Imaginaries in Boolean algebras.

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ABSTRACT

Given an infinite Boolean algebra B, we find a natural class of $\emptyset$-definable equivalence relations $\mathcal{E}_B$ such that every imaginary element from B^{eq} is interdefinable with an element from a sort determined by some equivalence relation from $\mathcal{E}_B$. It follows that B together with the family of sorts determined by $\mathcal{E}_B$ admits elimination of imaginaries in a suitable multisorted language. The paper generalizes author's earlier results concerning definable equivalence relations and weak elimination of imaginaries for Boolean algebras, obtained in [We3].

0 Introduction and preliminaries

Although no infinite Boolean algebra admits elimination of imaginaries, there exist infinite Boolean algebras admitting weak elimination of imaginaries. As proved in [We3], an infinite Boolean algebra B admits weak elimination of imaginaries iff the quotient Boolean algebra $B/I(B)$ consists of at most two elements (here $I(B)$ denotes the ideal of B consisting of all elements of the form $a \sqcup b$ with a atomless and b atomic). A special case of this result (namely: weak elimination of imaginaries for infinite Boolean algebras with finitely many atoms) plays a crucial role in studying definable sets of partially ordered o-minimal structures with ordering derived from a Boolean algebra.

C. Toffalori in [To] introduced two notions of o-minimality for partially ordered first-order structures. A partially ordered structure $\mathcal{M} = (M, \leq, \ldots)$ is called quasi o-minimal if every definable set $X \subseteq M$ is a finite Boolean combination of sets defined by inequalities of the form $x \leq a$ and $x \geq b$, where $a, b \in M$. If additionally the parameters appearing in these inequalities may be taken from the algebraic closure of the set of parameters needed to define X, then the structure $\mathcal{M} = (M, \leq, \ldots)$ is called o-minimal. It is easy to see that in case the ordering $\leq$ is linear, these two notions are equivalent to the usual o-minimality. C. Toffalori observed that if $\mathcal{M} = (M, \leq, \ldots)$ is quasi o-minimal and the ordering $\leq$ comes from some Boolean algebra B, then the number of atoms of B must be finite. By weak elimination of imaginaries for Boolean algebras with finitely atoms, the Toffalori's notions of o-minimality and quasi o-minimality coincide in case of Boolean ordered structures.

A natural counterpart of o-minimality (called q-minimality) for expansions of arbitrary Boolean algebras was introduced in author's PhD thesis. An expansion $(B, \ldots)$ of a Boolean algebra B to the language $L \supseteq L_{BA}$ is said to be q-minimal iff every L-definable subset of B is L_{BA}-definable, where $L_{BA} = \{\sqcap, \sqcup, ', 0, 1\}$ denotes the usual language of Boolean algebras. By results of [NW], for expansions of Boolean algebras with finitely many atoms, q-minimality coincides with Toffalori's notions of quasi o-minimality and o-minimality. As the model theoretic results obtained in [NW], [We1] and [We2] for Boolean ordered structures heavily rest on weak elimination of imaginaries of the underlying Boolean algebras, it is natural to expect that some form of elimination of imaginaries will be needed to investigate sets definable in q-minimal expansions of arbitrary Boolean algebras.

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In this paper, for every infinite Boolean algebra B, we find a natural multisorted language in which B admits elimination of imaginaries. The main theorem of the paper (i.e. Theorem 3.3) could be regarded as a preliminary step towards development of model theory of q-minimal expansions of arbitrary infinite Boolean algebras. Nevertheless, it might be also of independent interest. Research in this spirit has recently been conducted in the context of algebraically closed valued fields [HHM] and real closed valued fields [Me].

The paper is organized as follows. In §1 we introduce the most important tool of the paper, namely the notion of restricted elementary invariant of a Boolean algebra, and demonstrate its basic properties. In §2 we investigate certain equivalence relations in Boolean algebras and generalize Lemmas 2.2, 3.2 and 4.4 from [We3].

§3 contains the main result of the paper (Theorem 3.3). For a given infinite Boolean algebra B and a definable set $X \subseteq B^n$, we find an L_{BA}-formula $\psi(\overline{x}, \overline{z})$ such that $Y := \{\overline{d} \in B^{|\overline{z}|} : X = \psi(B, \overline{d})\}$ is a non-empty set of partitions of 1_B, and for any tuples $\overline{d} = d_0 \dots d_m$ and $\overline{e} = e_0 \dots e_m$ from Y, there is a unique permutation σ of the set $\{0, \dots, m\}$ such that the tuple $d_{\sigma(0)} \dots d_{\sigma(m)}$ is "roughly equal" to $e_0 \dots e_m$. By "roughly equal" we mean here that for any $i \leq m$ and $d \in \{d_i, e_i : i \leq m\}$, there is a number $\alpha < \omega$ for which $d \sqcap ((d_{\sigma(i)} \sqcap e_i') \sqcup (d'_{\sigma(i)} \sqcap e_i))$ belongs to the α-th elementary ideal $I_\alpha(B)$, while $d \notin I_\alpha(B)$. The proof extends author's methods from [We3] used in case of Boolean algebras with $|B/I(B)| \leq 2$.

The class of formulas of the form $\psi(\overline{x}, \overline{z})$ obtained in the proof of Theorem 3.3 and enjoying the properties described above determines a class $\mathcal{E}_B$ of $\emptyset$-definable equivalence relations on sets of partitions of 1_B. From Theorem 3.3 and the subsequent propositions we derive the fact that the multisorted structure $B(\mathcal{E}_B)$, obtained by adjoining to B all the sorts determined by equivalence relations from $\mathcal{E}_B$, admits elimination of imaginaries in a suitable multisorted language.

For the basics of model theory we refer the reader to [ChK] and to the first chapter of [Pi]; for elementary properties of Boolean algebras to the eighth chapter of [HBA].

Our notation concerning Boolean algebras is consistent with [We3]. We use the symbols $\sqcap$ (meet), $\sqcup$ (join), $'$ (complement), 0_B and 1_B to denote the Boolean operations and Boolean constants in a Boolean algebra B. Moreover, $a + b$ denotes the symmetric difference of elements $a, b \in B$. The language of Boolean algebras is $L_{BA} = \{\sqcap, \sqcup, ', 0, 1\}$. For $a \in B$ we consider the Boolean algebra $B|a := ([0, a]^B, \sqcap, \sqcup, '^a, 0_B, a)$, where $b'^a := b' \sqcap a$, and call it B restricted to a. Symbols a^+ and a^- denote a and a' respectively. If $Aa := A \cup \{a\} \subseteq B$, by $a \sqcap A$ we mean $\{a \sqcap b : b \in A\}$. Similarly we define $a + A$. In case $b_{\leq n} := b_0 \dots b_n$ is a tuple of elements of B, by $a \sqcap b_{\leq n}$ we denote the tuple $\langle a \sqcap b_i : i \leq n \rangle$. For $n < \omega$, $\eta \in \{+, -\}^{n+1}$ and $a_{\leq n} \subseteq B$ we define $a^\eta_{\leq n} := a_0^{\eta(0)} \sqcap \dots \sqcap a_n^{\eta(n)}$. An element $a \in B$ is called atomic, if for every $b \in (0_B, a]$ there is an atom c such that $c \leq b$; a is said to atomless if there are no atoms $b \leq a$. For instance, 0_B is always atomic and atomless. A Boolean algebra B is called atomic [atomless] is 1_B is atomic [atomless]. The set of all atoms of B is denoted by $At(B)$.

For every $n < \omega$, we define the Boolean algebra $B^{(n)}$ and an ideal $I_n(B) \subseteq B$ by the following conditions: $I_0(B) = \{0_B\}$, $B^{(n)} = B/I_n(B)$ and $I_{n+1}(B) = \pi_n^{-1}(I(B^{(n)}))$, where $\pi_n : B \longrightarrow B^{(n)}$ denotes the canonical projection. The elementary invariant of B (notation: $\mathrm{Inv}(B)$) is a triple $\langle \alpha, \beta, \gamma \rangle$ defined as follows:

$$\alpha = \min\left(\{k < \omega : B^{(k)} \text{ is trivial}\} \cup \{\omega\}\right),$$

$$\beta = \begin{cases} 0 & \text{if } \alpha \in \{0, \omega\} \text{ or } (0 < \alpha < \omega \text{ and } B^{(\alpha-1)} \text{ is atomic}) \\ 1 & \text{otherwise,} \end{cases}$$

$$\gamma = \begin{cases} 0 & \text{if } \alpha \in \{0, \omega\} \\ \min(|At(B^{(\alpha-1)})|, \omega) & \text{otherwise.} \end{cases}$$

The set of all triples $\langle \alpha, \beta, \gamma \rangle$ such that $\langle \alpha, \beta, \gamma \rangle = \mathrm{Inv}(B)$ for some Boolean algebra B is equal to

$$\mathrm{INV} := \{\langle 0,0,0\rangle, \langle \omega, 0, 0\rangle\} \cup \{\langle \alpha, \beta, \gamma\rangle : 0 < \alpha < \omega, \beta \in \{0,1\}, \gamma \leq \omega \text{ and } \beta + \gamma > 0\}.$$

If $\overline{a}\overline{b} = a_{\leq n} \subseteq B$, then $\mathrm{tp}(\overline{a}/\overline{b})$ is completely determined by the elementary invariants of Boolean algebras $B|d$, where d is an atom in the Boolean subalgebra generated by $\overline{a}\overline{b}$. Two Boolean algebras B_1 and B_2 are elementarily equivalent iff $\mathrm{Inv}(B_1) = \mathrm{Inv}(B_2)$.

Let B be a Boolean algebra. For every $a \in B$ and $n < \omega$ we define the (possibly empty) set $\Pi_n(a)$ of $n+1$-partitions as follows:

$$\Pi_n(a) = \{b_{\leq n} \subseteq (0_B, a] : b_0 \sqcup \ldots \sqcup b_n = a \text{ and } b_i \sqcap b_j = 0_B \text{ for } i < j \leq n\}.$$

We say that $\overline{b}$ is a partition of $a \in B$ iff $\overline{b} \in \Pi_n(a)$ for some $n < \omega$.

If $j < \omega$ and $1 \leq s < \omega$, then there are L_{BA}-formulas $\varepsilon_j(x)$, $\psi_{j,s}(x)$ and $\mathrm{at}_j(x)$ such that for every Boolean algebra B and every element $a \in B$, the following conditions hold (see [HBA], p. 292):

- $B \models \varepsilon_j(a)$ iff $a \in I_j(B)$ iff $(B|a)^{(j)}$ is trivial,
- $B \models \psi_{j,s}(a)$ iff $(B|a)^{(j)}$ contains at least s atoms,
- $B \models \mathrm{at}_j(a)$ iff $(B|a)^{(j)}$ is atomic.

Lemma 0.1 *[HBA, Lemma 18.8] For every* $\langle \alpha, \beta, \gamma \rangle \in \mathrm{INV}$, *there is a (possibly infinite) set of* L_{BA}*-formulas* $\Sigma_{\alpha,\beta,\gamma}(x)$ *such that*

- *every formula in* $\Sigma_{\alpha,\beta,\gamma}(x)$ *is of the form* $\varepsilon_j(x)$, $\neg\varepsilon_j(x)$, $\psi_{j,s}(x)$, $\neg\psi_{j,s}(x)$, $\mathrm{at}_j(x)$ *or* $\neg\mathrm{at}_j(x)$, *where* $j < \omega$ *and* $1 \leq s < \omega$, *and*
- *for every Boolean algebra* B *and an element* $a \in B$,

$$B \models \Sigma_{\alpha,\beta,\gamma}(a) \text{ if and only if } \mathrm{Inv}(B|a) = \langle \alpha, \beta, \gamma \rangle.$$

Lemma 0.2 *Assume that* B *is an infinite Boolean algebra,* $\overline{a} = a_{\leq r}$ *is a partition of* 1_B *and* $\varphi(x_{\leq n}, y_{\leq r}) \in L_{BA}$ *(*$x_{\leq n}$ *and* $y_{\leq r}$ *are abbreviations for* $x_0 \ldots x_n$ *and* $y_0 \ldots y_r$ *respectively). Then* $\varphi(B, \overline{a})$ *is a finite Boolean combination of sets defined by formulas of the form:* $\varepsilon_j(a_i \sqcap x^\eta_{\leq n})$, $\psi_{j,s}(a_i \sqcap x^\eta_{\leq n})$, $\mathrm{at}_j(a_i \sqcap x^\eta_{\leq n})$, *where* $i \leq r$, $\eta \in \{+,-\}^{n+1}$, $j < \omega$ *and* $1 \leq s < \omega$.

Proof. Assume that B, $a_{\leq r}$ and $\varphi(x_{\leq n}, y_{\leq r})$ satisfy assumptions of the lemma. For every $b_{\leq n} \subseteq B$, $\mathrm{tp}(b_{\leq n}/a_{\leq r})$ is determined by the elementary invariants $\mathrm{Inv}(B|a_i \sqcap b^\eta_{\leq n})$, where $i \leq r$ and $\eta \in \{+,-\}^{n+1}$. From this and Lemma 0.1 our assertion follows. ■

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1 Restricted elementary invariants

For every Boolean algebra B and a triple $\langle \alpha, \beta, \gamma \rangle \in \mathrm{INV}$, the set $\Sigma_{\alpha,\beta,\gamma}(B)$ from Lemma 0.1 is type-definable, but not necessarily definable. This means that in certain cases it is not possible to express all elementary properties of an element of a Boolean algebra using a single formula. For example, if B is the Boolean algebra of all subsets of ω, then the set

$$\{a \in B : B|a, B|a' \text{ are infinite}\} = \{a \in B : \mathrm{Inv}(B|a) = \mathrm{Inv}(B|a') = \langle 1, 0, \omega \rangle\}$$

is not definable. In spite of this obstacle, in §3 of [We3] we were able to prove weak elimination of imaginaries for infinite atomic Boolean algebras counting the number of atoms below elements up to a certain level determined by the formula under consideration.

In the proof of Theorem 3.3, given an infinite Boolean algebra B and a set $X \subseteq B^n$ definable over a tuple $\overline{a} \subseteq B$, we give a canonical construction of an $\overline{a}$-definable family of partitions of 1_B such that over each of them the given set is definable. To ensure that such a class is definable over $\overline{a}$ and not too large, a single formula expressing a sufficient amount of its elementary properties could be useful. In order to realize this idea, in the following definition we introduce the concept of a restricted elementary invariant of a Boolean algebra.

