R$  is of class  $\mathcal{C}^1$ .

Assume that  $\frac{\partial f}{\partial x_n}$  has a finite limit value<sup>4</sup> at (almost) each point of  $\varphi$  ( for example, when  $\frac{\partial f}{\partial x_n}$  is bounded ).

Then there is a closed nowhere dense subset  $Z$  of  $\varphi$  such that  $f$  extends to a  $\mathcal{C}^1$ -function

$$f : S \cup (\varphi \setminus Z) \rightarrow R$$

to  $S \cup (\varphi \setminus Z)$  as a  $\mathcal{C}^1$ -submanifold with boundary.

*Proof.* It is left to the reader as an exercise (cf [vdD]).

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<sup>4</sup>An element  $\alpha \in \bar{R}$  is a limit value of a function  $g : S \rightarrow R$  at  $a \in \bar{S}$  iff there is an arc  $\gamma : (0, 1) \rightarrow S$  such that  $\lim_{t \rightarrow 0} \gamma(t) = a$  and  $\lim_{t \rightarrow 0} g(\gamma(t)) = \alpha$ .

**Lemma 3.** *Let  $L, M, N, P \in \mathbb{R}$  be positive and let*

$$G = \{(x', x_n) : x' \in \Delta, \varphi_1(x') < x_n < \varphi_2(x')\}$$

*be a semi- $M$ -cell in  $\mathbb{R}^n$  such that  $\Delta$  is an  $N$ -cell in  $\mathbb{R}^{n-1}$ ,  $\varphi_i : \Delta \rightarrow \mathbb{R}$ , for each  $i \in \{1, 2\}$ , and the following conditions are satisfied almost everywhere in  $\Delta$ :*

$$(2.1) \quad \left| \frac{\partial \varphi_1}{\partial x_j} \right| \leq M, \quad \text{for each } j \in \{1, \dots, n-1\};$$

$$(2.2) \quad \left| \frac{\partial \varphi_1}{\partial x_{n-1}} \right| < L < \left| \frac{\partial \varphi_2}{\partial x_{n-1}} \right|;$$

$$(2.3) \quad \frac{\left| \frac{\partial \varphi_2}{\partial x_j} \right|}{\left| \frac{\partial \varphi_2}{\partial x_{n-1}} \right|} \leq P, \quad \text{for each } j \in \{1, \dots, n-1\};$$

$$(2.4) \quad \operatorname{sgn} \frac{\partial \varphi_2}{\partial x_{n-1}} = \operatorname{const.}$$

*Then  $G$  admits an almost decomposition*

$$G \simeq S_1 \cup \dots \cup S_k,$$

*where every  $S_\nu$  is an  $\tilde{M}$ -cell, possibly after transposition  $(x_{n-1}, x_n)$ , where  $\tilde{M}$  is a positive constant depending only on  $L, M, N$  and  $P$ .*

*Proof.* Put

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega, \sigma(x'') < x_{n-1} < \rho(x'')\}.$$

One can assume that

$$(2.5) \quad \frac{\partial \varphi_2}{\partial x_{n-1}} > 0;$$

the other case will follow by a modification. Because of (2.2) and (2.5), it is clear that  $\sigma : \Omega \rightarrow \mathbb{R}$ . By a subdivision of  $\Omega$  one can assume that  $\sigma$  is of class  $\mathcal{C}^1$  and that (2.2) is satisfied almost everywhere on every segment  $\{(x'', x_{n-1}) : \sigma(x'') < x_{n-1} < \rho(x'')\}$ , where  $x'' \in \Omega$  and that  $\varphi_i$  admit continuous extensions

$$\varphi_i : \Delta \cup \sigma \rightarrow \mathbb{R} \quad (i = 1, 2)$$

and

$$\varphi_2 : \Delta \cup \rho \rightarrow \mathbb{R} \cup \{+\infty\}$$

such that  $\varphi_2(\rho) \subset \mathbb{R}$  or  $\varphi_2(\rho) = \{+\infty\}$ .

By Proposition 2,  $\varphi_1$  is of class  $\mathcal{C}^1$  almost everywhere on  $\sigma$ . Put

$$\psi(x'', x_{n-1}) = \varphi_1(x'', \sigma(x'')) + L(x_{n-1} - \sigma(x'')), \quad \text{for } (x'', x_{n-1}) \in \Delta.$$

Then  $\psi$  is an  $\max(M + MN + LN, L)$ -function and  $\varphi_1 < \psi < \varphi_2$ .

Now  $G \simeq S_1 \cup S_2$ , where  $S_1 = \{(x', x_n) : \varphi_1(x') < x_n < \psi(x')\}$  and  $S_2 = \{(x'', x_{n-1}, x_n) : x'' \in \Omega, \Phi_1(x'', x_n) < x_{n-1} < \Phi_2(x'', x_n)\}$ , where

$$\Phi_2(x'', x_n) = \begin{cases} \psi^{-1}(x'', x_n) = L^{-1}(x_n - \varphi_1(x'', \sigma(x''))) + \sigma(x''), & \text{if } \varphi_1(x'', \sigma(x'')) < x_n < \psi(x'', \rho(x'')) \\ \rho(x''), & \text{if } \psi(x'', \rho(x'')) \leq x_n < \varphi_2(x'', \rho(x'')) \end{cases}$$

and

$$\Phi_1(x'', x_n) = \begin{cases} \sigma(x''), & \text{if } \varphi_1(x'', \sigma(x'')) < x_n \leq \varphi_2(x'', \sigma(x'')) \\ \varphi_2^{-1}(x'', x_n), & \text{if } \varphi_2(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x'')), \end{cases}$$

where  $\psi^{-1}$  and  $\varphi_2^{-1}$  stand for inversions with respect to  $x_{n-1}$ .

