ON RUDIN–KEISLER PREORDERS IN SMALL THEORIES

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Rudin–Keisler preorders play a key role in the classification of countable models of small theories as a tool for distributions of prime models over tuples [1, 2]. In the paper, we consider variations and properties of Rudin–Keisler preorders in small theories.

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We consider complete first-order theories T with infinite models. Additionally we assume that T are small, i. e., they have countably many types ($|S(T)| = \omega$). So for any type $q \in S(T)$ and its realization $\bar{a}$, there exists a model $\mathcal{M}(\bar{a})$, being prime over $\bar{a}$. Since all prime models over realizations of q are isomorphic, we often denote such by $\mathcal{M}_q$.

Let p and q be types in $S(T)$. We say that the type p *is dominated by a type* q, or p *does not exceed* q *under the Rudin–Keisler preorder* (written $p \leq_{\rm RK} q$), if $\mathcal{M}_q \models p$, that is, $\mathcal{M}_p$ is an elementary submodel of $\mathcal{M}_q$ (written $\mathcal{M}_p \preceq \mathcal{M}_q$). Besides, we say that a model $\mathcal{M}_p$ *is dominated by a model* $\mathcal{M}_q$, or $\mathcal{M}_p$ *does not exceed* $\mathcal{M}_q$ *under the Rudin–Keisler preorder*, and write $\mathcal{M}_p \leq_{\rm RK} \mathcal{M}_q$.

Syntactically, the condition $p \leq_{\rm RK} q$ (and hence also $\mathcal{M}_p \leq_{\rm RK} \mathcal{M}_q$) is expressed thus: there exists a formula $\varphi(\bar{x},\bar{y})$ such that the set $q(\bar{y}) \cup \{\varphi(\bar{x},\bar{y})\}$ is consistent and $q(\bar{y}) \cup \{\varphi(\bar{x},\bar{y})\} \vdash p(\bar{x})$. Since we deal with a small theory, $\varphi(\bar{x},\bar{y})$ can be chosen so that, for any formula $\psi(\bar{x},\bar{y})$, the

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set $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$ being consistent implies that $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\} \vdash \psi(\bar{x}, \bar{y})$. In this event the formula $\varphi(\bar{x}, \bar{y})$ is said to be (q, p)-*principal*.

Types p and q are said to be *domination-equivalent*, *realization-equivalent*, *Rudin–Keisler equivalent*, or RK-*equivalent* (written $p \sim_{\rm RK} q$) if $p \leq_{\rm RK} q$ and $q \leq_{\rm RK} p$. Besides, models $\mathcal{M}_p$ and $\mathcal{M}_q$ are said to be *domination-equivalent*, *Rudin–Keisler equivalent*, or RK-*equivalent* (written $\mathcal{M}_p \sim_{\rm RK} \mathcal{M}_q$).

As in [3], types p and q are said to be *strongly domination-equivalent*, *strongly realization-equivalent*, *strongly Rudin–Keisler equivalent*, or *strongly* RK-*equivalent* (written $p \equiv_{\rm RK} q$) if, for some realizations $\bar{a}$ and $\bar{b}$ of p and q accordingly, both ${\rm tp}(\bar{b}/\bar{a})$ and ${\rm tp}(\bar{a}/\bar{b})$ are principal. Models $\mathcal{M}_p$ and $\mathcal{M}_q$ are said to be *strongly domination-equivalent*, *strongly Rudin–Keisler equivalent*, or *strongly* RK-*equivalent* (written $\mathcal{M}_p \equiv_{\rm RK} \mathcal{M}_q$).

Clearly, domination relations form preorders, and (strong) domination-equivalence relations are equivalence relations. Here, $\mathcal{M}_p \equiv_{\rm RK} \mathcal{M}_q$ implies $\mathcal{M}_p \sim_{\rm RK} \mathcal{M}_q$.

If $\mathcal{M}_p$ and $\mathcal{M}_q$ are not domination-equivalent then they are non-isomorphic. Moreover, non-isomorphic models may be found among domination-equivalent ones.

For the illustration, we consider the following *Ehrenfeucht examples* [4] of theories T_n, $n \in \omega$, with $I(T_n, \omega) = n \geq 3$.