Definition 1.1 *Assume that B is a Boolean algebra with $\mathrm{Inv}(B) = \langle \alpha, \beta, \gamma \rangle$ and α_0, γ_0 are positive integers. We define the restricted $\langle \alpha_0, \gamma_0 \rangle$-elementary invariant of B as follows:*

$$\mathrm{Inv}_{\alpha_0,\gamma_0}(B) := \begin{cases} \langle \alpha, \beta, \min(\gamma, \gamma_0) \rangle & \text{for} \quad \alpha \leq \alpha_0, \\ \langle \alpha_0 + 1, 0, 0 \rangle & \text{for} \quad \alpha > \alpha_0. \end{cases}$$

The set of all restricted $\langle \alpha_0, \gamma_0 \rangle$-elementary invariants of Boolean algebras will be denoted by $\mathrm{INV}_{\alpha_0,\gamma_0}$. It is easy to see that

$$\mathrm{INV}_{\alpha_0,\gamma_0} = \{\langle 0,0,0 \rangle, \langle \alpha_0 + 1, 0, 0 \rangle\} \cup \{\langle \alpha, \beta, \gamma \rangle : 1 \leq \alpha \leq \alpha_0, \beta \in \{0,1\}, \gamma \leq \gamma_0 \text{ and } \beta + \gamma > 0\}.$$

The crucial difference between elementary invariants and their restricted counterpart is that in the latter case for every Boolean algebra B, the set of elements $a \in B$ such that $B|a$ has a given restricted elementary invariant is definable (in fact, $\emptyset$-definable).

Lemma 1.2 *If $\alpha_0, \gamma_0 \in \mathbb{N}_+$ and $\langle \alpha, \beta, \gamma \rangle \in \mathrm{INV}_{\alpha_0,\gamma_0}$, then there is an L_{BA}-formula $\sigma^{\alpha_0,\gamma_0}_{\langle \alpha,\beta,\gamma \rangle}(x)$ such that*

- *$\sigma^{\alpha_0,\gamma_0}_{\langle \alpha,\beta,\gamma \rangle}(x)$ is a conjunction of formulas of the form $\varepsilon_j(x^\eta_{\leq n})$, $\neg\varepsilon_j(x^\eta_{\leq n})$, $\psi_{j,s}(x^\eta_{\leq n})$, $\neg\psi_{j,s}(x^\eta_{\leq n})$, $\mathrm{at}_j(x^\eta_{\leq n})$ and $\neg\mathrm{at}_j(x^\eta_{\leq n})$, where $j \leq \alpha_0$, $s \leq \gamma_0$ and $\eta \in \{+,-\}^{n+1}$, and*
- *for every Boolean algebra B and every element $a \in B$,*

$$B \models \sigma^{\alpha_0,\gamma_0}_{\langle \alpha,\beta,\gamma \rangle}(a) \text{ iff } \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a) = \langle \alpha, \beta, \gamma \rangle.$$

Proof. Assume that $1 \leq \alpha_0, \gamma_0 < \omega$. Below, for every $\langle \alpha, \beta, \gamma \rangle \in \mathrm{INV}_{\alpha_0,\gamma_0}$ we define the formula $\sigma^{\alpha_0,\gamma_0}_{\langle \alpha,\beta,\gamma \rangle}(x)$ satisfying our demands.

$$\sigma^{\alpha_0,\gamma_0}_{\langle 0,0,0 \rangle}(x) = \varepsilon_0(x) = (x = 0),$$
$$\sigma^{\alpha_0,\gamma_0}_{\langle \alpha_0+1,0,0 \rangle}(x) = \neg\varepsilon_{\alpha_0}(x).$$

For $1 \leq \alpha \leq \alpha_0$ we define:

$$\begin{aligned}
\sigma^{\alpha_0,\gamma_0}_{\langle\alpha,1,0\rangle}(x) &= \varepsilon_\alpha(x) \wedge \neg\varepsilon_{\alpha-1}(x) \wedge \neg\mathrm{at}_{\alpha-1}(x) \wedge \neg\psi_{\alpha-1,1}(x), \\
\sigma^{\alpha_0,\gamma_0}_{\langle\alpha,1,\gamma_0\rangle}(x) &= \varepsilon_\alpha(x) \wedge \neg\varepsilon_{\alpha-1}(x) \wedge \neg\mathrm{at}_{\alpha-1}(x) \wedge \psi_{\alpha-1,\gamma_0}(x), \text{ and} \\
\sigma^{\alpha_0,\gamma_0}_{\langle\alpha,0,\gamma_0\rangle}(x) &= \varepsilon_\alpha(x) \wedge \neg\varepsilon_{\alpha-1}(x) \wedge \mathrm{at}_{\alpha-1}(x) \wedge \psi_{\alpha-1,\gamma_0}(x).
\end{aligned}$$

Finally, in case $1 \leq \alpha \leq \alpha_0$ and $1 \leq \gamma < \gamma_0$ we define:

$$\begin{aligned}
\sigma^{\alpha_0,\gamma_0}_{\langle\alpha,0,\gamma\rangle}(x) &= \varepsilon_\alpha(x) \wedge \neg\varepsilon_{\alpha-1}(x) \wedge \mathrm{at}_{\alpha-1}(x) \wedge \psi_{\alpha-1,\gamma}(x) \wedge \neg\psi_{\alpha-1,\gamma+1}(x), \\
\sigma^{\alpha_0,\gamma_0}_{\langle\alpha,1,\gamma\rangle}(x) &= \varepsilon_\alpha(x) \wedge \neg\varepsilon_{\alpha-1}(x) \wedge \neg\mathrm{at}_{\alpha-1}(x) \wedge \psi_{\alpha-1,\gamma}(x) \wedge \neg\psi_{\alpha-1,\gamma+1}(x).
\end{aligned}$$

■

Every tuple in a Boolean algebra B determines a partition of 1_B. Sometimes we will consider tuples of restricted elementary invariants of B restricted to elements of such a partition.

Definition 1.3 *Assume that B is a Boolean algebra, $a_{\leq n} \subseteq B$ and $\alpha_0, \gamma_0 \in \mathbb{N}_+$. The restricted $\langle\alpha_0, \gamma_0\rangle$-elementary invariant of $a_{\leq n}$ is defined as follows:*

$$\mathrm{Inv}_{\alpha_0,\gamma_0}(a_{\leq n}) = \langle \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a^\eta_{\leq n}) : \eta \in \{+,-\}^{n+1}\rangle.$$

In particular, $\mathrm{Inv}_{\alpha_0,\gamma_0}(a) = \langle \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a), \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a')\rangle$.

Lemma 1.4 *Assume that $\alpha_0, \gamma_0 \in \mathbb{N}_+$ and for every $\eta \in \{+,-\}^{n+1}$, $\langle\alpha_\eta, \beta_\eta, \gamma_\eta\rangle$ is a triple from $\mathrm{INV}_{\alpha_0,\gamma_0}$. Then there is an L_{BA}-formula $\varphi(x_{\leq n})$ such that*

- *$\varphi(x_{\leq n})$ is a conjunction of formulas of the form $\varepsilon_j(x^\eta_{\leq n})$, $\neg\varepsilon_j(x^\eta_{\leq n})$, $\psi_{j,s}(x^\eta_{\leq n})$, $\neg\psi_{j,s}(x^\eta_{\leq n})$, $\mathrm{at}_j(x^\eta_{\leq n})$ and $\neg\mathrm{at}_j(x^\eta_{\leq n})$, where $j \leq \alpha_0$, $s \leq \gamma_0$ and $\eta \in \{+,-\}^{n+1}$, and*
- *for every Boolean algebra B and $a_{\leq n} \subseteq B$,*

$$B \models \varphi(a_{\leq n}) \text{ iff } \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a^\eta_{\leq n}) = \langle\alpha_\eta, \beta_\eta, \gamma_\eta\rangle \text{ for } \eta \in \{+,-\}^{n+1}.$$

Proof. The formula

$$\varphi(x_{\leq n}) = \bigwedge_{\eta\in\{+,-\}^{n+1}} \sigma^{\alpha_0,\gamma_0}_{\langle\alpha_\eta,\beta_\eta,\gamma_\eta\rangle}(x^\eta_{\leq n})$$

satisfies our demands. ■

In the following two lemmas we outline some of the basic properties of restricted elementary invariants.

Lemma 1.5 *Assume that $\alpha_0, \alpha_1, \gamma_0, \gamma_1$ are positive integers, B is a Boolean algebra and $a, b \in B$.*
(a) If $\mathrm{Inv}(B|a) = \mathrm{Inv}(B|b)$, then $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|a) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b)$.
(b) If $\langle\alpha_0, \gamma_0\rangle \leq \langle\alpha_1, \gamma_1\rangle$, where $\leq$ denotes the lexicographic ordering of $\mathbb{N}_+ \times \mathbb{N}_+$, then

$$\mathrm{Inv}_{\alpha_1,\gamma_1}(B|a) = \mathrm{Inv}_{\alpha_1,\gamma_1}(B|b) \Longrightarrow \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b).$$

Lemma 1.6 *Assume that α_0, γ_0 are positive integers, B is a Boolean algebra, $a, b, c, d \in B$, and $a_{\leq n}, b_{\leq n} \subseteq B$.*

(a) If $a_{\leq n}$ is a partition of 1_B, then $\mathrm{Inv}_{\alpha_0,\gamma_0}(a \sqcap a_{\leq n})$ is completely determined by the sequence $\langle \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a \sqcap a_i) : i \leq n\rangle$.

(b) If $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|a) = \langle \alpha_1, \beta_1, \gamma_1\rangle$, $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b) = \langle \alpha_2, \beta_2, \gamma_2\rangle$ and $a \sqcap b = 0_B$, then

$$\mathrm{Inv}_{\alpha_0,\gamma_0}(B|a \sqcup b) = \begin{cases} \langle \alpha_1, \beta_1, \gamma_1\rangle & if \quad \alpha_1 > \alpha_2 \\ \langle \alpha_1, \max(\beta_1, \beta_2), \min(\gamma_0, \gamma_1 + \gamma_2)\rangle & if \quad \alpha_1 = \alpha_2 \\ \langle \alpha_2, \beta_2, \gamma_2\rangle & if \quad \alpha_1 < \alpha_2. \end{cases}$$

(c) If $a \sqcap b = 0_B$, $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|a \sqcap c) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a \sqcap d)$ and $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap c) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap d)$, then $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a \sqcup b) \sqcap c) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a \sqcup b) \sqcap d)$.

(d) If $a \sqcap b = 0_B$, $\mathrm{Inv}_{\alpha_0,\gamma_0}(a \sqcap a_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(a \sqcap b_{\leq n})$ and $\mathrm{Inv}_{\alpha_0,\gamma_0}(b \sqcap a_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(b \sqcap b_{\leq n})$, then $\mathrm{Inv}_{\alpha_0,\gamma_0}((a \sqcup b) \sqcap a_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}((a \sqcup b) \sqcap b_{\leq n})$.

2 Equivalence relations in Boolean algebras

Consider an infinite Boolean algebra B and elements $a, b, c \in B$ such that $a \sqcap c, b \sqcap c > 0_B$, $c \leq a \sqcup b$ and $a \sqcap b = 0_B$. In [We3] we were dealing with the problem of obtaining one partition of $a \sqcup b$ from another by a finite series of modifications below a, b or c. Lemmas 2.2 and 3.2 from [We3] could be expressed in the form of the following statement.

Fact 2.1 *Assume that $\gamma_0 \in \mathbb{N}_+$, $\mathrm{Inv}(B|a \sqcup b) \in \{\langle 1, 1, 0\rangle, \langle 1, 0, \omega\rangle\}$, $a_{\leq n}, b_{\leq n} \subseteq B$, for every $\eta \in \{+, -\}^{n+1}$, $\mathrm{Inv}_{1,\gamma_0}((a \sqcup b) \sqcap a^{\eta}_{\leq n}) = \mathrm{Inv}_{1,\gamma_0}((a \sqcup b) \sqcap b^{\eta}_{\leq n})$, and $(a \sqcup b)' \sqcap a_{\leq n} = (a \sqcup b)' \sqcap b_{\leq n}$. There there are tuples $a^0_{\leq n} = a_{\leq n}, a^1_{\leq n}, \ldots, a^k_{\leq n} = b_{\leq n} \subseteq B$ such that*

- $(\forall i < k)(\exists d \in \{a, b, c\})(d' \sqcap a^i_{\leq n} = d' \sqcap a^{i+1}_{\leq n})$ *and*
- $(\forall i < k)(\forall d \in \{a, b, c\})(\mathrm{Inv}_{1,\gamma_0}(d \sqcap a^i_{\leq n}) = \mathrm{Inv}_{1,\gamma_0}(d \sqcap a^{i+1}_{\leq n}))$.

It turns out that an analogical result cannot be obtained in case $c \in I_\alpha(B)$ and $a, b \notin I_\alpha(B)$ for some $\alpha < \omega$. Informally speaking, there is too little space below c to transform one tuple into another by modifying it only below a, b or c. Lemma 4.4 from [We3] deals with modifications of tuples below elements in Boolean algebras of elementary invariant equal to $\langle 2, 0, 1\rangle$, in which case one needs additional assumptions. In this section we isolate some reasonable conditions on a, b, c under which analogues of Lemmas 2.2, 3.2 and 4.4 from [We3] can be proved in arbitrary Boolean algebras and relevant methods generalized.

Definition 2.2 *Assume that B is a Boolean algebra and $a, b \in B$. We say that a is large in b iff*

$$(\forall n < \omega)(a \sqcap b \in I_n(B) \iff b \in I_n(B)).$$

Lemma 2.3 *Assume that B is a Boolean algebra and $a, b, c \in B$.*

(a) If $a \geq b$, then a is large in b.

(b) 0_B is large in a iff $a = 0_B$.

(c) If $a \leq b \leq c$, a is large in b and b is large in c, then a is large in c.

(d) If b is large in 1_B, then a is large in b iff b is large in a.

(e) If b is not large in a, then $\mathrm{Inv}(B|a) = \mathrm{Inv}(B|a \sqcap b')$, and consequently, $B|a \equiv B|a \sqcap b'$.

(f) If $a > 0_B$ and $b_{\leq n} \in \Pi_n(1_B)$, then b_i is large in a for some $i \leq n$.

