THE FIRST ORDER THEORY OF UNIVERSAL SPECIALIZATIONS

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1. INTRODUCTION

This paper concentrates on understanding the first order theory of universal specializations of Zariski structures. Models of the theory are pairs, a Zariski structure and an elementary extension with a map (specialization) from the extension to the structure that preserves positive quantifier free formulas. The reader will find that this context generalizes both the study of algebraically closed valued fields (see [2]) and sheds light on the theory of Zariski structures. It is also a natural setting for studying compact complex manifolds with a standard part map.

We determine the first order theory of universal specializations and prove that it is model complete. Also, we prove that the ground Zariski structure in its core language is stably embedded in models of the theory, which has nice consequences for the theory of Zariski structures.

The structure of the paper is as follows. In the rest of Section 1 we will give the basic definitions of Zariski structures. In Section 2 we will define specializations and prove that anything that can be defined in a Zariski structure using a universal specialization which projects onto it, can be defined within the original language (Corollary 2.11). Section 3 studies the consequences of Corollary 2.11 and is therefore a de-tour from the main goals of the paper, justified in the geometric implications that we can prove for the ground Zariski structure. Section 4 includes the main results of the paper: first order properties of universal specializations, including an axiomatization and a relative quantifier elimination. Finally, Appendix A will work a bit on realizations of types over universal specializations and their projections.

1.1. **Definitions and basic results.** We will work in a mixture of the framework of model theory and Zariski structures. As such, we will in general work with a first order language $\mathcal{L}$ and $\mathcal{L}$-theories T. A *monster model* of a theory T will be a large universal domain which is a model of T and which will be κ-saturated (this means that every consistent set of sentences of size less than κ will be realized in $\mathcal{C}$) for all κ which we will care about. We will take $\mathcal{C}$ as our universal domain and assume that every set, model and tuple we mention is a subset, submodel or belongs to $\mathcal{C}$; we will also abbreviate $\mathcal{C} \models \varphi(\bar{a})$ by $\models \varphi(\bar{a})$ for any tuple $\bar{a}$ in $\mathcal{C}$.

A theory T will be said to be *stable* if there is no $\mathcal{L}$-formula $\varphi(x, y)$ and no infinite tuples $\langle a_i \rangle$ and $\langle b_i \rangle$ such that $\mathcal{C} \models \varphi(a_i, b_j)$ if and only if $i < j$.

We can now move into the basics of Zariski structures. We will work with a language $\mathcal{L}$ and an $\mathcal{L}$-structure M. We will define a topology on M^n as follows: given any primitive $(n+k)$-ary relation R (for each n) and any k-tuple a we will define the set of realizations in M^n defined by R after inserting the elements in a

as parameters as a closed set. We will then define closed sets as a set of elements defined in such way by a positive quantifier free boolean combinations of primitive $\mathcal{L}$-relations (we will define such a boolean combination to be a positive quantifier free $\mathcal{L}$-formula).

The axioms we require for the topology (some of which are consequences of the definition) are the following:

(1) Intersections of any family of closed sets is a closed set.
(2) Finite unions of closed sets are closed.
(3) The domain M of the structure is closed.
(4) The graph of equality is closed.
(5) Any singleton of the domain is closed.
(6) Cartesian products of closed sets are closed.
(7) The image of a closed set under a permutation of coordinates is closed.
(8) For $a \in M^k$ and S, a closed subset of M^{k+l} defined by a predicate $S(x, y)$ the set $S(a, M^l)$ is closed.

We will define the constructible sets to be all possible boolean combination of closed sets (so quantifier free $\mathcal{L}$-definable sets).

We will say that $S \subset_{cl} X$ if S is the intersection of a closed set with X. Similarly, we will say that $U \subset_{op} X$ if U is the intersection of an open set with X.

We will say that X *is irreducible* if given any closed S_1, S_2, if $X \subset S_1 \cup S_2$ then either $X \subset S_1$ or $X \subset S_2$.

Zariski structures will be structures with a given topology which satisfies items (1)–(8) above, and such that there is a dimension function dim from definable sets into the natural numbers satisfying the following:

- (DP) **Dimension of a point**: The dimension of a point is 0.
- (DU) **Dimension of unions**: $\dim(S_1 \cup S_2) = \max\{\dim(S_1), \dim(S_2)\}$.
- (SI) **Strong irreducibility**: For any irreducible $S \subset_{cl} U \subset_{op} M^n$ and any closed $S_1 \subsetneq S$ we have $\dim(S_1) < \dim(S)$.
- (AF) **Addition formula**: For any irreducible closed $S \subset_{cl} U \subset_{op} M^n$ and a projection map $M^n \to M^m$,
$$\dim(S) = \dim pr(S) + \min_{a \in pr(S)} \dim\left(pr^{-1}(a) \cap S\right).$$
- (FC) **Fibre condition**: $S \subset_{cl} U \subset_{op} M^n$ and a projection map $M^n \to M^m$ there exists $V \subseteq_{op} pr(S)$ such that
$$\min_{a \in pr(S)} \dim\left(pr^{-1}(a) \cap S\right) = \dim\left(pr^{-1}(v) \cap S\right)$$
for any $v \in V \cap pr(S)$.
- (SP) **Semi-properness** (of projection mappings): Given a closed irreducible subset $S \subset_{cl} M^n$ and a projection map $pr : M^n \to M^k$, there is a proper closed subset $F \subset \overline{prS}$ such that $\overline{prS} \setminus F \subset prS$.

The following is Theorem 3.2.1 in [7] (it follows quite easily from (SP)).

Fact 1.1. *A Zariski structure M admits elimination of quantifiers (so that any definable subset $Q \subseteq M$ is constructible). Equivalently, the projection of a constructible set is constructible.*

We will define the dimension of a type as the infimum of the dimensions of all closed subsets defined by formulas in the type. Given $A \supset B$ and a tuple a we will say that a is independent from A over B (or $tp(a/B)$ does not fork over A) if $\dim(a/B) = \dim(a/A)$.

We will say that given a closed set S a point $a \in S$ is a generic point in S if for any S' closed such that $a \in S'$ we have $S \subseteq S'$.