Example. Let T_n be the theory of a structure $\mathcal{M}^n$, formed from the structure $\langle \mathbb{Q}; < \rangle$ by adding of constants c_k, $k \in \omega$, such that $\lim\limits_{k\to\infty} c_k = \infty$, and by unary predicates $P_0, \ldots, P_{n-3}$ which form a partition of the set $\mathbb{Q}$ of rationals, with

$$\models \forall x, y\, ((x < y) \to \exists z\, ((x < z) \wedge (z < y) \wedge P_i(z))),\ i = 0, \ldots, n-3.$$

The theory T_n has exactly n pairwise non-isomorphic models:

(a) a prime model $\mathcal{M}^n$ ($\lim\limits_{k\to\infty} c_k = \infty$);

(b) prime models $\mathcal{M}^n_i$ over realizations of types $p_i(x) \in S^1(\varnothing)$, isolated by sets of formulas $\{c_k < x \mid k \in \omega\} \cup \{P_i(x)\}$, $i = 0, \ldots, n-3$ ($\lim\limits_{k\to\infty} c_k \in P_i$);

(c) a saturated model $\overline{\mathcal{M}}^n$ (the limit $\lim\limits_{k\to\infty} c_k$ is irrational).

The models $\mathcal{M}^n_{p_0}, \ldots, \mathcal{M}^n_{p_{n-3}}$ are domination-equivalent but pairwise non-isomorphic. $\Box$

A syntactic characterization for the model isomorphism between $\mathcal{M}_p$ and $\mathcal{M}_q$ is given by the following proposition. It asserts that an existence of isomorphism between $\mathcal{M}_p$ and $\mathcal{M}_q$ is equivalent to the strong domination-equivalence of that models.

Proposition 1 [1, 2, 3]. *For any types $p(\bar{x})$ and $q(\bar{y})$ of a small theory T, the following conditions are equivalent:*

(1) *models $\mathcal{M}_p$ and $\mathcal{M}_q$ are isomorphic;*

(2) *models $\mathcal{M}_p$ and $\mathcal{M}_q$ are strongly domination-equivalent;*

(3) *there exist (p,q)- and (q,p)-principal formulas $\varphi_{p,q}(\bar{y},\bar{x})$ and $\varphi_{q,p}(\bar{x},\bar{y})$ respectively, such that the set*

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y},\bar{x}), \varphi_{q,p}(\bar{x},\bar{y})\}$$

is consistent;

(4) *there exists a (p,q)- and (q,p)-principal formula $\varphi(\bar{x},\bar{y})$, such that the set*

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi(\bar{x},\bar{y})\}$$

is consistent.

Proof. (1) $\Rightarrow$ (3). Let $\mathcal{M}(\bar{a})$ and $\mathcal{M}(\bar{b})$ be prime models over realizations $\bar{a}$ and $\bar{b}$ of types $p(\bar{x})$ and $q(\bar{y})$, respectively.

If there is an isomorphism between $\mathcal{M}(\bar{a})$ and $\mathcal{M}(\bar{b})$, the existence of (p,q)- and (q,p)-principal formulas $\varphi_{p,q}(\bar{y},\bar{x})$ and $\varphi_{q,p}(\bar{x},\bar{y})$, satisfying the condition that

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y},\bar{x}), \varphi_{q,p}(\bar{x},\bar{y})\}$$

is consistent, follows from the facts that $\mathcal{M}(\bar{a})$ and $\mathcal{M}(\bar{b})$ realize just principal types over $\bar{a}$ and $\bar{b}$, respectively, and $\mathcal{M}(\bar{a}) = \mathcal{M}(\bar{b}')$ for some tuple $\bar{b}'$ realizing type $q(\bar{y})$.

