PIECEWISE DEFINABLE C^rG TRIVIALITY AND DEFINABLE C^rG COMPACTIFICATION

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Abstract. Let G be a definably compact definable C^r group and $1 \le r < \infty$. Let X be a definable C^rG submanifold of a representation of G and Y a definable C^r submanifold of R^n. We prove that every G invariant surjective submersive definable C^r map $f: X \to Y$ is piecewise definably C^rG trivial.

1. Introduction

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Everything is considered in $\mathcal{N}$, the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{N}$", each definable map is assumed to be continuous and $1 \le r < \infty$ unless otherwise stated.

General references on o-minimal structures are [2], [3], also see [12].

Definable C^r manifolds are studied in [11], [1], and definable C^rG manifolds are studied in [5], [10]. If R is the field $\mathbb{R}$ of real numbers, then definable C^rG manifolds are considered in [9], [8], [7] [6].

Let f be a G invariant surjective submersive definable C^r map from a definable C^rG manifold X to a definable C^r manifold Y. We say that f is *definably C^rG trivial* if there exist a definable C^rG diffeomorphism $k: X \to Y \times f^{-1}(a)$ with $f = p \circ k$, where $a \in X$ and p denotes the projection $Y \times f^{-1}(a) \to Y$. We call f *piecewise definably C^rG trivial* if there exist a finite partition $\{C_i\}_i$ of Y into definable C^r submanifolds such that each $f|f^{-1}(C_i)$ is definably C^rG trivial.

A definable C^r manifold X possibly with boundary is *definably compact* if for every $a, b \in R \cup \{\infty\} \cup \{-\infty\}$ with $a < b$ and for every definable map $f: (a, b) \to X$, $\lim_{x \to a+0} f(x)$ and $\lim_{x \to b-0} f(x)$ exist in X.

If $R = \mathbb{R}$, then for any definable C^r submanifold X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general a definably compact definable C^r manifold is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \le x \le 1\}$ is definably compact but not compact.

Theorem 1.1. *Let G be a definably compact definable C^r group and $1 \le r < \infty$. Let X be a definable C^rG submanifold of a representation of G and Y a definable C^r submanifold of R^n. Then every G invariant surjective submersive definable C^r map $f: X \to Y$ is piecewise definably C^rG trivial.*

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If R is the field $\mathbb{R}$ of real numbers, Theorem 1.1 is proved in [9].

A non-definably compact definable C^rG manifold is *definably compactifiable as a definable C^rG manifold* if it is definably C^rG diffeomorphic (definably G homeomorphic if $r = 0$) to the interior of some definably compact definable C^rG manifold with boundary.

Theorem 1.2. *Let G be a definably compact definable C^r group and $2 \leq r < \infty$. Then every definable C^rG submanifold X of a representation Ω of G such that $\Omega - \{0\}$ has one orbit type and $0 \notin X$ is either definably compact or definably compactifiable as a definable $C^{r-1}G$ manifold.*

If $R = \mathbb{R}$, then a stronger result of Theorem 1.2 is proved in [9].

In the rest of Introduction, we assume $R = \mathbb{R}$.

Let $L > 0$ and $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ definable sets. A definable map $f : X \to Y$ is a *definable L-Lipschitz map* if for any $x, y \in X$, f satisfies the inequality $||f(x) - f(y)|| \leq L||x - y||$.

Theorem 1.3. *Let G be a compact definable C^2 group, X a definable C^2G submanifold of a representation of G such that X has one orbit type. Let Y a definable C^1 submanifold of $\mathbb{R}^m$, $f : X \to Y$ a G invariant definable L-Lipschitz map, $e : X \to (0, \infty)$ a G invariant definable function and $\epsilon > 0$. Then there exists a G invariant definable C^1 $(L+\epsilon)$-Lipschitz map $h : X \to Y$ such that $||h - f|| < e$ on X.*

2. Proof of results.

Let $U \subset R^n, V \subset R^m$ be definable open sets and $f : U \to V$ a definable map. We say that f is a *definable C^r map* if f is of class C^r. A definable C^r map is a *definable C^r diffeomorphism* if f is a C^r diffeomorphism.

