LIPSCHITZ CELL DECOMPOSITION IN O-MINIMAL STRUCTURES

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Abstract. A main tool in studying topological properties of sets definable in o-minimal structures is the Cell Decomposition Theorem. This paper proposes its metric counterpart.

1.Introduction.

Fix any o-minimal structure on a real closed field R (for the definition and fundamental properties of o-minimal structures the reader is referred to [vdD]). Let n be a positive integer.

A subset S of R^n will be called an (*open*) *cell* in R^n iff

$$S = \{(x', x_n) \in R^n : x' \in \Delta,\ \varphi_1(x') < x_n < \varphi_2(x')\}, \tag{1.1}$$

where $x' = (x_1, \dots, x_{n-1})$, Δ is an open definable subset of R^{n-1}, every $\varphi_i\,(i \in \{1,2\})$ is either a definable continuous function $\varphi_i : \Delta \longrightarrow R$ or $\varphi_i \equiv -\infty$ or $\varphi_i \equiv +\infty$ and, for each $x' \in \Delta$, $\varphi_1(x') < \varphi_2(x')$.

For any positive $M \in R$, a definable continuous function $\varphi : \Delta \longrightarrow R$ defined on an open subset Δ of R^{n-1} will be called an *M-function* iff

$$\Big|\frac{\partial \varphi}{\partial x_j}(a)\Big| \le M \qquad (j \in \{1, \dots, n-1\}), \tag{1.2}$$

at each point $a \in \Delta$ in a neighborhood of which φ is of class $\mathcal{C}^1$.

An cell S in R^n will be called an *M-cell* (a *semi-M-cell*) iff, for each $i \in \{1,2\}$ (for at least one $i \in \{1,2\}$), if φ_i is finite, it is an M-function. A cell S in R^n will be called a *regular M-cell* iff it is any open interval in the case $n = 1$ and, in the

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case $n>1$, for each $i\in\{1,2\}$, if φ_i is finite it is an M-function of class $\mathcal{C}^1$ on Δ and the projection Δ of S into R^{n-1} is a regular M-cell in R^{n-1}.

An M-cell will be called an *M-disc* iff it is any open interval in the case $n=1$ and, in the case $n>1$, the both $\varphi_i\,(i\in\{1,2\})$ are finite and admit continuous extensions

$$\varphi_i:\overline{\Delta}\longrightarrow R \tag{1.3}$$

onto the closure of Δ in R^{n-1}, and

$$\varphi_1=\varphi_2 \quad \text{on} \quad \partial\Delta. \tag{1.4}$$

Proposition 1. *Let S be a regular M-cell in R^n and let $\varphi:S\longrightarrow R$ be an L-function $(\,L>0\,)$ of class $\mathcal{C}^1$.*

Then

(1) *for any two different points $a,b\in S$, there is a definable continuous mapping*

$$\lambda=(\lambda_1,\dots,\lambda_n):[0\,,|a-b|]\longrightarrow S$$

such that $\lambda(0)=a,\lambda(|a-b|)=b$ and $|\lambda_j'(t)|\le(j-1)!M^{j-1}$, for any $j\in\{1,\dots,n\}$ and any t such that $\lambda_j'(t)$ exists[1]*;*

(2) *for any two points $a,b\in S$,*

$$|\varphi(a)-\varphi(b)|\le n!M^{n-1}L|a-b|.$$

Proof. (1) Let S be as in (1.1). Arguing by induction and assuming that $a'\neq b'$, one can find a mapping

$$\omega=(\omega_1,\dots,\omega_{n-1}):[0\,,|a'-b'|]\longrightarrow\Delta$$

such that $\omega(0)=a',\omega(|a'-b'|)=b'$ and $|\omega_j'(\tau)|\le(j-1)!M^{j-1}$, for any $j\in\{1,\dots,n-1\}$ and any τ such that $\omega_j'(\tau)$ exists. Let $\varepsilon>0$ be such that

$$\varphi_1(\omega(\tau))+\varepsilon<\varphi_2(\omega(\tau))-\varepsilon,\quad \text{for any}\quad \tau\in[0\,,|a'-b'|],$$

and

$$\varphi_1(a')+\varepsilon<a_n<\varphi_2(a')-\varepsilon\quad\text{and}\quad\varphi_1(b')+\varepsilon<b_n<\varphi_2(b')-\varepsilon.$$

Now, it suffices to put

$$\lambda_j(t)=\omega_j(t\frac{|a'-b'|}{|a-b|}),\quad\text{for}\quad j\in\{1,\dots,n-1\},$$

and

$$\lambda_n(t)=max\{\varphi_1(\omega(t\frac{|a'-b'|}{|a-b|}))+\varepsilon,min\{\varphi_2(\omega(t\frac{|a'-b'|}{|a-b|}))-\varepsilon,a_n+t\frac{b_n-a_n}{|a-b|}\}\}.$$

(2) follows from (1), by the Mean Value Theorem (see [vdD, Chapter 7, (2.3)]).

[1] $|a-b|=\sqrt{\sum_{j=1}^{n}(a_j-b_j)^2}$

Kurdyka-Parusiński Theorem ([K, P]). *Any open definable subset G of R^n has a finite decomposition*

$$G = S_1 \cup \cdots \cup S_k \cup \Sigma,$$

where every S_ν is a regular M_n-cell in some linear coordinate system in R^n and Σ is nowhere dense, M_n being a constant depending only on n.

The aim of the present article is to show that in fact permutations of coordinates are sufficient in the above theorem. We will prove simultaneously by induction on n the following three theorems.

Theorem $1_n\,(2_n,\, 3_n)$**.** *Any open definable subset G of R^n has a finite decomposition*

$$G = S_1 \cup \cdots \cup S_k \cup \Sigma, \tag{1.5}$$

where every S_ν is an M_{1n}-cell (M_{2n}-disc, a regular M_{3n}-cell) in R^n after a permutation of coordinates and Σ is nowhere dense, M_{1n} (M_{2n}, M_{3n}) being a constant ≥ 1 depending only on n.

For simplicity we will often skip the adjective *definable*, when considering subsets of spaces R^n and mappings between such subsets. Also, we adopt the following conventions. A local property (w) of a mapping $f : A \longrightarrow R^m$, where $A \subset R^n$, is said to be satisfied *almost everywhere* iff there is a closed subset E of A such that $\dim E < \dim A$ and (w) is satisfied at each point of $A \setminus E$. A finite sequence $B_1, \ldots, B_k$ of subsets of a set $A \subset R^n$ is said to be an *almost decomposition* of A iff $B_\nu\,(\nu = 1, \ldots, k)$ are pairwise disjoint and $\dim\big(A \setminus (B_1 \cup \cdots \cup B_k)\big) < \dim A$. This will be denoted by writing

$$G \simeq B_1 \cup \cdots \cup B_k.$$

Since Theorem 2_n easily implies both Theorems 1_n and 3_n, it suffices to derive first Theorem 1_n from Theorem 2_{n-1} and then Theorem 2_n from Theorems 1_n, 2_{n-1} and 3_{n-1}. From now on, we will assume that $n \geq 2$ is fixed.