(3) $\Rightarrow$ (1). Assume that there exist (p,q)- and (q,p)-principal formulas $\varphi_{p,q}(\bar{y},\bar{x})$ and $\varphi_{q,p}(\bar{x},\bar{y})$ such that the set

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y},\bar{x}), \varphi_{q,p}(\bar{x},\bar{y})\}$$

is consistent. We argue to show that $\mathcal{M}_p$ and $\mathcal{M}_q$ are isomorphic, where $\mathcal{M}_p = \mathcal{M}(\bar{a})$, $\mathcal{M}_q = \mathcal{M}(\bar{b})$, $\models p(\bar{a})$, $\models q(\bar{b})$. Since $\varphi_{p,q}(\bar{y},\bar{x})$ is (p,q)-principal and $\varphi_{q,p}(\bar{x},\bar{y})$ is (q,p)-principal, we have

$$p(\bar{x}) \cup \{\varphi_{p,q}(\bar{y},\bar{x})\} \equiv r_1(\bar{x},\bar{y}) \in S(\varnothing),$$

$$q(\bar{y}) \cup \{\varphi_{q,p}(\bar{x},\bar{y})\} \equiv r_2(\bar{y},\bar{x}) \in S(\varnothing).$$

As $p(\bar{x}) \cup \{\varphi_{p,q}(\bar{y},\bar{x})\} \cup q(\bar{y}) \cup \{\varphi_{q,p}(\bar{x},\bar{y})\}$ is consistent, so $r_1(\bar{x},\bar{y}) = r_2(\bar{y},\bar{x})$. Let $\models r_1(\bar{a}\hat{\ }\bar{b}')$, $\models r_2(\bar{b}\hat{\ }\bar{a}')$, where $\bar{b}' \in M_q$, $\bar{a}' \in M_p$, then

$$\mathcal{M}_p = \mathcal{M}_{r_1} = \mathcal{M}(\bar{a}\hat{\ }\bar{b}') \simeq \mathcal{M}(\bar{b}\hat{\ }\bar{a}') = \mathcal{M}_{r_2} = \mathcal{M}_q.$$

It follows by that $(\mathcal{M}_p, \bar{a})$ is a prime model of theory $T \cup p(\bar{c}_1)$, $(\mathcal{M}_p, \bar{a}, \bar{b}')$ is a prime model of theory $T \cup r_1(\bar{c}_1, \bar{c}_2)$, $(\mathcal{M}_q, \bar{b})$ is a prime model of theory

$T \cup q(\bar{c}_2)$, $(\mathcal{M}_q, \bar{a}', \bar{b})$ is a prime model of theory $T \cup r_1(\bar{c}_1, \bar{c}_2)$, and that any constant expansion of prime model is a prime model of new theory.

(3) $\Rightarrow$ (4). Having (p,q)- and (q,p)-principal formulas $\varphi_{p,q}(\bar{y}, \bar{x})$ and $\varphi_{q,p}(\bar{x}, \bar{y})$, and consistent set

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y}, \bar{x}), \varphi_{q,p}(\bar{x}, \bar{y})\},$$

we get a required (p,q)- and (q,p)-principal formula $\varphi(\bar{x}, \bar{y}) \rightleftharpoons \varphi_{p,q}(\bar{y}, \bar{x}) \wedge \varphi_{q,p}(\bar{x}, \bar{y})$.

The directions (4) $\Rightarrow$ (3) and (4) $\Leftrightarrow$ (2) are obvious. $\square$

Denote by $\mathrm{RK}(T)$ the set $\mathbf{PM}$ of isomorphism types of models $\mathcal{M}_p$, $p \in S(T)$, on which the relation of domination is induced by $\leq_{\mathrm{RK}}$, a relation deciding domination among $\mathcal{M}_p$, that is, $\mathrm{RK}(T) = \langle \mathbf{PM}; \leq_{\mathrm{RK}} \rangle$. We say that isomorphism types $\mathbf{M}_1, \mathbf{M}_2 \in \mathbf{PM}$ are *domination-equivalent* (written $\mathbf{M}_1 \sim_{\mathrm{RK}} \mathbf{M}_2$) if so are their representatives.

Clearly, the preordered set $\mathrm{RK}(T)$ has a least element, which is an isomorphism type of a prime model.