Definition 2.1. *A Hausdorff space X is an n-dimensional definable C^r manifold if there exist a finite open cover $\{U_i\}_{i=1}^k$ of X, finite open sets $\{V_i\}_{i=1}^k$ of R^n, and a finite collection of homeomorphisms $\{\phi_i : U_i \to V_i\}_{i=1}^k$ such that for any i, j with $U_i \cap U_j \neq \emptyset$, $\phi_i(U_i \cap U_j)$ is definable and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. This pair $(\{U_i\}_{i=1}^k, \{\phi_i : U_i \to V_i\}_{i=1}^k)$ of sets and homeomorphisms is called a definable C^r coordinate system. We can define a definable C^r manifold with boundary.*

Let G be a definable group. Let f be a G invariant surjective definable map from a definable G set X to a definable set Y. We say that f is *definably G trivial* if there exist a definable G homeomprphism $k : X \to Y \times f^{-1}(a)$ with $f = p \circ k$, where $a \in X$ and p denotes the projection $Y \times f^{-1}(a) \to Y$.

By a way similar to the proof of 2.5 [9], we have the following theorem.

Theorem 2.2. *Let G be a definably compact group, X a definable G set, Y a definable set and $f : X \to Y$ a G invariant definable map. Then there exists a finite partition $\{C_i\}_i$ of Y into definable sets such that each $f|f^{-1}(C_i) : f^{-1}(C_i) \to C_i$ is definably G trivial.*

By the C^r cell decomposition theorem (e.g. 7.3.3.2 [2]), we have the following lemma.

Lemma 2.3. *Let X, Y be definable C^r submanifolds of R^n, R^m, respectively, and $1 \leq r < \infty$. For every definable map $f : X \to Y$, there exists a definable open subset Z such that $f|Z : Z \to Y$ is a definable C^r map and* $\dim(X - Z) < \dim X$.

Proof of Theorem 1.1. We proceed by induction on $\dim X$. If $\dim X = 0$, X is a finite set. Thus the result is clear. Assume $\dim X = l > 0$. By Theorem 2.2, there exists a finite partition $\{D_j\}$ of Y into definable sets such that each $f|f^{-1}(D_j) : f^{-1}(D_j) \to D_j$ is definably G trivial. Applying a C^r cell decomposition of Y compatible with $\{D_j\}$ and replacing them, we may assume that each D_j is a definable C^r submanifold of Y.

Let $X_j = f^{-1}(D_j)$. Then since f is G invariant and submersive, X_j is a definable C^rG submanifold of X. If $\dim X_j < l$, then $f|X_j : X_j \to D_j$ is piecewise definably C^rG trivial by the inductive hypothesis. We now consider the case where $\dim X_j = l$. Note that $f|X_j : X_j \to D_j$ is a submersion.

Since $f|X_j : X_j \to D_j$ is definably G trivial, there exists a definable G map $h_j : X_j \to F_j$ such that $(f|X_j, h_j) : X_j \to D_j \times F_j$ is a definable G homeomorphism, where $F_j = f^{-1}(a_j), a_j \in D_j$. Note that F_j is a definable C^rG submanifold of X since f is submersive. Applying Lemma 2.3 to h_j, we have a G invariant definable closed subset X'_j of X_j such that $\dim X'_j < l$ and $h_j|X_j - X'_j : X_j - X'_j \to h_j(X_j - X'_j) \subset F_j$ is a definable C^rG map. Since $X_j - X'_j$ is open and G invariant in X_j, $f(X_j - X'_j)$ is a G invariant definable open subset of $f(X_j)$. Hence $(f, h_j)|X_j - X'_j : X_j - X'_j \to f(X_j - X'_j) \times h_j(X_j - X'_j)$ is a definable C^rG map. Applying the same argument to the inverse of $(f, h_j)|X_j - X'_j$, we obtain a G invariant definable closed subset W_j of $X_j - X'_j$ and a G invariant definable closed subset W'_j of $f(X_j - X'_j) \times h_j(X_j - X'_j)$ such that $\dim W_j, \dim W'_j < l$ and $(f, h_j)|(X_j - X'_j - W_j) : X_j - X'_j - W_j \to ((f(X_j - X'_j) \times h_j(X_j - X'_j)) - W'_j)$ is a definable C^rG diffeomorphism. Let $\{U^t_j\}$ be a C^r cell decomposition of $X_j - X'_j - W_j$. Since $(f, h_j)(W_j) = W'_j$, each $(f, h_j)|U^t_l : U^t_l \to f(U^t_j) \times h_j(U^t_l)$ is a definable C^rG diffeomorphism.