2. A preparation.

Lemma 1. *If $G \subset R^{n-1}$ is open and $E \subset \partial G$ is closed of dimension $< n-2$ and Theorem 2_{n-1} is true, then G has an almost decomposition*

$$G \simeq \Delta_1 \cup \cdots \cup \Delta_p,$$

where every Δ_ν, after a permutation of coordinates in R^{n-1}, is an M_{2n-1}-disc:

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega,\, \sigma_\nu(x'') < x_{n-1} < \rho_\nu(x'')\}^2,$$

such that the both (graphs of)[3] *σ_ν and ρ_ν are disjoint with E.*

[2] $x'' = (x_1, \ldots, x_{n-2})$

[3] We will identify functions with their graphs.

Proof. Take the projections

$$\pi_j : R^{n-1} \ni (x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}) \in R^{n-2},$$

for $j \in \{1, \dots, n-1\}$, and set

$$Z = \text{the closure of } \bigcup_j \pi_j^{-1}(\pi_j(E)).$$

Then $\dim Z \le n-2$ and it suffices to use Theorem 2_{n-1} to $G \setminus Z$.

As a corollary one easily gets (see [vdD]) the following

Lemma 2. *If $G \subset R^{n-1}$ is open and $\varphi : G \longrightarrow R$ is continuous, then G has an almost decomposition*

$$G \simeq \Delta_1 \cup \dots \cup \Delta_p\,,$$

where every Δ_ν, after a permutation of coordinates in R^{n-1}, is an M_{2n-1}-disc

$$\Delta_\nu = \{(x'', x_{n-1}) : \, x'' \in \Omega, \, \sigma_\nu(x'') < x_{n-1} < \rho_\nu(x'')\}$$

such that $\varphi|\Delta_\nu$ has a continuous extension

$$\varphi_\nu : \Delta_\nu \cup \sigma_\nu \cup \rho_\nu \longrightarrow \overline{R} = R \cup \{-\infty, +\infty\}$$

such that $\varphi_\nu(\sigma_\nu) \subset R$ or $\varphi_\nu(\sigma_\nu) = \{-\infty\}$, or $\varphi_\nu(\sigma_\nu) = \{+\infty\}$ and the same for ρ_ν.

Proposition 2. *Let $f : S \longrightarrow R$ be a definable $\mathcal{C}^1$-function defined on a cell*

$$S = \{(x', x_n) \in R^n : x' \in \Delta, \varphi(x') < x_n < \psi(x')\}$$

in R^n such that $\varphi : \Delta \longrightarrow R$ is of class $\mathcal{C}^1$.

Assume that $\dfrac{\partial f}{\partial x_n}$ has a finite limit value[4] *at (almost) each point of φ (for example, when $\dfrac{\partial f}{\partial x_n}$ is bounded).*

Then there is a closed nowhere dense subset Z of φ such that f extends to a $\mathcal{C}^1$-function

$$f : S \cup (\varphi \setminus Z) \longrightarrow R$$

to $S \cup (\varphi \setminus Z)$ as a $\mathcal{C}^1$-submanifold with boundary.

Proof. It is left to the reader as an exercise (cf [vdD]).

[4] An element $\alpha \in \overline{R}$ is a limit value of a function $g : S \longrightarrow R$ at $a \in \overline{S}$ iff there is an arc $\gamma : (0,1) \longrightarrow S$ such that $\lim_{t \to 0} \gamma(t) = a$ and $\lim_{t \to 0} g(\gamma(t)) = \alpha$.

Lemma 3. *Let $L, M, N, P \in R$ be positive and let*

$$G = \{(x', x_n) : x' \in \Delta,\ \varphi_1(x') < x_n < \varphi_2(x')\}$$

be a semi-M-cell in R^n such that Δ is an N-cell in R^{n-1}, $\varphi_i : \Delta \longrightarrow R$, for each $i \in \{1, 2\}$, and the following conditions are satisfied almost everywhere in Δ:

$$\left|\frac{\partial \varphi_1}{\partial x_j}\right| \leq M, \quad \text{for each } j \in \{1, \dots, n-1\}; \tag{2.1}$$

$$\left|\frac{\partial \varphi_1}{\partial x_{n-1}}\right| < L < \left|\frac{\partial \varphi_2}{\partial x_{n-1}}\right|; \tag{2.2}$$

$$\frac{\left|\dfrac{\partial \varphi_2}{\partial x_j}\right|}{\left|\dfrac{\partial \varphi_2}{\partial x_{n-1}}\right|} \leq P, \quad \text{for each } j \in \{1, \dots, n-1\}; \tag{2.3}$$

$$sgn \frac{\partial \varphi_2}{\partial x_{n-1}} = const. \tag{2.4}$$

Then G admits an almost decomposition

$$G \simeq S_1 \cup \dots \cup S_k,$$

where every S_ν is an $\tilde{M}$-cell, possibly after transposition (x_{n-1}, x_n), where $\tilde{M}$ is a positive constant depending only on L, M, N and P.

Proof. Put

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega, \sigma(x'') < x_{n-1} < \rho(x'')\}.$$

One can assume that

$$\frac{\partial \varphi_2}{\partial x_{n-1}} > 0\,; \tag{2.5}$$

the other case will follow by a modification. Because of (2.2) and (2.5), it is clear that $\sigma : \Delta \longrightarrow R$. By a subdivision of Ω one can assume that σ is of class $\mathcal{C}^1$ and that (2.2) is satisfied almost everywhere on every segment $\{(x'', x_{n-1}) : \sigma(x'') < x_{n-1} < \rho(x'')\}$, where $x'' \in \Omega$ and that φ_i admit continuous extensions

$$\varphi_i : \Delta \cup \sigma \longrightarrow R \qquad (i = 1, 2)$$

and

$$\varphi_2 : \Delta \cup \rho \longrightarrow R \cup \{+\infty\}$$

such that $\varphi_2(\rho) \subset R$ or $\varphi_2(\rho) = \{+\infty\}$.

By Proposition 2, φ_1 is of class $\mathcal{C}^1$ almost everywhere on σ. Put

$$\psi(x'', x_{n-1}) = \varphi_1(x'', \sigma(x'')) + L(x_{n-1} - \sigma(x'')), \quad \text{for } (x'', x_{n-1}) \in \Delta.$$

Then ψ is an $\max(M + MN + LN, L)$-function and $\varphi_1 < \psi < \varphi_2$.