Proposition 2 [1, 2]. *If $I(T, \omega) < \omega$ then $\mathrm{RK}(T)$ is a finite preordered set whose factor set $\mathrm{RK}(T)/\!\sim_{\mathrm{RK}}$, with respect to domination-equivalence $\sim_{\mathrm{RK}}$, forms a partially ordered set with a greatest element.*

Proof. That $\mathbf{PM}$ is a finite set is obvious, and the fact that $\mathrm{RK}(T)/\!\sim_{\mathrm{RK}}$ contains a greatest element follows from the existence of a powerful type which dominates any type in $S(T)$. $\square$

Obviously, a small theory T is ω-categorical iff $|\mathrm{RK}(T)| = 1$.

In the above-given Ehrenfeucht examples of theories T_n with $I(T_n, \omega) = n$, each preordered set $\mathrm{RK}(T_n)$ consists of the least element and $(n-2)$ domination-equivalent elements corresponding to the models $\mathcal{M}^n_{p_0}, \ldots, \mathcal{M}^n_{p_{n-3}}$. Thus all ordered sets $\mathrm{RK}(T_n)/\!\sim_{\mathrm{RK}}$ are two-element and linearly ordered.

The following theorem describes preordered sets $\mathrm{RK}(T)$ for small theories T.

Theorem 1 [2, 5]. (1) *For any small theory T, the preordered set $\mathrm{RK}(T)$ is at most countable, upward directed, and has a least element.*

(2) *For any finite or countable, preordered, upward directed set $\langle X; \leq \rangle$ having a least element, there exists a small theory T, for which $\mathrm{RK}(T) \simeq \langle X; \leq \rangle$.*

Proof. (1) That $|\mathrm{RK}(T)| \leq \omega$ follows from the property of T being small. The property for the preordered set $\mathrm{RK}(T) = \langle \mathbf{PM}; \leq_{\mathrm{RK}} \rangle$ to be upward directed is implied by the following: if $\mathbf{M}_1$ and $\mathbf{M}_2$ are isomorphism types of $\mathbf{PM}$ corresponding to models $\mathcal{M}(\bar{a}_1)$ and $\mathcal{M}(\bar{a}_2)$, then types $\mathrm{tp}(\bar{a}_1)$ and $\mathrm{tp}(\bar{a}_2)$ are dominated by $q = \mathrm{tp}(\bar{a}_1\hat{\ }\bar{a}_2)$; hence, $\mathbf{M}_1 \leq_{\mathrm{RK}} \mathbf{M}$ and $\mathbf{M}_2 \leq_{\mathrm{RK}} \mathbf{M}$, where $\mathbf{M}$ is the isomorphism type of $\mathcal{M}_q$. The least element in $\mathrm{RK}(T)$ is the isomorphism type of the prime model.

(2) In view of [2, Theorem 3.4.1], there is no loss of generality in assuming that the set X is countable. A small theory T with $\mathrm{RK}(T) \simeq \langle X; \leq\rangle$ is constructed similarly to how were the theories constructed in proving [2, Theorem 3.4.1], with the theory of unary predicates $P_1, \ldots, P_{|X|-1}$ replaced by a theory of pairwise disjoint unary predicates P_i, $i \in \omega$, each containing infinitely many elements. □

Now we consider the relation $\leq_{\mathrm{RK}}$, being defined on the set $S(T)$ of complete types of small theory T. Denote the structure $\langle S(T); \leq_{\mathrm{RK}}\rangle$ by $\mathrm{RKT}(T)$.

Since for each type $p \in S(T)$ there is a model $\mathcal{M}_p$, and countably many types (for instance, $\mathrm{tp}(\bar{a}), \mathrm{tp}(\bar{a}\hat{\ }\bar{a}), \ldots$ with $\models p(\bar{a})$) forms isomorphic models, being prime over realizations of these types, the structure $\mathrm{RKT}(T)$ can be obtained from $\mathrm{RK}(T)$ by replacement of each element by countably many pairwise $\sim_{\mathrm{RK}}$-equivalent elements, where $\sim_{\mathrm{RK}} = \leq_{\mathrm{RK}} \cap \geq_{\mathrm{RK}}$. Thus Theorem 1 implies

Corollary 1. (1) *For any small theory T, the preordered set $\mathrm{RKT}(T)$ is countable, upward directed, has the least $\sim_{\mathrm{RK}}$-class, and each $\sim_{\mathrm{RK}}$-class consists of countably many elements.*