2. Universal specializations

Let M_0 be a (Noetherian) Zariski structure and let M be an elementary extension of M_0. A map $\pi : M \to M_0$ will be called a *specialization* if for every formula over $\emptyset$ $S(\bar{x})$ defining an M-closed set and every $\bar{a} \in M$ with $\bar{a}$ in the domain of π, we have

$$M \models S(\bar{a}) \Leftrightarrow M_0 \models S(\pi(\bar{a})).$$

This article is about the theory of certain class of specializations. We will need to work both with the original language of the Zariski structure (which we will denote with $\mathcal{L}$) and with the theory $\mathcal{L}^\pi := \mathcal{L} \cup \{\pi\}$ where π will represent the map described above. During this section, all types will be assumed to be $\mathcal{L}$-types.

A specialization $\pi : M \to M_0$ will be called a *κ-universal specialization* if $|M_0| < \kappa$, M is κ-saturated, and given any $M' \succeq M \succeq M_0$, any $A \subset M'$ with $|A| < \kappa$ and a specialization $\pi_A : M \cup A \to M_0$ extending π, there is an embedding $\alpha : A :\to M$ over $A \cap M$ such that

$$\pi_A := (\pi \circ \alpha)|A.$$

A *universal specialization* is defined to be a ω-universal specialization.

We will need some definitions and facts which can be found in either [4] or [7].

Definition 2.1. *The* locus *of c over A is the smallest A-closed set containing c. The rank of c over A is the dimension of the locus of c over A.*

The following is an easy adaptation of Proposition 2.2.15 and Lemma 2.2.17 in [7]. The extra condition of the first item is explicit in the proof of Proposition 2.2.15 in [7].

Fact 2.2. *Let κ be any cardinal number. Then the following hold.*

- *Let M_0 be any Zariski geometry with $|M_0| < \kappa$. Then there exists a κ-universal specialization $\pi : N \to M_0$. Even more, given any specialization $\pi_M : M \to M_0$ we can get N so that $N \succ M$ and such that π extends π_M.*
- *Let $\pi_M : M \to M^\pi$ be a specialization with $|M| < \kappa$. Then there exists N and $\pi : N \to M$ such that both $\pi : N \to M$ and $\pi_M \circ \pi : N \to M^\pi$ are κ-universal specializations.*

The following is once again, a very easy consequence of the results in [7].

Claim 2.3. *Let $\pi : M \to M_0$ be a specialization on a Noetherian Zariski structure. Then, for any positive quantifier free $C(x, y)$ and any $a \in M$*

$$dim(C(x, a)) \leq dim(C(x, a^\pi)).$$

Proof. This follows by induction on $\dim(C(x, y))$ using condition (FC$'$) –see Lemma 3.2.4 in [7]. $\square$

We will work in general with a geometric Zariski structure and a specialization $M \to M_0$. For any $A \subset M$ and any ($\mathcal{L}$)-type $p(x) \in S(A)$ we will denote by $p|M$

the (unique) non forking extension from $p(x)$ to M. If $B \subset M$ is any other subset of M, $p(x)|B$ is the restriction of $p|M$ to B.

Given any type $p(x)$ over M we denote by p^+ the set of realizations of positive formulas in $p(x)$ and we let

$$p^{+,\pi} := \{C(x, d^\pi) | C(x, d) \in p^+\}.$$

Recall that $\dim(p(x)) := \inf\{\dim(C(x, a)) | C(x, a) \in p(x)\}$.

The following is a corollary of the definitions and Claim 2.3.

Lemma 2.4. *Let M be a Noetherian Zariski structure. Then, for any $p(x)$ we have*

$$\dim(p(x)) \leq \dim(p^{+,\pi}).$$

For stating Theorem 2.6, which is the main result of this section, we will need the following definition.

Definition 2.5. *Given types $p(x)$ and $q(x)$ we will say that $p(x)$* forces *$q(x)$ if every instance of every positive formula in $p(x)$ appears in $q(x)$.*

Theorem 2.6. *Let M be a Noetherian Zariski structure and let $\pi : M \to M_0$ be a specialization and let $p(x)$ be any type over M which does not fork over M_0, and let $m^\pi \in M_0$ be an element such that $p(x)|M_0$ forces $\mathrm{tp}(m^\pi/M_0)$. Then for any m realizing $p(x)$ one can extend π to a specialization π' of $M \cup \{m\}$ such that $\mathrm{tp}(m/M) = p(x)$ and $\pi'(m) = m^\pi$.*

Proof. Let m be any element realizing $p(x)$ and suppose that the specialization $\pi' : M \cup \{m\} \to M^\pi$ defined as an extension of π which sends m to m^π is not a specialization.

By definition there is a positive formula $C(x, y)$ and some $n \in M$ such that $C(x, n) \in p(x)$ and $M_0 \models \neg C(m^\pi, n^\pi)$. We will see that this contradicts the dimensions of the types.

For notation purposes, let $q(x) := p(x)|M_0$. By definition

$$\dim(p(x)) = \dim(q(x))$$

and $\mathrm{tp}(m^\pi/M_0)$ contains all the positive formulas that $q(x)$ does, so in particular, $C(x, n^\pi) \notin q^+(x)$. Now, $p(x)$ extends $q(x)$ so $q^{+,\pi}(x) \subset p^{+,\pi}(x)$. Since both $p(x)$ and $q(x)$ are types over a models, $q^{+,\pi}(x)$ is irreducible and, since

$$C(x, n^\pi) \in p^{+,\pi}(x) \setminus q^{+,\pi}(x),$$

we must have, by Lemma 2.4,

$$\dim(p(x)) = \dim\left(p^{+,\pi}(x)\right) \lneq \dim\left(q^{+,\pi}(x)\right) = \dim(q(x)),$$

a contradiction. □

Corollary 2.7. *Let M_0 be a Noetherian Zariski structure and let $\pi : N \to M_0$ be a universal specialization. Let M be a model such that $M_0 \prec M \prec N$, let $p(x)$ be any type over M and let $m^\pi \in M_0$ be an element such that $p(x)|M_0$ forces $\mathrm{tp}(m^\pi/M_0)$. Then there is some $m \in M$ such that $\mathrm{tp}(m/M_0) = p(x)$ and $\pi(m) = m^\pi$.*

Lemma 2.8. *Let M_0 be a Noetherian Zariski structure, let $\pi : N \to M_0$ be a universal specialization, and let $M \subset N$ be a submodel of N containing M_0. Let b be a tuple contained in some $M_b \succ M$ such that there is a specialization $\pi_b : M_b \to M_0$ such that $\pi|_M = \pi_b|_M$. Then there is some $b' \in N$ and a specialization such that $tp(b'/M) = tp(b/M)$ and $\pi(b') = \pi_b(b)$.*

Proof. Let $D(x, c)$ be the locus of b over M and let b' satisfy the (unique) non forking extension of $tp(b/M)$ to N. Let $b^\pi = \pi(b)$. By definition of universal specializations it is enough to show that π' defined as the extension of π sending b' to b^π is a specialization. Assuming otherwise, we get in particular that $C(b', m)$ and $\neg C(b^\pi, m^\pi)$ for some $m \in N$. By construction we know that $D(x, c) \Rightarrow C(x, m)$.