Take a C^r cell decoposition $\{E_k\}$ of $f(X'_j \cup W_j)$. Then each $f^{-1}(E_k)$ is a definable C^rG submanifold of X and $f|f^{-1}(E_k) : f^{-1}(E_k) \to E_k$ satisfies the inductive hypothesis. Hence it is piecewise definably C^rG trivial. □

Theorem 2.4 ([1])**.** *Let A be a definable closed subset of R^n and $0 \le r < \infty$. Then there exists a definable C^r function $f : R^n \to R$ such that $A = f^{-1}(0)$.*

Theorem 2.5 ([10])**.** *Let G be a definably compact definable C^r group, H a definable C^r subgroup of G, X an affine definable C^rG manifold and $1 \le r < \infty$. Suppose that every orbit in X has type G/H. Then the orbit space X/G admits a unique structure of affine definable C^{r-1} manifold such that:*

(1) *The orbit map $\pi : X \to X/G$ is a definable C^{r-1} map.*
(2) *For any definable C^{r-1} manifold Y and a map $h : X/G \to Y$, h is a definable C^{r-1} map if and only if so is $h \circ \pi$.*

Proposition 2.6. *Let G be a definably compact definable C^r group, X a definable C^rG submanifold of a representation Ω of G such that $\Omega - \{0\}$ has one orbit type and $0 \notin X$ and $2 \le r < \infty$. Then X is definably $C^{r-1}G$ imbeddable into $\Omega \times R^2$ such that X is bounded and $\overline{X} - X$ consists of at most one point, $\overline{X}$ denotes the closure of X.*

Proof. We may assume that X is non-definably compact. Then $\overline{X} - X$ is a G invariant definable closed subset of Ω. Let $\pi : \Omega - \{0\} \to (\Omega - \{0\})/G (\subset R^s)$ be the orbit map. Then π is definably proper. Thus $\pi(\overline{X} - X)$ is a definable closed subset of R^s. By Theorem 2.4, there exists a definable C^r function $f : R^s \to R$ with $\pi((\overline{X} - X) = f^{-1}(0)$. By Theorem 2.5, π is a definable C^{r-1} map. Thus replacing X by the graph of $1/(f \circ \pi)$, we may assume that X is a definable $C^{r-1}G$ submanifold of $\Omega \times R$ which is closed in $\Omega \times R$. Using the stereographic projection $s : \Omega \times R \to S(\Omega \times R)$, $s(X)$ satisfies our conditions, where $S(\Omega \times R)$ denote the unit sphere of $\Omega \times R$. $\square$

Proposition 2.7. *Let X be a definable C^r submanifold of R^n and $\{U_i\}_{i=1}^l$ a finite definable open cover of X and $1 \le r < \infty$. Then there exist definable C^r functions $\lambda_1, \dots, \lambda_l : X \to R$ such that $0 \le \lambda_i \le 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

We call $\{\lambda_i\}$ in Proposition 2.7 a *definable C^r partition of unity subordinate to* $\{U_i\}$.