Now $G \simeq S_1 \cup S_2$, where $S_1 = \{(x', x_n) : \varphi_1(x') < x_n < \psi(x')\}$ and $S_2 = \{(x'', x_{n-1}, x_n) : x'' \in \Omega, \Phi_1(x'', x_n) < x_{n-1} < \Phi_2(x'', x_n)\}$, where

$$\Phi_2(x'', x_n) = \begin{cases} \psi^{-1}(x'', x_n) = L^{-1}(x_n - \varphi_1(x'', \sigma(x''))) + \sigma(x''), \\ \qquad\qquad \text{if } \varphi_1(x'', \sigma(x'')) < x_n < \psi(x'', \rho(x'')) \\ \\ \rho(x''), \quad \text{if } \psi(x'', \rho(x'')) \le x_n < \varphi_2(x'', \rho(x'')) \end{cases}$$

and

$$\Phi_1(x'', x_n) = \begin{cases} \sigma(x''), & \text{if } \varphi_1(x'', \sigma(x'')) < x_n \le \varphi_2(x'', \sigma(x'')) \\ \varphi_2^{-1}(x'', x_n), & \text{if } \varphi_2(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x'')), \end{cases}$$

where ψ^{-1} and φ_2^{-1} stand for inversions with respect to x_{n-1}.

Lemma 4. *Let $A \subset R^{n-1}$ be open and let $M \in R, M > 0$. Let $f_\alpha : A \longrightarrow R$ ($\alpha \in \{1, \dots, k+l\}$) be M-functions on A each of which has a continuous extension to $\overline{A}$:*

$$f_\alpha : \overline{A} \longrightarrow R.$$

Assume that for each $a \in \partial A$ there are $\alpha \le k$ and $\beta > k$ such that $f_\beta(a) \le f_\alpha(a)$.

Then the set

$$S = \{(x', x_n) \in A \times R : \max_{1 \le \alpha \le k} f_\alpha(x') < x_n < \min_{k < \beta \le k+l} f_\beta(x')\}$$

is an M-disc in R^n.

Proof. Indeed,

$$S = \{(x', x_n) \in B \times R : \max_{1 \le \alpha \le k} f_\alpha(x') < x_n < \min_{k < \beta \le k+l} f_\beta(x')\},$$

where B is the natural projection of S to A. It is clear that $\max\limits_{1 \le \alpha \le k} f_\alpha = \min\limits_{k < \beta \le k+l} f_\beta$ on ∂B and the lemma follows.

Lemma 5. *Let $\alpha_1, \alpha_2 \in \overline{R}, \alpha_1 < \alpha_2$ and let $f, g, h : (\alpha_1, \alpha_2) \longrightarrow R$ be three continuous definable functions such that*

(2.6) $$g \le f \quad \text{on } (\alpha_1, \alpha_2);$$

(2.7) *for each $i \in \{1, 2\}$, if $\alpha_i \in R$, then* $\lim\limits_{t \to \alpha_i} g(t) = \lim\limits_{t \to \alpha_i} h(t) \in R$;

$$(2.8) \qquad sgn f'(t) = const \quad \textit{almost everywhere in } (\alpha_1, \alpha_2),$$

and there is $\epsilon > 0$ *such that*

$$(2.9) \qquad |f'(t)| \geq |g'(t)| + \epsilon \textit{ and } |f'(t)| > |h'(t)| \quad \textit{almost everywhere in } (\alpha_1, \alpha_2).$$

Then $h < f$ *on* (α_1, α_2).

Proof. One can assume that $f'(t) > 0$. Then $\alpha_1 \in R$, since otherwise by (2.9), $\lim_{t \to -\infty} (f(t) - g(t)) = -\infty$, a contradiction with (2.6). By (2.9), $f - h$ is strictly increasing and, by (2.6) and (2.7),

$$\lim_{t \to \alpha_1} (f(t) - h(t)) \geq \lim_{t \to \alpha_1} (g(t) - h(t)) = 0.$$

Hence, $f - h > 0$ on (α_1, α_2).

3. Reduction of Theorem 1_n to a special case of semi-M-cells.

By the standard cell decomposition theorem (see [vdD]) and since

$$R^n = \bigcup_{j=1}^{n} \{(x_1, \dots, x_n) \in R^n : |x_k| \leq |x_j|, \text{for any } k \neq j\},$$

it suffices to derive Theorem 1_n for any cell G in R^n such that

$$(3.1) \qquad G = \{(x', x_n) : x' \in \Delta, \, \varphi_1(x') < x_n < \varphi_2(x')\},$$

where $\varphi_i : \Delta \longrightarrow R$ $(i = 1, 2)$ are continuous.

For given positive $L, P \in R$ such a cell G will be called an (L, P)*-cell* (*with respect to the variable* x_r), where $r \in \{1, \dots, n-1\}$, iff

$$(3.2) \qquad \left| \frac{\partial \varphi_i}{\partial x_r} \right| \geq L \quad \text{and} \quad \frac{\left| \frac{\partial \varphi_i}{\partial x_j} \right|}{\left| \frac{\partial \varphi_i}{\partial x_r} \right|} \leq P \,,$$

almost everywhere on Δ, for $i \in \{1, 2\}$, $j \in \{1, \dots, n-1\}$.

Proposition 3.

(1) *Any open cell* $G \subset R^n$ *has an almost decomposition*

$$(3.3) \qquad G \simeq S_1 \cup \dots \cup S_k,$$

where every S_ν *is either a semi-*M_n*-cell or an* (L_n, P_n)*-cell after a permutation of coordinates, where positive constants* M_n, L_n *and* P_n *depend only on* n.

(2) *If a cell* G *is an* (L, P)*-cell, then* G *has an almost decomposition* (3.3) *with only semi-*M*-cells, where a constant* M *depends only on* n, L *and* P.

To prove Proposition 3 we first have the following

Lemma 6. *Let H be an open subset of R^n and let E be a closed subset of ∂H such that* $\dim E < n-1$. *Let $r_i \in \{1,\dots,n-1\}$ $(i \in \{1,2\})$. Assume that $L, P \in R$ are positive and such that, for each $a \in \partial H \setminus E$:*

$(3.4-i)$ *there exists a neighborhood U of a in R^n such that $\partial H \cap U$ is (the graph of) a $\mathcal{C}^1$-function $\psi : V \longrightarrow R$ defined on an open $V \subset R^{n-1}$ and such that*

$$\left|\frac{\partial \psi}{\partial x_{r_i}}\right| \geq L \quad \textit{and} \quad \frac{\left|\dfrac{\partial \psi}{\partial x_j}\right|}{\left|\dfrac{\partial \psi}{\partial x_{r_i}}\right|} \leq P \quad \textit{on } V \textit{ for } j \in \{1,\dots,n-1\},$$

for $i=1$ or $i=2$.

Then:

(1) *H admits an almost decomposition*

$$H \simeq S_1 \cup \cdots \cup S_k, \tag{3.5}$$

where every S_{nu} is either a semi-$\max(L^{-1},P)$*-cell or a* $(P^{-1}, \max(L^{-1},P))$*-cell in R^n after transposition (x_{r_i}, x_n).*

(2) *If $r_1 = r_2 = r, H$ has such an almost decomposition* (3.5) *that every S_ν is a* $\max(L^{-1},P)$*-cell after transposition (x_r, x_n).*

Proof of Lemma 6. After transposition (x_{r_1}, x_n) take a $\mathcal{C}^1$-cell decomposition compatible with each of the sets

$$\Lambda_i = \{a \in \partial H \setminus E : a \text{ satisfies } (3.4-i)\}$$

$(i=1,2)$ and with E. This gives an almost decomposition

$$H \simeq S_1 \cup \cdots \cup S_k,$$

where every cell S_ν is of the form

$$S_\nu = \{(x', x_n) : x' \in \Delta_\nu, \varphi_{1\nu}(x') < x_n < \varphi_{2\nu}(x')\},$$

such that, for $i \in \{1,2\}$, either $\varphi_{i\nu} \subset \Lambda_1$ or $\varphi_{i\nu} \subset \Lambda_2$, or $\varphi_{i\nu} \equiv -\infty$, or $\varphi_{i\nu} \equiv +\infty$.