Let (N_2, π_2) be a universal specialization of M_0 such that N_2 is an elementary extension of M_b, and such that π_2 extends π_b.

Notice that the element m in N witnesses that $\mathrm{tp}(m/M)|M_0$ forces $\mathrm{tp}(m^\pi/M_0)$, so by Corollary 2.7 there is $m' \in N_2$ with $tp(m'/M) = tp(m/M)$ and such that $\pi_2(m') = m^\pi$. This implies in particular that $D(x, c) \Rightarrow C(x, m')$ which implies $N_2 \models C(b, m')$ contradicting the fact that π_2 is a specialization. □

Theorem 2.9. *Let M be a Noetherian Zariski structure, let $\pi : M \to M^\pi$ be a specialization with $|M| < \kappa$ and let $\pi_N : N \to N^\pi$ be a κ-universal specialization such that $M^\pi \prec N^\pi$. Assume also that for some $M^0 \subset N \cap M$ and π_N agrees with π in the intersection. Then there is an embedding $\sigma : M \to N$ such that $\pi_N \circ \sigma = \pi$ and σ is the identity in M^0.*

Proof. Let $\langle M^i \rangle_{i \in \kappa}$. Since we have prime models over any set (see Fact 4.1), we can find a sequence of models $\langle M^\lambda \rangle_{\lambda \in \kappa}$ such that

- $M^\mu \prec M^\lambda$ for $\mu < \lambda$,
- $\dim(M^{\lambda+1}/M^\lambda) = 1$, and
- $\bigcup_{i \in \kappa} M_i = M$.

It is clearly enough to show that, defining $\pi^\lambda := \pi|M^\lambda$, if we have an embedding $\sigma^\lambda : M^\lambda \to N$ such that $\pi^N \circ \sigma^\lambda = \pi^\lambda$ then we can find an extension $\sigma^{\lambda+1}$ of σ^λ such that $\sigma^{\lambda+1} : M^{\lambda+1} \to N$ is an embedding such that $\pi^N \circ \sigma^{\lambda+1} = \pi^{\lambda+1}$. For notation purposes we will assume that $M^\lambda \subset N$ and σ^λ is the identity.

Let $\bar{m} := \langle m_i \rangle$ be an enumeration of $M^{\lambda+1} \setminus M^\lambda$ and let $p^\lambda(\bar{x}) := \mathrm{tp}(\bar{m}/M^\lambda)$. Given any ordinals $i_1, \ldots, i_n$, let $p_{i_1,\ldots,i_n}$ be the restriction of p^λ to the variables $x_{i_1}, \ldots, x_{i_n}$. Let $m_i^\pi := \pi(m_i)$.

Notice that the following claim is a trivial application of Lemma 2.8.

Claim 2.10. *For any $i_1, \ldots, i_n$, there is a tuple $\bar{\alpha} \in N$ satisfying $p_{i_1,\ldots,i_n}$ which specializes in $m^\pi_{i_1}, \ldots, m^\pi_{i_n}$. In particular, $p^{+,\pi}_{i_1,\ldots,i_n}|M_0$ forces $\mathrm{tp}(m^\pi_{i_1}, \ldots, m^\pi_{i_n}/M_0)$.*

Let $Th_N(N)$ be the elementary theory of N. Notice that by Claim 2.10 and compactness the theory

$$Th_N(N) \cup \{\text{“}\pi' \text{ is a specialization which extends } \pi\text{”}\} \cup \{p^\lambda(\bar{\beta}')\} \cup \{\pi'(\beta'_i) = m_i\}_{i \in |\beta|}$$

(in the language of $\mathcal{L}^\pi \cup \{n \mid n \in N\} \cup \{\beta'_i \mid i \in |\beta'|\}$) is consistent and a model M' would be an elementary extension of N with a specialization π' into a supermodel of M_0 such that the restriction of π' into $N \cup \{\bar{\beta}\}$ is a specialization into M_0. By κ-universality (all λ's are less than κ), we can find some $\bar{\beta}$ in N such that $\pi_N(\bar{\beta}) = \bar{m}^\pi$. Defining $\sigma_{\lambda+1}(M_{\lambda+1} \setminus M_\lambda)$ to be $\bar{\beta}$, we have proved the induction step. □

The reader might find the previous result quite close to the notion of model completeness. We will in fact prove model completeness in Section 4, and Theorem 2.9 will be the main ingredient. We will, however, concentrate for a while on the consequences that Theorem 2.6 has on the ground Zariski structure.

Corollary 2.11. *Let M be a Noetherian Zariski structure, let $\pi : M \to M_0$ be a universal specialization, and let $\mathcal{M}$ be the $\mathcal{L}^\pi$-model of the specialization. Then M_0 (which is defined by the formula $\varphi(x) := \exists y \pi(y) = x$) is stably embedded in $\mathcal{M}$.*

Proof. Since the restriction of π to M_0 is the identity, any $\mathcal{L}$-automorphism of M_0 is a $\mathcal{L}^\pi$-automorphism of M_0. Thus, Theorem 2.9 implies that any $\mathcal{L}^\pi$-automorphism of M_0 extends to an embedding from M to M. By a simple back and forth method, this is clearly enough to prove stable embeddedness. □

More generally, the following follows from Theorem 2.9.