Proof of Proposition 2.7. As in the proof of Proposition 2.6, we may assume that X is closed in R^n. Hence every $R^n - U_i$ is a definable closed subset of R^n. By Theorem 2.4, we have a definable C^r function $h_i : R^n \to R$ with $h_i^{-1}(0) = R^n - U_i$. For every i, define $V_i = \{x \in X | h_i(x) > \frac{1}{2} \max_{1 \le j \le l} h_j(x)\}$. Then $\{V_i\}_{i=1}^l$ is a definable open cover of X and the closure $\overline{V_i}$ of V_i in X lies in U_i. By Theorem 2.4, there exists a definable C^r function $h'_i : R^n \to R$ with $h_i'^{-1}(0) = R^n - V_i$. Hence $\lambda_i := h'_i / \sum_{i=1}^l h'_i, 1 \le i \le l$, are the required definable C^r functions. $\square$

Proposition 2.8. *Let X be a definable C^rG submanifold closed in a representation Ω of G such that $\Omega - \{0\}$ has one orbit type and $0 \notin X$ and $\{U_i\}_{i=1}^l$ a finite G invariant definable open cover of X and $2 \le r < \infty$. Then there exist G invariant definable C^{r-1} functions $\lambda_1, \dots, \lambda_l : X \to R$ such that $0 \le \lambda_i \le 1$, supp $\lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.*

We say that $\{\lambda_i\}$ in Proposition 2.8 an *equivariant definable C^{r-1} partition of unity subordinate to* $\{U_i\}$

Proof of Proposition 2.8. By Theorem 2.5, the orbit map $\pi : \Omega - \{0\} \to (\Omega - \{0\})/G \subset R^s$ is a definable C^{r-1} map. Since $\pi | X : X \to X/G$ is open, $\{\pi(U_i)\}_{i=1}^l$ is a finite definable open covering of a definable C^{r-1} manifold X/G. Note that $\pi(X)$ is closed in R^s because X is closed in Ω. By Proposition 2.7, we can find a definable partition of unity $\{\overline{\lambda_i}\}_{i=1}^l$ subordinate to $\{\pi(U_i)\}_{i=1}^l$. Thus $\lambda_1 := \overline{\lambda_1} \circ \pi, \dots, \lambda_l := \overline{\lambda_l} \circ \pi$ are the required G invariant definable C^{r-1} functions. $\square$

Proof of Theorem 1.2. Assume that X is non-definably compact. By Proposition 2.6, we can find a representation Ω of G and a definable $C^{r-1}G$ imbedding $i : X \to \Omega$ such that $i(X)$ is bounded and $\overline{i(X)} - i(X) = \{0\}$, where $\overline{i(X)}$ denotes the closure of X in Ω.

Let $f : i(X) \to R, f(x) = \frac{1}{||x||}$, where $||x||$ denotes the standard norm of x in Ω. By Theorem 1.1, there exist a positive element $k \in R$ and a definable $C^{r-1}G$ diffeomorphism $h := (f, h_1) : f^{-1}((k, \infty)) \to (k, \infty) \times f^{-1}(k)$. If k is sufficiently large, then $f^{-1}([0, k])$ is a definably compact $C^{r-1}G$ manifold with boundary. Hence using h and by construction of $i(X)$ and Proposition 2.8, $i(X)$ is definably $C^{r-1}G$ diffeomorphic to $f^{-1}([0, k))$ which is the interior of $f^{-1}([0, k])$. $\square$

Proof of Theorem 1.3. Since X is a definable C^2 manifold with one orbit type and by Theorem 2.5, X/G is a definable C^1 submanifold in some $\mathbb{R}^n$ and the orbit map $\pi : X \to X/G$ is a definable C^1 map. Since f, e are G invariant, they induce a definable C^1 map $\overline{f} : X/G \to Y$ and a definable function $\overline{e} : X/G \to \mathbb{R}$ such that $f = \pi \circ \overline{f}, e = \pi \circ \overline{e}$. Since f is L-Lipschitz, $\overline{f}$ is L'-Lipschitz for some $L' > 0$. By [4], there exists a definable

C^1 $(L'+\epsilon')$-Lipschitz map $\overline{h}: X/G \to Y$ such that $||\overline{h}-\overline{f}|| < \overline{e}$. Therefore $h = \pi \circ \overline{h}$ is the required definable C^1 $(L+\epsilon)$-Lipschitz map $X \to Y$. □

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