Lemma 2.4 *Assume that $\alpha_0, \gamma_0 \in \mathbb{N}_+$ and a, b, d are elements of a Boolean algebra B such that a is large in b, d is not large in b, and $b > a, d$. Then there is an element $e < a$ such that e is not large in a and $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|d) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|e)$.*

Proof. Let $\mathrm{Inv}(B|a) = \langle \alpha_1, \beta_1, \gamma_1 \rangle$, $\mathrm{Inv}(B|b) = \langle \alpha_2, \beta_2, \gamma_2 \rangle$ and $\mathrm{Inv}(B|d) = \langle \alpha_3, \beta_3, \gamma_3 \rangle$. Our assumptions guarantee that $\alpha_1 = \alpha_2 > \alpha_3$, so $\alpha := \max(\alpha_0, \alpha_3) < \omega$. Define $\gamma := \max(\gamma_0, \gamma_3)$ for $\gamma_3 < \omega$ and $\gamma := \gamma_0$ in case $\gamma_3 = \omega$. Let $\mathrm{Inv}_{\alpha,\gamma}(B|d) = \langle \alpha', \beta', \gamma' \rangle$. Note that $a' \sqcap d$ is not large in $a \sqcup d$. By Lemma 2.3(e), $\mathrm{Inv}(B|a \sqcup d) = \mathrm{Inv}(B|a)$ and $B|a \sqcup d \equiv B|a$. Since $B|a \sqcup d \models \sigma^{\alpha,\gamma}_{\langle \alpha',\beta',\gamma' \rangle}(d) \wedge \varepsilon_{\alpha_3}(d)$, there is $e < a$ such that $B|a \models \sigma^{\alpha,\gamma}_{\langle \alpha',\beta',\gamma' \rangle}(e) \wedge \varepsilon_{\alpha_3}(e)$. Hence e is not large in a, and $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|d) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|e)$. ■

For $n < \omega$ and $0 < \alpha_0, \gamma_0 < \omega$ we denote by $E_{n,\alpha_0,\gamma_0}(u_{\leq n}, v_{\leq n}, z)$ the following L_{BA}-formula.

$$E_{n,\alpha_0,\gamma_0}(u_{\leq n}, v_{\leq n}, z) = \bigwedge_{\eta \in \{+,-\}^{n+1}} \Bigg[\bigwedge_{j \leq \alpha_0} \Big(\varepsilon_j(z \sqcap u^\eta_{\leq n}) \longleftrightarrow \varepsilon_j(z \sqcap v^\eta_{\leq n}) \Big) \wedge$$
$$\bigwedge_{j \leq \alpha_0} \bigwedge_{s \leq \gamma_0} \Big(\psi_{j,s}(z \sqcap u^\eta_{\leq n}) \longleftrightarrow \psi_{j,s}(z \sqcap v^\eta_{\leq n}) \Big) \wedge$$
$$\bigwedge_{j \leq \alpha_0} \Big(\mathrm{at}_j(z \sqcap u^\eta_{\leq n}) \longleftrightarrow \mathrm{at}_j(z \sqcap v^\eta_{\leq n}) \Big) \Bigg] \wedge z' \sqcap u_{\leq n} = z' \sqcap v_{\leq n}.$$

It is clear that for every $a \in B$, $E_{n,\alpha_0,\gamma_0}(u_{\leq n}, v_{\leq n}, a)$ defines an equivalence relation $E^a_{n,\alpha_0,\gamma_0}$ on B^{n+1}. Note that $E^{0_B}_{n,\alpha_0,\gamma_0}$ is the equality on B^{n+1}.

Definition 2.5 *Assume that $\alpha_0, \gamma_0 \in \mathbb{N}_+$, a is a non-zero element of a Boolean algebra B and $a_{\leq n} b_{\leq n} \subseteq B$. A tuple $b_{\leq n} \subseteq B$ is said to be an $\langle \alpha_0, \gamma_0 \rangle$-modification of $a_{\leq n}$ below a iff*

$$a' \sqcap a_{\leq n} = a' \sqcap b_{\leq n} \text{ and } \mathrm{Inv}_{\alpha_0,\gamma_0}(a \sqcap a_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(a \sqcap b_{\leq n}).$$

Note that in the setting of the above definition, $b_{\leq n}$ is an $\langle \alpha_0, \gamma_0 \rangle$-modification of $a_{\leq n}$ below a iff $B \models E_{n,\alpha_0,\gamma_0}(a_{\leq n}, b_{\leq n}, a)$. If $b_{\leq n}$ is an $\langle \alpha_0, \gamma_0 \rangle$-modification of $a_{\leq n}$ below a and $b \geq a$, then it is also an $\langle \alpha_0, \gamma_0 \rangle$-modification of $a_{\leq n}$ below b.

Definition 2.6 *A non-zero element a of a Boolean algebra B is called simple iff one of the following conditions hold.*

- *$(B|a)^{(m)}$ is non-trivial and atomless for some $m < \omega$,*
- *$(B|a)^{(m)}$ is non-trivial and atomic for some $m < \omega$,*
- *$\mathrm{Inv}(B|a) = \langle \omega, 0, 0 \rangle$.*

Note that an element a of a Boolean algebra B is simple iff

$$Inv(B|a) \in \{\langle \omega, 0, 0 \rangle\} \cup \{\langle \alpha, 1, 0 \rangle : 0 < \alpha < \omega\} \cup \{\langle \alpha, 0, \gamma \rangle : 0 < \alpha < \omega, 0 < \gamma \leq \omega\}.$$

Lemma 2.7 *Assume that a, b, c are non-zero elements of an infinite Boolean algebra B*

(a) If $a \geq b$, and b is large in a, then a is simple iff b is simple.

(b) If a, b, c are simple, $a \sqcap b = 0_B$, c is large both in a and b, and $a \sqcup b$ is large in c, then $a \sqcup b$ is simple.

Proof. Since (a) is easy, we only prove (b). Let a, b, c be simple elements of a Boolean algebra B such that $a \sqcap b = 0_B$, c is large both in a and b, and $a \sqcup b$ is large in c. We consider three cases.

Case 1. $Inv(B|c) = \langle \omega, 0, 0 \rangle$. Since $a \sqcup b$ is large in c, also $Inv(B|(a \sqcup b) \sqcap c) = \langle \omega, 0, 0 \rangle$. Hence $Inv(B|a \sqcup b) = \langle \omega, 0, 0 \rangle$.

Case 2. $Inv(B|c) = \langle \alpha, 1, 0 \rangle$, where $0 < \alpha < \omega$. The element $a \sqcup b$ is large in c, so $Inv(B|(a \sqcup b) \sqcap c) = \langle \alpha, 1, 0 \rangle$. But then $Inv(B|a \sqcap c) = \langle \alpha, 1, 0 \rangle$ or $Inv(B|b \sqcap c) = \langle \alpha, 1, 0 \rangle$. Suppose for instance that the first possibility holds.

As c is large in a, we get $Inv(B|a) = \langle \alpha, 1, 0 \rangle$. Note that either $Inv(B|b \sqcap c) = \langle \alpha, 1, 0 \rangle$ or $b \sqcap c \in I_\alpha(B)$. In the first case $Inv(B|b) = \langle \alpha, 1, 0 \rangle$. If $b \sqcap c \in I_\alpha(B)$, then (since c is large in b) also $b \in I_\alpha(B)$. In both cases $Inv(B|a \sqcup b) = \langle \alpha, 1, 0 \rangle$.

Case 3. $Inv(B|c) = \langle \alpha, 0, \gamma \rangle$, where $0 < \alpha < \omega$ and $0 < \gamma \leq \omega$. An argument similar to that used in Case 2 yields that $Inv(B|a \sqcup b) = \langle \alpha, 0, \gamma \rangle$.

In all possible cases the element $a \sqcup b$ is simple. This finishes the proof. ■

Theorem 2.8 *Assume that $n < \omega$, a, b, c are simple non-zero elements of a Boolean algebra B, $a \sqcap b = 0_B$, $c \leq a \sqcup b$ and c is large in each of the elements a, b. Denote by $\langle \alpha_1, \beta_1, \gamma_1 \rangle$ and $\langle \alpha_2, \beta_2, \gamma_2 \rangle$ the elementary invariants of $B|a$ and $B|b$ respectively and let α_0, γ_0 be positive integers satisfying the following conditions.*

- *For i=1,2: if $\alpha_i < \omega$, then $\alpha_0 \geq \alpha_i$.*
- *For i=1,2: if $\alpha_i < \omega$, $\beta_i = 0$ and $\gamma_i < \omega$, then $\gamma_0 \geq \gamma_i$.*
- *If $\alpha_1 = \alpha_2 < \omega$, $\beta_1 = \beta_2 = 0$ and $\gamma_1, \gamma_2 < \omega$, then $\gamma_0 \geq \gamma_1 + \gamma_2$.*

Then the following conditions hold.

(a) If $\sim$ is an equivalence relation on B^{n+1} containing $E^a_{n,\alpha_0,\gamma_0} \cup E^b_{n,\alpha_0,\gamma_0} \cup E^c_{n,\alpha_0,\gamma_0}$ and $a_{\leq n} \in B^{n+1}$, then there is $b_{\leq n} \sim a_{\leq n}$ such that for every $\eta \in \{+,-\}^{n+1}$, $a \sqcap b^\eta_{\leq n} = 0_B$ or $a^\eta_{\leq n}$ is large in a.

(b) Any equivalence relation on B^{n+1} containing $E^a_{n,\alpha_0,\gamma_0} \cup E^b_{n,\alpha_0,\gamma_0} \cup E^c_{n,\alpha_0,\gamma_0}$ contains $E^{a \sqcup b}_{n,\alpha_0,\gamma_0}$.

Proof. Note that by Lemma 2.7, the element $a \sqcup b$ is simple. Throughout the proof, for the sake of notational simplicity, $\langle \alpha_0, \gamma_0 \rangle$-modifications will be called shortly "modifications".

(a) Let $\sim$ be an equivalence relation on B^{n+1} containing $E^a_{n,\alpha_0,\gamma_0} \cup E^b_{n,\alpha_0,\gamma_0} \cup E^c_{n,\alpha_0,\gamma_0}$. Define

$$E = \{\eta \in \{+,-\}^{n+1} : a^\eta_{\leq n} \text{ is not large in } a\}.$$

By Lemma 2.4, there are pairwise disjoint elements $d_\eta \in [0_B, a \sqcap c)$, $\eta \in E$ such that for every $\eta \in E$, $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|d_\eta) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a \sqcap a^\eta_{\leq n})$ and d_η is not large in $a \sqcap c$. Let $d := \bigsqcup_{\eta \in E} d_\eta$ and let $\eta_1 \in \{+,-\}^{n+1}$ be such that $a^{\eta_1}_{\leq n}$ is large in a. We define the tuple $a^1_{\leq n}$ by the following conditions:

- $a' \sqcap a^1_{\leq n} = a' \sqcap a_{\leq n}$,
- $a \sqcap (a^1_{\leq n})^\eta = d_\eta$ for $\eta \in E$,
- $a \sqcap (a^1_{\leq n})^{\eta_1} = a \sqcap d' \sqcap \left(a^{\eta_1}_{\leq n} \sqcup \bigsqcup_{\eta \in E} a^\eta_{\leq n} \right)$,
- $a \sqcap (a^1_{\leq n})^\eta = a \sqcap d' \sqcap a^\eta_{\leq n}$ for $\eta \notin E \cup \{\eta_1\}$.

$a^1_{\leq n}$ is a modification of $a_{\leq n}$ below a, and for every $\eta \in \{+,-\}^{n+1}$, if $(a^1_{\leq n})^\eta$ is not large in a, then $a \sqcap (a^1_{\leq n})^\eta < a \sqcap c$. Repeating this argument with a and $a \sqcap c$ replaced by c and $b \sqcap c$ (respectively), we obtain $b_{\leq n}$, a modification of $a^1_{\leq n}$ below c satisfying our demands.

(b) Let $\sim$ be an equivalence relation on B^{n+1} containing $E^a_{n,\alpha_0,\gamma_0} \cup E^b_{n,\alpha_0,\gamma_0} \cup E^c_{n,\alpha_0,\gamma_0}$.

Without loss of generality we can assume that b is large in $a \sqcup b$. Let $a_{\leq n}, b_{\leq n} \subseteq B$ be tuples such that $(a \sqcup b)' \sqcap a_{\leq n} = (a \sqcup b)' \sqcap b_{\leq n}$ and $\mathrm{Inv}_{\alpha_0,\gamma_0}((a \sqcup b) \sqcap a_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}((a \sqcup b) \sqcap b_{\leq n})$. We have to show that $a_{\leq n} \sim b_{\leq n}$. We will obtain $b_{\leq n}$ from $a_{\leq n}$ in a series of modifications of $a_{\leq n}$ below a, b or c. We consider 5 cases.

Case 1. There is $m < \omega$ such that $(B|a)^{(m)}$ and $(B|b)^{(m)}$ are both non-trivial and atomless.

Note that the elementary invariant of each of the Boolean algebras $B|a$, $B|b$, $B|c$, $B|a \sqcup b$, $B|a \sqcap c$ and $B|b \sqcap c$ is equal to $\langle m+1, 1, 0\rangle$. Our assumptions guarantee that $\alpha_0 \geq m+1$, so the restricted $\langle \alpha_0, \gamma_0\rangle$-elementary invariant of each of the listed Boolean algebras is also equal to $\langle m+1, 1, 0\rangle$.

There is $\eta_0 \in \{+,-\}^{n+1}$ such that $a^{\eta_0}_{\leq n}$ is large in $a \sqcup b$. Then also $b^{\eta_0}_{\leq n}$ is large in $a \sqcup b$. By (a), without loss of generality we can assume that for every $\eta \in \{+,-\}^{n+1}$, either $a \sqcap a^\eta_{\leq n} = 0_B$ or $a^\eta_{\leq n}$ is large in a. There is $a^1_{\leq n}$, a modification of $a_{\leq n}$ below a or below b such that $(a^1_{\leq n})^{\eta_0}$ is large in c. If for example $a^{\eta_0}_{\leq n}$ is large in a and not large in c, then there is an $\eta_1 \in \{+,-\}^{n+1} \setminus \{\eta_0\}$ such that $a^{\eta_1}_{\leq n}$ is large in $a \sqcap c$, and the tuple $a^1_{\leq n}$ may be defined by the following conditions: $a' \sqcap a^1_{\leq n} = a' \sqcap a_{\leq n}$, $a \sqcap (a^1_{\leq n})^{\eta_0} = a \sqcap a^{\eta_1}_{\leq n}, a \sqcap (a^1_{\leq n})^{\eta_1} = a \sqcap a^{\eta_0}_{\leq n}$ and $a \sqcap (a^1_{\leq n})^{\eta} = a \sqcap a^\eta_{\leq n}$ for $\eta \notin \{\eta_0, \eta_1\}$.