One can assume that for each i either $\varphi_{i\nu} \subset \Lambda_1$ or $\varphi_{i\nu} \subset \Lambda_2$, since otherwise S_ν is trivially a semi-$\max(L^{-1},P)$-cell.

If $\varphi_{i\nu} \subset \Lambda_1$, for at least one i, then S_ν is a semi-$\max(L^{-1},P)$-cell.

If $\varphi_{i\nu} \subset \Lambda_2$, for each $i \in \{1,2\}$, and $r_1 \neq r_2$, then it is easy to check that S_ν is an $(L, \max(L^{-1},P))$-cell with respect to x_{r_2}.

Proof of Proposition 3. One can assume that G is as in (3.1). The proof will be by descending induction on the number

$$\langle G \rangle = \sum_{i=1}^{2} \sharp\left\{ j : \left|\frac{\partial \varphi_i}{\partial x_j}\right| < 1 + 2M_{2n-1} \quad \text{almost everywhere on } \Delta \right\}.$$

If $\langle G\rangle = 2(n-1), G$ is a $(1+2M_{2n-1})$-cell, so assume that $\langle G\rangle < 2(n-1)$. Observe that if $\tilde{\Delta} \subset \Delta$ is open, then for $\tilde{G} = G \cap (\tilde{\Delta} \times R)$, $\langle \tilde{G}\rangle \geq \langle G\rangle$. Hence, one can assume that every φ_i is $\mathcal{C}^1$ and

$$\text{for each } j \in \{1, \dots, n-1\}, \quad \operatorname{sgn}\frac{\partial \varphi_i}{\partial x_j} = \text{const on } \Delta; \tag{3.6}$$

$$\text{for each } j \in \{1, \dots, n-1\}, \quad \text{either} \quad \left|\frac{\partial \varphi_i}{\partial x_j}\right| < 1 + 2M_{2n-1} \tag{3.7}$$

$$\text{or} \quad \left|\frac{\partial \varphi_i}{\partial x_j}\right| > 1 + 2M_{2n-1}, \quad \text{or} \quad \left|\frac{\partial \varphi_i}{\partial x_j}\right| = 1 + 2M_{2n-1} \quad \text{on} \quad \Delta$$

and there is $r_i \in \{1, \dots, n-1\}$ such that

$$\text{for each } j \in \{1, \dots, n-1\}, \quad \left|\frac{\partial \varphi_i}{\partial x_j}\right| \leq \left|\frac{\partial \varphi_i}{\partial x_{r_i}}\right| \quad \text{on} \quad \Delta. \tag{3.8}$$

Moreover, one can assume that

$$\left|\frac{\partial \varphi_i}{\partial x_{r_i}}\right| \geq 4M_{2n-1}(1 + 2M_{2n-1}), \qquad \text{for } i \in \{1, 2\}, \tag{3.9}$$

since otherwise G is a semi-$4M_{2n-1}(1 + 2M_{2n-1})$-cell. Besides, by Lemma 2, one can assume that

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega,\ \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an M_{2n-1}-disc and every φ_i has a continuous extension

$$\varphi_i : \Delta \cup \sigma \cup \rho \longrightarrow \overline{R}$$

such that $\varphi_i(\sigma) \subset R$ or $\varphi_i(\sigma) = \{-\infty\}$ or $\varphi_i(\sigma) = \{+\infty\}$, and the same for ρ.

Observe that if

$$\frac{\partial \varphi_1}{\partial x_{n-1}} \cdot \frac{\partial \varphi_2}{\partial x_{n-1}} \leq 0,$$

then clearly G is a semi-M_{2n-1}-cell after transposition (x_{n-1}, x_n), so without any loss of generality one can assume that

$$\frac{\partial \varphi_i}{\partial x_{n-1}} > 0 \quad \text{on} \quad \Delta, \quad \text{for } i \in \{1, 2\}.$$

By (3.7), one can distinguish the following three cases:

$$\left|\frac{\partial \varphi_i}{\partial x_{n-1}}\right| \leq 1 + 2M_{2n-1}, \quad \text{for} \quad i \in \{1, 2\}; \tag{3.10}$$

$$\left|\frac{\partial \varphi_i}{\partial x_{n-1}}\right| \geq 1 + 2M_{2n-1}, \quad \text{for} \quad i \in \{1, 2\}; \tag{3.11}$$

$$\left|\frac{\partial\varphi_1}{\partial x_{n-1}}\right| < 1+2M_{2n-1} \quad \text{and} \quad \left|\frac{\partial\varphi_2}{\partial x_{n-1}}\right| > 1+2M_{2n-1} \quad \text{(or vice-versa).} \tag{3.12}$$

Case (3.10) Here we will be using only that every $\varphi_i : \Delta\cup\sigma\cup\rho \longrightarrow R$ is continuous and there is a closed nowhere dense $Z\subset\Delta$ such that φ_i is $\mathcal{C}^1$ on $\Delta\setminus Z$ and

$$\left|\frac{\partial\varphi_i}{\partial x_{n-1}}\right| \le 1+2M_{2n-1}, \quad \text{on} \quad \Delta\setminus Z; \tag{3.13}$$

$$\left|\frac{\partial\varphi_i}{\partial x_j}\right| \le 3\left|\frac{\partial\varphi_i}{\partial x_{r_i}}\right| \quad \text{on} \quad \Delta\setminus Z \qquad (j=1,\dots,n-1) \tag{3.14}$$

and

$$\left|\frac{\partial\varphi_i}{\partial x_{r_i}}\right| \ge 2M_{2n-1}(1+2M_{2n-1}) \quad \text{on } \Delta\setminus Z. \tag{3.15}$$