Corollary 2.12. *Let M be a Noetherian Zariski structure, let $\pi^M : M \to \pi(M)$ be a specialization, and $\pi^N : N \to \pi^N(N)$ be a $|M|^+$-universal specialization. Assume also that the $\mathcal{L}$-theories of $\pi_N(N)$ and $\pi_M(M)$ coincide and has elimination of quantifiers and that there is an $\mathcal{L}$-homomorphism φ from $\pi^M(M)$ into $\pi^N(N)$. Then there is an $\mathcal{L}^\pi$ embedding from (M, π^M) into (N, π^N) which extends φ.*

Proof. It is clear that we may assume that φ is the identity and $\pi^M(M) \subseteq \pi^N(N)$. By elimination of quantifiers we know that $\pi^M(M) \prec \pi^N(N)$. The corollary now follows from Theorem 2.9. □

3. Consequences of stable embeddedness

This is a short section which follow from results in [7] using Corollaries 2.11 and 2.12. It is based on results proved in [4] for Noetherian Zariski geometries although, since we are not assuming pre-smoothness, the results are much more limited and also require modifying of some of the definitions.

Throughout this section, Let M be a Noetherian Zariski structure with language $\mathcal{L}$.

Definition 3.1. *Recall the following definition from* [7]*:*

- *We will say that F* is an irreducible covering of *D if $F(x, y) \subseteq_{cl} V \subseteq_{op} M^n \times M^k$ is an irreducible set and D be the projection of F into the x (first n) coordinates.*
- *If F is a irreducible covering of D then we will define r to be the* dimension of a generic fiber of F *if*
$$r = min_{a \in D} \dim(F(a, M)).$$
- *We will define $a \in D$ to be* regular for F *if* $\dim(F(a, y)) = r$*. We will call $reg(F/D)$ the set of all regular points in D over F.*
- *We will say that F is a* finite covering *of D if it is an irreducible covering of D and for all $a \in D$ the set $F(a, M) := \{y \in M \mid F(a, y)\}$ is finite.*

Definition 3.2. *Given a specialization $\pi : M^* \to M$ and a point $a \in M$ we define the set $\nu_a := \{x \mid \pi(x) = a\}$.*

In general, in a one dimensional Zariski geometry (this is, assuming pre-smoothness) one has the following statement.

Statement 1. *Let $\pi : M^* \to M$ be a universal specialization and let F be a finite covering of D with $(a, b) \in F$ and a a regular point for F. Then, given any $a' \in \nu_a$ there is $b' \in \nu_b$ such that $F(a', b')$.*

This is in general not true without assuming pre-smoothness. This implies that one needs to modify some of the definitions of [7] since one cannot hope to have a uniform behavior of the generic points. In particular we will define multiplicity as follows.

Definition 3.3. *Let F be a finite covering of D in a and let $\models F(a,b)$. We will define multiplicity of (a,b) in F over D as*

$$\mathrm{mult}_b(a, F/D) := \max\{|F(a', M^*) \cap \nu_b|\}$$

where $\pi : M^* \to M$ is a universal specialization of M, $\nu_b := \pi^{-1}(b)$, and $a' \in \nu_a := \pi^{-1}(a)$ is any point of D over M.

If $\models F(a,b)$ then we will say that F is unramified at point *(a,b) if the multiplicity of (a,b) in F over D in one.*

The following follows from Corollary 2.12.

Theorem 3.4. *Let F be a finite covering of D in a and let $\models F(a,b)$ and a be a regular point in F. Then for any $k \in \mathbb{N}$ the sets*

$$\{(a,b) \mid \mathrm{mult}_b(a, F/D) = k\}$$

and

$$\{(a,b) \mid \mathrm{mult}_b(a, F/D) \leq k\}$$

are $\mathcal{L}$-definable in M. In particular, $\mathrm{mult}_b(a, F/D)$ *is independent of the choice of M^* given in Definition 3.3.*

Proof. Let M^* and N^* be two universal specializations of M, let $\mathrm{mult}_b^M(a, F/D)$ and $\mathrm{mult}_b^N(a, F/D)$ the multiplicities "according" to M^* and N^*, respectively, and let $N^*_+ \succ N^*$ be a $|M^*|^+$-universal specialization of M. By Corollary 2.12 there is an immersion from M^* to N^*_+ which commutes with the projections. In particular, there are at least $\mathrm{mult}_b^M(a, F/D)$ many points in $F(a', N^*_+) \cap \nu_b$ so by definition $\mathrm{mult}_b^M(a, F/D) \leq \mathrm{mult}_b^N(a, F/D)$. By symmetry the other inequality holds so that $\mathrm{mult}_b^M(a, F/D) = \mathrm{mult}_b^N(a, F/D)$ for any two universal specializations of M.

The theorem is now easy from Corollary 2.11. By (FC) we know that $a \in reg(F/D)$ is definable, and clearly $\{a \mid \mathrm{mult}_b(a, F/D) = k\}$ and $\{a \mid \mathrm{mult}_b(a, F/D) \leq k\}$ are subsets of M definable in the $\mathcal{L}^\pi$-model (M^*, π, M). Corollary 2.11 now yields the theorem. □

The following results originally appear in [7] as an introduction to intersection and numerical equivalence for pre-smooth families of curves in Zariski structures, which seems impossible to achieve without assuming pre-smoothness (and Statement 1). However, we will include the definability results with the hope that they might eventually be used for defining an adequate notion of tangency for curves in contexts which do not assume pre-smoothness.

Definition 3.5. *Let P and L be definable irreducible sets and I be a irreducible closed subset of $P \times L$ such that the projection of I to the L-coordinates is L. We will call such I a* family of closed subsets of P, *viewing each $l \in L$ as an index for the family $\{p \in P \mid I(l,p)\}$.*

In general, given a family I of closed subsets of P we will identify l with $\{p \in P \mid I(p,l)\}$ and say, for example, that $p \in l$ if $I(p,l)$ holds and that $l \in I$ if $l \in L$.

Definition 3.6. *Let L_1 and L_2 be irreducible families of closed subsets of an irreducible set P. We say that L_1* and L_2 generically have finite intersections *if for any generic pair $(l_1, l_2) \in L_1 \times L_2$ the intersection $l_1 \cap l_2$ is either empty or finite.*

Let L_1 and L_2 be irreducible families of closed subsets which generically have finite intersections, and let $p \in P, l_1 \in L_1, l_2 \in L_2$ be such that $l_1 \cap l_2$ is finite. We will define the index of intersection of l_1 and l_2 at point p with respect to L_1 and L_2 *to be*

$$ind_p(l_1, l_2/L_1, L_2) := \max\{|l_1' \cap l_2' \cap \nu_p|\}$$

for (l_1', l_2') in $\nu_{l_1} \times \nu_{l_2}$ for some universal specialization $\pi : M^* \to M$.

