ON HRUSHOVSKI FUSIONS FOR PREDICATES AND THEIR ENVELOPES

S.V. Sudoplatov*

Sobolev Institute of Mathematics,
4, Acad. Koptyug avenue, Novosibirsk, 630090, Russia;
Novosibirsk State Technical University,
20, K.Marx avenue, Novosibirsk, 630092, Russia
e-mail: sudoplat@math.nsc.ru

E.Hrushovski [1] showed that, for any predicate symbol R, taking a linear *prerank* function of form

$$y_R(\mathcal{A}) = |A| - \alpha \cdot e_R(\mathcal{A}),$$

(where $\mathcal{A}$ is a structure $\langle A; R\rangle$, $\alpha \in \mathbb{R}^+$, $e_R(\mathcal{A})$ is a number of tuples in R, being interpreted in $\mathcal{A}$) and a class K_R of finite structures $\mathcal{A}$ with $y_R(\mathcal{A}) \geq 0$, one get a saturated countable K_R-generic (in the sense of [2, 3]) structure $\mathcal{M}_R = \langle M_R; R\rangle$ with a stable theory.

For predicate symbols $R_1, \ldots, R_n, \ldots$, taking a prerank function

$$y_{R_1,\ldots,R_n,\ldots}(\mathcal{A}) = |A| - \sum_i \alpha_i \cdot e_{R_i}(\mathcal{A}),$$

and a class $K_{R_1,\ldots,R_n,\ldots}$ of finite structures $\mathcal{A}$ of language $\{R_1, \ldots, R_n, \ldots\}$ with $y_{R_1,\ldots,R_n,\ldots}(\mathcal{A}) \geq 0$, one again get a saturated countable $K_{R_1,\ldots,R_n,\ldots}$-generic structure $\mathcal{M}$ with a stable theory and such that $\mathcal{M} \upharpoonright R_i = \mathcal{M}_{R_i}$. That structure $\mathcal{M}$ is called the *Hrushovski fusion* of structures $\mathcal{M}_{R_i}$ (cf. [3, 4, 5]). Considering $\mathcal{M}$ itself, $\mathcal{M}$ and the restrictions $\mathcal{M} \upharpoonright \{R_{i_1}, \ldots, R_{i_k}, \ldots\}$ are called the *Hrushovski structures*. If for a Hrushovski structure $\mathcal{M}$, arities for all relations R_i are at most r, coefficients α_i are chosen in Herwig style [6] (cf. [3, Chapter 4]) and additionally with $\alpha_i \cdot (k_p)^r < \varepsilon_p$, as well as the special

*The work is supported by RFFI (grant 09-01-00336-a), by the Council for Grants (under RF President) and State Aid of Fundamental Science Schools via project NSh-3669.2010.1, and by the grant of Novosibirsk State Technical University.

b_n^p are lower bounds for p-approximations of $y_{R_1,\dots,R_n,\dots}$ (with $\sum\limits_{i=1}^{p}$ instead of $\sum\limits_{i}$), then $\mathcal{M}$ is called a *Hrushovski–Herwig structure.*

Notice that each Hrushovski structure is countable. Notice also that, by construction, the Hrushovski structures in [3] are Hrushovski–Herwig structures.

The following definition generalizes the notion of (binary) envelope in [3], being used for constructions of generic Ehrenfeucht theories.

Definition. Let $R(x_1,\dots,x_n)$ be an atomic formula, $\bar{x}$ be a tuple with coordinates in $\{x_1,\dots,x_n\}$. The formula $\exists\bar{x}\,R(x_1,\dots,x_n)$ (as well as the correspondent relations in structures) is called the *$\bar{x}$-projection* or the *$\bar{x}$-envelope* of $R(x_1,\dots,x_n)$. A tuple $(Q_1,\dots,Q_m)$ is an *envelope* of the relation R or *R-envelope* if, for each coordinate x_i in $R(x_1,\dots,x_n)$, there is a tuple $\bar{x}$ without x_i such that some Q_j is an $\bar{x}$-envelope of R. The envelope $(Q_1,\dots,Q_m)$ is *k-ary* if each Q_j is a k-ary relation. Any 2-ary envelope is called *binary.* If $(Q_1,\dots,Q_m)$ is an envelope of R, the relation R is called the *bush* in $(Q_1,\dots,Q_m)$.