**Lemma 4.** *Let  $A \subset R^{n-1}$  be open and let  $M \in R, M > 0$ . Let  $f_\alpha : A \rightarrow R$  ( $\alpha \in \{1, \dots, k+l\}$ ) be  $M$ -functions on  $A$  each of which has a continuous extension to  $\bar{A}$ :*

$$f_\alpha : \bar{A} \rightarrow R.$$

*Assume that for each  $a \in \partial A$  there are  $\alpha \leq k$  and  $\beta > k$  such that  $f_\beta(a) \leq f_\alpha(a)$ .*

*Then the set*

$$S = \{(x', x_n) \in A \times R : \max_{1 \leq \alpha \leq k} f_\alpha(x') < x_n < \min_{k < \beta \leq k+l} f_\beta(x')\}$$

*is an  $M$ -disc in  $R^n$ .*

*Proof.* Indeed,

$$S = \{(x', x_n) \in B \times R : \max_{1 \leq \alpha \leq k} f_\alpha(x') < x_n < \min_{k < \beta \leq k+l} f_\beta(x')\},$$

where  $B$  is the natural projection of  $S$  to  $A$ . It is clear that  $\max_{1 \leq \alpha \leq k} f_\alpha = \min_{k < \beta \leq k+l} f_\beta$  on  $\partial B$  and the lemma follows.

**Lemma 5.** *Let  $\alpha_1, \alpha_2 \in \bar{R}, \alpha_1 < \alpha_2$  and let  $f, g, h : (\alpha_1, \alpha_2) \rightarrow R$  be three continuous definable functions such that*

$$(2.6) \quad g \leq f \quad \text{on } (\alpha_1, \alpha_2);$$

$$(2.7) \quad \text{for each } i \in \{1, 2\}, \text{ if } \alpha_i \in R, \text{ then } \lim_{t \rightarrow \alpha_i} g(t) = \lim_{t \rightarrow \alpha_i} h(t) \in R;$$

$$(2.8) \quad \operatorname{sgn} f'(t) = \operatorname{const} \quad \text{almost everywhere in } (\alpha_1, \alpha_2),$$

and there is  $\epsilon > 0$  such that

$$(2.9) \quad |f'(t)| \geq |g'(t)| + \epsilon \text{ and } |f'(t)| > |h'(t)| \quad \text{almost everywhere in } (\alpha_1, \alpha_2).$$

Then  $h < f$  on  $(\alpha_1, \alpha_2)$ .

*Proof.* One can assume that  $f'(t) > 0$ . Then  $\alpha_1 \in R$ , since otherwise by (2.9),  $\lim_{t \rightarrow -\infty} (f(t) - g(t)) = -\infty$ , a contradiction with (2.6). By (2.9),  $f - h$  is strictly increasing and, by (2.6) and (2.7),

$$\lim_{t \rightarrow \alpha_1} (f(t) - h(t)) \geq \lim_{t \rightarrow \alpha_1} (g(t) - h(t)) = 0.$$

Hence,  $f - h > 0$  on  $(\alpha_1, \alpha_2)$ .

### 3. Reduction of Theorem 1<sub>n</sub> to a special case of semi- $M$ -cells.

By the standard cell decomposition theorem (see [vdD]) and since

$$R^n = \bigcup_{j=1}^n \{(x_1, \dots, x_n) \in R^n : |x_k| \leq |x_j|, \text{ for any } k \neq j\},$$

it suffices to derive Theorem 1<sub>n</sub> for any cell  $G$  in  $R^n$  such that

$$(3.1) \quad G = \{(x', x_n) : x' \in \Delta, \varphi_1(x') < x_n < \varphi_2(x')\},$$

where  $\varphi_i : \Delta \rightarrow R$  ( $i = 1, 2$ ) are continuous.

For given positive  $L, P \in R$  such a cell  $G$  will be called an  $(L, P)$ -cell (with respect to the variable  $x_r$ ), where  $r \in \{1, \dots, n-1\}$ , iff

$$(3.2) \quad \left| \frac{\partial \varphi_i}{\partial x_r} \right| \geq L \quad \text{and} \quad \frac{\left| \frac{\partial \varphi_i}{\partial x_j} \right|}{\left| \frac{\partial \varphi_i}{\partial x_r} \right|} \leq P,$$

almost everywhere on  $\Delta$ , for  $i \in \{1, 2\}$ ,  $j \in \{1, \dots, n-1\}$ .

#### Proposition 3.

(1) Any open cell  $G \subset R^n$  has an almost decomposition

$$(3.3) \quad G \simeq S_1 \cup \dots \cup S_k,$$

where every  $S_\nu$  is either a semi- $M_n$ -cell or an  $(L_n, P_n)$ -cell after a permutation of coordinates, where positive constants  $M_n, L_n$  and  $P_n$  depend only on  $n$ .

(2) If a cell  $G$  is an  $(L, P)$ -cell, then  $G$  has an almost decomposition (3.3) with only semi- $M$ -cells, where a constant  $M$  depends only on  $n, L$  and  $P$ .

To prove Proposition 3 we first have the following

**Lemma 6.** *Let  $H$  be an open subset of  $R^n$  and let  $E$  be a closed subset of  $\partial H$  such that  $\dim E < n - 1$ . Let  $r_i \in \{1, \dots, n - 1\}$  ( $i \in \{1, 2\}$ ). Assume that  $L, P \in R$  are positive and such that, for each  $a \in \partial H \setminus E$ :*

(3.4 –  $i$ ) *there exists a neighborhood  $U$  of  $a$  in  $R^n$  such that  $\partial H \cap U$  is (the graph of) a  $C^1$ -function  $\psi : V \rightarrow R$  defined on an open  $V \subset R^{n-1}$  and such that*

$$\left| \frac{\partial \psi}{\partial x_{r_i}} \right| \geq L \quad \text{and} \quad \frac{\left| \frac{\partial \psi}{\partial x_j} \right|}{\left| \frac{\partial \psi}{\partial x_{r_i}} \right|} \leq P \quad \text{on } V \text{ for } j \in \{1, \dots, n - 1\},$$

for  $i = 1$  or  $i = 2$ .

Then:

(1)  $H$  admits an almost decomposition

$$(3.5) \quad H \simeq S_1 \cup \dots \cup S_k,$$

where every  $S_\nu$  after transposition  $(x_{r_1}, x_n)$  is either a semi- $\max(L^{-1}, P)$ -cell or a  $(P^{-1}, \max(L^{-1}, P))$ -cell in  $R^n$  with respect to  $x_{r_2}$ .