In this situation, using the fact that the Boolean algebras $(B|a)^{(m)}$ and $(B|b)^{(m)}$ are atomless, we easily obtain tuples $a^2_{\leq n}, a^3_{\leq n}, a^4_{\leq n} \subseteq B$ such that

- $a^2_{\leq n}$ is a modification of $a^1_{\leq n}$ below c, $(a^2_{\leq n})^{\eta_0}$ is large both in $a \sqcap c$ and $b \sqcap c$, and for every $\eta \in \{+,-\}^{n+1}$, either $a \sqcap (a^2_{\leq n})^\eta = 0_B$ or $(a^2_{\leq n})^\eta$ is large in a.
- $a^3_{\leq n}$ is a modification of $a^2_{\leq n}$ below a, $a \sqcap c' < (a^3_{\leq n})^{\eta_0}$ and $(a^3_{\leq n})^{\eta_0}$ is large in $a \sqcap c$.
- $a^4_{\leq n}$ is a modification of $a^3_{\leq n}$ below c, $a < (a^4_{\leq n})^{\eta_0}$ and $(a^4_{\leq n})^{\eta_0}$ is large in $b \sqcap c$ (so in b).

The tuples $a_{\leq n}$ and $a^4_{\leq n}$ are $\sim$-equivalent. Similarly we can find $b^1_{\leq n} \sim b_{\leq n}$ such that $(b^1_{\leq n})^{\eta_0}$ is large in b and $(b^1_{\leq n})^{\eta_0} > a$. Now, $b^1_{\leq n}$ is clearly a modification of $a^4_{\leq n}$ below b, so $a_{\leq n} \sim b_{\leq n}$.

Case 2. There is $m < \omega$ such that $(B|a)^{(m)}$, $(B|b)^{(m)}$ are both finite, non-trivial and atomic.

Note that $\mathrm{Inv}(B|a) = \langle m+1, 0, k\rangle$ and $\mathrm{Inv}(B|b) = \langle m+1, 0, l\rangle$ for some $k, l \in \mathbb{N}_+$. Our assumptions guarantee that $\alpha_0 \geq m+1$ and $\gamma_0 \geq k+l$. There is a tuple $c_{<k+l} \subseteq (0_B, a \sqcup b]$ of pairwise disjoint elements such that

- $\mathrm{Inv}(B|c_i) = \langle m+1, 0, 1\rangle$ for every $i < k+l$,
- $(\forall i < k+l)(\exists \eta_0, \eta_1 \in \{+,-\}^{n+1})(c_i \leq a^{\eta_0}_{\leq n} \sqcap b^{\eta_1}_{\leq n})$, and
- $(\forall i < k+l-1)(c_i \sqcup c_{i+1} \leq a$ or $c_i \sqcup c_{i+1} \leq b$ or $c_i \sqcup c_{i+1} \leq c)$.

Then $(c_0 \sqcup \ldots \sqcup c_{k+l-1})'$ is not large in $a \sqcup b$ and there is a unique permutation σ of the set $\{0, \ldots, k+l-1\}$ such that for any $i < k+l$ and $\eta \in \{+,-\}^{n+1}$,

$$c_i \leq a^{\eta}_{\leq n} \text{ iff } c_{\sigma(i)} \leq b^{\eta}_{\leq n}.$$

$\sigma = \tau_1 \circ \ldots \circ \tau_s$, where $\tau_1, \ldots, \tau_s$ are transpositions of the form $(i, i+1)$, $i < k+l-1$. Let $a^0_{\leq n} = a_{\leq n}$ and let $a^r_{\leq n}$ for $1 \leq r \leq s$ be the tuple defined by the following conditions:

- $(c_0 \sqcup \ldots \sqcup c_{k+l-1})' \sqcap a^r_{\leq n} = (c_0 \sqcup \ldots \sqcup c_{k+l-1})' \sqcap a_{\leq n}$ and
- for any $i < k+l$ and $\eta \in \{+,-\}^{n+1}$, $c_i \leq a^{\eta}_{\leq n}$ iff $c_{(\tau_{s-r+1} \circ \ldots \circ \tau_s)(i)} \leq (a^r_{\leq n})^{\eta}$.

Our choice of $c_{<k+l}$ guarantees that for every $r < s$, $a^{r+1}_{\leq n}$ is a modification of $a^r_{\leq n}$ below a, b or c. Thus $a_{\leq n} \sim a^s_{\leq n}$ and for every $\eta \in \{+,-\}^{n+1}$, $(a^s_{\leq n})^{\eta} + b^{\eta}_{\leq n}$ is not large in $a \sqcup b$. By (a) we can find $a^{s+1}_{\leq n} \sim a^s_{\leq n}$ and $b^1_{\leq n} \sim b_{\leq n}$ such that for every $\eta \in \{+,-\}^{n+1}$,

- $a \sqcap (a^{s+1}_{\leq n})^{\eta} = 0_B$ or $(a^{s+1}_{\leq n})^{\eta}$ is large in a, and
- $a \sqcap (b^{s+1}_{\leq n})^{\eta} = 0_B$ or $(b^1_{\leq n})^{\eta}$ is large in a.

$b^1_{\leq n}$ is a modification of $a^{s+1}_{\leq n}$ below b. Therefore $a_{\leq n} \sim b_{\leq n}$.

Case 3. There is $m < \omega$ such that $(B|a)^{(m)}$, $(B|b)^{(m)}$ are both non-trivial and atomic, and $(B|a \sqcup b)^{(m)}$ is infinite.

Without loss of generality we can assume that $(B|b)^{(m)}$ is infinite. Note that

$$\mathrm{Inv}(B|a) = \langle m+1, 0, k \rangle \text{ and } \mathrm{Inv}(B|b) = \langle m+1, 0, \omega \rangle,$$

where k is a positive integer or ω. Our assumptions guarantee that $\alpha_0 \geq m+1$ and if $k < \omega$, then $\gamma_0 \geq k$. There is $\eta_0 \in \{+,-\}^{n+1}$ such that the Boolean algebra $(B|b \sqcap b^{\eta_0}_{\leq n})^{(m)}$ is infinite. Then

$$\mathrm{Inv}(B|b \sqcap b^{\eta_0}_{\leq n}) = \mathrm{Inv}(B|(a \sqcup b) \sqcap b^{\eta_0}_{\leq n}) = \langle m+1, 0, \omega \rangle \text{ and}$$
$$\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap b^{\eta_0}_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a \sqcup b) \sqcap b^{\eta_0}_{\leq n}) = \langle m+1, 0, \gamma_0 \rangle.$$

If $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap a^{\eta_0}_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap b^{\eta_0}_{\leq n})$, then there is $a^5_{\leq n}$, a modification of $a_{\leq n}$ below b such that the Boolean algebra $B|b \sqcap (a^5_{\leq n})^{\eta_0}$ is infinite. If $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap a^{\eta_0}_{\leq n}) \neq \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap b^{\eta_0}_{\leq n})$, then $a^{\eta_0}_{\leq n}$ is large in a. There is $a^1_{\leq n} \subseteq B$, a modification of $a_{\leq n}$ below a such that $(a^1_{\leq n})^{\eta_0}$ is large in $a \sqcap c$. Similarly, there are tuples $a^2_{\leq n}, a^3_{\leq n} \subseteq B$ such that

- $a^2_{\leq n}$ is a modification of $a^1_{\leq n}$ below b,
- $((a^2_{\leq n})^{\eta_0})'$ is large in $b \sqcap c$
- $a^3_{\leq n}$ is a modification of $a^2_{\leq n}$ below c,
- $|At((B|b \sqcap (a^3_{\leq n})^{\eta_0})^{(m)})| = |At((B|b \sqcap (a^2_{\leq n})^{\eta_0})^{(m)})| + 1.$

Repeating this procedure, after $\leq \gamma_0$ steps we can obtain $a^4_{\leq n} \sim a^3_{\leq n}$ such that

$$\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap (a^4_{\leq n})^{\eta_0}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap b^{\eta_0}_{\leq n}).$$

There is $a^5_{\leq n}$, a modification of $a^4_{\leq n}$ below b such that the Boolean algebra $(B|b \sqcap (a^5_{\leq n})^{\eta_0})^{(m)}$ is infinite.

Now, we will show that there is $a^{10}_{\leq n} \sim a^5_{\leq n}$ such that the Boolean algebra $(B|a \sqcap ((a^{10}_{\leq n})^{\eta_0})')^{(m)}$ is finite. This is trivial in case $At((B|a)^{(m)})$ is finite. If $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|a \sqcap (a^5_{\leq n})^{\eta_0}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a \sqcup b) \sqcap (a^5_{\leq n})^{\eta_0})$, then $a^{10}_{\leq n}$ is a result of a single modification of $a^5_{\leq n}$ below a. If $(B|a)^{(m)}$ is infinite and $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|a \sqcap (a^5_{\leq n})^{\eta_0}) \neq \mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a \sqcup b) \sqcap (a^5_{\leq n})^{\eta_0})$, then there are tuples $a^6_{\leq n}, a^7_{\leq n}, a^8_{\leq n} \subseteq B$ such that

- $a^6_{\leq n}$ is a modification of $a^5_{\leq n}$ below a and $((a^6_{\leq n})^{\eta_0})'$ is large in $a \sqcap c$,
- $a^7_{\leq n}$ is a modification of $a^6_{\leq n}$ below b and $(a^7_{\leq n})^{\eta_0}$ is large in $b \sqcap c$,
- $a^8_{\leq n}$ is a modification of $a^7_{\leq n}$ below c,
- $|At((B|a \sqcap (a^8_{\leq n})^{\eta_0})^{(m)})| = |At((B|a \sqcap (a^7_{\leq n})^{\eta_0})^{(m)})| + 1$, and
- $At((B|b \sqcap (a^8_{\leq n})^{\eta_0})^{(m)})$ is infinite

Repeating this procedure, after $\leq \gamma_0$ steps we can find $a^9_{\leq n} \sim a^8_{\leq n}$ such that $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|a \sqcap (a^9_{\leq n})^{\eta_0}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a \sqcup b) \sqcap (a^9_{\leq n})^{\eta_0})$ and the Boolean algebra $(B|b \sqcap (a^9_{\leq n})^{\eta_0})^{(m)}$ is infinite. Then there is $a^{10}_{\leq n}$, a modification of $a^9_{\leq n}$ below a, such that the Boolean algebra $(B|a \sqcap ((a^{10}_{\leq n})^{\eta_0})')^{(m)}$ is finite.

A similar argument yields $a^{11}_{\leq n} \sim a^{10}_{\leq n}$ such that $((a^{11}_{\leq n})^{\eta_0})'$ is not large in a. By (a) there is $a^{12}_{\leq n} \sim a^{11}_{\leq n}$ such that $(a^{12}_{\leq n})^{\eta_0} > a$ and $(B|b \sqcap a^{12}_{\leq n})^{(m)}$ is infinite.

Similarly we can find $b^1_{\leq n} \sim b_{\leq n}$ such that $(b^1_{\leq n})^{\eta_0} > a$ and $(B|b \sqcap (b^1_{\leq n})^{\eta_0})^{(m)}$ is infinite. It is easy to see that $b_{\leq n}$ is a modification of $a^{12}_{\leq n}$ below b. Thus $a_{\leq n} \sim b_{\leq n}$.

Case 4. $\mathrm{Inv}(B|a) = \mathrm{Inv}(B|b) = \langle \omega, 0, 0 \rangle$.

Note that

$$\mathrm{Inv}(B|a \sqcup b) = \mathrm{Inv}(B|a \sqcap c) = \mathrm{Inv}(B|b \sqcap c) = \mathrm{Inv}(B|c) = \langle \omega, 0, 0 \rangle.$$

There is $\eta_0 \in \{+,-\}^{n+1}$ such that $b^{\eta_0}_{\leq n}$ is large in b. Then

$$\mathrm{Inv}(B|b \sqcap b^{\eta_0}_{\leq n}) = \langle \omega, 0, 0 \rangle \text{ and } \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap b^{\eta_0}_{\leq n}) = \langle \alpha_0 + 1, 0, 0 \rangle.$$

If $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap a^{\eta_0}_{\leq n}) = \langle \alpha_0 + 1, 0, 0 \rangle$, then there is $a^2_{\leq n}$, a modification of $a_{\leq n}$ below b such that $(a^2_{\leq n})^{\eta_0}$ is large in b. So assume that $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap a^{\eta_0}_{\leq n}) \neq \langle \alpha_0 + 1, 0, 0 \rangle$. Then $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|a \sqcap a^{\eta_0}_{\leq n}) = \langle \alpha_0 + 1, 0, 0 \rangle$, and there is $a^1_{\leq n}$, a modification of $a_{\leq n}$ below a such that $(a^1_{\leq n})^{\eta_0}$ is large in $a \sqcap c$. Similarly, there is $a^2_{\leq n}$, a modification of $a^1_{\leq n}$ below c such that $(a^2_{\leq n})^{\eta_0}$ is large in $a \sqcap c$ and large in $b \sqcap c$.

Now, modifying $a^2_{\leq n}$ below a we obtain $a^3_{\leq n}$ such that $(a^3_{\leq n})^{\eta_0}$ is large in $a \sqcap c$ and $(a^3_{\leq n})^{\eta_0} > a \sqcap c'$. A modification of $a^3_{\leq n}$ below c gives $a^4_{\leq n}$ such that $(a^4_{\leq n})^{\eta_0}$ is large in $b \sqcap c$ and $(a^4_{\leq n})^{\eta_0} > a$.

In a similar manner one can find $b^1_{\leq n} \sim b_{\leq n}$ such that $(b^1_{\leq n})^{\eta_0}$ is large in b and $(b^1_{\leq n})^{\eta_0} > a$. Note that $b^1_{\leq n}$ is a modification of $a^4_{\leq n}$ below b. Therefore $a_{\leq n} \sim b_{\leq n}$.

Case 5. There is $m < \omega$ such that $a \in I_{m+1}(B) \setminus I_m(B)$ and $b \notin I_{m+1}(B)$.

Let $\mathrm{Inv}(B|b) = \langle \alpha, \beta, \gamma \rangle$. Our assumptions guarantee that

- $\alpha_0 \geq m+1$, $\alpha \geq m+2$,
- exactly one of β, γ is equal 0,
- if $\alpha < \omega$, then $\alpha_0 \geq \alpha$,
- if $(B|a)^{(m)}$ is finite, then $\gamma_0 \geq |At((B|a)^{(m)})|$,
- if $\alpha < \omega$ and $(B|b)^{(\alpha-1)}$ is finite, then $\gamma_0 \geq |At((B|b)^{(\alpha-1)})|$, and
- $\mathrm{Inv}(B|b) = \mathrm{Inv}(B|a \sqcup b)$.

