Some results and problems on complex germs with definable Mittag-Leffler stars

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Abstract

Working in an o-minimal expansion of the real field, we investigate when a germ (around 0 say) of a complex analytic function has a definable analytic continuation to its Mittag-Leffler star.

As an application we show that any algebro-logarithmic function that is complex analytic in a neighbourhood of the origin in $\mathbb{C}$ has an analytic continuation to all but finitely many points in $\mathbb{C}$.

It is my pleasure to dedicate this paper to Anand Pillay on the occasion of his 60th birthday.

1 Introduction

This paper is motivated by the conjecture of Zilber stating that every $\mathbb{C}_{exp}$-definable subset of $\mathbb{C}$ is either countable or co-countable. Here, $\mathbb{C}_{exp}$ is the expansion of the ring of complex numbers by the complex exponential function. As far as I know, even sets of the form

$$\{z \in \mathbb{C} : \exists w \in \mathbb{C}\ F(z, w) = 0\} \qquad (*)$$

where $F(z, w)$ is a (two variable) term of the language $\mathcal{L}(\mathbb{C}_{exp})$ have not been shown to satisfy Zilber's conjecture.

Our approach to this particular case is as follows. Let us suppose that

$$F(0,0) = 0 \neq \frac{\partial F}{\partial w}(0,0).$$

Then by the implicit function theorem there exists $\epsilon > 0$ and a complex analytic function $\phi : \Delta(0;\epsilon) \to \mathbb{C}$ (where, in general, $\Delta(a;r)$ denotes the disk centred at $a \in \mathbb{C}$ and having radius r) such that for all $z \in \Delta(0;\epsilon)$, we have $F(z, \phi(z)) = 0$. We must show that the set $(*)$ is co-countable and it seems reasonable to conjecture that the function element ϕ has an analytic continuation (which necessarily preserves the equation $F(z, \phi(z)) = 0$) to all but countably many points in the complex plane. Indeed, one can fairly easily show that if one proves a suitably generalized version of this *analytic continuation conjecture* (in which w is allowed to be an n-tuple of variables and F an n-tuple of terms in the $1+n$ variables z, w, and where the countably many exceptional points have a certain specific form) then Zilber's conjecture (even for subsets of $\mathbb{C}$ defined by formulas of the language $\mathcal{L}_{\omega_1,\omega}(\mathbb{C}_{exp})$) would follow.

Let us now consider issues of definabilty. The approach to Zilber's conjecture suggested above transcends $\mathcal{L}(\mathbb{C}_{exp})$-definablility (at least, if Zilber's conjecture is true!): one cannot define *restricted* functions $\phi : \Delta(0;\epsilon) \to \mathbb{C}$ without the resource of the real line and the usual metric. So we follow the Peterzil-Starchenko idea of doing complex analysis definably in a suitable o-minimal structure via the usual identifications $\mathbb{C} \sim \mathbb{R} \oplus i\mathbb{R} \sim \mathbb{R} \times \mathbb{R}$. Actually, we will only be considering a fixed o-minimal expansion $\widetilde{\mathbb{R}}$ of the ordered field of real numbers $\overline{\mathbb{R}}$, so many of the subtleties of [4] will not be required here. But the uniform finiteness of the winding number for definable functions will be, and this was inspired by the Peterzil-Starchenko approach.

My aim in this paper, then, is to consider definable analytic continuation relative to an o-minimal expansion $\widetilde{\mathbb{R}}$ of $\overline{\mathbb{R}}$. I shall only consider continuations along straight line paths emanating from the origin in $\mathbb{C}$, so let me discuss this now. The mathematical theory (i.e. without definability considerations) may be found in [1] and [3], but in very few modern texts as far as I can see.