(2) If  $r_1 = r_2 = r$ ,  $H$  has such an almost decomposition (3.5) that every  $S_\nu$  is a  $\max(L^{-1}, P)$ -cell after transposition  $(x_r, x_n)$ .

*Proof of Lemma 6.* After transposition  $(x_{r_1}, x_n)$  take a  $C^1$ -cell decomposition compatible with each of the sets

$$\Lambda_i = \{a \in \partial H \setminus E : a \text{ satisfies (3.4 – } i)\}$$

( $i = 1, 2$ ) and with  $E$ . This gives an almost decomposition

$$H \simeq S_1 \cup \dots \cup S_k,$$

where every cell  $S_\nu$  is of the following form

$$S_\nu = \{\varphi_{1\nu}(x_1, \dots, \hat{x}_{r_1}, \dots, x_n) < x_{r_1} < \varphi_{2\nu}(x_1, \dots, \hat{x}_{r_1}, \dots, x_n)\},$$

such that, for  $i \in \{1, 2\}$ , either  $\varphi_{i\nu} \subset \Lambda_1$  or  $\varphi_{i\nu} \subset \Lambda_2$ , or  $\varphi_{i\nu} \equiv -\infty$ , or  $\varphi_{i\nu} \equiv +\infty$ .

One can assume that for each  $i$  either  $\varphi_{i\nu} \subset \Lambda_1$  or  $\varphi_{i\nu} \subset \Lambda_2$ , since otherwise  $S_\nu$  is trivially a semi- $\max(L^{-1}, P)$ -cell.

If  $\varphi_{i\nu} \subset \Lambda_1$ , for at least one  $i$ , then  $S_\nu$  is a semi- $\max(L^{-1}, P)$ -cell.

If  $\varphi_{i\nu} \subset \Lambda_2$ , for each  $i \in \{1, 2\}$ , and  $r_1 \neq r_2$ , then it is easy to check that  $S_\nu$  is an  $(P, \max(L^{-1}, P))$ -cell with respect to  $x_{r_2}$ .

*Proof of Proposition 3.* One can assume that  $G$  is as in (3.1). The proof will be by descending induction on the number

$$\langle G \rangle = \sum_{i=1}^2 \# \left\{ j : \left| \frac{\partial \varphi_i}{\partial x_j} \right| < 1 + 2M_{2n-1} \quad \text{almost everywhere on } \Delta \right\}.$$



If  $\langle G \rangle = 2(n-1)$ ,  $G$  is a  $(1+2M_{2n-1})$ -cell, so assume that  $\langle G \rangle < 2(n-1)$ . Observe that if  $\tilde{\Delta} \subset \Delta$  is open, then for  $\tilde{G} = G \cap (\tilde{\Delta} \times R)$ ,  $\langle \tilde{G} \rangle \geq \langle G \rangle$ . Hence, one can assume that every  $\varphi_i$  is  $\mathcal{C}^1$  and

$$(3.6) \quad \text{for each } j \in \{1, \dots, n-1\}, \quad \text{sgn} \frac{\partial \varphi_i}{\partial x_j} = \text{const on } \Delta;$$

$$(3.7) \quad \text{for each } j \in \{1, \dots, n-1\}, \quad \text{either} \quad \left| \frac{\partial \varphi_i}{\partial x_j} \right| < 1 + 2M_{2n-1}$$

$$\text{or} \quad \left| \frac{\partial \varphi_i}{\partial x_j} \right| > 1 + 2M_{2n-1}, \quad \text{or} \quad \left| \frac{\partial \varphi_i}{\partial x_j} \right| = 1 + 2M_{2n-1} \quad \text{on } \Delta$$

and there is  $r_i \in \{1, \dots, n-1\}$  such that

$$(3.8) \quad \text{for each } j \in \{1, \dots, n-1\}, \quad \left| \frac{\partial \varphi_i}{\partial x_j} \right| \leq \left| \frac{\partial \varphi_i}{\partial x_{r_i}} \right| \quad \text{on } \Delta.$$

Moreover, one can assume that

$$(3.9) \quad \left| \frac{\partial \varphi_i}{\partial x_{r_i}} \right| \geq 4M_{2n-1}(1 + 2M_{2n-1}), \quad \text{for } i \in \{1, 2\},$$

since otherwise  $G$  is a semi- $4M_{2n-1}(1 + 2M_{2n-1})$ -cell. Besides, by Lemma 2, one can assume that

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega, \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an  $M_{2n-1}$ -disc and every  $\varphi_i$  has a continuous extension

$$\varphi_i : \Delta \cup \sigma \cup \rho \longrightarrow \bar{R}$$

such that

$$\varphi_i(\sigma) \subset R \text{ or } \varphi_i(\sigma) = \{-\infty\} \text{ or } \varphi_i(\sigma) = \{+\infty\}, \text{ and the same for } \rho.$$

Observe that if

$$\frac{\partial \varphi_1}{\partial x_{n-1}} \cdot \frac{\partial \varphi_2}{\partial x_{n-1}} \leq 0,$$

then clearly  $G$  is a semi- $M_{2n-1}$ -cell after transposition  $(x_{n-1}, x_n)$ , so without any loss of generality one can assume that

$$\frac{\partial \varphi_i}{\partial x_{n-1}} > 0 \quad \text{on } \Delta, \quad \text{for } i \in \{1, 2\}.$$

We will first show how to reduce our proposition to the case of any  $(L, P)$ -cell with respect to any variable  $x_r$ , so assume Proposition 3 true for any  $(L, P)$ -cell.