There is $\eta_0 \in \{+,-\}^{n+1}$ such that $b^{\eta_0}_{\leq n}$ is large in b. Then $a \sqcap b^{\eta_0}_{\leq n}$ is not large in $(a \sqcup b) \sqcap b^{\eta_0}_{\leq n}$ and

$$\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap b^{\eta_0}_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a \sqcup b) \sqcap b^{\eta_0}_{\leq n}).$$

Since $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a\sqcup b)\sqcap a^{\eta_0}_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a\sqcup b)\sqcap b^{\eta_0}_{\leq n})$, $(a\sqcup b)\sqcap a^{\eta_0}_{\leq n} \notin I_{m+1}(B)$ and $a\sqcap a^{\eta_0}_{\leq n}$ is not large in $(a\sqcup b)\sqcap a^{\eta_0}_{\leq n}$. Hence $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b\sqcap a^{\eta_0}_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|(a\sqcup b)\sqcap a^{\eta_0}_{\leq n})$ and $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b\sqcap a^{\eta_0}_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b\sqcap b^{\eta_0}_{\leq n})$. There is $a^1_{\leq n}$, a modification of $a_{\leq n}$ below b, such that $b\sqcap c\sqcap (a^1_{\leq n})^{\eta_0} \notin I_{m+1}(B)$.

Now, we are ready to show that there is $c_{\leq n} \sim a^1_{\leq n}$ such that $c^{\eta_0}_{\leq n} > a$ and $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b\sqcap c^{\eta_0}_{\leq n}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap (a^1_{\leq n})^{\eta_0})$. Obviously, if $(a^1_{\leq n})^{\eta_0} > a$, then we can take $c_{\leq n} := a^1_{\leq n}$. If for every $\eta_1 \neq \eta_0$, $(a^1_{\leq n})^{\eta_1}$ is not large in a, then $c_{\leq n}$ exists by (a). So assume that there is $\eta_1 \neq \eta_0$ such that $(a^1_{\leq n})^{\eta_1}$ is large in a. Let $a^2_{\leq n}$ be a modification of $a^1_{\leq n}$ below a such that $(a^2_{\leq n})^{\eta_1}$ is large in $a \sqcap c$.

Case 5a: $(B|a)^{(m)}$ is atomless.

By (a) without loss of generality we can assume that

$$\mathrm{Inv}(B|a \sqcap (a^2_{\leq n})^{\eta}) = \mathrm{Inv}(B|a) = \mathrm{Inv}(B|a \sqcap c) = \langle m+1, 1, 0 \rangle \text{ for every } \eta \in \{+,-\}^{n+1}$$

with $a \sqcap (a^2_{\leq n})^{\eta} > 0_B$.

Let $d = a \sqcap c \sqcap (a^2_{\leq n})^{\eta_1}$ and let $e \in (0_B, b \sqcap c \sqcap (a^2_{\leq n})^{\eta_0})$ be an element such that $\mathrm{Inv}(B|e) = \langle m+1, 1, 0 \rangle$. Define $a^3_{\leq n}$, a modification of $a^2_{\leq n}$ below c by the following conditions:

- $(d \sqcup e)' \sqcap a^3_{\leq n} = (d \sqcup e)' \sqcap a^2_{\leq n}$,
- $(d \sqcup e) \sqcap (a^3_{\leq n})^{\eta_0} = d$, and
- $(d \sqcup e) \sqcap (a^3_{\leq n})^{\eta_1} = e$.

There is $a^4_{\leq n}$, a modification of $a^3_{\leq n}$ below a such that $(a^3_{\leq n})^{\eta_0} > a \sqcap c'$ and $(a^3_{\leq n})^{\eta_0}$ is large in $a \sqcap c$. Repeating our previous argument we find $a^5_{\leq n}$, a modification of $a^4_{\leq n}$ below c such that $(a^5_{\leq n})^{\eta_0} > a$ and $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap (a^5_{\leq n})^{\eta_0}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|b \sqcap (a^1_{\leq n})^{\eta_0})$. So $c_{\leq n} := a^5_{\leq n}$ satisfies our demands.

Case 5b: $(B|a)^{(m)}$ is atomic.

Choose $d \in (0_B, a \sqcap c \sqcap (a^2_{\leq n})^{\eta_1}]$ and $e \in (0_B, b \sqcap c \sqcap (a^2_{\leq n})^{\eta_0})$ such that $\mathrm{Inv}(B|d) = \mathrm{Inv}(B|e) = \langle m+1, 0, 1\rangle$. Define $a^3_{\leq n}$, a modification of $a^2_{\leq n}$ below c by the following conditions:

- $(d \sqcup e)' \sqcap a^3_{\leq n} = (d \sqcup e)' \sqcap a^2_{\leq n}$,
- $(d \sqcup e) \sqcap (a^3_{\leq n})^{\eta_0} = d$, and
- $(d \sqcup e) \sqcap (a^3_{\leq n})^{\eta_1} = e$.

If $(B|a)^{(m)}$ is finite, then, repeating this procedure, one can find $a^4_{\leq n} \sim a^3_{\leq n}$ such that $((a^4_{\leq n})^{\eta_0})'$ is not large in a. Then, by (a) there is $c_{\leq n}$ satisfying our demands.

If $(B|a)^{(m)}$ is infinite, then again, repeating the procedure, one can find $a^4_{\leq n} \sim a^3_{\leq n}$ such that $(B|a \sqcap (a^4_{\leq n})^{\eta_0})^{(m)}$ has at least γ_0 atoms. There is $a^5_{\leq n}$, a modification of $a^4_{\leq n}$ below a such that the Boolean algebra $(B|a \sqcap ((a^4_{\leq n})^{\eta_0})')^{(m)}$ is finite. As previously, there are: $a^6_{\leq n} \sim a^5_{\leq n}$ such that $((a^6_{\leq n})^{\eta_0})'$ is not large in a, and (by (a)) there is $c_{\leq n} \sim a^6_{\leq n}$ satisfying our demands.

By arguments similar to those applied above, we can find $b^1_{\leq n} \sim b_{\leq n}$ such that $(b^1_{\leq n})^{\eta_0}$ is large in b and $(b^1_{\leq n})^{\eta_0} > a$. It is clear that $b^1_{\leq n}$ is a modification of $a^6_{\leq n}$ below b. So $b_{\leq n} \sim a_{\leq n}$. ■

3 The main theorem

In [We3], given an infinite Boolean algebra B with $|B/I(B)| \leq 2$ and a consistent formula $\varphi(\overline{x}, \overline{a}) \in L_{BA}(B)$, we demonstrated how to find a formula $\psi(\overline{x}, z_{\leq m}) \in L_{BA}$ for which $\{d_{\leq m} : \varphi(B, \overline{a}) = \psi(B, d_{\leq m})\}$ is a non-empty and finite subset of $\Pi_m(1_B)$. In the proofs of Lemmas 2.3, 3.3, 4.3 and 4.5 in [We3], we obtained the mentioned result for Boolean algebras with elementary invariants $\langle 1,1,n\rangle$ $(n < \omega)$, $\langle 1,0,\omega\rangle$, $\langle 1,1,\omega\rangle$ and $\langle 2,0,1\rangle$ respectively. In this section we generalize our case-by-case analysis to arbitrary Boolean algebras. The following lemmas will be used in the proof of the main theorem.

Lemma 3.1 *Assume that B is a non-trivial Boolean algebra and $d_{\leq m}, e_{\leq m}$ are partitions of 1_B. Then the following conditions are equivalent.*

(a) There is a unique permutation σ of the set $\{0, \ldots, m\}$ such that

$$(\forall i \leq m)(\forall d \in d_{\leq m} e_{\leq m})(d_{\sigma(i)} + e_i \textit{ is not large in } d).$$

(b) (a) without uniqueness of σ.

(c) For every $i \leq m$, e_i is large in at most one element from $d_{\leq m}$ and d_i is large in at most one element from $e_{\leq m}$.

Proof. For $m = 0$ the lemma holds trivially, so assume that $m \geq 1$ and suppose that the lemma holds in all Boolean algebras for tuples of length m.

(b)$\Longrightarrow$(c). Suppose for a contradiction that there are $i, j, l \leq m$, $j \neq l$ such that e_i is large both in d_j and in d_l. Let σ be a permutation of the set $\{0, \ldots, m\}$. If $\sigma(i) = j$, then $d_{\sigma(i)} + e_i = d_j + e_i \geq d_l \sqcap e_i$, so $d_{\sigma(i)} + e_i$ is large in d_l. Similarly, if $\sigma(i) = l$, then $d_{\sigma(i)} + e_i$ is large in d_j. If $\sigma(i) \notin \{j, l\}$, then $d_{\sigma(i)} + e_i \geq (d_j \sqcup d_l) \sqcap e_i$ and $d_{\sigma(i)} + e_i$ is large both in d_j and d_l.

(c)$\Longrightarrow$(a) Fix a Boolean algebra B and $d_{\leq m}, e_{\leq m} \subseteq B$, partitions of 1_B for which (c) holds. Choose $i_0 \leq m$ such that e_{i_0} is large in 1_B. There is a unique $j_0 \leq m$ such that d_{j_0} is large in e_{i_0}. Then e_{i_0} is the only element of the partition $e_{\leq m}$ which is large in d_{j_0}. Thus $d_{j_0} + e_{i_0}$ is not large in d_{j_0} nor in e_{i_0}. (c) guarantees that d_{j_0} is not large in e_i for $i \neq i_0$. Note that $(d_{j_0} + e_{i_0}) \sqcap e_i = d_{j_0} \sqcap e_i$ for $i \neq i_0$. Consequently, $d_{j_0} + e_{i_0}$ is not large in e_i for $i \neq i_0$. Similarly we prove that $d_{j_0} + e_{i_0}$ is not large in d_j for $j \neq j_0$. In this way have shown that if $d \in d_{\leq m} e_{\leq m}$, then $d_{j_0} + e_{i_0}$ is not large in d.

Now, for $i, j \leq m$, $i \neq i_0$, $j \neq j_0$ we define:

$$d^1_j = (d_{j_0} \sqcup e_{i_0})' \sqcap d_j \text{ and } e^1_i = (d_{j_0} \sqcup e_{i_0})' \sqcap e_i.$$

The tuples $\overline{d} := \langle d^1_i : i \leq m, i \neq i_0 \rangle$ and $\overline{e} := \langle e^1_j : j \leq m, j \neq j_0 \rangle$ are partitions of $(d_{j_0} \sqcup e_{i_0})'$. If $i \neq i_0$ and $j \neq j_0$, then

$$d^1_j \sqcap e^1_i = (d_{j_0} \sqcup e_{i_0})' \sqcap d_j \sqcap e_i = d_j \sqcap e_i.$$

So for any $i, j \leq m$, $i \neq i_0$, $j \neq j_0$,

$$d^1_j \text{ is large in } e^1_i \text{ iff } d_j \text{ is large in } e_i,$$

and similarly

$$e^1_i \text{ is large in } d^1_j \text{ iff } e_i \text{ is large in } d_j.$$

This implies that

- for every $i \leq m$, $i \neq i_0$, e^1_i is large in at most one element from $\langle d^1_j : j \leq m, j \neq j_0 \rangle$, and
- for every $j \leq m$, $j \neq j_0$, d^1_j is large in at most one element from $\langle e^1_i : i \leq m, i \neq i_0 \rangle$.

Therefore, by the inductive hypothesis applied to the Boolean algebra $B|(d_{j_0} \sqcup e_{i_0})'$, there is a bijection

$$\tau : \{i \leq m : i \neq i_0\} \longrightarrow \{j \leq m : j \neq j_0\}$$

such that $d^1_{\tau(i)} + e^1_i$ is not large in c whenever $i \leq m$, $i \neq i_0$ and $c \in \overline{d}\overline{e}$.

We know that if $i \leq m$ and $i \neq i_0$, then none of the elements d_{j_0}, $d^1_{\tau(i)} + e^1_i$ is large in e_i, in which case

$$(d_{\tau(i)} + e_i) \sqcap e_i = ((d^1_{\tau(i)} + e^1_i) \sqcap e_i) \sqcup (d_{j_0} \sqcap e_i).$$

So $d_{\tau(i)} + e_i$ is not large in e_i.

We also know that d'_{j_0} is not large in e_{i_0}. Moreover, if $i \leq m$ and $i \neq i_0$, then

$$(d_{\tau(i)} + e_i) \sqcap e_{i_0} = d_{\tau(i)} \sqcap e_{i_0} \leq d'_{j_0} \sqcap e_{i_0}.$$

Hence, $d_{\tau(i)} + e_i$ is not large in e_{i_0}.

If i, l are distinct numbers from $\{0, \ldots, m\} \setminus \{i_0\}$, then

$$(d_{\tau(i)} + e_i) \sqcap e_l = (d^1_{\tau(i)} + e^1_i) \sqcap e_l.$$

But $d^1_{\tau(i)} + e^1_i$ is not large in e_l, so $d_{\tau(i)} + e_i$ is not large in e_l.

In this way we have shown that $d_{\tau(i)} + e_i$ is not large in e_l whenever $i, l \le m$ and $i \ne i_0$. Similarly one can prove that $d_{\tau(i)} + e_i$ is not large in d_j for $i, j \le m$ and $j \ne j_0$. Let σ be a permutation of the set $\{0, \dots, m\}$ defined by the conditions:

$$\sigma(i_0) = j_0 \text{ and } \sigma(i) = \tau(i) \text{ for } i \in \{0, \dots, m\} \setminus \{i_0\}.$$

Our above arguments combined together show that $d_{\sigma(i)} + e_i$ is not large in d whenever $i \le m$ and $d \in d_{\le m} e_{\le m}$. For the uniqueness of σ, observe that if $j \ne \sigma(i)$, then e_i is not large in d_j, and $d_j + e_i$ is large in d_j. This finishes the proof. ■

Lemma 3.2 *Assume that B is a Boolean algebra and $d_{\le m}, e_{\le m} \subseteq B$ are partitions of 1_B such that at least one of the following conditions (a), (b) holds.*

(a) There are $i, j, l \le m$, $j \ne l$ such that e_i is large both in d_j and d_l.

(b) There are $i, j, l \le m$, $j \ne l$ such that d_i is large both in e_j and e_l.

Then at least one of the following conditions (a'), (b') holds.

(a') There are $i', j', l' \le m$, $j' \ne l'$ such that $e_{i'}$ is large both in $d_{j'}$ and $d_{l'}$, and $d_{j'} \sqcup d_{l'}$ is large in $e_{i'}$.

(b') There are $i', j', l' \le m$, $j' \ne l'$ such that $d_{i'}$ is large both in $e_{j'}$ and $e_{l'}$, and $e_{j'} \sqcup e_{l'}$ is large in $d_{i'}$.