Put

$$H=\{(x'',x_{n-1},x_n)\in G : \varphi_2(x'',\sigma(x''))<x_n<\varphi_1(x'',\rho(x''))\} =$$
$$\{(x',x_n)\in R^n : x'\in D,\ \Phi_1(x')<x_n<\Phi_2(x')\},$$

where

$$D=\{(x'',x_{n-1})\in\Delta : \varphi_2(x'',\sigma(x''))<\varphi_1(x'',\rho(x''))\},$$
$$\Phi_1(x'',x_{n-1})=\max(\varphi_2(x'',\sigma(x'')),\varphi_1(x'',x_{n-1}))$$

and

$$\Phi_2(x'',x_{n-1})=\min(\varphi_2(x'',x_{n-1}),\varphi_1(x'',\rho(x''))).$$

Observe that $\Phi_1=\Phi_2$ on $(\partial D)\cap(\Delta\cup\sigma\cup\rho)$, so almost everywhere on ∂D. Besides, if $\varphi_2(x'',\sigma(x''))\not\equiv-\infty$, we have by Proposition 2

$$\frac{\partial}{\partial x_j}\varphi_2(x'',\sigma(x''))=\frac{\partial\varphi_2}{\partial x_j}(x'',\sigma(x''))+\frac{\partial\varphi_2}{\partial x_{n-1}}(x'',\sigma(x''))\frac{\partial\sigma}{\partial x_j}(x''),$$

almost everywhere on Ω, for $j\in\{1,\dots,n-2\}$. Hence, by (3.13) and (3.15)

$$\left|\frac{\partial}{\partial x_j}\varphi_2(x'',\sigma(x''))\right| \le \frac{7}{2}\left|\frac{\partial\varphi_2}{\partial x_{r_2}}(x'',\sigma(x''))\right|$$

and

$$\left|\frac{\partial}{\partial x_{r_2}}\varphi_2(x'',\sigma(x''))\right| \ge \frac{1}{2}\left|\frac{\partial\varphi_2}{\partial x_{r_2}}(x'',\sigma(x''))\right| \ge M_{2n-1}(1+2M_{2n-1}).$$

Consequently,

$$\frac{\left|\frac{\partial}{\partial x_j}\varphi_2(x'',\sigma(x''))\right|}{\left|\frac{\partial}{\partial x_{r_2}}\varphi_2(x'',\sigma(x''))\right|} \le 7, \qquad \text{for any } j\in\{1,\dots,n-1\}.$$

In the same way, if $\varphi_1(x'',\rho(x''))\not\equiv+\infty$, we have almost everywhere on D

$$\left|\frac{\partial}{\partial x_{r_1}}\varphi_1(x'',\rho(x''))\right|\ge M_{2n-1}(1+2M_{2n-1})$$

and

$$\frac{\left|\dfrac{\partial}{\partial x_j}\varphi_1(x'',\rho(x''))\right|}{\left|\dfrac{\partial}{\partial x_{r_1}}\varphi_1(x'',\rho(x''))\right|}\le 7,\qquad \text{for any } j\in\{1,\dots,n-1\}.$$

By Lemma 6 (1), H admits an almost decomposition

$$H\simeq S_1\cup\cdots\cup S_k, \tag{3.16}$$

where every S_ν is either a semi-7-cell or a $(\frac{1}{7},7)$-cell in R^n after transposition (x_{r_1},x_n).

Since $G\setminus\overline{H}$ easily almost decomposes into a finite union of semi-M_{2n-1}-cells after transposition (x_{n-1},x_n), (3.16) extends to a similar decomposition of G. Now, repeating the same argument for any $(\frac{1}{7},7)$-cell S_ν in the place of G with Lemma 6 (2) ends the proof in this case.

Case (3.11) Let φ_i^{-1} denotes the inversion of φ_i with respect to x_{n-1} $(i\in\{1,2\})$.

Observe that if $\left|\dfrac{\partial\varphi_i}{\partial x_j}\right|<1+2M_{2n-1}$, then

$$\left|\frac{\partial\varphi_i^{-1}}{\partial x_j}\right|=\frac{\left|\dfrac{\partial\varphi_i}{\partial x_j}\right|}{\left|\dfrac{\partial\varphi_i}{\partial x_{n-1}}\right|}<1<1+2M_{2n-1}$$

and, moreover,

$$\left|\frac{\partial\varphi_i^{-1}}{\partial x_n}\right|=\frac{1}{\left|\dfrac{\partial\varphi_i}{\partial x_{n-1}}\right|}<1<1+2M_{2n-1}.$$

Hence,

$$\sharp\left\{j:\left|\frac{\partial\varphi_i}{\partial x_j}\right|<1+2M_{2n-1}\right\}<\sharp\left\{\nu:\left|\frac{\partial\varphi_i^{-1}}{\partial x_\nu}\right|<1+2M_{2n-1}\right\}\quad\text{for } i\in\{1,2\}.$$

Then, after transposition (x_{n-1},x_n), G is the following cell

$$G^*=\{(x'',x_n,x_{n-1}):x''\in\Omega,\quad \varphi_1(x'',\sigma(x''))<x_n<\varphi_2(x'',\rho(x'')),$$
$$\chi_1(x'',x_n)<x_{n-1}<\chi_2(x'',x_n)\},$$

where

$$\chi_1(x'',x_n)=\begin{cases}\sigma(x''), & \text{if } \varphi_1(x'',\sigma(x''))<x_n\le\varphi_2(x'',\sigma(x''))\\ \varphi_2^{-1}(x'',x_n), & \text{if } \varphi_2(x'',\rho(x''))<x_n<\varphi_2(x'',\rho(x''))\end{cases}$$

and

$$\chi_2(x'',x_n) = \begin{cases} \varphi_1^{-1}(x'',x_n)\,, & \text{if } \varphi_1(x'',\sigma(x'')) < x_n < \varphi_1(x'',\rho(x'')) \\ \rho(x'')\,, & \text{if } \varphi_1(x'',\rho(x'')) \le x_n < \varphi_2(x'',\rho(x'')). \end{cases}$$

Since $\langle G^*\rangle > \langle G\rangle$, the induction hypothesis gives the desired decomposition.

Case (3.12) Then $\varphi_1(\sigma) \subset R$ and define

$$\psi(x'',x_{n-1}) = \varphi_1(x'',\sigma(x'')) + (1+2M_{2n-1})(x_{n-1}-\sigma(x'')),$$

for $(x'',x_{n-1}) \in \Delta$. Now G splits into two cells:

$$S_1 = \{(x',x_n) : x' \in \Delta, \quad \varphi_1(x') < x_n < \psi(x')\}$$

and

$$S_2 = \{(x',x_n) : x' \in \Delta, \quad \psi(x') < x_n < \varphi_2(x')\}.$$

Observe that

$$\frac{\partial\psi}{\partial x_j} = \frac{\partial\varphi_1}{\partial x_j} + \left[\frac{\partial\varphi_1}{\partial x_{n-1}} - (1+2M_{2n-1})\right]\frac{\partial\sigma}{\partial x_j},$$

for $j \in \{1,\dots,n-2\}$, almost everywhere on Δ.

Hence, by (3.8), (3.12) and (3.9),

$$\left|\frac{\partial\psi}{\partial x_j}\right| \le \left|\frac{\partial\varphi_1}{\partial x_{r_1}}\right| + 2M_{2n-1}(1+2M_{2n-1}) \le \frac{3}{2}\left|\frac{\partial\varphi_1}{\partial x_{r_1}}\right|$$

and

$$\left|\frac{\partial\psi}{\partial x_{r_1}}\right| \ge \left|\frac{\partial\varphi_1}{\partial x_{r_1}}\right| - 2M_{2n-1}(1+2M_{2n-1}) \ge \frac{1}{2}\left|\frac{\partial\varphi_1}{\partial x_{r_1}}\right| \ge 2M_{2n-1}(1+2M_{2n-1}).$$

Therefore,

$$\frac{\left|\dfrac{\partial\psi}{\partial x_j}\right|}{\left|\dfrac{\partial\psi}{\partial x_{r_1}}\right|} \le 3,$$

for any $j \in \{1,\dots,n-2\}$. Thus S_1 satisfies the conditions (3.13)–(3.15) and the case (3.10) applies.