(2) *For any countable, preordered, upward directed set $\langle X; \leq\rangle$ having the least $(\leq \cap \geq)$-class and such that each $(\leq \cap \geq)$-class is countable, there exists a small theory T, for which $\mathrm{RKT}(T) \simeq \langle X; \leq\rangle$.*

P. Tanović noticed that the factorization of $\mathrm{RKT}(T)$ by the equivalence relation $\equiv_{\mathrm{RK}}$ forms a structure which is isomorphic to $\mathrm{RK}(T)$:

$$\mathrm{RKT}(T)/\equiv_{\mathrm{RK}}\ \simeq \mathrm{RK}(T).$$

Indeed, in view of Proposition 1, for any type $p \in S(T)$, the set of types, that are strongly RK-equivalent to p, corresponds to the model $\mathcal{M}_p$. And for types p and q in $S(T)$, being not strongly RK-equivalent, $p \leq_{\mathrm{RK}} q$ iff $\mathcal{M}_p \leq_{\mathrm{RK}} \mathcal{M}_q$.

In particular, $\mathrm{RK}(T)$ is finite iff $\mathrm{RKT}(T)/\equiv_{\mathrm{RK}}$ is finite.

Since for any theory T, the inclusion $\equiv_{\mathrm{RK}} \subseteq \sim_{\mathrm{RK}}$ holds, the finiteness of $\mathrm{RK}(T)$ implies that $\mathrm{RKT}(T)/\sim_{\mathrm{RK}}$ is finite (and $|\mathrm{RK}(T)| = 1$ iff $|\mathrm{RKT}(T)/\sim_{\mathrm{RK}}| = 1$, that means the ω-categoricity of theory). At the same time there are theories T with infinite $\mathrm{RK}(T)$ and finite $\mathrm{RKT}(T)/\sim_{\mathrm{RK}}$, since by Theorem 1 there are infinite preordered sets $\langle X;\leq\rangle$ being isomorphic to $\mathrm{RK}(T)$ and having only finitely many $\sim_{\mathrm{RK}}$-classes.

Extend the relation $\leq_{\mathrm{RK}}$, being defined on the set $S(T)$ of complete types of the small theory T, to the set ${}^{\subseteq}S(T)$ of all types (including incomplete types) of T. For types $p,q\in{}^{\subseteq}S(T)$ we set $p\leq_{\mathrm{RK}}q$, if any model, realizing q, realizes p.

Notice that the relation $\leq_{\mathrm{RK}}$ on ${}^{\subseteq}S(T)$ is induced by the according relation on $S(T)$:

Proposition 3. *For types $p,q\in{}^{\subseteq}S(T)$, $p\leq_{\mathrm{RK}}q$ holds iff, for any type $q'\in S(T)$, containing q, there exists a type $p'\in S(T)$ such that $p'\supseteq p$ and $p'\leq_{\mathrm{RK}}q'$.*

Proof. Assume that, for the types $p,q\in{}^{\subseteq}S(T)$, $p\leq_{\mathrm{RK}}q$ holds and $q'\in S(T)$ is a completion of q. Since q is realized in the model $\mathcal{M}_{q'}$, the conjecture of proposition implies that p is realized in that model by a tuple $\bar{a}$. The type $p'\rightleftharpoons\mathrm{tp}(\bar{a})$ is a required completion of p such that $p'\leq_{\mathrm{RK}}q'$.

Now we assume that, for any completion $q'\in S(T)$ of the type q, there exists a completion $p'\in S(T)$ of p such that $p'\leq_{\mathrm{RK}}q'$. Consider an arbitrary model $\mathcal{M}$, realizing q by a tuple $\bar{a}$, and the completion $q'=\mathrm{tp}(\bar{a})$ of q. By assumption, some completion $p'\in S(T)$ of p is realized in $\mathcal{M}(\bar{a})$ and so in $\mathcal{M}$. Hence, the model $\mathcal{M}$ realizes p and we have $p\leq_{\mathrm{RK}}q$. $\Box$

Thus, the relation $\leq_{\mathrm{RK}}$ on ${}^{\subseteq}S(T)$ is reduced to the relation $\leq_{\mathrm{RK}}$ on the set $S(T)$ and to possible combinations of complete types, forming type-definable sets.