Proof. Suppose for example that (a) holds and let $i_0 = i$. If $d_j \sqcup d_l$ is not large in e_{i_0}, then there is $j_0 \le m$, $j_0 \ne j, l$ such that d_{j_0} is large in e_{i_0}. If e_{i_0} is large in d_{j_0}, then (a') holds for $i' := i_0$, $j' := j$ and $l' := j_0$. Otherwise, there is $i_1 \le m$, $i_1 \ne i_0$ such that e_{i_1} is large in d_{j_0}. If d_{j_0} is large in e_{i_1}, then (b') holds for $i' := j_0$, $j' := i_0$ and $i' := i_1$. Continuing this construction, after finitely many steps we obtain $i', j', l' \le m$ for which (a') or (b') holds. ■

Theorem 3.3 *Assume that B is an infinite Boolean algebra, $\varphi(\overline{x}, \overline{y}) \in L_{BA}$, $\overline{x} = x_{\le n}$, $\overline{y} = y_{\le r}$, $\overline{a} = a_{\le r} \subseteq B$ and $\varphi(B, \overline{a}) \ne \emptyset$. There is an L_{BA}-formula $\psi(\overline{x}, z_{\le m})$ such that*

- *$\{d_{\le m} \in B^{m+1} : \varphi(B, \overline{a}) = \psi(B, d_{\le m})\}$ is a non-empty subset of $\Pi_m(1_B)$, and*
- *for any $d_{\le m}, e_{\le m} \subseteq B$, if $\psi(B, d_{\le m}) = \psi(B, e_{\le m}) = \varphi(B, \overline{a})$, then $d_{\le m}, e_{\le m} \in \Pi_m(1_B)$ and there is a unique permutation σ of the set $\{0, \dots, m\}$ such that for any $l \le m$ and $d \in d_{\le m} e_{\le m}$, the element $d_{\sigma(l)} + e_l$ is not large in d.*

Proof. Suppose that B, $\varphi(\overline{x}, \overline{y})$ and $\overline{a} = a_{\le r}$ satisfy our assumptions. Without loss of generality we can assume that $a_{\le r}$ is a partition of 1_B whose all elements are simple. By Lemma 0.2, we can also assume that $\varphi(\overline{x}, \overline{a})$ is a finite Boolean combination of formulas of the form

$$\varepsilon_j(a_i \sqcap x^\eta_{\le n}), \psi_{j,s}(a_i \sqcap x^\eta_{\le n}) \text{ and } \mathrm{at}_j(a_i \sqcap x^\eta_{\le n}),$$

where $i \le r$, $\eta \in \{+, -\}^{n+1}$, $j < \omega$ and $1 \le s < \omega$. For every $i \le r$, denote by $\langle \alpha^i, \beta^i, \gamma^i \rangle$ the elementary invariant of the Boolean algebra $B|a_i$. Fix positive integers α_0 and γ_0 so that the following conditions are satisfied.

- If $j > \alpha_0$, then the formulas $\varepsilon_j, \mathrm{at}_j$ do not occur in $\varphi(\overline{x}, \overline{a})$.
- If $j > \alpha_0$ or $s > \gamma_0$, then $\psi_{j,s}$ does not occur in $\varphi(\overline{x}, \overline{a})$.
- For every $i \le r$, if $\alpha^i < \omega$, then $\alpha_0 \ge \alpha^i$ and $\gamma_0 \ge \sum_{\{s : \alpha^s = \alpha^i, \gamma^s < \omega\}} \gamma^i$.

Define the following formula:

$$\theta(z,\overline{y}) = \forall u_{\le n}\forall v_{\le n}\left[E_{n,\alpha_0,\gamma_0}(u_{\le n}, v_{\le n}, z) \longrightarrow (\varphi(u_{\le n},\overline{y}) \longleftrightarrow \varphi(v_{\le n},\overline{y}))\right].$$

Of course, by our choice of α_0 and γ_0, $B \models \theta(a_i, \overline{a})$ for every $i \le r$. Fix $c_{\le m}$, a partition of 1_B, satisfying the following conditions:

- if $i \le r$, then $a_i \le c_l$ for some $l \le m$,
- $B \models \theta(c_l, \overline{a})$ for every l,
- for every $l \le m$, c_l is simple,
- if $l_1 < l_2 \le m$ and $B \models \theta(c_{l_1} \sqcup c_{l_2}, \overline{a})$, then $c_{l_1} \sqcup c_{l_2}$ is not simple.

For every tuple $\overline{d} \in B^{n+1}$ we define a function

$$f_{\overline{d}} : \{+,-\}^{n+1} \times \{0,\ldots,m\} \longrightarrow \mathrm{INV}_{\alpha_0,\gamma_0}$$

as follows:

$$f_{\overline{d}}(\eta, l) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|a_l \sqcap d^{\eta}_{\le n}),$$

where $\eta \in \{+,-\}^{n+1}$ and $l \le m$. Let

$$\{f_0,\ldots,f_t\} = \{f_{\overline{d}} : \overline{d} \in B^{n+1} \text{ and } B \models \varphi(\overline{d},\overline{a})\},$$

and for every $i \le t$ define the following L_{BA}-formula:

$$\varrho^i(x_{\le n}, z_{\le m}) = \bigwedge_{\eta\in\{+,-\}^{n+1}} \bigwedge_{l\le m} \sigma^{\alpha_0,\gamma_0}_{f_i(\eta,l)}(z_l \sqcap x^{\eta}_{\le n}).$$

It is easy to see that

$$B \models \varrho^i(\overline{d}, c_{\le m}) \text{ iff } f_{\overline{d}} = f_i$$

for every $i \le t$ and $\overline{d} \in B^{n+1}$. Let

$$\varrho(x_{\le n}, z_{\le m}) = \bigvee_{i\le t} \varrho^i(x_{\le m}, z_{\le m}).$$

Claim 1. $\varrho(B, c_{\le m}) = \varphi(B, \overline{a})$.

Proof of Claim 1. Fix a tuple $\overline{d} = d_{\le n} \in B^{n+1}$ and suppose that $B \models \varphi(\overline{d},\overline{a})$. Then $f_{\overline{d}} = f_i$ for some $i \le t$. For every $\eta \in \{+,-\}^{n+1}$, $l \le m$ and $\langle\alpha,\beta,\gamma\rangle \in \mathrm{INV}_{\alpha_0,\gamma_0}$ we have that

$$f_i(\eta,l) = f_{\overline{d}}(\eta,l) = \langle\alpha,\beta,\gamma\rangle \Longleftrightarrow B \models \sigma^{\alpha_0,\gamma_0}_{\langle\alpha,\beta,\gamma\rangle}(c_l \sqcap d^{\eta}_{\le m}).$$

This means that

$$B \models \sigma^{\alpha_0,\gamma_0}_{f_i(\eta,l)}(c_l \sqcap d^{\eta}_{\le m})$$

for every $\eta \in \{+,-\}^n$ and $l \le m$. Hence $B \models \varrho^i(\overline{d}, c_{\le m})$ and $B \models \varrho(\overline{d}, c_{\le m})$.

In this way we have shown that $\varphi(B,\overline{a}) \subseteq \varrho(B,c_{\leq m})$. To prove the reverse inclusion, suppose that $B \models \varrho(\overline{d},c_{\leq m})$. Then $B \models \varrho^i(d_{\leq n},c_{\leq m})$ for some $i \leq t$, which means that $f_{\overline{d}} = f_i$. Choose a tuple $\overline{e} \in B^{n+1}$ such that $f_i = f_{\overline{e}}$ and $B \models \varphi(\overline{e},\overline{a})$. The equality $f_{\overline{d}} = f_{\overline{e}}$ implies that

$$B \models \bigwedge_{l\leq m} \bigwedge_{\eta\in\{+,-\}^{n+1}} \bigwedge_{\langle\alpha,\beta,\gamma\rangle\in \mathrm{INV}_{\alpha_0,\gamma_0}} \left(\sigma^{\alpha_0,\gamma_0}_{\langle\alpha,\beta,\gamma\rangle}(c_l \sqcap \overline{d}^{\eta}) \longleftrightarrow \sigma^{\alpha_0,\gamma_0}_{\langle\alpha,\beta,\gamma\rangle}(c_l \sqcap \overline{e}^{\eta})\right).$$

Define the tuples $\overline{d}_0,\ldots,\overline{d}_{m+1}$ by the following conditions:

- $\overline{d}_0 = \overline{d}$,
- $c_l \sqcap \overline{d}_{l+1} = c_l \sqcap \overline{e}$ for every $l \leq m$ and
- $c_l' \sqcap \overline{d}_{l+1} = c_l' \sqcap \overline{d}_l$ for every $l \leq m$.

It is clear that $\overline{d}_{m+1} = \overline{e}$ and $B \models E_{n,\alpha_0,\gamma_0}(\overline{d}_l,\overline{d}_{l+1},c_l)$ for every $l \leq m$. Hence $B \models \varphi(\overline{d}_l,\overline{a}) \longleftrightarrow \varphi(\overline{d}_{l+1},\overline{a})$ for $l \leq m$. Since $B \models \varphi(\overline{d}_{m+1},\overline{a})$, we infer that $B \models \varphi(\overline{d},\overline{a})$, which finishes the proof of Claim 1.

Let

$$\theta'(z,z_{\leq m}) = \forall u_{\leq n}\forall v_{\leq n}\left[E_{n,\alpha_0,\gamma_0}(u_{\leq n},v_{\leq n},z) \longrightarrow (\varrho(u_{\leq n},z_{\leq m}) \longleftrightarrow \varrho(v_{\leq n},z_{\leq m}))\right].$$

It is evident from Claim 1 that $\theta'(B,c_{\leq m}) = \theta(B,\overline{a})$. Put

$$\begin{aligned}\psi(\overline{x},z_{\leq m}) \;=\; & \varrho(\overline{x},z_{\leq m}) \wedge z_{\leq m} \in \Pi_m(1) \wedge \bigwedge_{l\leq m} \theta'(z_l,z_{\leq m}) \wedge \\ & \bigwedge_{\{l_1<l_2\leq m: c_{l_1}\sqcup c_{l_2} \text{ is simple }\}} \neg\theta'(z_{l_1}\sqcup z_{l_2},z_{\leq m}) \wedge \bigwedge_{l\leq m} \sigma^{\alpha_0,\gamma_0}_{\mathrm{Inv}_{\alpha_0,\gamma_0}(B|c_l)}(z_l).\end{aligned}$$

Obviously, $\psi(B,c_{\leq m}) = \varrho(B,c_{\leq m}) = \varphi(B,\overline{a})$. The definition of $\psi(\overline{x},z_{\leq m})$ assures that for every $d_{\leq m} \in B^{m+1}$, if $\varphi(B,\overline{a}) = \psi(B,d_{\leq m})$, then $d_{\leq m} \in \Pi_m(1_B)$.

Claim 2. If $d_{\leq m} \in B^{m+1}$ and $\varphi(B,\overline{a}) = \psi(B,d_{\leq m})$, then each element of the tuple $d_{\leq m}$ is simple.

Proof of claim 2. The claim is obvious if $\mathrm{Inv}(B) \neq \langle\omega,0,0\rangle$, because then $\alpha_0 \geq \max(\alpha^0,\ldots,\alpha^m)$ and $\mathrm{Inv}_{\alpha_0,\gamma_0}(c_{\leq m}) = \mathrm{Inv}_{\alpha_0,\gamma_0}(d_{\leq m})$. So assume that $\mathrm{Inv}(B) = \langle\omega,0,0\rangle$ and let

$$\begin{aligned} J &= \{i \leq m : \mathrm{Inv}(B|c_i) = \langle\omega,0,0\rangle\},\\ J' &= \{i \leq m : \mathrm{Inv}(B|d_i) = \langle\omega,0,0\rangle\}.\end{aligned}$$

Again, our assumptions guarantee that if $j \notin J$, then $\alpha_0 \geq \alpha^j$ and $\mathrm{Inv}_{\alpha_0,\gamma_0}(B|c_j) = \mathrm{Inv}_{\alpha_0,\gamma_0}(B|d_j)$ whenever $j \notin J$. This implies that if $j \notin J$, then d_j is simple and $\mathrm{Inv}(B|d_j) \neq \langle\omega,0,0\rangle$, so $J' \subseteq J$. To finish the proof of the claim, we only have to show that $J' = J$. This is clear if $|J| = 1$. Suppose that $|J| \geq 2$ and $J' \subsetneq J$. Then there are $i \in J'$ and $j,l \in J$, $j \neq l$ such that d_i is large in c_j and in c_l. The element $c_j \sqcup c_l$ is simple and $B \models \neg\theta(c_j \sqcup c_l,\overline{a})$. Define an equivalence relation $\sim$ on B^{n+1} as follows:

$$g_{\leq n} \sim h_{\leq n} \iff B \models \varphi(g_{\leq n},\overline{a}) \longleftrightarrow \varphi(h_{\leq n},\overline{a}).$$

Since $B \models \theta(c_j, \overline{a}) \wedge \theta(c_l, \overline{a})$, $\sim$ contains $E^{c_j}_{n,\alpha_0,\gamma_0} \cup E^{c_l}_{n,\alpha_0,\gamma_0}$. We know that $\psi(B, d_{\leq m}) = \varphi(B, \overline{a})$, so $\theta'(B, d_{\leq m}) = \theta(B, \overline{a})$. Since $B \models \theta'(d_i, d_{\leq m})$, we have that $B \models \theta(d_i, \overline{a})$, and $\sim$ contains $E^{d_i}_{n,\alpha_0,\gamma_0}$. Hence $\sim$ contains $E^{d_i \sqcap (c_j \sqcup c_l)}_{n,\alpha_0,\gamma_0}$. By Lemma 2.8, $\sim$ contains $E^{c_j \sqcup c_l}_{\alpha_0,\gamma_0}$, which means that $B \models \theta(c_j \sqcup c_l, \overline{a})$, a contradiction.

Now assume that $d_{\leq m}, e_{\leq m} \in B^{n+1}$ and $\varphi(B, \overline{a}) = \psi(B, d_{\leq m}) = \psi(B, e_{\leq m})$. Then the tuples $d_{\leq m}, e_{\leq m}$ are partitions of 1_B into simple elements. By Lemma 3.1, we will be done if we prove the following claim.

Claim 3. There is a permutation σ of the set $\{0, \ldots, m\}$ such that for any $i \leq m$ and $d \in d_{\leq m}e_{\leq m}$, $d_{\sigma(i)} + c_i$ is not large in d.