Since
$$R(x_1,\dots,x_n)\vdash\exists\bar{x}\,R(x_1,\dots,x_n),$$
any envelope $(Q_1,\dots,Q_m)$ forms a formula
$$\varphi(x_1,\dots,x_n)\rightleftharpoons\bigwedge_{j=1}^{m}\exists\bar{x}^j\,R(x_1,\dots,x_n),$$
where $\exists\bar{x}^j\,R(x_1,\dots,x_n)$ corresponds to Q_j, such that
$$R(x_1,\dots,x_n)\vdash\varphi(x_1,\dots,x_n).$$

Now we consider an influence of bushes R and their envelopes $(Q_1,\dots,Q_m)$ with respect to Hrushovski construction, as well as Hrushovski fusions for R and $(Q_1,\dots,Q_m)$.

A bush R (an envelope $(Q_1,\dots,Q_m)$) is a *Hrushovski(–Herwig) bush* (a *Hrushovski(–Herwig) envelope*) if its structure $\mathcal{M}$ is a generic Hrushovski(–Herwig) structure.

The following examples show that an existence of K_R-generic Hrushovski structure can not guarantee that envelopes of R are Hrushovski envelopes.

Example. Let $R(x,y_1,\dots,y_n)$ be an infinite bush (with a K_R-generic Hrushovski structure) for a binary envelope $(Q_1,\dots,Q_n)$, where
$$\exists y_1,\dots,y_{i-1},y_{i+1},\dots,y_n\,R(x,y_1,\dots,y_n)\equiv Q_i(x,y_i),$$

$i = 1, \ldots, n$, $Q_j = Q_k$ for some $j \neq k$, and Q_j satisfies the *pairwise intersection property* [3], i. e.,

$$\models \forall x, y \exists z (Q_j(z, x) \wedge Q_j(z, y)).$$

Then the binary R-envelope $(Q_1, \ldots, Q_n, Q_{n+1})$, where Q_{n+1} corresponds to the formula

$$\exists x, y_1, \ldots, y_{j-1}, y_{j+1} \ldots, y_{k-1}, y_{k+1}, \ldots, y_n\, R(x, y_1, \ldots, y_n),$$

has the coordinate Q_{n+1} that forms a complete graph (having $\frac{n(n-1)}{2}$ edges for n-element subgraphs). Thus there is no Hrushovski structures with respect to the class $K_{Q_{n+1}}$, being generated by finite restrictions of a Q_{n+1}-structure. If $n > 2$ one can replace Q_j and Q_k by Q_{n+1} and again get an non-Hrushovski R-envelope. $\square$

Modifying Example one can get a K_R-generic Hrushovski structure with a binary envelope $(Q_1, \ldots, Q_n)$, not having the pairwise intersection property but with $O(k^2)$ edges for k-element subgraphs with respect to some Q_i producing a non-Hrushovski generic structure.

Recall [1, 3] that a finite substructure $\mathcal{A}$ in a Hrushovski structure $\mathcal{M}$ is *strong* or *self-sufficient* if $y.(\mathcal{A}) \leq y.(\mathcal{B})$ for any finite $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B} \subseteq M$.

A Hrushovski relation R is *self-sufficient* any tuple $(a_1, \ldots, a_n)$ in R forms a self-sufficient set $\{a_1, \ldots, a_n\}$.

Theorem. *If $(Q_1, \ldots, Q_m)$ be a Hrushovski-Herwig envelope for an n-ary bush and consisting of self-sufficient relations, then $(Q_1, \ldots, Q_m)$ has a Hrushovski-Herwig bush R forming a Hrushovski fusion of generic structures $\mathcal{M}_{Q_i}$, $i = 1, \ldots, m$, and $\mathcal{M}_R$.*

Proof. Let

$$y_{Q_1,\ldots,Q_m}(\mathcal{A}) = |A| - \sum_{i=1}^{m} \alpha_i \cdot e_{Q_i}(\mathcal{A}),$$

be the prerank function for a generic model $\mathcal{M}_{Q_1,\ldots,Q_m}$. Take α_{m+1}, being in Herwig construction [3, 6] and additionally with $\alpha_{m+1} \cdot (k_p)^n < \varepsilon_p$, allow to permutate coordinates of tuples in R, preserving R, and to put $(a_1, \ldots, a_n)$ in R if that tuple belongs to a bush of $(Q_1, \ldots, Q_m)$).