Theorem 3.7. *Let L_1 and L_2 be irreducible families of closed subsets which generically have finite intersections, and let $p \in P, l_1 \in L_1, l_2 \in L_2$ be such that $l_1 \cap l_2$ is finite. Then for any $k \in \mathbb{N}$ the sets*

$$\{(l_1, l_2, p) \mid ind_p(l_1, l_2/L_1, L_2) = k\}$$

and

$$\{(l_1, l_2, p) \mid ind_p(l_1, l_2/L_1, L_2) \leq k\}$$

are $\mathcal{L}$-definable in M.

Even more, $ind_p(l_1, l_2/L_1, L_2)$ in Definition 3.6 is independent of the choice of M^*.

Proof. This follows once again from Corollaries 2.11 and 2.12; the proof is analogous to the proof of Theorem 3.4. □

The following multiplicity based tangency notion on curves now becomes definable in the original $\mathcal{L}$-structure M (by Corollary 2.11).

Definition 3.8. *Let L_1 and L_2 be families of curves. We will say that two curves l_1, l_2 with $\langle l_1, l_2\rangle \in L_1 \times L_2$ are tangent at a point p if $p \in l_1 \cap l_2$ and either p belongs to an infinite irreducible component of $l_1 \cap l_2$ or $ind_p(l_1, l_2/L_1, L_2) \geq 2$. Equivalently, if $|l_1' \cap l_2' \cap \nu_p| \geq 2$ in some $\langle l_1', l_2'\rangle \in \nu_{l_1} \times \nu_{l_2}$ in a universal specialization M^* of M.*

The proof that the definition is independent of the choice of M^* follows once again using Corollary 2.12 as in the proof of Theorem 3.4.

Finally, we complete this section with the following theorem, which is a version of the Implicit Function Theorem proved in [7] for one dimensional Noetherian Zariski geometries. Once again we should point out the limitations. In the original context, Statement 1 yields that any local function is in fact a bijection between neighborhoods. This is not true in the general context of Zariski structures since in many cases the local functions are neither injective nor surjective. In particular, this implies for example that local functions cannot necessarily be (even locally) inverted. The main problem, however, is that without Statement 1 one cannot show that this sets are large enough, which limits the power of the results considerably.

Definition 3.9. *Let $F \subseteq D \times M^k$ be a definable relation and let $(a, b) \in F$. We say that F* defines a local function from $\nu_a \cap D$ to ν_b *if for some universal specialization $\pi : M^* \to M$, the function $F \mid (\nu_a \times \nu_b)$ is the graph of a function from $\nu_a \cap D$ to ν_b.*

We will say that F defines a local function on D *if for every $(a, b) \in F$ with $a \in D$, F defines a local function from $\nu_a \cap D$ to ν_b.*

Theorem 3.10. *Let $D \subset M^n$ be irreducible and let $F \subseteq D \times M^r$ be an irreducible finite covering of D, $\dim(F) = \dim(D)$.*

Then the following sets are $\mathcal{L}$-definable.

- *The set $F_0 \subseteq F$ of points (a, b) such that a is regular and F defines a local function on (a, b).*
- *The set $D_1 \subseteq D$ such that $F \cap (D_1 \cap M^r)$ defines a local function on D_1 is $\mathcal{L}$-definable in M.*

In particular, defining a local function does not depend on the specialization π chosen in Definition 3.9.

Proof. By (FC) the set of regular points is definable in the structure (M^*, M).

Notice that the set

$$\{(a, b) \mid \forall a' \in \nu_a \cap D \forall b' \in \nu_b \, |\{F(a', M^*)\} \cap \nu_b| = 1\}$$

is definable in (M^*, M) for some (any) universal specialization $\pi : M^* \to M$, and by Corollary 2.11 it is definable in the $\mathcal{L}$-structure M.

By definition, for any such a, b there is exactly one choice of $b' \in \nu_b$ for any $a' \in \nu_a$. This is clearly the graph of a function. □

Notice that one can require in the definition for the local function to be one to one, and therefore invertible when restricted to the infinitesimal neighborhoods. However, since we cannot prove (as one could in theories for which Statement 1 holds) that any of these definable sets projects into a dense subset of D, it is not even clear that the sets are non empty.

4. First order theory of universal specializations

In this section we will describe the $\mathcal{L}^\pi$-axiomatization of universal specializations. Throughout this section we will need to constantly work with both languages ($\mathcal{L}$ and $\mathcal{L}^\pi$) and we will be specific about which one we mean when we think that any confusion may arise.

We will assume that the reader is familiar with the concepts of completeness and model completeness in first order logic.

We will assume that $\mathcal{L}$ is a countable language and that T is the $\mathcal{L}$- theory of a $\aleph_1$-saturated Zariski structure. Before we continue, we will need to recall some of the basic facts about such a theory T. We will also assume that the reader is familiar with the concepts of Morley rank, and prime and constructible models which can be found in [7] and [5] respectively.

Fact 4.1. *Let $\mathcal{L}$ be a countable language and let T be the $\mathcal{L}$- theory of a $\aleph_1$-saturated Zariski structure. Let M be any model of T.*

Then the following hold:

1. *M has Morley rank 1. Even more, given any closed Q in M^n we have that $MR(Q) \leq \dim(Q)$.*
2. *Given any set $A \subset M$, there is a prime model $M(A) \prec M$. This model is constructible and unique up to isomorphism.*

Proof. The first two items can be found in [7] (Theorems 3.2.1 and 3.2.8). In fact, The existence of prime models is a well known fact for all theories of finite Morley rank, and the precise statement of (3) is the content of [5]. □

For the rest of this section, we will assume that we are working with a $\aleph_1$-saturated Zariski structure M. Given a set A in a Zariski geometry we will define $\mathrm{mcl}(A)$ as the prime model of A.

Lemma 4.2. *Let $\pi_M : M \to M_0$ and $\pi_N : N \to N_0$ be κ and λ-universal specializations (respectively) where κ and λ are greater than the cardinality of the language. Then given any finite tuples $\bar{a} \in M$ and $\bar{b} \in N$, if the $\mathcal{L}^\pi$-quantifier free types $\mathrm{qftp}^M_{\mathcal{L}^\pi}(\mathrm{mcl}(\bar{a})) = \mathrm{qftp}^N_{\mathcal{L}^\pi}(\mathrm{mcl}(\bar{b}))$ then the $\mathcal{L}^\pi$-types $\mathrm{tp}^M_{\mathcal{L}^\pi}(\mathrm{mcl}(\bar{a})) = \mathrm{tp}^N_{\mathcal{L}^\pi}(\mathrm{mcl}(\bar{b}))$.*

Proof. Notice that the second item follows immediately from the first one.