Proof of Claim 3. The claim is trivial for $m = 0$, so let $m \geq 1$ and suppose for a contradiction that the claim does not hold for m. Then, by Lemma 3.1, there are $i, j, l \leq m$, $j \neq l$ such that e_i is large in d_j and in d_l, or d_i is large in e_j and in e_l. By Lemma 3.2, without loss of generality we can assume that e_i is large in d_j and in d_l, and $d_j \sqcup d_l$ is large in e_i. Since d_j, d_l, e_i are all simple, by Lemma 2.7, also $d_j \sqcup d_l$ is simple. As in the proof of Claim 2 we have that

$$B \models \theta(d_j, \overline{a}) \wedge \theta(d_l, \overline{a}) \wedge \theta(e_i \sqcap (d_j \sqcup d_l), \overline{a}) \wedge \neg\theta(d_j \sqcup d_l, \overline{a}).$$

Let $\sim$ be the equivalence relation on B^{n+1} defined as in the proof of Claim 2. The relation $\sim$ contains $E^{d_j}_{n,\alpha_0,\gamma_0} \cup E^{d_l}_{\alpha_0,\gamma_0} \cup E^{(d_j \sqcup d_l) \sqcap e_i}_{\alpha_0,\gamma_0}$. By Theorem 2.8 for $a := d_j$, $b := d_l$ and $c := (d_j \sqcup d_l) \sqcap e_i$, $\sim$ contains also $E^{d_j \sqcup d_l}_{n,\alpha_0,\gamma_0}$, which means that $B \models \theta(d_j \sqcup d_l, \overline{a})$, a contradiction.

This finishes the proof of the theorem. ■

Proposition 3.4 *Assume that B is an infinite Boolean algebra and $n \in \mathbb{N}_+$. Then the following conditions are equivalent.*

(a) $|B/I_n(B)| \leq 2$.

(b) If $r \in \mathbb{N}_+$ and $X \subseteq B^r$ is a non-empty definable set, then there is an L_{BA}-formula $\psi(\overline{x}, z_{\leq m})$ such that

- *$\{d_{\leq m} \in B^{m+1} : X = \psi(B, d_{\leq m})\}$ is a nonempty subset of $\Pi_m(1_B)$, and*
- *for any $d_{\leq m}, e_{\leq m} \in B^{m+1}$, if $X = \psi(B, d_{\leq m}) = \psi(B, e_{\leq m})$, then there is a unique permutation σ of the set $\{0, \ldots, m\}$ such that $d_{\sigma(i)} + e_i \in I_{n-1}(B)$ and $d_{\sigma(i)} + e_i$ is not large in d whenever $i \leq m$ and $d \in d_{\leq m}e_{\leq m}$.*

Proof. (a)$\Longrightarrow$(b). Let B be an infinite Boolean algebra with $|B/I_n(B)| \leq 2$, $r \in \mathbb{N}_+$ and $X \subseteq B^r$ a non-empty definable set. Fix an L_{BA}-formula $\psi(\overline{x}, z_{\leq m})$ satisfying the assertion of Theorem 3.3 and partitions $d_{\leq m}, e_{\leq m}$ of 1_B for which $X = \psi(B, d_{\leq m}) = \psi(B, e_{\leq m})$. Denote by σ the unique permutation of the set $\{0, \ldots, m\}$ such that for any $i \leq m$ and $d \in d_{\leq m}e_{\leq m}$, the element $d_{\sigma(i)} + e_i$ is not large in d. Below we consider two cases.

Case 1. $B = I_n(B)$. For any $i, j \leq m$, $(d_{\sigma(i)} + e_i) \sqcap d_j \in I_{n-1}(B)$. So $d_{\sigma(i)} + e_i \in I_{n-1}(B)$.

Case 2. $|B/I_n(B)| = 2$. In this case there are a unique $i_0 \leq m$ such that $e_{i_0} \notin I_n(B)$, and a unique $j_0 \leq m$ such that $d_{j_0} \notin I_n(B)$. We claim that $\sigma(i_0) = j_0$. Suppose for a contradiction that this is not true. Then $(d_{\sigma(i_0)} + e_{i_0}) \sqcap d_{j_0} \notin I_n(B)$, which means that $d_{\sigma(i_0)} + e_{i_0}$ is large in d_{j_0} contradicting Theorem 3.3. So $\sigma(i_0) = j_0$.

The element $d_{j_0} + e_{i_0}$ is not large in e_i for $i \leq m$. Thus, if $i \neq i_0$, then $(d_{j_0} + e_{i_0}) \sqcap e_i \in I_{n-1}(B)$. This implies that $d_{j_0} \sqcap e'_{i_0} = (d_{j_0} + e_{i_0}) \sqcap e'_{i_0} \in I_{n-1}(B)$. A similar argument shows that $d'_{j_0} \sqcap e_{i_0} \in I_{n-1}(B)$. Hence $d_{j_0} + e_{i_0} \in I_{n-1}(B)$.

Now, let $l \leq m$ and $l \neq i_0$. As previously, $(d_{\sigma(l)} + e_l) \sqcap e_i \in I_{n-1}(B)$ for $i \neq i_0$. Also

$$(d_{\sigma(l)} + e_l) \sqcap e_{i_0} = d'_{j_0} \sqcap d_{\sigma(l)} \sqcap e_{i_0} \leq d'_{j_0} \sqcap e_{i_0} \in I_{n-1}(B).$$

Hence $d_{\sigma(l)} + e_l \in I_{n-1}(B)$.

(b)$\Longrightarrow$(a). Let B be a Boolean algebra with $|B/I_n(B)| \geq 4$ and $a \in B$ an element for which $a + I_n(B)$ is not $\emptyset$-definable. We can assume that a satisfies the following additional conditions.

- In case $B/I_n(B)$ is atomic, $a + I_n(B)$ is an atom of $B/I_n(B)$.
- In case $B/I_n(B)$ is not atomic, $a + I_n(B)$ is atomless as an element of $B/I_n(B)$ and $a' + I_n(B)$ is not atomic as an element of $B/I_n(B)$.

Our choice of a guarantees that the elements $0_B, a, a', 1_B$ belong to distinct cosets of $I_n(B)$. In particular, $a, a' \notin I_n(B)$. Consider the following $\{a\}$-definable set:

$$X = \{c \in B : a + c \in I_n(B)\} = \{c \in B : B \models \varepsilon_n(a + c)\}.$$

Fix an L_{BA}-formula $\psi(\overline{x}, z_{\leq m})$ such that $\{e_{\leq m} \in B^{m+1} : X = \psi(B, e_{\leq m})\}$ is a nonempty subset of $\Pi_m(1_B)$ and a partition $d_{\leq m} \in \Pi_m(1_B)$ for which $X = \psi(B, d_{\leq m})$. Since X is not $\emptyset$-definable, $m \geq 1$. We will show how to find a partition $e_{\leq m} \in \Pi_m(1_B)$ such that $X = \psi(B, e_{\leq m})$ and for every permutation σ of the set $\{0, \ldots, m\}$, there are $i \leq m$ and $d \in d_{\leq m} e_{\leq m}$ such that $d_{\sigma(i)} + e_i$ is large in d or $d_{\sigma(i)} + e_i \notin I_{n-1}(B)$. We consider three cases.

Case 1. There is $i \leq m$ such that $d_i \sqcap a, d'_i \sqcap a \notin I_n(B)$ or $d_i \sqcap a', d'_i \sqcap a' \notin I_n(B)$.

Assume for example that the first alternative holds. Fix $i < j \leq m$ such that $d_i \sqcap a, d_j \sqcap a \notin I_n(B)$. There is $d \in (0_B, a \sqcap d_i)$ such that $d \in I_n(B) \setminus I_{n-1}(B)$. Then

$$\mathrm{Inv}(B|a \sqcap d_i \sqcap d') = \mathrm{Inv}(B|a \sqcap d_i) \text{ and } \mathrm{Inv}(B|a \sqcap (d_j \sqcup d)) = \mathrm{Inv}(B|a \sqcap d_j).$$

Define $e_i = d'_i \sqcap d'$, $e_j = d_j \sqcup d$ and $e_l = d_l$ for $l \notin \{i, j\}$. The tuple $e_{\leq m}$ is a partition of 1_B and $\mathrm{tp}(e_{\leq m}/a) = \mathrm{tp}(d_{\leq m}/a)$. Consequently, $X = \psi(B, e_{\leq m})$. Let σ be a permutation of the set $\{0, \ldots m\}$. If $\sigma(i) = i$, then

$$d_{\sigma(i)} + e_i = d_i + (d_i \sqcap d') = d \notin I_{n-1}(B).$$

If $\sigma(i) = j$, then

$$d_{\sigma(i)} + e_i = d_j + (d_i \sqcap d') > d_j, \text{ so } d_{\sigma(i)} + e_i \notin I_{n-1}(B).$$

Finally, if $\sigma(i) \notin \{i, j\}$, then

$$d_{\sigma(i)} + e_i = d_l + e_i > e_i, \text{ so } d_{\sigma(i)} + e_i \notin I_{n-1}(B).$$

Case 2. There is $i \leq m$ such that $d'_i \in I_n(B)$.

Fix $j \leq m, j \neq i$. Let $c = a \sqcup d_j$. Then $a + c = d_j \sqcap a' \in I_n(B)$ and the formula $\varepsilon_n(c + x)$ defines X. There is $d \in (0_B, (a \sqcap d_i) \sqcup d_j)$ such that $\mathrm{Inv}(B|d) = \mathrm{Inv}(B|d_j)$ and the elements $d \sqcap d'_j$

and $d' \sqcap d_j$ are both large in $d \sqcup d_j$. Define: $e_i = (d_i \sqcup d_j) \sqcap d'$, $e_j = d$ and $e_l = d_l$ for $l \notin \{i, j\}$. The tuple $e_{\leq m}$ is a partition of 1_B and $\mathrm{tp}(cd_{\leq m}) = \mathrm{tp}(ce_{\leq m})$. Consequently, $X = \psi(B, e_{\leq m})$. Let σ be a permutation of the set $\{0, \ldots, m\}$. If $\sigma(j) = j$, then $d_{\sigma(j)} + e_j = d_j + d > d' \sqcap d_j$, so $d_{\sigma(j)} + e_j$ is large in d_j. If $\sigma(j) = i$, then $d_{\sigma(j)} + e_j = d_i + d > d' \sqcap d_i$, so $d_{\sigma(j)} + e_j$ is large in d_i. If $\sigma(j) = l \notin \{i, j\}$, then $d_{\sigma(j)} + e_j = d_l + d > d_l$, so $d_{\sigma(j)} + e_j$ is large in d_l.

Case 3. There are $i < j \leq m$ such that $d_i + a \in I_n(B)$ and $d_j + a' \in I_n(B)$.

There is $d \in (0_B, a \sqcap d_i)$ such that $d \in I_n(B) \setminus I_{n-1}(B)$. Define $c = a \sqcap d'$, $e_i = d' \sqcap d_i$, $e_j = d_j \sqcup d$ and $e_l = d_l$ for $l \notin \{i, j\}$. Then $e_{\leq m}$ is a partition of 1_B and $\mathrm{tp}(ad_{\leq m}) = \mathrm{tp}(ce_{\leq m})$. Consequently, the set defined by the formula $\varepsilon_n(c + x)$ is equal to $\psi(B, e_{\leq m})$. Since $a + c \in I_n(B)$, the formula $\varepsilon_n(x + c)$ defines X and $X = \psi(B, e_{\leq m})$. Proceeding as in Case 1, one can show that if σ is a permutation of the set $\{0, \ldots, m\}$ and $l \leq n$, then $d_{\sigma(l)} + e_l \notin I_{n-1}(B)$. ■

Corollary 3.5 *[We3, Theorem 4.8] Assume that B is an infinite Boolean algebra. Then the following conditions are equivalent.*
(a) $|B/I(B)| \leq 2$.
(b) If $r \in \mathbb{N}_+$ and $X \subseteq B^r$ is a non-empty definable set, then there is an L_{BA}-formula $\psi(\overline{x}, z_{\leq m})$ such that

- *$\{d_{\leq m} \in B^{m+1} : X = \psi(B, d_{\leq m})\}$ is a nonempty subset of $\Pi_m(1_B)$, and*
- *for any $d_{\leq m}, e_{\leq m} \in B^{m+1}$, if $X = \psi(B, d_{\leq m}) = \psi(B, e_{\leq m})$, $d_{\leq m}$ is a permutation of $e_{\leq m}$.*

Proposition 3.6 *Assume that B is an infinite Boolean algebra with $\mathrm{Inv}(B) = \langle \omega, 0, 0 \rangle$, $r \in \mathbb{N}_+$ and $X \subseteq B^r$ is a definable set. Then there are a positive integer α and an L_{BA}-formula $\psi(\overline{x}, z_{\leq m})$ such that*

- *$\{d_{\leq m} \in B^{m+1} : X = \psi(B, d_{\leq m})\}$ is a nonempty subset of $\Pi_m(1_B)$, and*
- *for any $d_{\leq m}, e_{\leq m} \in B^{m+1}$, if $X = \psi(B, d_{\leq m}) = \psi(B, e_{\leq m})$, then there is a unique permutation σ of the set $\{0, \ldots, m\}$ such that for any $i \leq m$ and $d \in d_{\leq m}e_{\leq m}$, $d_{\sigma(i)} + e_i$ is not large in d and $d_{\sigma(i)} + e_i \in I_\alpha(B)$.*

Proof. Fix a nonempty set $X \subseteq B^r$ defined by $\varphi(\overline{x}, \overline{a}) \in L_{BA}(B)$. Let $\psi(\overline{x}, z_{\leq m})$ be the formula obtained as in proof of Theorem 3.3. We claim that $\psi(\overline{x}, z_{\leq m})$ satisfies the assertion of Proposition 3.6. Assume for a contradiction that for every positive integer α, there are partitions $d^\alpha_{\leq m}, e^\alpha_{\leq m} \in \Pi_m(1_B)$ such that

- $X = \psi(B, d^\alpha_{\leq m}) = \psi(B, e^\alpha_{\leq m})$ and
- for every permutation σ of the set $\{0, \ldots, m\}$, there are $i \leq m$ and $d \in d^\alpha_{\leq m}e^\alpha_{\leq m}$ such that $d^\alpha_{\sigma(i)} + e^\alpha_i \notin I_\alpha(B)$.

Then there are $B_1 \succ B$ and partitions $d_{\leq m}, e_{\leq m} \in \Pi_m(1_{B_1})$ such that $\psi(B_1, \overline{a}) = \psi(B_1, d_{\leq m}) = \psi(B_1, e_{\leq m})$ and for every permutation σ of $\{0, \ldots m\}$, there are $i \leq m$ and $d \in d_{\leq m}e_{\leq m}$ such that $\mathrm{Inv}(B|d \sqcap (d_{\sigma(i)} + e_i)) = \langle \omega, 0, 0 \rangle$. The latter means that $d_{\sigma(i)} + e_i$ is large in d, which contradicts Theorem 3.3. ■

4 Elimination of imaginaries

Assume that $\mathcal{M}$ is a multisorted structure for a multisorted language L and let $n \in \mathbb{N}_+$. For any finite collection $M_0, \ldots, M_n$ of sorts in $\mathcal{M}$, any $\emptyset$-definable set $X \subseteq M_0 \times \ldots \times M_n$ and any $\emptyset$-definable equivalence relation E on X, denote by S_E the set of all equivalence classes of E and by f_E the function from X onto S_E sending $\overline{a} \in X$ to $[\overline{a}]_E$. In case E is the equality on a sort N from $\mathcal{M}$, we identify N with N/E. The structure $\mathcal{M}$ together with all sorts S_E and functions f_E will be denoted by $\mathcal{M}^{eq}$. If $\mathcal{E}$ is a family of $\emptyset$-definable equivalence relations on $\emptyset$-definable subsets of products of finite collections of sorts in $\mathcal{M}$, then $\mathcal{M}(\mathcal{E})$ denotes the multisorted structure $\mathcal{M}$ together with all sorts S_E and functions f_E, where $E \in \mathcal{E}$. The multisorted language of $\mathcal{M}(\mathcal{E})$ will be denoted by $L(\mathcal{E})$.