By (3.7), one can distinguish the following three cases:

$$(3.10) \quad \left| \frac{\partial \varphi_i}{\partial x_{n-1}} \right| \leq 1 + 2M_{2n-1}, \quad \text{for } i \in \{1, 2\};$$

$$(3.11) \quad \left| \frac{\partial \varphi_i}{\partial x_{n-1}} \right| \geq 1 + 2M_{2n-1}, \quad \text{for } i \in \{1, 2\};$$

$$(3.12) \quad \left| \frac{\partial \varphi_1}{\partial x_{n-1}} \right| < 1 + 2M_{2n-1} \quad \text{and} \quad \left| \frac{\partial \varphi_2}{\partial x_{n-1}} \right| > 1 + 2M_{2n-1} \quad (\text{or vice-versa}).$$

*Case (3.10)* In fact we will be using only that every  $\varphi_i : \Delta \cup \sigma \cup \rho \rightarrow R$  is continuous and there is a closed nowhere dense  $Z \subset \Delta$  such that  $\varphi_i$  is  $\mathcal{C}^1$  on  $\Delta \setminus Z$  and

$$(3.13) \quad \left| \frac{\partial \varphi_i}{\partial x_{n-1}} \right| \leq 1 + 2M_{2n-1}, \quad \text{on } \Delta \setminus Z;$$

$$(3.14) \quad \left| \frac{\partial \varphi_i}{\partial x_j} \right| \leq 3 \left| \frac{\partial \varphi_i}{\partial x_{r_i}} \right| \quad \text{on } \Delta \setminus Z \quad (j = 1, \dots, n-1)$$

and

$$(3.15) \quad \left| \frac{\partial \varphi_i}{\partial x_{r_i}} \right| \geq 2M_{2n-1}(1 + 2M_{2n-1}) \quad \text{on } \Delta \setminus Z.$$

Put

$$H = \{(x'', x_{n-1}, x_n) \in G : \varphi_2(x'', \sigma(x'')) < x_n < \varphi_1(x'', \rho(x''))\} = \\ \{(x', x_n) \in R^n : x' \in D, \Phi_1(x') < x_n < \Phi_2(x')\},$$

where

$$D = \{(x'', x_{n-1}) \in \Delta : \varphi_2(x'', \sigma(x'')) < \varphi_1(x'', \rho(x''))\}, \\ \Phi_1(x'', x_{n-1}) = \max(\varphi_2(x'', \sigma(x'')), \varphi_1(x'', x_{n-1}))$$

and

$$\Phi_2(x'', x_{n-1}) = \min(\varphi_2(x'', x_{n-1}), \varphi_1(x'', \rho(x''))).$$

Observe that  $\Phi_1 = \Phi_2$  on  $(\partial D) \cap (\Delta \cup \sigma \cup \rho)$ , so almost everywhere on  $\partial D$ . Besides, by Proposition 2,  $\varphi_2(x'', \sigma(x'')) \not\equiv -\infty$  and

$$\frac{\partial}{\partial x_j} \varphi_2(x'', \sigma(x'')) = \frac{\partial \varphi_2}{\partial x_j}(x'', \sigma(x'')) + \frac{\partial \varphi_2}{\partial x_{n-1}}(x'', \sigma(x'')) \frac{\partial \sigma}{\partial x_j}(x''),$$

almost everywhere on  $\Omega$ , for  $j \in \{1, \dots, n-2\}$ . Hence, by (3.13) and (3.15)

$$\left| \frac{\partial}{\partial x_j} \varphi_2(x'', \sigma(x'')) \right| \leq \frac{7}{2} \left| \frac{\partial \varphi_2}{\partial x_{r_2}}(x'', \sigma(x'')) \right|$$

and

$$\left| \frac{\partial}{\partial x_{r_2}} \varphi_2(x'', \sigma(x'')) \right| \geq \frac{1}{2} \left| \frac{\partial \varphi_2}{\partial x_{r_2}}(x'', \sigma(x'')) \right| \geq M_{2n-1}(1 + 2M_{2n-1}).$$

Consequently,

$$\frac{\left| \frac{\partial}{\partial x_j} \varphi_2(x'', \sigma(x'')) \right|}{\left| \frac{\partial}{\partial x_{r_2}} \varphi_2(x'', \sigma(x'')) \right|} \leq 7, \quad \text{for any } j \in \{1, \dots, n-1\}.$$

In the same way,  $\varphi_1(x'', \rho(x'')) \not\equiv +\infty$  and almost everywhere on  $D$

$$\left| \frac{\partial}{\partial x_{r_1}} \varphi_1(x'', \rho(x'')) \right| \geq M_{2n-1}(1 + 2M_{2n-1})$$

and

$$\frac{\left| \frac{\partial}{\partial x_j} \varphi_1(x'', \rho(x'')) \right|}{\left| \frac{\partial}{\partial x_{r_1}} \varphi_1(x'', \rho(x'')) \right|} \leq 7, \quad \text{for any } j \in \{1, \dots, n-1\}.$$

By Lemma 6 (1),  $H$  admits an almost decomposition

$$(3.16) \quad H \simeq S_1 \cup \dots \cup S_k,$$

where every  $S_\nu$  is either a semi-7-cell or a  $(\frac{1}{7}, 7)$ -cell in  $R^n$  after transposition  $(x_{r_1}, x_n)$ .

Since  $G \setminus \overline{H}$  easily almost decomposes into a finite union of semi- $M_{2n-1}$ -cells after transposition  $(x_{n-1}, x_n)$ , (3.16) extends to a similar decomposition of  $G$ .

*Case (3.11)* Let  $\varphi_i^{-1}$  denotes the inversion of  $\varphi_i$  with respect to  $x_{n-1}$  ( $i \in \{1, 2\}$ ).

Observe that if  $\left| \frac{\partial \varphi_i}{\partial x_j} \right| < 1 + 2M_{2n-1}$ , then

$$\left| \frac{\partial \varphi_i^{-1}}{\partial x_j} \right| = \frac{\left| \frac{\partial \varphi_i}{\partial x_j} \right|}{\left| \frac{\partial \varphi_i}{\partial x_{n-1}} \right|} < 1 < 1 + 2M_{2n-1}$$

and, moreover,

$$\left| \frac{\partial \varphi_i^{-1}}{\partial x_n} \right| = \frac{1}{\left| \frac{\partial \varphi_i}{\partial x_{n-1}} \right|} < 1 < 1 + 2M_{2n-1}.$$

Hence,

$$\#\left\{ j : \left| \frac{\partial \varphi_i}{\partial x_j} \right| < 1 + 2M_{2n-1} \right\} < \#\left\{ \nu : \left| \frac{\partial \varphi_i^{-1}}{\partial x_\nu} \right| < 1 + 2M_{2n-1} \right\} \quad \text{for } i \in \{1, 2\}.$$

Again it suffices to decompose the cell  $H$  defined as in Case (3.10). Observe that after transposition  $(x_{n-1}, x_n)$ ,  $H$  is the following cell

$$H = \{(x'', x_n, x_{n-1}) : x'' \in \Omega, \quad \varphi_1(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x'')), \\ \varphi_2^{-1}(x'', x_n) < x_{n-1} < \varphi_1^{-1}(x'', x_n)\}.$$

Since  $\langle H \rangle > \langle G \rangle$ , the induction hypothesis gives the desired decomposition.