2 The Mittag-Leffler star

So consider any complex analytic function $\phi : \Delta(0;\epsilon) \to \mathbb{C}$. The *Mittag-Leffler star* of ϕ (henceforth just the *star* of ϕ), denoted S_ϕ, is defined to be the set of all $z \in \mathbb{C}$ such that there exists an open set $U_z \subseteq \mathbb{C}$ with $\Delta(0;r) \cup [0,z] \subseteq U_z$ and a complex analytic function $\psi : U_z \to \mathbb{C}$ with $\psi \restriction \Delta(0;r) = \phi$. (Here, $[0,z]$ denotes the straight line segment in $\mathbb{C}$ from 0 to z, i.e. $[0,z] := \{tz : 0 \leq t \leq 1\}$.)

It can be shown (see [3] Volume 3) that S_ϕ is an open, connected and simply connected set containing $\Delta(0;r)$, and that there exists a unique complex analytic function $\star\phi : S_\phi \to \mathbb{C}$ with $\star\phi \restriction \Delta(0;r) = \phi$. I call $\star\phi$ the *star function* of ϕ. Also, a point z such that $z \notin S_\phi$ but satisfying $[0,w] \subseteq S_\phi$ for all $w \in [0,z] \setminus \{z\}$ will be called a *singular point* of ϕ.

In general one can say very little else about the nature of the set S_ϕ (it could, for example, be bounded). However, we have the following result in the definable situation.

Theorem 1. *Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of the ordered field of real numbers $\overline{\mathbb{R}}$ and assume that $\widetilde{\mathbb{R}}$ has analytic cell decomposition. Let $\phi : \Delta(0;r) \to \mathbb{C}$ be a complex analytic function and suppose that its star function $\star\phi : S_\phi \to \mathbb{C}$ (and hence its star S_ϕ) is definable (in $\widetilde{\mathbb{R}}$). Then ϕ has only finitely many singular points.*

Proof. Let $\phi : \Delta(0;r) \to \mathbb{C}$ and $\star\phi : S_\phi \to \mathbb{C}$ be as in the hypotheses of the theorem and suppose that ϕ has infinitely many singular points. Then by analytic cell decomposition there would exist (possibly after rotating $\mathbb{C}$ about 0) a 2-cell C of the form

$$C = \{x + iy : a < x < b, f(x) < y < g(x)\},$$

where $f, g : (a,b) \to \mathbb{R}$ are definable real analytic functions, such that $C \subseteq S_\phi$ and such that $\star\phi \restriction C$ has no analytic continuation to any open set in $\mathbb{C}$ containing a point of $graph(g)$. We may further assume (by refining the original cell decomposition) that either for all $z \in C$, $|\star\phi(z)| < 1$ or for all $z \in C$, $|\star\phi(z)| > 1$.

Let us consider the first case. By o-minimality there is a finite set $s \subseteq graph(g)$ such that $\star\phi$ has a definable continuous extension (which we also denote by $\star\phi$) to $(C \cup (graph(g))) \setminus s$. By analytic cell decomposition again, there exists a', b' with $a < a' < b' < b$ such that $\star\phi \circ g \restriction (a',b')$ is a definable real analytic function. We now obtain a contradiction by a using a classical argument (as described in, for example, [2], Chapter IX). Namely, fix $x_0 \in (a', b')$ and let $\epsilon > 0$ be chosen small enough so that $a' < x_0 - \epsilon < x_0 < x_0 + \epsilon < b'$ and also so that the (real) Taylor series of both g and $\star\phi \circ g$ extend (via the same power series) to (not necessarily definable) complex analytic functions $G : \Delta(x_0; \epsilon) \to \mathbb{C}$ and $\Phi : \Delta(x_0; \epsilon) \to \mathbb{C}$ respectively.

Define the complex analytic function $H : \Delta(x_0; \epsilon) \to \mathbb{C}$ by $H(z) := z + iG(z)$. Since the Taylor coefficients of G are real it follows that $H'(x_0) \neq 0$ and hence (by reducing ϵ if necessary) that H is a holomorphic homeomorphism from $\Delta(x_0; \epsilon)$ onto an open set, U say. Further, H maps the interval $(x_0 - \epsilon, x_0 + \epsilon)$ onto $graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))$.