On the other hand, if $j \in \{1,\dots,n-2\}$ and

$$\left|\frac{\partial\varphi_1}{\partial x_j}\right| < 1+2M_{2n-1},$$

then

$$\left|\frac{\partial\psi^{-1}}{\partial x_j}\right| = \frac{\left|\dfrac{\partial\psi}{\partial x_j}\right|}{\left|\dfrac{\partial\psi}{\partial x_{n-1}}\right|} \le \frac{\left|\dfrac{\partial\varphi_1}{\partial x_j}\right| + 2M_{2n-1}(1+2M_{2n-1})}{1+2M_{2n-1}} < 1+2M_{2n-1};$$

hence,

$$\sharp\left\{j : \left|\frac{\partial\varphi_1}{\partial x_j}\right| < 1+2M_{2n-1}\right\} \le \sharp\left\{\nu : \left|\frac{\partial\psi^{-1}}{\partial x_\nu}\right| < 1+2M_{2n-1}\right\},$$

while

$$\sharp\left\{j : \left|\frac{\partial\varphi_2}{\partial x_j}\right| < 1+2M_{2n-1}\right\} < \sharp\left\{\nu : \left|\frac{\partial\varphi_2^{-1}}{\partial x_\nu}\right| < 1+2M_{2n-1}\right\}$$

and we finish by the induction hypothesis as in Case (3.11).

4. Theorem 1_n for a semi-M-cell.

Proposition 4. *Any semi-M-cell G in R^n (where $M > 0$) admits an almost decomposition*

$$G \simeq S_1 \cup \cdots \cup S_k, \tag{4.1}$$

where every S_ν is an M'-cell after a permutation of coordinates and $M' \geq 1$ is a constant depending only on M and n.

Proof. One can assume that G is in the form (3.1), where $\varphi_i : \Delta \longrightarrow R$ $(i = 1, 2)$ are continuous and

$$\left|\frac{\partial \varphi_1}{\partial x_j}\right| < M \quad \text{almost everywhere on } \Delta, \text{ for } j \in \{1, \ldots, n-1\}. \tag{4.2}$$

Indeed, in the case $\varphi_1 \equiv -\infty$ or $\varphi_1 \equiv +\infty$ reduces to the above by assuming first that Δ is an M_{2n-1}-disc and applying next transposition (x_{n-1}, x_n).

The proof will be by descending induction on the number

$$[G] = \sharp\left\{ j : \quad \left|\frac{\partial \varphi_2}{\partial x_j}\right| \leq M_{2n-1} \quad \text{almost everywhere on } \Delta \right\}.$$

If $[G] = n-1$, G is a $\max(M, M_{2n-1})$-cell, so assume that $[G] < n-1$. Notice that if $\tilde{\Delta} \subset \Delta$, then for $\tilde{G} = G \cap (\tilde{\Delta} \times R)$, $[\tilde{G}] \geq [G]$.

Fix any $L > \max(M, M_{2n-1})$ and any $M^* > M + (L+M)M_{2n-1}$. Dividing Δ, one can assume that every φ_i is $\mathcal{C}^1$ on Δ and

$$\text{for each } j \in \{1, \ldots, n-1\}, \quad \operatorname{sgn}\frac{\partial \varphi_i}{\partial x_j} = \text{const}; \tag{4.3}$$

$$\text{for each } j \in \{1, \ldots, n-1\}, \quad \left|\frac{\partial \varphi_2}{\partial x_j}\right| > L \quad \text{on } \Delta \quad \text{or} \quad \left|\frac{\partial \varphi_2}{\partial x_j}\right| \leq L \quad \text{on } \Delta \tag{4.4}$$

and

$$\text{there exists } r \in \{1, \ldots, n-1\} \quad \text{such that} \quad \left|\frac{\partial \varphi_2}{\partial x_r}\right| \geq \left|\frac{\partial \varphi_2}{\partial x_j}\right| \tag{4.5}$$

for each $j \in \{1, \ldots, n-1\}$, and either $\left|\frac{\partial \varphi_2}{\partial x_r}\right| \geq M^*$ or $\left|\frac{\partial \varphi_2}{\partial x_r}\right| \leq M^*$ on Δ.

Clearly, one can assume that

$$\left|\frac{\partial \varphi_2}{\partial x_r}\right| \geq M^* \qquad \text{on } \Delta. \tag{4.6}$$

Finally, by Theorem 2_{n-1} and Lemma 2, one can assume that

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega, \quad \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an M_{2n-1}-disc in R^{n-1} and every φ_i admits a continuous extension

$$\varphi_i : \Delta \cup \sigma \cup \rho \longrightarrow \overline{R}$$

such that $\varphi_i(\sigma) \subset R$ or $\varphi_i(\sigma) = \{-\infty\}$, or $\varphi_i(\sigma) = \{+\infty\}$, and the same for ρ. Because of (4.2), $\varphi_1 : \Delta \cup \sigma \cup \rho \longrightarrow R$.

Case I: $\left|\dfrac{\partial \varphi_2}{\partial x_{n-1}}\right| > L$ on Δ.

Assume that $\dfrac{\partial \varphi_2}{\partial x_{n-1}} > L$; the case $\dfrac{\partial \varphi_2}{\partial x_{n-1}} < -L$ will follow by a modification. Consider the following function

$$\psi(x'', x_{n-1}) = \varphi_1(x'', \sigma(x'')) + L(x_{n-1} - \sigma(x'')), \tag{4.7}$$

for $(x'', x') \in \Delta$.