*Case (3.12)* Then  $\varphi_1(\sigma) \subset R$  and define

$$\psi(x'', x_{n-1}) = \varphi_1(x'', \sigma(x'')) + (1 + 2M_{2n-1})(x_{n-1} - \sigma(x'')),$$

for  $(x'', x_{n-1}) \in \Delta$ . Now  $G$  splits into two cells:

$$S_1 = \{(x', x_n) : x' \in \Delta, \quad \varphi_1(x') < x_n < \psi(x')\}$$

and

$$S_2 = \{(x', x_n) : x' \in \Delta, \quad \psi(x') < x_n < \varphi_2(x')\}.$$

Observe that

$$\frac{\partial \psi}{\partial x_j} = \frac{\partial \varphi_1}{\partial x_j} + \left[ \frac{\partial \varphi_1}{\partial x_{n-1}} - (1 + 2M_{2n-1}) \right] \frac{\partial \sigma}{\partial x_j},$$

for  $j \in \{1, \dots, n-2\}$ , almost everywhere on  $\Delta$ .

Hence, by (3.8), (3.12) and (3.9),

$$\left| \frac{\partial \psi}{\partial x_j} \right| \leq \left| \frac{\partial \varphi_1}{\partial x_{r_1}} \right| + 2M_{2n-1}(1 + 2M_{2n-1}) \leq \frac{3}{2} \left| \frac{\partial \varphi_1}{\partial x_{r_1}} \right|$$

and

$$\left| \frac{\partial \psi}{\partial x_{r_1}} \right| \geq \left| \frac{\partial \varphi_1}{\partial x_{r_1}} \right| - 2M_{2n-1}(1 + 2M_{2n-1}) \geq \frac{1}{2} \left| \frac{\partial \varphi_1}{\partial x_{r_1}} \right| \geq 2M_{2n-1}(1 + 2M_{2n-1}).$$

Therefore,

$$\frac{\left| \frac{\partial \psi}{\partial x_j} \right|}{\left| \frac{\partial \psi}{\partial x_{r_1}} \right|} \leq 3,$$

for any  $j \in \{1, \dots, n-2\}$ . Thus  $S_1$  satisfies the conditions (3.13)–(3.15) and the case (3.10) applies.

On the other hand, if  $j \in \{1, \dots, n-2\}$  and

$$\left| \frac{\partial \varphi_1}{\partial x_j} \right| < 1 + 2M_{2n-1},$$

then

$$\left| \frac{\partial \psi^{-1}}{\partial x_j} \right| = \frac{\left| \frac{\partial \psi}{\partial x_j} \right|}{\left| \frac{\partial \psi}{\partial x_{n-1}} \right|} \leq \frac{\left| \frac{\partial \varphi_1}{\partial x_j} \right| + 2M_{2n-1}(1 + 2M_{2n-1})}{1 + 2M_{2n-1}} < 1 + 2M_{2n-1};$$

hence,

$$\#\left\{j : \left|\frac{\partial\varphi_1}{\partial x_j}\right| < 1 + 2M_{2n-1}\right\} \leq \#\left\{\nu : \left|\frac{\partial\psi^{-1}}{\partial x_\nu}\right| < 1 + 2M_{2n-1}\right\},$$

while

$$\#\left\{j : \left|\frac{\partial\varphi_2}{\partial x_j}\right| < 1 + 2M_{2n-1}\right\} < \#\left\{\nu : \left|\frac{\partial\varphi_2^{-1}}{\partial x_\nu}\right| < 1 + 2M_{2n-1}\right\}$$

and we finish by the induction hypothesis as in Case (3.11).

In the case of any  $(L, P)$ -cell with respect to  $x_r$  it is enough to repeat all the argument with suitable changes; in particular, one should put  $r_1 = r_2 = r$  and a coefficient  $P$  instead of 3 in (3.15). Moreover, one can assume that

$$\left|\frac{\partial\varphi_i}{\partial x_r}\right| \geq 2M_{2n-1} \left|\frac{\partial\varphi_i}{\partial x_{n-1}}\right|,$$

for each  $i \in \{1, 2\}$ , since otherwise we could assume the opposite inequality, which easily gives a representation of  $G$  as a semi- $2M_{2n-1}max(L^{-1}, P)$ -cell.

#### 4. Theorem $1_n$ for a semi- $M$ -cell.

**Proposition 4.** *Any semi- $M$ -cell  $G$  in  $R^n$  (where  $M > 0$ ) admits an almost decomposition*

$$(4.1) \quad G \simeq S_1 \cup \dots \cup S_k,$$

where every  $S_\nu$  is an  $M'$ -cell after a permutation of coordinates and  $M' \geq 1$  is a constant depending only on  $M$  and  $n$ .

*Proof.* One can assume that  $G$  is in the form (3.1), where  $\varphi_i : \Delta \rightarrow R$  ( $i = 1, 2$ ) are continuous and

$$(4.2) \quad \left|\frac{\partial\varphi_1}{\partial x_j}\right| < M \quad \text{almost everywhere on } \Delta, \text{ for } j \in \{1, \dots, n-1\}.$$

Indeed, in the case  $\varphi_1 \equiv -\infty$  or  $\varphi_1 \equiv +\infty$  reduces to the above by assuming first that  $\Delta$  is an  $M_{2n-1}$ -disc and applying next transposition  $(x_{n-1}, x_n)$ .

The proof will be by descending induction on the number

$$[G] = \#\left\{j : \left|\frac{\partial\varphi_2}{\partial x_j}\right| \leq M_{2n-1} \quad \text{almost everywhere on } \Delta\right\}.$$

If  $[G] = n - 1$ ,  $G$  is a  $max(M, M_{2n-1})$ -cell, so assume that  $[G] < n - 1$ . Notice that if  $\tilde{\Delta} \subset \Delta$ , then for  $\tilde{G} = G \cap (\tilde{\Delta} \times R)$ ,  $[\tilde{G}] \geq [G]$ .