Now consider the function

$$\Psi := \star\phi - \Phi \circ H^{-1} : (C \cup graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))) \cap U \to \mathbb{C}.$$

By our construction Ψ is continuous, holomorphic on $C \cap U$, and identically zero on the analytic curve $graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))$ which forms (a nontrivial) part of the boundary of $C \cap U$. This implies (see [2], page 303, exercise 6) that Ψ is identically zero throughout $(C \cup graph(g \restriction (x_0 - \epsilon, x_0 + \epsilon))) \cap U$. It follows that $\Phi \circ H^{-1}$ provides an analytic continuation of $\star\phi$ to the open set $C \cup U$. But this is a contradiction since U contains the point $x_0 + ig(x_0)$ of $graph(g)$.

The case that $| \star \phi(z)| > 1$ for all $z \in C$ is dealt with by applying the above argument to the function $\frac{1}{\star\phi}$ and then inverting the analytic continuation. (The proof actually shows that $\star\phi$ is necessarily locally bounded at all but finitely many points of $graph(g)$.)

This completes the proof of Theorem 1. $\square$

Later I shall show that the collection of all those complex analytic germs having a definable Mittag-Leffler star has a reasonably rich structure, at least if $\widetilde{\mathbb{R}}$ does. This is in contrast to those germs having definable entire, or definable meromorphic, extensions which, as one can easily show, are (for any o-minimal $\widetilde{\mathbb{R}}$) just polynomials or rational functions respectively.

The proof of Theorem 1 shows that any definable complex analytic function whose domain is an open cell C in $\mathbb{C}$ has an analytic continuation (though not necessarily a definable one) across the boundary of C at all but finitely many points. I leave the reader to combine this remark with Theorem 1 itself to give a proof of the following result.

Theorem 2. *Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of the ordered field of real numbers $\overline{\mathbb{R}}$ and assume that $\widetilde{\mathbb{R}}$ has analytic cell decomposition. Let $\phi : \Delta(0; r) \to \mathbb{C}$ be a complex analytic function and suppose that its star function $\star\phi : S_\phi \to \mathbb{C}$ (and hence its star S_ϕ) is definable (in $\widetilde{\mathbb{R}}$). Then for all but finitely many points $z \in \mathbb{C}$ there exists a continuous, piecewise linear path begining at 0 and terminating at z (and, in fact, consisting of at most two line segments) along which ϕ has an analytic continuation.*

However, I am unable to settle the following question.

Open Problem 1

Let ϕ and $\star\phi$ be as in the hypotheses of Theorems 1 and 2. Does there exist a finite set $s \subseteq \mathbb{C} \setminus \{0\}$ such that ϕ has an analytic continuation along all continuous, definable paths that begin at 0 and avoid s?

Before finishing this section I should mention the Mittag-Leffler Star Theorem. This provides a remarkable series expansion for $\star\phi$ which converges uniformly to $\star\phi$ on compact subsets of S_ϕ. It is completely analagous to the Taylor expansion on the disk of convergence of ϕ in the sense that the only dependence of the series on the germ ϕ is a fixed (i.e. independent of ϕ) linear one on the numbers $\phi(0), \phi'(0), \ldots, \phi^{(n)}(0), \ldots$. I will not need this result and so I will not expand on this remark. The interested reader may consult [3] for further information and proofs.

3 The ring of definable star functions

I now fix an o-minimal expasion $\widetilde{\mathbb{R}}$ of $\overline{\mathbb{R}}$ and I assume that $\widetilde{\mathbb{R}}$ has analytic cell decomposition. Notions of definablility are relative to $\widetilde{\mathbb{R}}$ and are *without* parameters.