We will prove the first item first. By symmetry (followed by an easy back and forth argument), it is enough to show that given any $\bar{a}$ and $\bar{b}$ as in the statement of the theorem and given any $c \in M$, there is some $d \in M$ such that

$$\mathrm{qftp}^M(\mathrm{mcl}(c\bar{a})) = \mathrm{qftp}^N(\mathrm{mcl}(d\bar{b})).$$

so that we can define a $\mathcal{L}$-embedding σ mapping $M_0 := \mathrm{mcl}(\bar{a})$ onto $\mathrm{mcl}(\bar{b})$.

If $c \in \mathrm{mcl}(\bar{a})$ there is nothing to prove. Otherwise, notice that, because $M_c := \mathrm{mcl}(c\bar{a})$ is a model of the theory, then

$$(M_c, \pi(M_c), \pi_M|_{M_c})$$

is a specialization. Since M_c has the same cardinality as the language, Theorem 2.9 implies that we extend σ to an embedding from M_c into N. Taking d the image of c completes the proof. □

Remark 4.3. *Notice that Lemma 4.2 is quantifier elimination up to the prime models. It is not possible to reduce this all the way to the original language: Consider for example the case where our original theory T is an algebraically closed field. The theory of specializations of fields is close to the two sorted theory of algebraically closed valued fields (ACVF), where we have sorts for both fields and a relation symbol for divisibility (see for example* [6]*). In this example it is not hard to show that the type of an element x over a set A is implied by the type of the* field *generated by xA over the type of the field generated by A. It is not hard to show that going to the type of the field is necessary: If the specialization sends x to 0, then we have a lot of freedom where to send $1/x \cdot a$ for all $a \in A$ which specializes in 0. It is possible that one could change* mcl *in Lemma 4.2 for either* acl *or (unlikely)* dcl*, but we have neither a proof nor a counterexample for this.*

As may be deduced by the result in Lemma 4.2, in order to move towards axiomatization and other properties we will need to move from types over an arbitrary set A to extensions over $\mathrm{mcl}(A)$. The following claim follows from the definitions of saturation and constructibility.

Claim 4.4. *Let M be any ω-saturated model of a theory $\mathcal{L}'$. Then the following hold:*

- *Let $p(x)$ be a $\mathcal{L}'$-type over some set A, and let $q(x) \in S_{\mathcal{L}'}(B)$ be a type extending $p(x)$ such that B is constructible over A. Then, if $p(x)$ is realized in M, $q(x)$ is too.*
- *Let $p(x)$ be a $\mathcal{L}'$-type over some set A and let $q(x, \bar{x})$ be a type extending $p(x)$ such that for all $x_i \in \bar{x}$ the type $q(x, \bar{x})$ implies that x_i is constructible over x. Then, if $p(x)$ is realized in M, $q(x)$ is too.*

Theorem 4.5. *Let $\pi : N \to N^\pi$ be a κ-saturated model of the theory of specializations. Then the following are equivalent:*

(1) *N is a κ-universal specialization.*
(2) *Given any finite set $A \in N$ and any type $p(x)$ in $S^1(A)$ which is realized in some $\mathcal{L}^\pi$-structure $\mathcal{M}$ containing A satisfying the theory of specializations, then $p(x)$ is realized in N.*
(3) *Given any finite set $A \in N$ and any quantifier free complete 1-type $p^{qf}(x)$ over $\mathrm{mcl}(A)$ which is realized in some $\mathcal{L}^\pi$-structure $\mathcal{M}$ containing A satisfying the theory of specializations, then $p^{qf}(x)$ is realized in N.*

Proof. The implication (2) implies (3) is clear. The converse follows from Claim 4.4 and the fact that $\mathrm{mcl}(A)$ is constructible over A, and Lemma 4.2.

The implication (1) $\Rightarrow$ (2) follows from the definition of universal specialization and Theorem 2.9:

Let $p(x) \in S(\mathrm{mcl}(A))$ be a quantifier free type with A finite. Let (M, π_M) with $M \supset A$ be a specialization where $p(x)$ is realized by some element d and $|M| < \kappa$. By saturation (and completeness of the $\mathcal{L}$-theory of $\pi(M)$) there is an embedding from $\pi_M(M) \to \pi(N)$ which is the identity on $\pi(A)$, so we may assume without loss of generality that $\pi(M) \subset \pi(N)$. By Theorem 2.9 there is a $\mathcal{L}^\pi$-embedding from M into N preserving A, so $p(x)$ must be realized in N.

The proof of (2) $\Rightarrow$ (1) follows from a construction which is almost identical to the one used in the proof of Theorem 2.9. Let $M' \succ N$ be an elementary extension and let $A \subset M'$ with $|A| < \kappa$ and a specialization π_A from A to $\pi(N)$ which agrees with π on $A \cap N$. Replacing A with $\mathrm{mcl}(A)$ we may assume that A is a model M which specializes in $\pi(N)$.

As in that proof of Theorem 2.9, we start by an "enumeration" of M. Let $\langle M^\lambda \rangle_{\lambda \in |M|}$ such that

- $M^\mu \prec M^\lambda$ for $\mu < \lambda$,
- $\dim(M^{\lambda+1}/M^\lambda) = 1$,
- $\bigcup_{i \in \kappa} M^i = M$, and
- $M_0 = M \cap N$.

Once again, it is enough to show that if $\sigma^\lambda : M^\lambda \to N$ is such that $\pi \circ \sigma^\lambda = \pi^\lambda$, then we can find an extension $\sigma^{\lambda+1}$ of σ^λ such that $\sigma^{\lambda+1} : M^{\lambda+1} \to N$ is an embedding with $\pi \circ \sigma^{\lambda+1} = \pi^{\lambda+1}$. Let $\sigma^\lambda(M^\lambda) = N^\lambda$.

Given such an embedding σ^λ, let c be an element in $M^{\lambda+1}$ be such that $\dim(c/M^\lambda) =$ 1 (so that $M^{\lambda+1} := \mathrm{mcl}(c, M_\lambda)$). Notice that Claim 4.4 $\sigma(\mathrm{tp}(M^{\lambda+1}/M^\lambda))$ is realized in N if and only if $\sigma(\mathrm{tp}(c/M^\lambda)$ is.