Fix any  $L > max(M, M_{2n-1})$  and any  $M^* > M + (L + M)M_{2n-1}$ . Dividing  $\Delta$ , one can assume that every  $\varphi_i$  is  $C^1$  on  $\Delta$  and

$$(4.3) \quad \text{for each } j \in \{1, \dots, n-1\}, \quad \text{sgn} \frac{\partial\varphi_i}{\partial x_j} = \text{const};$$

$$(4.4) \quad \text{for each } j \in \{1, \dots, n-1\}, \quad \left| \frac{\partial \varphi_2}{\partial x_j} \right| > L \quad \text{on } \Delta \quad \text{or} \quad \left| \frac{\partial \varphi_2}{\partial x_j} \right| \leq L \quad \text{on } \Delta$$

and

$$(4.5) \quad \text{there exists } r \in \{1, \dots, n-1\} \quad \text{such that} \quad \left| \frac{\partial \varphi_2}{\partial x_r} \right| \geq \left| \frac{\partial \varphi_2}{\partial x_j} \right|$$

$$\text{for each } j \in \{1, \dots, n-1\}, \text{ and either } \left| \frac{\partial \varphi_2}{\partial x_r} \right| \geq M^* \quad \text{or} \quad \left| \frac{\partial \varphi_2}{\partial x_r} \right| \leq M^* \quad \text{on } \Delta.$$

Clearly, one can assume that

$$(4.6) \quad \left| \frac{\partial \varphi_2}{\partial x_r} \right| \geq M^* \quad \text{on } \Delta.$$

Finally, by Theorem  $2_{n-1}$  and Lemma 2, one can assume that

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega, \quad \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an  $M_{2n-1}$ -disc in  $R^{n-1}$  and every  $\varphi_i$  admits a continuous extension

$$\varphi_i : \Delta \cup \sigma \cup \rho \longrightarrow \bar{R}$$

such that  $\varphi_i(\sigma) \subset R$  or  $\varphi_i(\sigma) = \{-\infty\}$ , or  $\varphi_i(\sigma) = \{+\infty\}$ , and the same for  $\rho$ . Because of (4.2),  $\varphi_1 : \Delta \cup \sigma \cup \rho \longrightarrow R$ .

$$\text{Case I:} \quad \left| \frac{\partial \varphi_2}{\partial x_{n-1}} \right| > L \quad \text{on } \Delta.$$

Assume that  $\frac{\partial \varphi_2}{\partial x_{n-1}} > L$ ; the case  $\frac{\partial \varphi_2}{\partial x_{n-1}} < -L$  will follow by a modification. Consider the following function

$$(4.7) \quad \psi(x'', x_{n-1}) = \varphi_1(x'', \sigma(x'')) + L(x_{n-1} - \sigma(x'')),$$

for  $(x'', x') \in \Delta$ .

Then  $\varphi_1 < \psi < \varphi_2$  and  $G \simeq S_1 \cup S_2$ , where

$$S_1 = \{(x', x_n) : x' \in \Delta, \quad \varphi_1(x') < x_n < \psi(x')\}$$

is an  $M^*$ -cell and

$$S_2 = \{(x', x_n) : x' \in \Delta, \quad \psi(x') < x_n < \varphi_2(x')\}$$

can be interpreted after transposition  $(x_{n-1}, x_n)$  as

$$S_2 = \{(x'', x_{n-1}, x_n) : x'' \in \Omega, \varphi_1(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x''))\},$$

$$\theta_2(x'', x_n) < x_{n-1} < \theta_1(x'', x_n)\},$$

where

$$\theta_2(x'', x_n) = \begin{cases} \sigma(x''), & \text{if } \varphi_1(x'', \sigma(x'')) < x_n \leq \varphi_2(x'', \sigma(x'')) \\ \varphi_2^{-1}(x'', x_n), & \text{if } \varphi_2(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x'')) \end{cases}$$

and

$$\theta_1(x'', x_n) = \begin{cases} \psi^{-1}(x'', x_n), & \text{if } \varphi_1(x'', \sigma(x'')) < x_n \leq \psi(x'', \rho(x'')) \\ \rho(x''), & \text{if } \psi(x'', \rho(x'')) < x_n < \varphi_2(x'', \rho(x'')), \end{cases}$$

and where  $\varphi_2^{-1}$  (and  $\psi^{-1}$ ) stands for the inversion of  $\varphi_2$  (and  $\psi$ ) with respect to  $x_{n-1}$ . Now, if  $j \in \{1, \dots, n-2\}$  and

$$\left| \frac{\partial \varphi_2}{\partial x_j} \right| \leq M_{2n-1},$$

then

$$\left| \frac{\partial \varphi_2^{-1}}{\partial x_j} \right| = \frac{\left| \frac{\partial \varphi_2}{\partial x_j} \right|}{\left| \frac{\partial \varphi_2}{\partial x_{n-1}} \right|} < \left| \frac{\partial \varphi_2}{\partial x_j} \right| \leq M_{2n-1}$$

and, moreover,

$$\left| \frac{\partial \varphi_2^{-1}}{\partial x_n} \right| = \frac{1}{\left| \frac{\partial \varphi_2}{\partial x_{n-1}} \right|} < \frac{1}{L} < M_{2n-1}.$$

Hence,  $[S_2] > [G]$  and the induction hypothesis ends the proof in this case.

$$\text{Case II: } \left| \frac{\partial \varphi_2}{\partial x_{n-1}} \right| \leq L \quad \text{on } \Delta.$$

By (4.6) and (4.3), one can assume without any loss of generality that

$$\frac{\partial \varphi_2}{\partial x_r} \geq M^*, \quad \frac{\partial \varphi_2}{\partial x_{n-1}} > 0 \quad \text{and} \quad \frac{\partial \varphi_1}{\partial x_{n-1}} > 0;$$

other possibilities will follow by simple modifications.