I denote by $\widetilde{\mathcal{G}}$ the collection of all definable, complex analytic germs at 0, i.e. the collection of definable, complex analytic functions $f : U \to \mathbb{C}$ (where U is a (definable) open neighbourhood of 0), where two such functions are identified if there is some open neighbourhood of 0 on which they agree. I will, however, not distinguish notationally betwen functions and their germs.

It is clear that $\widetilde{\mathcal{G}}$ is an integral domain (under pointwise operations) and a differential ring (under the usual derivative $\frac{d}{dz}$). We are interested in its subset consisting of those $\phi \in \widetilde{\mathcal{G}}$ having a definable star function $\star\phi : S_\phi \to \mathbb{C}$.

It is not immediately obvious this is a subring of $\widetilde{\mathcal{G}}$: it could be the case, for example, that $\star\phi : S_\phi \to \mathbb{C}$ and $\star\psi : S_\psi \to \mathbb{C}$ are definable but that the domain of $\star(\phi + \psi)$ (i.e. $S_{(\phi+\psi)}$) is strictly larger than $S_\phi \cap S_\psi$. So we would need to show that the extension of the (obviously definable) function $\star\phi + \star\psi : S_\phi \cap S_\psi \to \mathbb{C}$ to $S_{(\phi+\psi)}$ is definable.

To resolve this rather annoying difficulty, we first let $\widetilde{\mathcal{S}}$ be the collection of all definable, open subsets of $\mathbb{C}$ of the form $\mathbb{C} \setminus \bigcup_{j=1}^{n}[a_j, \infty)$, where $a_1, \ldots, a_n$ are (necessarily definable) nonzero complex numbers (and where, for $a \in \mathbb{C}$, $[a, \infty) := \{ta : 1 \leq t\}$).
Now let

$$\widetilde{\mathcal{M}} := \{\phi \in \widetilde{\mathcal{G}} : \phi \,\text{has a (definable) representative}\; \bar{\phi} : U \to \mathbb{C} \text{ for some } U \in \widetilde{\mathcal{S}}\}.$$

Now it is certainly clear that $\widetilde{\mathcal{M}}$ is a subring of $\widetilde{\mathcal{G}}$ since $\widetilde{\mathcal{S}}$ is closed under intersection. We would like to show that $\widetilde{\mathcal{M}}$ may be identified with the collection of those $\phi \in \widetilde{\mathcal{G}}$ having a definable star function. Such a result is in the spirit of those in section 2.7 of [4] but does not seem to follow directly from them. So instead we argue as follows.
Consider, more generally, any definable complex analytic function $F : U \to \mathbb{C}$ where U is a (definable) open subset of $\mathbb{C}$ of co-dimension at most 1 (in the sense of the o-minimal structure $\widetilde{\mathbb{R}}$). Let $\mathcal{E}(F)$ denote the collection of all (not necessarily definable) complex analytic functions $G : V \to \mathbb{C}$ with $U \subseteq V \subseteq \mathbb{C}$, V open, and $G \restriction U = F$. Now if $G_i : V_i \to \mathbb{C}$ are in $\mathcal{E}(F)$ for $i = 1, 2$, and $z \in V_1 \cap V_2$ then $G_1(z) = G_2(z)$. This is because for some $\epsilon > 0$, $\Delta(z; \epsilon) \subseteq V_1 \cap V_2$ and $\Delta(z; \epsilon) \cap U$ is a nonempty open set (as U has codimension 1) on which G_1 and G_2 agree (with F). It follows that G_1 and G_2, being complex analytic, agree throughout $\Delta(z; \epsilon)$ and hence in particular that $G_1(z) = G_2(z)$.
It now follows that all functions in $\mathcal{E}(F)$ have a common extension, $H : W \to \mathbb{C}$ say, which also lies in $\mathcal{E}(F)$. Further, H is definable. To see this let $A \subseteq \mathbb{C} \times \mathbb{C}$ be the closure of the graph of $F : U \to \mathbb{C}$. Then one easily shows that for all $z, w \in \mathbb{C}$, $H(z) = w$ if and only if $\langle z, w \rangle \in A$ and for some $\epsilon > 0$, $A \cap (\Delta(z; \epsilon) \times \mathbb{C})$ is the graph of a continuously (complex) differentiable function with domain $\Delta(z; \epsilon)$, and this is a definable condition.