Let $p(x) := \mathrm{tp}(c/M^\lambda)$. By hypothesis $p^\sigma(x) := \sigma^\lambda(p(x))|A$ is realized in N for any finite $A \subset \sigma^\lambda(M^\lambda) = N^\lambda$, so by compactness $p^\sigma(x)$ is consistent with the elementary theory of N (in $\mathcal{L}^*(N^\lambda)$). Since $|N_\lambda| < \kappa$ for all λ, by saturation $p^\sigma(x)$ is realized in (N, π); by Claim 4.4 $\sigma(\mathrm{tp}(M^{\lambda+1}/\lambda))$ is realized in N, which gives us an $\mathcal{L}$-embedding from $M^{\lambda+1} = \mathrm{mcl}(c, M^\lambda)$ into N extending σ^λ. □

Notice that given A, the set of types which can be realized in a specialization containing A, are precisely those which are consistent with the quantifier free type of A. But the $\mathcal{L}^\pi$-quantifier free types over some set A are all of the form

$$\{C(x,a) \mid C(x,y) \in \mathcal{L}, a \in A\} \cup \{\pi(x) = z\} \cup \{C(x,a^\pi) \mid C(x,y) \in \mathcal{L}, a^\pi \in \pi(A)\}.$$

Let Σ' be the set of all quadruples of quantifier free $\mathcal{L}$-formulas

$$(\varphi(\bar{x},\bar{y}), \theta(\bar{y}), \psi(\bar{z},\bar{w}), \mu(\bar{w}))$$

such that for some specialization $\pi : M \to M^{\pi}$ we have some $\bar{b}, \bar{a} \in M$ with

$$M \models \varphi(\bar{b},\bar{a}) \wedge \theta(\bar{a})$$

and

$$M^{\pi} \models \psi\left(\pi\left(\bar{b}\right), \pi\left(\bar{a}\right)\right) \wedge \mu\left(\pi\left(\bar{a}\right)\right).$$

Claim 4.6. *Let $(\varphi, \theta_0, \psi, \mu_0) \in \Sigma'$ and let a, b witness this. Then there exist finite subtypes of* $\mathrm{tp}(a)$ *and* $\mathrm{tp}(\pi(a))$ *respectively with $\theta \supset \theta_0$ and $\mu \supset \mu_0$ such that*

$$(1) \qquad \forall y\left[\theta(y) \wedge \mu(\pi(y)) \Rightarrow \exists x\left(\varphi(x,y) \wedge \psi(\pi(x), \pi(y))\right)\right]$$

is consistent in some $(M, \pi(M))$.

Proof. If not, we will have that the L-type of $(a, \pi(a))$ implies

$$\forall xy[\psi(\pi(x), \pi(y)) \to \neg\varphi(x,y)]$$

in every $(M, \pi(M))$. But this contradicts Theorem 4.5 and the assumptions on a, b. □

Notice that if a, b witness that $(\varphi, \theta_0, \psi, \mu_0)$ is in Σ', then a, b will also witness that $(\varphi, \theta, \psi, \mu)$ given by Claim 4.6 belongs to Σ' too.

Let $\Sigma \subset \Sigma'$ be the set of quadruples

$$(\varphi(\bar{x},\bar{y}), \theta(\bar{y}), \psi(\bar{z},\bar{w}) \, \mu(\bar{w}))$$

satisfying the assumptions of Claim 4.6.

We define

$$T^U := T \cup \{\forall y\left[\theta(y) \wedge \mu(\pi(y)) \Rightarrow \exists x\left(\varphi(x,y) \wedge \psi(\pi(x), \pi(y))\right)\right]\}_{(\varphi,\theta,\psi,\mu)\in\Sigma}.$$

By definition of Σ we know that every formula of the form

$$\forall y\left[\theta(y) \wedge \mu(\pi(y)) \Rightarrow \exists x\left(\varphi(x,y) \wedge \psi(\pi(x), \pi(y))\right)\right]$$

with $(\varphi, \theta, \psi, \mu) \in \Sigma$ must be satisfied in some specialization $(M, \pi(M))$ and by Theorem 4.5 they must all be satisfied in any universal specialization. It follows that T^U is consistent and contained in the theory of universal specializations.

Moreover, the following is immediate from Theorem 4.5 and compactness.

Theorem 4.7. *Let $\mathcal{M}$ be a $\mathcal{L}^{\pi}$-structure. Then the following hold.*

(1) *If $\mathcal{M}$ is a κ-saturated model of T^U then $\mathcal{M}$ is κ-universal.*
(2) *T^U is the model companion of the theory of specializations. Even more, a specialization (M, π^M) can be embedded into* any *$|M|^+$ saturated model of T^U.*
(3) *T^U is model complete.*
(4) *Any two models of T^U can be embedded into a single model of T^U (this is, T^U has the joint embedding property).*
(5) *T^U is a complete $\mathcal{L}^{\pi}$-theory.*

Proof. The first item follows from Theorem 4.5 and the definition of T^U.

For (ii), assume that (M, π^M) is a specialization. By elimination of quantifiers and completeness of $\mathrm{Th}_{\mathcal{L}}(\pi(M))$ (Fact 1.1) given any $|M|^+$ saturated model (N, π^N) of T^U there is an elementary embedding from $\pi^M(M)$ into $\pi^N(N)$ and by Corollary 2.12 this can be extended to a $\mathcal{L}^\pi$-embedding from (M, π^M) into (N, π^N).

(iii) also follows from Theorem 2.9 and (i): Let (M, π^M) be any model of T^U (so it is in particular a specialization) and let (N, π^N) and (K, π^K) be universal specializations containing (M, π^M) and such that (K, π^K) is $|N|^+$-saturated. By Theorem 2.9 we know that there is a submersion from (N, π^N) into (K, π^K) which is the identity in (M, π^M), which is enough to proof model completeness, so (iii) holds.

Let (M, π^M) and (N, π^N) be two universal specializations and let (K, π^K) be a $|M \cup N|^+$-saturated specialization of T^U. Since $\mathrm{Th}(\pi^M(M)) = \mathrm{Th}(\pi^N(N))$ is complete, there are elementary immersions from $\pi^M(M)$ and $\pi^N(N)$ into $\pi^K(K)$. By Corollary 2.12 this extends to immersions from (M, π^M) and (N, π^N) into (K, π^K) completing the proof of (iv).