Since  $M^* > L$ ,  $r \in \{1, \dots, n-2\}$ . By Proposition 2, we have almost everywhere on  $\Delta$ :

$$\frac{\partial}{\partial x_r} \varphi_2(x'', \sigma(x'')) = \left| \frac{\partial \varphi_2}{\partial x_r}(x'', \sigma(x'')) + \frac{\partial \varphi_2}{\partial x_{n-1}}(x'', \sigma(x'')) \frac{\partial \sigma}{\partial x_r}(x'') \right| \geq$$

$$M^* - LM_{2n-1},$$

while

$$\left| \frac{\partial}{\partial x_r} \varphi_1(x'', \sigma(x'')) \right| \leq M + MM_{2n-1} \quad \text{and} \quad \left| \frac{\partial}{\partial x_r} \varphi_1(x'', \rho(x'')) \right| \leq M + MM_{2n-1}.$$

Thus, by Lemma 5,

$$\varphi_2(x'', \sigma(x'')) > \varphi_1(x'', \rho(x'')) \quad \text{on } \Omega.$$

Hence,

$$G \simeq S_1 \cup S_2 \cup S_3,$$

where

$$S_1 = \{(x'', x_{n-1}, x_n) : (x'', x_{n-1}) \in \Delta, \varphi_1(x'', x_{n-1}) < x_n < \varphi_1(x'', \rho(x''))\},$$

$$S_2 = \{(x'', x_{n-1}, x_n) : x'' \in \Omega, \varphi_1(x'', \rho(x'')) < x_n < \varphi_2(x'', \sigma(x'')), \\ \sigma(x'') < x_{n-1} < \rho(x'')\}$$

and

$$S_3 = \{(x'', x_{n-1}, x_n) : (x'', x_{n-1}) \in \Delta, \varphi_2(x'', \rho(x'')) < x_n < \varphi_2(x'', x_{n-1})\}.$$

$S_1$  is an  $M^*$ -cell, while  $S_2$  is an  $M_{2n-1}$ -cell after transposition  $(x_{n-1}, x_n)$ . We will investigate  $S_3$ . Put

$$\tilde{\Delta} = \{(x'', x_n) : x'' \in \Omega, \varphi_2(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x''))\}.$$

Now,

$$S_3 = \{(x'', x_{n-1}, x_n) : (x'', x_n) \in \tilde{\Delta}, \varphi_2^{-1}(x'', x_n) < x_{n-1} < \rho(x'')\},$$

where  $\varphi_2^{-1}$  stands for the inversion of  $\varphi_2$  with respect to  $x_{n-1}$ .

We will use Lemma 3 to get a desired decomposition of  $S_3$ . Observe first that

$$\frac{\partial \varphi_2^{-1}}{\partial x_r} = \frac{\frac{\partial \varphi_2}{\partial x_r}}{\frac{\partial \varphi_2}{\partial x_{n-1}}} \geq \frac{\frac{\partial \varphi_2}{\partial x_r}}{L} \geq \frac{M^*}{L} > \frac{M + (L + M)M_{2n-1}}{L} > M_{2n-1} \geq \left| \frac{\partial \rho}{\partial x_r} \right|$$

and

$$\frac{\left| \frac{\partial \varphi_2^{-1}}{\partial x_j} \right|}{\left| \frac{\partial \varphi_2^{-1}}{\partial x_r} \right|} = \frac{\left| \frac{\partial \varphi_2}{\partial x_j} \right|}{\left| \frac{\partial \varphi_2}{\partial x_r} \right|} \leq 1, \quad \text{for } j \in \{1, \dots, n-2\},$$

and

$$\frac{\left| \frac{\partial \varphi_2^{-1}}{\partial x_n} \right|}{\left| \frac{\partial \varphi_2^{-1}}{\partial x_r} \right|} = \frac{1}{\left| \frac{\partial \varphi_2^{-1}}{\partial x_r} \right|} \leq \frac{1}{M^*} < 1.$$

Now it suffices to check that  $\Delta$  has an almost decomposition into  $N$ -cells with respect to the variable  $x_r$ , where a constant  $N$  depends only on  $M, L, M^*$  and  $M_{2n-1}$ . We will check this using Lemma 6 (2).



We have almost everywhere on  $\Omega$ :

$$\frac{\partial}{\partial x_r} \varphi_2(x'', \sigma(x'')) \geq \frac{\partial \varphi_2}{\partial x_r}(x'', \sigma(x'')) \left(1 - \frac{LM_{2n-1}}{M^*}\right) \geq M^* - LM_{2n-1}$$

and

$$\frac{\left| \frac{\partial}{\partial x_j} \varphi_2(x'', \sigma(x'')) \right|}{\left| \frac{\partial}{\partial x_r} \varphi_2(x'', \sigma(x'')) \right|} \leq \frac{\left| \frac{\partial \varphi_2}{\partial x_j}(x'', \sigma(x'')) + \frac{\partial \varphi_2}{\partial x_{n-1}}(x'', \sigma(x'')) \frac{\partial \sigma}{\partial x_j}(x'') \right|}{\left| \frac{\partial \varphi_2}{\partial x_r}(x'', \sigma(x'')) \right| \frac{M(1 + M_{2n-1})}{M^*}} \leq \frac{M^*}{M}.$$

The same is true for  $\rho$  in the place of  $\sigma$ . Moreover, by the assumption of Case II,

$$|\varphi_2(x'', \sigma(x'')) - \varphi_2(x'', \rho(x''))| \leq |\sigma(x'') - \rho(x'')| \quad \text{on } \Omega.$$

Hence,

$$\lim_{x'' \rightarrow a''} [\varphi_2(x'', \sigma(x'')) - \varphi_2(x'', \rho(x''))] = 0,$$

for any  $a'' \in \partial\Omega$ , so the assumptions of Lemma 6 (2) are satisfied.