Now suppose that $\phi \in \widetilde{\mathcal{M}}$, represented by $\bar{\phi} : U \to \mathbb{C}$ with $U \in \widetilde{\mathcal{S}}$. Since sets in $\widetilde{\mathcal{S}}$ obviously have co-dimension at most 1, we may apply the argument above to $F = \bar{\phi}$ and let $H : W \to \mathbb{C}$ be the resulting maximal extension. Then as the function $\star\phi : S_\phi \to \mathbb{C}$ lies in $\mathcal{E}(\bar{\phi})$ it follows that $S_\phi \subseteq W$ and $H \restriction S_\phi = \star\phi$. Now it may be the case that the inclusion here is proper (e.g. if $\phi(z) = (1-z)^{-1}$, then $S_\phi = \mathbb{C} \setminus [1, \infty)$ whereas $W = \mathbb{C} \setminus \{1\}$), but, given W it is very easy to define the singular points of ϕ, and hence also the set S_ϕ. Since $\star\phi$ is just the restriction of H to S_ϕ its definabilty also follows, as required.

The rest of this paper is devoted to proving that $\widetilde{\mathcal{M}}$ is closed under various operations. Our first observation is now clear.

Theorem 3. *$\widetilde{\mathcal{M}}$ is a subring (in fact, a differential subring) of $\widetilde{\mathcal{G}}$.*

We also have the following.

Theorem 4. *$\widetilde{\mathcal{M}}$ is algebraically closed in $\widetilde{\mathcal{G}}$.*

Proof. Firstly, if $f \in \widetilde{\mathcal{M}}$ is invertible in $\widetilde{\mathcal{G}}$ (i.e. if $f(0) \neq 0$) then it is invertible in $\widetilde{\mathcal{M}}$. For if $domain(f) = U \in \widetilde{\mathcal{S}}$, let $Z_f := \{a \in U : f(a) = 0\}$. Since Z_f is a discrete set, it is finite (by o-minimality). So if we set $V := U \setminus \bigcup_{a \in Z_f} [a, \infty)$ then $V \in \widetilde{\mathcal{S}}$ and $\frac{1}{f} : V \to \mathbb{C}$ is a definable, complex analytic function. Hence $\frac{1}{f} \in \widetilde{\mathcal{M}}$.

So to prove the theorem it is now sufficient to consider a monic polynomial

$$P(w) = w^n + f_1 \cdot w^{n-1} + \cdots + f_n$$

where $f_1, \ldots, f_n \in \widetilde{\mathcal{M}}$, which has a root, ϕ say, in $\widetilde{\mathcal{G}}$. We may assume that P is irreducible over (the field of fractions of) $\widetilde{\mathcal{M}}$ and, in particular, that its discriminant, D say, is a nonzero element of $\widetilde{\mathcal{M}}$. It follows, as above, that we can find a set $U \in \widetilde{\mathcal{S}}$ such that both $U \subseteq \bigcap_{j=1}^n domain(f_j)$ and $D(z) \neq 0$ for all $z \in U \setminus \{0\}$.