We say that a multisorted structure $\mathcal{M}$ admits elimination of imaginaries iff for any finite collection $M_0, \ldots, M_n$ of sorts in $\mathcal{M}$, any $\emptyset$-definable set $X \subseteq M_0 \times \ldots \times M_n$ and any $\emptyset$-definable equivalence relation E on X, there is a $\emptyset$-definable function f from X into a product of some finite collection of sorts in $\mathcal{M}$ such that for any $\overline{a}, \overline{b} \in X$, $\mathcal{M} \models E(\overline{a}, \overline{b})$ iff $f(\overline{a}) = f(\overline{b})$. We say that a complete theory for a multisorted language admits elimination of imaginaries iff every model of T does. Clearly, for any multisorted structure $\mathcal{M}$, if $Th(\mathcal{M})$ admits elimination of imaginaries, then also $\mathcal{M}$ does.

Proposition 4.1 *Assume that $\mathcal{M}$ is a multisorted L-structure and $\mathcal{E}$ is a family of $\emptyset$-definable equivalence relations on $\emptyset$-definable subsets of products of finite collections of sorts in $\mathcal{M}$. Then the following conditions are equivalent.*

(a) $\mathcal{M}(\mathcal{E})$ admits elimination of imaginaries.

(b) If $M_0, \ldots, M_n$ is a collection of sorts in $\mathcal{M}$, $X \subseteq M_0 \times \ldots \times M_n$ is a $\emptyset$-definable (in $\mathcal{M}$) set and E is a $\emptyset$-definable (in $\mathcal{M}$) equivalence relation on X, then there exist a collection of sorts $N_0, \ldots, N_k$ in $\mathcal{M}(\mathcal{E})$ and a $\emptyset$-definable (in $\mathcal{M}(\mathcal{E})$) function $f : X \longrightarrow N_0 \times \ldots \times N_k$ such that for any $\overline{a}, \overline{b} \in X$,

$$\mathcal{M} \models E(\overline{a}, \overline{b}) \text{ iff } f(\overline{a}) = f(\overline{b}).$$

Proof. The implication from (a) to (b) is trivial. To prove that (b) implies (a), assume that $\mathcal{E}$ is a family of $\emptyset$-definable (in $\mathcal{M}$) equivalence relations on $\emptyset$-definable (in $\mathcal{M}$) subsets of products of finite collections of sorts in $\mathcal{M}$. Fix a finite collection $M_0, \ldots, M_n$ of sorts in $\mathcal{M}(\mathcal{E})$, a $\emptyset$-definable (in $\mathcal{M}(\mathcal{E})$) set $X \subseteq M_0 \times \ldots \times M_n$ and a parameter-free formula $E(x_0, \ldots, x_n, y_0, \ldots, y_n) \in L(\mathcal{E})$ defining an equivalence relation E on X. Then, for every $i \leq n$, we can find a finite collection $M_{i,0}, \ldots, M_{i,l_i}$ of sorts in $\mathcal{M}$, a $\emptyset$-definable (in $\mathcal{M}$) set $X_i \subseteq M_{i,0} \times \ldots \times M_{i,l_i}$ and a $\emptyset$-definable (in $\mathcal{M}$) equivalence relation E_i on X_i such that $M_i = X_i/E_i = f_{E_i}[X_i]$, where $f_{E_i}(\overline{x}) = [\overline{x}]_{E_i}$ whenever $\overline{x} \in X_i$. Let

$$Z = \{\overline{a}_0 \ldots \overline{a}_n \in X_0 \times \ldots \times X_n : \langle f_{E_0}(\overline{a}_0), \ldots, f_{E_n}(\overline{a}_n) \rangle \in X\}.$$

Clearly, the set Z is $\emptyset$-definable in $\mathcal{M}$. There is an L-formula $\psi(\overline{u}_0, \ldots, \overline{u}_n, \overline{v}_0, \ldots, \overline{v}_n)$ such that for every $i \leq n$, $|\overline{u}_i| = |\overline{v}_i| = l_i + 1$ and for any $\overline{a}_{\leq n}, \overline{b}_{\leq n} \in Z$,

$$\mathcal{M} \models \psi(\overline{a}_{\leq n}, \overline{b}_{\leq n}) \text{ iff } \mathcal{M}(\mathcal{E}) \models E(f_{E_0}(\overline{a}_0), \ldots, f_{E_n}(\overline{a}_n), f_{E_0}(\overline{b}_0), \ldots, f_{E_n}(\overline{b}_n)).$$

The formula $\psi(\overline{u}_{\leq n}, \overline{v}_{\leq n})$ defines an equivalence relation F on Z. By (b) there is a collection $N_0, \ldots, N_k$ of sorts in $\mathcal{M}(\mathcal{E})$ and a $\emptyset$-definable (in $\mathcal{M}(\mathcal{E})$) function

$$g : Z \longrightarrow N_0 \times \ldots \times N_k$$

such that for any $\overline{a}, \overline{b} \in Z$, $\mathcal{M} \models \psi(\overline{a}, \overline{b})$ iff $g(\overline{a}) = g(\overline{b})$.

Now, define a function $f : X \longrightarrow N_0 \times \ldots \times N_k$ by the following condition: $f(\overline{a}) = \overline{b}$ iff there is $\overline{u}_{\leq n} \in X_0 \times \ldots \times X_n$ such that $\overline{a} = \langle f_{E_0}(\overline{u}_0), \ldots, f_{E_n}(\overline{u}_n) \rangle$ and $g(\overline{u}_{\leq n}) = \overline{b}$. Note that f is $\emptyset$-definable in $\mathcal{M}(\mathcal{E})$ and for any $\overline{a}_1, \overline{a}_2 \in X$, $f(\overline{a}_1) = f(\overline{a}_2)$ iff $\mathcal{M}(\mathcal{E}) \models E(\overline{a}_1, \overline{a}_2)$. This finishes the proof. ■

Assume that B is an infinite Boolean algebra. The proof of Theorem 3.3 provides us with an algorithm how, given an L_{BA}-formula $\varphi(\overline{x}, \overline{y}) \in L_{BA}$ together with a tuple of parameters $\overline{a} \in B^{|\overline{y}|}$, to find a canonical L_{BA}-formula $\psi_{\varphi,\overline{a}}(\overline{x}, \overline{z})$ and a tuple of parameters $\overline{c} \in B^{|\overline{z}|}$ such that $\varphi(B, \overline{a}) = \psi(B, \overline{c})$ and $\overline{c}$ is a partition of 1_B determined up to almost a permutation (see the definition below).

Definition 4.2 *Assume that B is an infinite Boolean algebra, $m < \omega$ and $d_{\leq m}, e_{\leq m}$ are partitions of 1_B. We say that $e_{\leq m}$ is almost a permutation of $d_{\leq m}$ if there exists a unique permutation σ of the set $\{0, \ldots, m\}$ such that for any $l \leq m$ and any $d \in d_{\leq m} e_{\leq m}$, the element $d_{\sigma(l)} + e_l$ is not large in d.*

For every L_{BA}-formula $\psi_{\varphi,\overline{a}}(\overline{x}, \overline{z})$ obtained in the way described above, denote by $E_{\psi_{\varphi,\overline{a}}}$ the $\emptyset$-definable equivalence relation on $B^{|\overline{z}|}$ defined by the following condition.

> The tuples $d_{\leq m}, e_{\leq m} \in B^{m+1}$ are $F_{\psi_{\varphi,\overline{a}}}$-equivalent iff either $d_{\leq m}, e_{\leq m} \notin \Pi_m(1_B)$, or $d_{\leq m}, e_{\leq m} \in \Pi_m(1_B)$ and $\psi_{\varphi,\overline{a}}(B, d_{\leq m}) = \psi_{\varphi,\overline{a}}(B, e_{\leq m})$.

Let $\mathcal{E}_B$ be the family of all equivalence relations $F_{\psi_{\varphi,\overline{a}}}$ on cartesian powers of B appearing as a result of the above procedure for all possible L_{BA}-formulas $\varphi(\overline{x}, \overline{y})$ and all tuples $\overline{a} \in B^{|\overline{y}|}$.

After these introductory definitions and remarks, we are in a position to state and prove our concluding result.

Theorem 4.3 *Let B be an infinite Boolean algebra. The multisorted structure $B(\mathcal{E}_B)$, where $\mathcal{E}_B$ denotes the family of $\emptyset$-definable equivalence relations on cartesian powers of B defined above, admits elimination of imaginaries.*

Proof. It is enough to show that B and $B(\mathcal{E}_B)$ satisfy condition (b) of Proposition 4.1.

Suppose that $n \in \mathbb{N}_+$, $X \subseteq B^n$ is a $\emptyset$-definable set and $E(\overline{x}, \overline{y})$ is an L_{BA}-formula defining an equivalence relation E on X. By Theorem 3.3, Proposition 3.4 and Proposition 3.6, for every $\overline{a} \in X$, we can find an L_{BA}-formula $\psi_{E,\overline{a}}(\overline{x}, z_{\leq m(\overline{a})})$ and such that

- $\{d_{\leq m(\overline{a})} \in B^{m(\overline{a})+1} : E(B, \overline{a}) = \psi_{E,\overline{a}}(B, d_{\leq m(\overline{a})})\}$ is a non-empty subset of $\Pi_{m(\overline{a})}(1_B)$, and
- for any $d_{\leq m(\overline{a})}, e_{\leq m(\overline{a})} \in \Pi_{m(\overline{a})}(1_B)$, if $\varphi(B, \overline{a}) = \psi_{E,\overline{a}}(B, d_{\leq m(\overline{a})}) = \psi_{E,\overline{a}}(B, e_{\leq m(\overline{a})})$, then $e_{\leq m(\overline{a})}$ is almost a permutation of $d_{\leq m(\overline{a})}$; moreover, if $\mathrm{Inv}(B) = \langle \alpha, 0, 1 \rangle$, where $2 \leq \alpha < \omega$, then $d_{\sigma(l)} + e_l \in I_{\alpha-2}(B)$.

For every $\overline{a} \in X$, denote by $X(\overline{a})$ the set of all tuples $\overline{b} \in X$ for which the following conditions hold.

- $\{d_{\leq m(\overline{a})} \in B^{m(\overline{a})+1} : E(B, \overline{b}) = \psi_{E,\overline{a}}(B, d_{\leq m(\overline{a})})\}$ is a non-empty subset of $\Pi_m(1_B)$, and
- for any $d_{\leq m(\overline{a})}, e_{\leq m(\overline{a})} \in \Pi_{m(\overline{a})}(1_B)$, if $E(B, \overline{b}) = \psi_{E,\overline{a}}(B, d_{\leq m(\overline{a})}) = \psi_{E,\overline{a}}(B, e_{\leq m(\overline{a})})$, then $e_{\leq m(\overline{a})}$ is almost a permutation of $d_{\leq m(\overline{a})}$.

For every $\overline{a} \in X$, $X(\overline{a})$ is a $\emptyset$-definable subset of X containing $\overline{a}$. Moreover, $X(\overline{a})$ is a union of some equivalence classes of E. For $\overline{a} \in X$, we consider the $\emptyset$-definable equivalence relation $F_{\psi_{E,\overline{a}}}$ on $B^{m(\overline{a})+1}$ defined before Theorem 4.3. Let $f_{\overline{a}} : X(\overline{a}) \longrightarrow B^{m(\overline{a})+1}/F_{\psi_{E,\overline{a}}}$ denote the function defined as follows:

$$f_{\overline{a}}(\overline{b}) = \overline{c}/F_{\psi_{E,\overline{a}}} \text{ iff } E(B, \overline{b}) = \psi_{E,\overline{a}}(B, \overline{c}).$$

Obviously, for any $\overline{b}_1, \overline{b}_2 \in X(\overline{a})$,

$$f_{\overline{a}}(\overline{b}_1) = f_{\overline{a}}(\overline{b}_2) \text{ iff } B \models E(\overline{b}_1, \overline{b}_2).$$

Since the formula $\psi_{E,\overline{a}}(\overline{x}, z_{\leq m(\overline{a})})$ constructed in the proof of Theorem 3.3 for the formula E and $\overline{a} \in X$ depends only on E and $\mathrm{Inv}(\overline{a})$, by an elementary compactness argument there are $\overline{a}_0, \ldots, \overline{a}_k \in X$ such that $X = X(\overline{a}_0) \cup \ldots \cup X(\overline{a}_k)$. Define a partition $X_0, \ldots, X_k$ of X as follows: $X_0 = X(\overline{a}_0)$ and $X_i = X(\overline{a}_i) \setminus (X(\overline{a}_0) \cup \ldots \cup X(\overline{a}_{i-1}))$ for $1 \leq i \leq k$ (without loss of generality the sets $X_0, \ldots, X_k$ are non-empty). Define a map

$$f : X \longrightarrow B^{m(\overline{a}_0)+1}/F_{\psi_{E,\overline{a}_0}} \times \ldots \times B^{m(\overline{a}_k)+1}/E_{\psi_{E,\overline{a}_k}}$$

as follows: if $\overline{b} \in X$, then $f(\overline{b}) = \langle f_0(\overline{b}), \ldots, f_k(\overline{b}) \rangle$, where

$$f_i(\overline{b}) = \begin{cases} f_{\overline{a}_i}(\overline{b}) & \text{if } \overline{b} \in X_i \\ \langle 0, \ldots, 0 \rangle / F_{\psi_{E,\overline{a}_i}} & \text{otherwise} \end{cases}$$

for any $i \leq k$. It is clear that f is $\emptyset$-definable in $\mathcal{B}(\mathcal{E}_B)$ and for any $\overline{b}_1, \overline{b}_2 \in X$,

$$B \models E(\overline{b}_1, \overline{b}_2) \text{ iff } f(\overline{b}_1) = f(\overline{b}_2).$$

In this way we have shown that B and $\mathcal{E}_B$ satisfy condition (b) of Proposition 4.1. Hence the multisorted $L(\mathcal{E}_B)$-structure $B(\mathcal{E}_B)$ admits elimination of imaginaries. ■

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