Then $\varphi_1 < \psi < \varphi_2$ and $G \simeq S_1 \cup S_2$, where

$$S_1 = \{(x', x_n) : x' \in \Delta, \quad \varphi_1(x') < x_n < \psi(x')\}$$

is an M^*-cell and

$$S_2 = \{(x', x_n) : x' \in \Delta, \quad \psi(x') < x_n < \varphi_2(x')\}$$

can be interpreted after transposition (x_{n-1}, x_n) as

$$S_2 = \{(x'', x_{n-1}, x_n) : x'' \in \Omega, \varphi_1(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x'')),$$
$$\theta_2(x'', x_n) < x_{n-1} < \theta_1(x'', x_n)\},$$

where

$$\theta_2(x'', x_n) = \begin{cases} \sigma(x''), & \text{if } \varphi_1(x'', \sigma(x'')) < x_n \leq \varphi_2(x'', \sigma(x'')) \\ \varphi_2^{-1}(x'', x_n), & \text{if } \varphi_2(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x'')) \end{cases}$$

and

$$\theta_1(x'', x_n) = \begin{cases} \psi^{-1}(x'', x_n), & \text{if } \varphi_1(x'', \sigma(x'')) < x_n \leq \psi(x'', \rho(x'')) \\ \rho(x''), & \text{if } \psi(x'', \rho(x'')) < x_n < \varphi_2(x'', \rho(x'')), \end{cases}$$

and where φ_2^{-1} (and ψ^{-1}) stands for the inversion of φ_2 (and ψ) with respect to x_{n-1}. Now, if $j \in \{1, \ldots, n-2\}$ and

$$\left|\frac{\partial \varphi_2}{\partial x_j}\right| \leq M_{2n-1},$$

then

$$\left|\frac{\partial \varphi_2^{-1}}{\partial x_j}\right| = \frac{\left|\dfrac{\partial \varphi_2}{\partial x_j}\right|}{\left|\dfrac{\partial \varphi_2}{\partial x_{n-1}}\right|} < \left|\frac{\partial \varphi_2}{\partial x_j}\right| \leq M_{2n-1}$$

and, moreover,

$$\left|\frac{\partial \varphi_2^{-1}}{\partial x_n}\right| = \frac{1}{\left|\dfrac{\partial \varphi_2}{\partial x_{n-1}}\right|} < \frac{1}{L} < M_{2n-1}.$$

Hence, $[S_2] > [G]$ and the induction hypothesis ends the proof in this case.

Case II: $\left|\dfrac{\partial \varphi_2}{\partial x_{n-1}}\right| \le L \quad \text{on} \quad \Delta.$

By (4.6) and (4.3), one can assume without any loss of generality that

$$\frac{\partial \varphi_2}{\partial x_r} \ge M^*\,, \quad \frac{\partial \varphi_2}{\partial x_{n-1}} > 0 \quad \text{and} \quad \frac{\partial \varphi_1}{\partial x_{n-1}} > 0\,;$$

other possibilities will follow by simple modifications.

Since $M^* > L$, $r \in \{1, \dots, n-2\}$. By Proposition 2, we have almost everywhere on Δ:

$$\frac{\partial}{\partial x_r}\varphi_2(x'', \sigma(x'')) = \left|\frac{\partial \varphi_2}{\partial x_r}(x'', \sigma(x'')) + \frac{\partial \varphi_2}{\partial x_{n-1}}(x'', \sigma(x''))\frac{\partial \sigma}{\partial x_r}(x'')\right| \ge$$

$$M^* - LM_{2n-1},$$

while

$$\left|\frac{\partial}{\partial x_r}\varphi_1(x'', \sigma(x''))\right| \le M + MM_{2n-1} \quad \text{and} \quad \left|\frac{\partial}{\partial x_r}\varphi_1(x'', \rho(x''))\right| \le M + MM_{2n-1}.$$

Thus, by Lemma 5,

$$\varphi_2(x'', \sigma(x'')) > \varphi_1(x'', \rho(x'')) \qquad \text{on} \quad \Omega.$$

Hence,

$$G \simeq S_1 \cup S_2 \cup S_3,$$

where

$$S_1 = \{(x'', x_{n-1}, x_n) : (x'', x_{n-1}) \in \Delta,\ \varphi_1(x'', x_{n-1}) < x_n < \varphi_1(x'', \rho(x''))\},$$

$$S_2 = \{(x'', x_{n-1}, x_n) : x'' \in \Omega,\ \varphi_1(x'', \rho(x'')) < x_n < \varphi_2(x'', \sigma(x'')),$$
$$\sigma(x'') < x_{n-1} < \rho(x'')\}$$

and

$$S_3 = \{(x'', x_{n-1}, x_n) : (x'', x_{n-1}) \in \Delta,\ \varphi_2(x'', \rho(x'')) < x_n < \varphi_2(x'', x_{n-1})\}.$$

S_1 is an M^*-cell, while S_2 is an M_{2n-1}-cell after transposition (x_{n-1}, x_n). We will investigate S_3. Put

$$\tilde{\Delta} = \{(x'', x_n) : x'' \in \Omega,\ \varphi_2(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x''))\}.$$

Now,

$$S_3 = \{(x'', x_{n-1}, x_n) : (x'', x_n) \in \tilde{\Delta},\ \varphi_2^{-1}(x'', x_n) < x_{n-1} < \rho(x'')\},$$

where φ_2^{-1} stands for the inversion of φ_2 with respect to x_{n-1}.

We will use Lemma 3 to get a desired decomposition of S_3. Observe first that

$$\frac{\partial \varphi_2^{-1}}{\partial x_r} = \frac{\dfrac{\partial \varphi_2}{\partial x_r}}{\dfrac{\partial \varphi_2}{\partial x_{n-1}}} \geq \frac{\dfrac{\partial \varphi_2}{\partial x_r}}{L} \geq \frac{M^*}{L} > \frac{M + (L+M)M_{2n-1}}{L} > M_{2n-1} \geq \left|\frac{\partial \rho}{\partial x_r}\right|$$

and

$$\frac{\left|\dfrac{\partial \varphi_2^{-1}}{\partial x_j}\right|}{\left|\dfrac{\partial \varphi_2^{-1}}{\partial x_r}\right|} = \frac{\left|\dfrac{\partial \varphi_2}{\partial x_j}\right|}{\left|\dfrac{\partial \varphi_2}{\partial x_r}\right|} \leq 1\,, \qquad \text{for } j \in \{1, \ldots, n-2\},$$

and

$$\frac{\left|\dfrac{\partial \varphi_2^{-1}}{\partial x_n}\right|}{\left|\dfrac{\partial \varphi_2^{-1}}{\partial x_r}\right|} = \frac{1}{\left|\dfrac{\partial \varphi_2^{-1}}{\partial x_r}\right|} \leq \frac{1}{M^*} < 1.$$

Now it suffices to check that Δ has an almost decomposition into N-cells with respect to the variable x_r, where a constant N depends only on M, L, M^* and M_{2n-1}. We will check this using Lemma 6 (2).

We have almost everywhere on Ω:

$$\frac{\partial}{\partial x_r}\varphi_2(x'', \sigma(x'')) \geq \frac{\partial \varphi_2}{\partial x_r}(x'', \sigma(x''))\left(1 - \frac{LM_{2n-1}}{M^*}\right) \geq M^* - LM_{2n-1}$$

and

$$\frac{\left|\dfrac{\partial}{\partial x_j}\varphi_2(x'', \sigma(x''))\right|}{\left|\dfrac{\partial}{\partial x_r}\varphi_2(x'', \sigma(x''))\right|} \leq \frac{\left|\dfrac{\partial \varphi_2}{\partial x_j}(x'', \sigma(x'')) + \dfrac{\partial \varphi_2}{\partial x_{n-1}}(x'', \sigma(x''))\dfrac{\partial \sigma}{\partial x_j}(x'')\right|}{\left|\dfrac{\partial \varphi_2}{\partial x_r}(x'', \sigma(x''))\right| \dfrac{M(1+M_{2n-1})}{M^*}} \leq \frac{M^*}{M}.$$

The same is true for ρ in the place of σ. Moreover, by the assumption of Case II,

$$|\varphi_2(x'', \sigma(x'')) - \varphi_2(x'', \rho(x''))| \leq |\sigma(x'') - \rho(x'')| \qquad \text{on} \qquad \Omega.$$

Hence,

$$\lim_{x'' \to a''} [\varphi_2(x'', \sigma(x'')) - \varphi_2(x'', \rho(x''))] = 0,$$

for any $a'' \in \partial\Omega$, so the assumptions of Lemma 6 (2) are satisfied.