It now follows from classical theory that ϕ has an analytic continuation to all points of U. (The continuation is single valued since U is simply connected.) In particular, $U \subseteq S_\phi$ and this continuation is necessarily equal to $\star\phi \restriction U$ (as both functions agree on an open neighbourhood of zero) and, further, $P(\star\phi) = 0$ (in the ring $\widetilde{\mathcal{M}}$). It remains to show that $\star\phi \restriction U$ is definable. However, this follows easily by considering a cell decomposition of $\mathbb{R}^3$ compatible with the (definable) set consisting of all $\langle x, y, t\rangle \in \mathbb{R}^3$ such that $x + iy \in U$ and for some $u \in \mathbb{R}$,

$$(t + iu)^n + f_1(x + iy) \cdot (t + iu)^{n-1} + \cdots + f_0(x + iy) = 0.$$

Then the graph of the real part of $\star\phi \restriction U$ is given by piecing together certain 0, 1 and 2 cells of this decomposition. The imaginary part of $\star\phi \restriction U$ is dealt with similarly and this completes the proof of the theorem. $\square$

It follows from Theorem 4 that if $\widetilde{\mathbb{R}} = \overline{\mathbb{R}}$ then $\widetilde{\mathcal{M}} = \widetilde{\mathcal{G}}$.

Open Problem 2

Does there exist an o-minimal expansion $\widetilde{\mathbb{R}}$ of $\overline{\mathbb{R}}$ in which some nonalgebraic, complex analytic germ $f : \Delta(0; r) \to \mathbb{C}$ is definable, but is such that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{G}}$?

Certainly the complex exponential function restricted to a disk $\Delta(0; r)$ could not be definable in such an $\widetilde{\mathbb{R}}$ since its star function is the entire exponential function which is not definable in any o-minimal structure.

From now on we assume that, for any $R > 0$, $\exp \restriction \{x + iy : -R < y < R\}$ is definable in $\widetilde{\mathbb{R}}$. This is equivalent to both the real exponential function $\exp \restriction \mathbb{R}$ and the restricted sine function $\sin : [0, 2\pi) \to \mathbb{R}$ being definable in $\widetilde{\mathbb{R}}$. The structure $\mathbb{R}_{an,\exp}$ is an example.

Theorem 5. *Let $f \in \widetilde{\mathcal{M}}$ and assume that $f(0) \neq 0$. Then any branch of* $\log f$ *(restricted to some set in $\widetilde{S}$) is in $\widetilde{\mathcal{M}}$.*

Proof. Clearly we may choose $r > 0$ small enough so that all determinations of $\log f \restriction \Delta(0; r)$ are in $\widetilde{\mathcal{G}}$. Let $L_f : \Delta(0; r) \to \mathbb{C}$ be such a determination. Let $U = domain(f)$, so that $U \in \widetilde{\mathcal{M}}$. Now arguing as before we may assume that $f(z) \neq 0$ for all $z \in U$. Thus, since U is simply connected, L_f extends to a single valued logarithm of f on all of U via the usual formula

$$L_f(z) = L_f(0) + \int_0^z \frac{f'(w)}{f(w)} dw \quad (z \in U)$$

where the integration is along the straight line segment $[0, z] \subseteq U$.

To see that $L_f : U \to \mathbb{C}$ is definable let us first note that the functions $|f| : U \to \mathbb{R}_{>0}$ and $\frac{f}{|f|} : U \to \{w \in \mathbb{C} : |w| = 1\}$ are definable, continuous functions. Since the real logarithm function from $\mathbb{R}_{>0}$ to $\mathbb{R}$ is definable, we obtain immediately that the real part of L_f ($= \log|f|$) is definable.

To deal with the imaginary part we note that as L_f is definable in some neighbourhood of 0, the number $L_f(0)$ is definable and hence so is its imaginary part, θ_0 say. Then for $z \in U$, the imaginary part of $L_f(z)$ is given by $\theta_z(z)$, where $\theta_z : [0, z] \to \mathbb{R}$ is the unique continuous function satisfying (a) $\theta_z(0) = \theta_0$, and (b) $\frac{f}{|f|}(w) = e^{i\theta_z(w)}$ for $w \in [0, z]$.

So we must show that $\theta_z(z)$ is a definable function.