## 5. Proof of Theorem 2<sub>n</sub> for any $M$ -cell.

Let

$$G = \{(x', x_n) : x' \in \Delta, \quad \varphi_1(x') < x_n < \varphi_2(x')\}$$

be any  $M$ -cell, where  $M \in R$ ,  $M \geq 1$ . Observe that all possible cases reduce to the case  $\varphi_i : \Delta \rightarrow R$  ( $i \in \{1, 2\}$ ). Indeed, suppose for example that  $\varphi_1 : \Delta \rightarrow R$  and  $\varphi_2 \equiv +\infty$ . Then one can assume first that  $\varphi_1$  is  $C^1$  on  $\Delta$  and, for each  $j \in \{1, \dots, n-1\}$ ,

$$\operatorname{sgn} \frac{\partial \varphi_1}{\partial x_j} = \operatorname{const} \quad \text{on } \Delta,$$

and next that

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega, \quad \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an  $M_{2n-1}$ -disc in  $R^{n-1}$  such that  $\varphi_1$  has a continuous extension

$$\varphi_1 : \Delta \cup \sigma \cup \rho \rightarrow R.$$

Then, assuming that  $\frac{\partial \varphi_1}{\partial x_{n-1}} > 0$ ,

$$G \simeq S_1 \cup S_2,$$

where

$$S_1 = \{(x'', x_{n-1}, x_n) : (x'', x_{n-1}) \in \Delta, \quad \varphi_1(x', x_{n-1}) < x_n < \varphi_1(x'', \rho(x''))\}$$

is an  $M(1 + M_{2n-1})$ -cell, while

$$S_2 = \{(x'', x_{n-1}, x_n) : x'' \in \Omega, \quad \varphi_1(x'', \rho(x'')) < x_n, \quad \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an  $M_{2n-1}$ -cell after transposition  $(x_{n-1}, x_n)$ .

Consequently, assume that  $\varphi_i : \Delta \rightarrow R$  ( $i \in \{1, 2\}$ ) and that they are  $\mathcal{C}^1$ . By Theorem 3 $_{n-1}$ , one can assume that  $\Delta$  is a regular  $M_{3n-1}$ -cell and then, by Proposition 1, that every  $\varphi_i$  has a continuous extension

$$\varphi_i : \bar{\Delta} \rightarrow R \quad (i \in \{1, 2\}).$$

Now, still keeping the last property, one can assume that

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega, \quad \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an  $M_{2n-1}$ -disc. Put

$$\lambda_1(x'', x_{n-1}) = \varphi_1(x'', \sigma(x'')) + 2M(x_{n-1} - \sigma(x'')),$$

$$\lambda_2(x'', x_{n-1}) = \varphi_1(x'', \rho(x'')) - 2M(x_{n-1} - \rho(x'')),$$

$$\lambda_3(x'', x_{n-1}) = \varphi_2(x'', \rho(x'')) + 2M(x_{n-1} - \rho(x'')),$$

and

$$\lambda_4(x'', x_{n-1}) = \varphi_2(x'', \sigma(x'')) - 2M(x_{n-1} - \sigma(x'')),$$

for any  $(x'', x_{n-1}) \in \Omega \times R$ . Every  $\lambda_i$  has a continuous extension to  $\bar{\Omega} \times R$  and is an  $M(1 + 3M_{2n-1})$ -function. Its inversion  $\lambda_i^{-1}$  with respect to  $x_{n-1}$  has a continuous extension to  $\bar{\Omega} \times R$  as well and is a  $\frac{1}{2}(1 + 3M_{2n-1})$ -function.

For any subset  $I \subset \{1, 2, 3, 4\}$ , put

$$S_I = \{(x', x_n) \in G : x_n < \lambda_i(x'), \text{ if } i \in I \quad \text{and} \quad \lambda_i(x') < x_n, \text{ if } i \notin I\}.$$

Then

$$G \simeq \bigcup_I S_I.$$

It suffices to show that every  $S_I$  is an  $M(1 + 3M_{2n-1})$ -disc after perhaps transposition  $(x_{n-1}, x_n)$ .

Fix any  $I \subset \{1, 2, 3, 4\}$ .

If  $\{1, 2\} \subset I$ , then

$$S_I = \{(x', x_n) \in \Delta \times R : \varphi_1(x') < x_n < \varphi_2(x'), \quad x_n < \lambda_i(x'), \text{ if } i \in I,$$

$$\lambda_i(x') < x_n, \text{ if } i \notin I\},$$

and  $\lambda_1 = \varphi_1$  on  $\sigma$ , while  $\lambda_2 = \varphi_1$  on  $\rho$  and Lemma 4 applies.

Similarly, when  $\{3, 4\} \cap I = \emptyset$ .

If  $\{1, 2\} \not\subset I$  and  $\{3, 4\} \cap I \neq \emptyset$ , we have  $1 \notin I$  and  $3 \in I$  or  $1 \notin I$  and  $4 \in I$  (or, similarly,  $2 \notin I$  and  $3 \in I$  or  $2 \notin I$  and  $4 \in I$ ).

Suppose first that  $1 \notin I$  and  $3 \in I$ . Then

$$(5.1) \quad S_I = \{(x'', x_{n-1}, x_n) : x'' \in \Omega, \quad \varphi_1(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x'')), \\ \sigma(x'') < x_{n-1} < \rho(x''), x_{n-1} < \lambda_i^{-1}(x'', x_n) \text{ if } i \in \tilde{I}, \lambda_i^{-1}(x'', x_n) < x_{n-1} \text{ if } i \notin \tilde{I}\},$$

where  $\tilde{I} \subset \{1, 2, 3, 4\}$  is defined by the formula:

$i \in \tilde{I}$  if and only if  $i \in I$  and  $i$  is even or  $i \notin I$  and  $i$  is odd.

Since

$$\lambda_1^{-1}(x'', \varphi_1(x'', \sigma(x''))) = \sigma(x'')$$

and

$$\lambda_3^{-1}(x'', \varphi_2(x'', \rho(x''))) = \rho(x''),$$

for each  $x'' \in \Omega$  and

$$\sigma(x'') = \rho(x''),$$

for each  $x'' \in \partial\Omega$ , we are done by Lemma 4.

Let now  $1 \notin I$  and  $4 \in I$ . Then (5.1) holds and since

$$\lambda_1^{-1}(x'', \varphi_1(x'', \sigma(x''))) = \sigma(x''), \quad \lambda_4^{-1}(x'', \varphi_2(x'', \sigma(x''))) = \sigma(x''),$$

for each  $x'' \in \Omega$  and  $\sigma(x'') = \rho(x'')$ , for each  $x'' \in \partial\Omega$ , we are again done due to Lemma 4.

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