Since even equivalent formulas form continuum many incomplete types (including or non-including to types the formulas of given set of equivalent formulas), it is natural to factorize the set ${}^{\subseteq}S(T)$ by the equivalence relation $\sim$ of reciprocal deducibility of types:

$$p(\bar{x})\sim q(\bar{x})\Leftrightarrow p(\bar{x})\vdash q(\bar{x})\text{ and }q(\bar{x})\vdash p(\bar{x}).$$

The relation $\leq_{\mathrm{RK}}$ is naturally transformed, by representatives, to the factor-set ${}^{\subseteq}S(T)/\sim$, and further it will be also denoted by $\leq_{\mathrm{RK}}$.

Notice the following properties of the relation $\leq_{\mathrm{RK}}$ on the set

$${}^{\subseteq}S(T)/\sim\;=\{\tilde{p}\mid p\in{}^{\subseteq}S(T)\}.$$

Proposition 4. *If $p,p',q,q'\in{}^{\subseteq}S(T)$, $p'\subseteq p$, $q\subseteq q'$, and $\tilde{p}\leq_{\mathrm{RK}}\tilde{q}$ then $\tilde{p'}\leq_{\mathrm{RK}}\tilde{q'}$.*

Proof is obvious. $\Box$

By the definition, we also have

Proposition 5. *The relation $\leq_{\rm RK}$ on the set ${}^{\subseteq}S(T)/\sim$ is preserved under expansions of theory: if $p,q\in{}^{\subseteq}S(T)$, $\tilde{p}\leq_{\rm RK}\tilde{q}$, and T' is an expansion of T then, for $p,q\in{}^{\subseteq}S(T')$, $\tilde{p}\leq_{\rm RK}\tilde{q}$ holds.*

Having $|S(T)|=\omega$, we get $|{}^{\subseteq}S(T)/\sim|\leq 2^{\omega}$. It is shown in [6], that any countable Boolean algebra $\mathcal{B}$ is *interval*, i.e., $\mathcal{B}$ is isomorphic to a Boolean algebra of subsets of linearly ordered set, being generated by intervals of form $(a,b]$. Now, take a countable saturated structure $\mathcal{M}$ with a small theory and, for some $n\in\omega\setminus\{0\}$, countably many pairwise different principal n-types $p_k(\bar{x})$ with isolating formulas $\varphi_k(\bar{x})$, $k\in\omega$. We get an interval Boolean algebra for the set of definable sets, countably many ultrafilters corresponding to types in $S(\varnothing)$, and continuum many filters corresponding to types $\{\neg\varphi_k(\bar{x})\mid k\in w\}$, $w\subseteq\omega$.

If for each $n\in\omega\setminus\{0\}$ there are finitely many pairwise different principal n-types $p_k(\bar{x})$, the theory is ω-categorical and it implies finitely many n-types $p(\bar{x})$ in $S(\varnothing)$, $n\in\omega\setminus\{0\}$, and so finitely many n-types $p(\bar{x})$ in ${}^{\subseteq}S(\varnothing)/\sim$.

Thus we get the following proposition.

Proposition 6. *Let T be a small theory. Then the following assertions hold.*

(1) *If T is ω-categorical, then $|{}^{\subseteq}S(T)/\sim|=\omega$.*

(2) *If T is not ω-categorical, then $|{}^{\subseteq}S(T)/\sim|=2^{\omega}$.*

Using Proposition 6 and combining the proof for Theorem 1 and Corollary 1, we get

Proposition 7. *The relation $\leq_{\rm RK}$ forms either countable or continual preordered set on ${}^{\subseteq}S(T)/\sim$, having unique $(\leq_{\rm RK}\cap\geq_{\rm RK})$-class (for countable ${}^{\subseteq}S(T)/\sim$) or being upward directed, having a least $\sim_{\rm RK}$-class (consisting of types that have isolated completions), where each $\sim_{\rm RK}$-class is countable.*