Now we consider the prerank function

$$y_{Q_1,\ldots,Q_m,R}(\mathcal{A}) = |A| - \sum_{i=1}^{m} \alpha_i \cdot e_{Q_i}(\mathcal{A}) - \alpha_{m+1} \cdot e_R(\mathcal{A}).$$

By choice of α_{m+1} for any n-element restriction $\mathcal{A}_0$ of structure in $K_{Q_1,\ldots,Q_m}$ there is a bush R of a copy of a restriction of $(Q_1,\ldots,Q_m)$ such that p-approximations of $y_{Q_1,\ldots,Q_m,R}(\mathcal{A})$ have lower bounds $b^p_{n'}$ for any non-empty restrictions $\mathcal{A}$ of $\mathcal{A}_0$.[1] Thus, a class $K_{Q_1,\ldots,Q_m,R}$ of finite $(Q_1,\ldots,Q_m,R)$-structures $\mathcal{A}$ with that lower bounds $b^p_{n'}$ for p-approximations is generic. Since each Q_i is self-sufficient, for each tuple $\bar{a}_i \in Q_i$, $i = 1,\ldots,m$, the structure $\mathcal{A}_i$ with the universe A_i, consisting of all elements in $\bar{a}_i$, admits (using standard arguments of Amalgamation Lemmas in [3]) the free amalgamation $\mathcal{B} *_{\mathcal{A}_i} \mathcal{A}'$ inside the class $K_{Q_1,\ldots,Q_m,R}$, where $\mathcal{A}'$ is a copy of $\mathcal{A}_0$ such that $\mathcal{A}_i$ is a strong restriction of $\mathcal{A}'$ to a tuple in Q_i, and $\mathcal{A}_i$ is a strong substructure of $\mathcal{B}$. Thus, in that amalgams, $\bar{a}$ belongs to the according projection of R. Taking a $K_{Q_1,\ldots,Q_m,R}$-generic structure $\mathcal{M}$, we get the small stable theory $\mathrm{Th}(\mathcal{M})$ and a required Hrushovski fusion with a bush R for the envelope $(Q_1,\ldots,Q_m)$. $\square$

References

[1] *Hrushovski E.* A stable $\aleph_0$-categorical pseudoplane / E. Hrushovski. — Jerusalem : Hebrew University, 1988. — (Preprint).

[2] *Sudoplatov S. V.* Syntactic approach to constructions of generic models / S. V. Sudoplatov // Algebra and Logic. — 2007. — Vol. 46, No. 2. — P. 134–146.

[3] *Sudoplatov S. V.* The Lachlan problem / S. V. Sudoplatov. — Novosibirsk : NSTU, 2009. — 336 p. [in Russian]

[4] *Hrushovski E.* A new strongly minimal set / E. Hrushovski // Ann. Pure and Appl. Logic. — 1993. — Vol. 62, No. 2. — P. 147–166.

[5] *Sudoplatov S. V.* On the finite closure property for fusions of generic classes / S. V. Sudoplatov // Algebra and Model Theory 7. Collection of papers / eds.: A. G. Pinus, K. N. Ponomaryov, S. V. Sudoplatov. — Novosibirsk : NSTU, 2009. — P. 131–140. [in Russian]

[6] *Herwig B.* Weight ω in stable theories with few types / B. Herwig // J. Symbolic Logic. — 1995. — Vol. 60, No. 2. — P. 353–373.

[1] The coefficients $(k_p)^n$ allow to add to R some or all permutations of any tuple in the bush R, if it is necessary, staying in the class $K_{Q_1,\ldots,Q_m,R}$ with lower bounds $b^p_{n'}$.