5. Proof of Theorem 2_n for any M-cell.

Let

$$G = \{(x', x_n) : x' \in \Delta\,, \quad \varphi_1(x') < x_n < \varphi_2(x')\}$$

be any M-cell, where $M \in R,\ M \geq 1$. Observe that all possible cases reduce to the case $\varphi_i : \Delta \longrightarrow R \quad (i \in \{1,2\})$. Indeed, suppose for example that $\varphi_1 : \Delta \longrightarrow R$ and $\varphi_2 \equiv +\infty$. Then one can assume first that φ_1 is $\mathcal{C}^1$ on Δ and, for each $j \in \{1, \ldots, n-1\}$,

$$\operatorname{sgn}\frac{\partial \varphi_1}{\partial x_j} = \text{const} \qquad \text{on } \Delta,$$

and next that

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega, \quad \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an M_{2n-1}-disc in R^{n-1} such that φ_1 has a continuous extension

$$\varphi_1 : \Delta \cup \sigma \cup \rho \longrightarrow R.$$

Then, assuming that $\dfrac{\partial \varphi_1}{\partial x_{n-1}} > 0$,

$$G \simeq S_1 \cup S_2,$$

where

$$S_1 = \{(x'', x_{n-1}, x_n) : (x'', x_{n-1}) \in \Delta,\ \varphi_1(x', x_{n-1}) < x_n < \varphi_1(x'', \rho(x''))\}$$

is an $M(1 + M_{2n-1})$-cell, while

$$S_2 = \{(x'', x_{n-1}, x_n) : x'' \in \Omega,\ \varphi_1(x'', \rho(x'')) < x_n,\ \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an M_{2n-1}-cell after transposition (x_{n-1}, x_n).

Consequently, assume that $\varphi_i : \Delta \longrightarrow R \quad (i \in \{1,2\})$ and that they are $\mathcal{C}^1$. By Theorem 3_{n-1}, one can assume that Δ is a regular M_{3n-1}-cell and then, by Proposition 1, that every φ_i has a continuous extension

$$\varphi_i : \overline{\Delta} \longrightarrow R \qquad (i \in \{1,2\}).$$

Now, still keeping the last property, one can assume that

$$\Delta = \{(x'', x_{n-1}) : x'' \in \Omega, \quad \sigma(x'') < x_{n-1} < \rho(x'')\}$$

is an M_{2n-1}-disc. Put

$$\lambda_1(x'', x_{n-1}) = \varphi_1(x'', \sigma(x'')) + 2M(x_{n-1} - \sigma(x'')),$$

$$\lambda_2(x'', x_{n-1}) = \varphi_1(x'', \rho(x'')) - 2M(x_{n-1} - \rho(x'')),$$

$$\lambda_3(x'', x_{n-1}) = \varphi_2(x'', \rho(x'')) + 2M(x_{n-1} - \rho(x'')),$$

and

$$\lambda_4(x'', x_{n-1}) = \varphi_2(x'', \sigma(x'')) - 2M(x_{n-1} - \sigma(x'')),$$

for any $(x'', x_{n-1}) \in \Omega \times R$. Every λ_i has a continuous extension to $\overline{\Omega} \times R$ and is an $M(1+3M_{2n-1})$-function. Its inversion λ_i^{-1} with respect to x_{n-1} has a continuous extension to $\overline{\Omega} \times R$ as well and is a $\frac{1}{2}(1+3M_{2n-1})$-function.

For any subset $I \subset \{1,2,3,4\}$, put

$$S_I = \{(x', x_n) \in G : x_n < \lambda_i(x'),\ \text{if } i \in I \quad \text{and} \quad \lambda_i(x') < x_n,\ \text{if } i \notin I\}.$$

Then

$$G \simeq \bigcup_I S_I.$$

It suffices to show that every S_I is an $M(1+3M_{2n-1})$-disc after perhaps transposition (x_{n-1}, x_n).

Fix any $I \subset \{1,2,3,4\}$.

If $\{1,2\} \subset I$, then

$$S_I = \{(x', x_n) \in \Delta \times R : \varphi_1(x') < x_n < \varphi_2(x'),\ x_n < \lambda_i(x'),\ \text{if } i \in I,$$
$$\lambda_i(x') < x_n, \text{ if } i \notin I\},$$

and $\lambda_1 = \varphi_1$ on σ, while $\lambda_2 = \varphi_1$ on ρ and Lemma 4 applies.

Similarly, when $\{3,4\} \cap I = \emptyset$.

If $\{1,2\} \not\subset I$ and $\{3,4\} \cap I \neq \emptyset$, we have $1 \notin I$ and $3 \in I$ or $1 \notin I$ and $4 \in I$ (or, similarly, $2 \notin I$ and $3 \in I$ or $2 \notin I$ and $4 \in I$).

Suppose first that $1 \notin I$ and $3 \in I$. Then

$$S_I = \{(x'', x_{n-1}, x_n) : x'' \in \Omega, \qquad \varphi_1(x'', \sigma(x'')) < x_n < \varphi_2(x'', \rho(x'')), \tag{5.1}$$
$$\sigma(x'') < x_{n-1} < \rho(x''),\ x_{n-1} < \lambda_i^{-1}(x'', x_n) \text{ if } i \in \tilde{I},\ \lambda_i^{-1}(x'', x_n) < x_{n-1} \text{ if } i \notin \tilde{I}\},$$

where $\tilde{I} \subset \{1,2,3,4\}$ is defined by the formula:

$i \in \tilde{I}$ if and only if $i \in I$ and i is even or $i \notin I$ and i is odd.

Since

$$\lambda_1^{-1}(x'', \varphi_1(x'', \sigma(x'')) = \sigma(x'')$$

and

$$\lambda_3^{-1}(x'', \varphi_2(x'', \rho(x'')) = \rho(x''),$$

for each $x'' \in \Omega$ and

$$\sigma(x'') = \rho(x''),$$

for each $x'' \in \partial\Omega$, we are done by Lemma 4.

Let now $1 \notin I$ and $4 \in I$. Then (5.1) holds and since

$$\lambda_1^{-1}(x'', \varphi_1(x'', \sigma(x'')) = \sigma(x''), \qquad \lambda_4^{-1}(x'', \varphi_2(x'', \sigma(x'')) = \sigma(x''),$$

for each $x'' \in \Omega$ and $\sigma(x'') = \rho(x'')$, for each $x'' \in \partial\Omega$, we are again done due to Lemma 4.

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