Proposition 8. *For any small theory T, the following conditions are equivalent:*

(1) *the structure* ${\rm RKT}(T)$ *has finitely many $\sim_{\rm RK}$-classes;*

(2) *the structure $\langle{}^{\subseteq}S(T)/\sim;\leq_{\rm RK}\rangle$ has finitely many $\sim_{\rm RK}$-classes.*

Proof. (1) $\Rightarrow$ (2). Let ${\rm RKT}(T)$ has n $\sim_{\rm RK}$-classes and $p_1,\ldots,p_n\in S(T)$ be pairwise non-$\sim_{\rm RK}$-equivalent, $P\rightleftharpoons\{p_1,\ldots,p_n\}$. Take an arbitrary type q in ${}^{\subseteq}S(T)$. Since any completion of q is $\sim_{\rm RK}$-equivalent to a type in P and there are only finitely many subsets of P, we have only finitely many possibilities for the $\sim_{\rm RK}$-equivalence of completions for q to types in P. Now we get the implication (1) $\Rightarrow$ (2), since $\sim$-classes $\tilde{q}$, for which completions of q are $\sim_{\rm RK}$-equivalent to the same types in P, are $\sim_{\rm RK}$-equivalent.

The implication (2) $\Rightarrow$ (1) is followed by inclusion $S(T) \subset {}^{\subseteq}S(T)$. $\square$

Further we assume that the small theory T is not ω-categorical and Isol is the set of all types in ${}^{\subseteq}S(T)$ having isolated completions.

Notice that, starting with some $n \in \omega \setminus \{0\}$, there exists (possibly incomplete) n-type $p_{\rm ni}(\bar{x})$, such that its realizations are exactly all possible realizations of non-principal n-types. That type is isolated by the set of formulas $\neg\varphi(\bar{x})$, where the formulas $\varphi(\bar{x})$ are principal. By the definition, the $\sim$-class $\tilde{p_{\rm ni}}$ is $\leq_{\rm RK}$-covering for the $\sim$-class, corresponding to isolated types, in the structure, being a restriction of $\langle {}^{\subseteq}S(T)/{\sim}; \leq_{\rm RK}\rangle$ to the set of n-types. Thus, if there exists a natural number n such that, on the set ${}^{\subseteq}S(T) \setminus {\rm Isol}$, each $\sim$-class is connected by $\geq_{\rm RK}$ with a $\sim$-class, corresponding to some n-type, then the structure $\langle {}^{\subseteq}S(T)/\sim; \leq_{\rm RK}\rangle$ has the least $\sim_{\rm RK}$-class and the least $\sim_{\rm RK}$-class among others. By the definition the reverse implication holds too.

Thus the following criterion for existence of two-element initial segment for the result of factorization of $\langle {}^{\subseteq}S(T)/{\sim}; \leq_{\rm RK}\rangle$ by the relation $\sim_{\rm RK}$.

Proposition 9. *The structure $\langle({}^{\subseteq}S(T) \setminus {\rm Isol})/\sim; \leq_{\rm RK}\rangle$ has the least $\sim_{\rm RK}$-class iff there exists $n \in \omega \setminus \{0\}$ such that, for each $\sim$-class*

$$\tilde{p} \in ({}^{\subseteq}S(T) \setminus {\rm Isol})/{\sim},$$

there exists a $\sim$-class

$$\tilde{q} \in ({}^{\subseteq}S(T) \setminus {\rm Isol})/{\sim},$$

where q is a n-type with $\tilde{q} \leq_{\rm RK} \tilde{p}$.

For finite structures ${\rm RK}(T)$ there exists a natural number n such that each type of T dominates some n-type (for n we can take the length of tuple realizing types p for models $\mathcal{M}_p$ that represent all isomorphism types in ${\rm RK}(T)$). Hence, Proposition 9 implies

Corollary 2. *If the structure* ${\rm RK}(T)$ *is finite then the structure*

$$\langle({}^{\subseteq}S(T) \setminus {\rm Isol})/{\sim}; \leq_{\rm RK}\rangle$$

has the least $\sim_{\rm RK}$-class.

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