For $z \in U$, let

$$A_z = \{t \in \mathbb{R} : 0 \leq t \leq 1 \text{ and } \frac{f}{|f|}(tz) = 1\}$$

Then A_z is, uniformly in z, a definable subset of $[0, 1]$. It follows by o-minimality that there exists $N > 0$ such that for all $z \in U$, either A_z contains at most N points, or else A_z contains an open interval. In the former case we clearly have that

$$\theta_0 - 2\pi(N + 1) \leq \theta_z(w) \leq \theta_0 + 2\pi(N + 1)$$

for all $w \in [0, z]$. This holds in the latter case too since then, by analyticity, f is real (and positive) throughout $[0, z]$, and hence θ_z is constant with value θ_0.

We now consider a cell decomposition of $\mathbb{R}^3$ compatible with the set

$$\{\langle x, y, \theta\rangle \in \mathbb{R}^3 : \ x + iy \in U,\ \theta_0 - 2\pi(N + 1) \leq \theta \leq \theta_0 + 2\pi(N + 1) \text{ and } \cos\theta + i \sin\theta = \tfrac{f(x+iy)}{|f(x+iy)|}\}.$$

(Notice that this set is definable by our assumptions on $\widetilde{\mathbb{R}}$.)

The graph of the function $z \longmapsto \theta_z(z)$ ($z \in U$) is now obtained by piecing together certain 0, 1 and 2 cells of this decomposition. □

By a similar argument one can also establish the following result.

Theorem 6. *If $f \in \widetilde{\mathcal{M}}$, $f(0) \neq 0$, and α is an exponent of $\widetilde{\mathbb{R}}$, then (any branch of) f^α also lies in $\widetilde{\mathcal{M}}$.*

I am reasonably confident that Theorems 4 and 5 have a common generalization as suggested by the following problem.

Open Problem 3
Let $f_1, \ldots, f_n \in \widetilde{\mathcal{M}} \setminus \{0\}$ and let $\tau_1(w), \ldots, \tau_n(w)$ be one variable terms of $\mathcal{L}(\mathbb{C}_{exp})$. Assume that $\phi \in \widetilde{\mathcal{G}}$ satisfies

$$\sum_{j=1}^{n} f_j(z) \cdot \tau_j(\phi(z)) = 0$$

for all $z \in domain(\phi)$. Is it the case that $\phi \in \widetilde{\mathcal{M}}$?

As for our original motivation, I conjecture a positive answer to the following.

Open Problem 4
Let $\tau(z, w)$ be a two variable term of $\mathcal{L}(\mathbb{C}_{exp})$ and let $\phi \in \widetilde{\mathcal{G}}$ be such that

$$\tau(z, \phi(z)) = 0$$

for all $z \in domain(\phi)$. Then does the star of ϕ have at most countably many singular points? If so, is each such point definable?

Finally, I state a result that makes no reference to definablity. It follows immediately from Theorems 4, 5, 6 and 2. However, I see no way of proving it without using the o-minimality of, say, $\mathbb{R}_{an,exp}$.

Theorem 7. *Let $\mathcal{O}$ denote the ring of complex analytic germs at the origin in $\mathbb{C}$. Let $\mathcal{H}$ be the smallest subset of $\mathcal{O}$ containing the polynomial ring $\mathbb{C}[z]$ and satisfying the following closure conditions:*
(i) $\mathcal{H}$ is a subring of $\mathcal{O}$ and is algebraically closed in $\mathcal{O}$;
(ii) if $f \in \mathcal{H}$ and $f(0) \neq 0$, then $\log f \in \mathcal{H}$;
(iii) if $f \in \mathcal{H}$, $f(0) \neq 0$ and $\alpha \in \mathbb{R}$, then $f^\alpha \in \mathcal{H}$.
Then for every germ $f \in \mathcal{H}$ there exists a finite set $s_f \subseteq \mathbb{C}$ such that for all $z \in \mathbb{C} \setminus s_f$ there exists a piecewise linear path starting at 0 and terminating at z along which f can be analytically continued.

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