Finally, it is known that (v) follows from (iii) and (iv) (see for example Proposition 3.5.11 in [1]). □

Appendix A. Types over universal specializations

Finally, we will show that the dimension of tuples over models and their projections work as expected when one starts with a universal specialization.

In this section we work inside a monster model $(\mathcal{C}, \pi, \mathcal{C}^\pi)$ of the theory of universal specializations.

We will start by the following slight generalization of Corollary 2.7.

Lemma A.1. *Let $M \to M^\pi$ be a universal specialization such that M^π is saturated. Let $C(x, a)$ be a closed set, let $p(x)$ be the (complete over M) type generated by $C(x, a)$, let $p^+(x), p^{+,\pi}$ be defined as in Section 2. Let $E(x, b^\pi)$ be the smallest closed set in $p^{+,\pi}$. Let M_0^π be a small submodel of M^π over which E is based. Let $m^\pi \in M^\pi$ be any element in M^π such that $\models E(m^\pi, b^\pi)$ and $m^\pi \perp_{b^\pi} M_0^\pi$ (so that in particular $tp(m^\pi/M_0^\pi) \supset p^{+,\pi}|M_0^\pi$).*

Then there is some $m \in M$ satisfying $C(x, a)$ which specializes in m^π.

Proof. Let m' be a realization of $p(x)$. By construction if $\models D(m', d)$ for some $d \in M$ then $\models D(x, d^\pi) \in p^{+,\pi}$ and $E(x, b^\pi) \Rightarrow D(x, d^\pi)$ so in particular $\models D(m^\pi, d^\pi)$. This means that $p(x)$ forces $\mathrm{tp}(m^\pi/M^\pi)$. Thus, the extension of π to $M^* \cup \{m'\}$ sending m' to π is a specialization. By universality we can find an m in M satisfying the hypothesis of the lemma. □

Lemma A.2. *Let $\pi|_M M \to M^\pi$ be a specialization of M^π (for some $M \preceq \mathcal{C}$), and let a, b be elements such that $\dim(ab/M) < \dim(a^\pi b^\pi/M^\pi)$. Let $\langle a_i b_i \rangle$ be a Morley sequence of ab over M. Then the projection $\langle a_i^\pi b_i^\pi \rangle$ is* not *a Morley sequence over M^π.*

Proof. Suppose otherwise, so that for every n the n-Morley sequence $\langle a_i b_i \rangle_{i \leq n}$ over M projects into the Morley sequence $\langle a_i^\pi b_i^\pi \rangle_{i \leq n}$.

Let $C(x, y, \bar{m})$ be the locus of ab over M and assume $\dim(C(x, y, \bar{m}) = k$ so that by hypothesis $C(x, y, \bar{m}^\pi)$ has dimension $k + l$ for some positive l. Notice that by hypothesis

$$\bigotimes_{i \leq n} C(x_i, y_i, \bar{m}) = nk$$

and

$$\bigotimes_{i \leq n} C(x_i y_i, \bar{m}^\pi) = n(k + l).$$

Let D and D' be the locus of $\bar{m}$ and $\bar{m}\pi$ (over $\emptyset$) respectively.

Notice that $\dim(C) = n + \dim(D)$ and $a_0^\pi, b_0^\pi, \ldots, a_n^\pi, b_n^\pi, \bar{m}^\pi$ is an exceptional point of C so

$$nk + \dim(D) \geq n(k + l) + \dim(D') + 1.$$

It follows that $\dim(D) - \dim(D') \geq nl + 1$. But D and D' are fixed and l is a fixed positive number, so for some n this will give a contradiction. □

Lemma A.3. *Let M be a universal specialization of M^π. Let c be any tuple which specializes in c^π. Then there are c_1, c_2 and c_1^π, c_2^π independent over M and M^π, respectively, such that $c_1 c_2 \to c_1^\pi c_2^\pi$ is a specialization compatible with $M \to M^\pi$.*

Proof. By stability, if $p(x)$ and $p^\pi(x)$ denote, respectively, $tp(c/M)$ and $tp(c^\pi/M^\pi)$ then there are unique types $p(x) \otimes p(y)$ and $p^\pi(x) \otimes p^\pi(y)$ and the statement of the lemma amounts to say that there is some $D(x, y, b) \in p(x) \otimes p(y)$ such that $D(x, y, b) \notin p^\pi(x) \otimes p^\pi(y)$.

Let $C(x, a)$ be the locus of $p(x)$ and we choose it so that $dim(C(x, a^\pi)) = n$ is as small as possible. So by construction we know, in particular, that

$$C(x, a) \wedge C(y, a) \Rightarrow D(x, y, b)$$

but

$$C(x, a^\pi) \wedge C(y, a^\pi) \not\Rightarrow D(x, y, b^\pi)$$

which by irreducibility implies that $dim(D(x, y, b^\pi)) < 2dim(C(x, a^\pi)) = 2n$.

Consider the set $\exists y D(x, y, b)$. This must be implied by $C(x, a)$ and by minimality of n we know that $\exists y D(x, y, b^\pi)$ must have the same dimension as $C(x, a)$. Now, since $dim(D(x, y, b)) < 2n$ we know that given any generic point m^π in $\exists y D(x, y, b^\pi)$ we have

$$dim(D(m^\pi, y, b^\pi)) = dim(D(x, y, b^\pi)) - n < n.$$

By Lemma A.1, there is some $m \in M$ such that:

- m specializes in m^π.
- $\models C(m, a)$.

Since $C(x, a) \otimes C(y, a) \Rightarrow D(x, y, b)$ we know that $p(y) \models D(m, y, b)$ so that by construction and definition of specializations we have $\models D(m^\pi, c^\pi, b^\pi)$. But since $\dim(D(m^\pi, x, b^\pi)) < n$, this contradicts minimality of n. □

Proposition A.4. *We are working within a monster model $\mathcal{C}^*$. Let M be a universal specialization of M^π. Then given any a, b in M we have that* $\dim(ab/M) \geq \dim(a^\pi b^\pi/M^\pi)$.

Proof. Let M, M^π and a, b be as in the statement of the proposition and assume that $\dim(ab/M) < \dim(a^\pi b^\pi/M^\pi)$. Applying Lemma A.3 repeatedly, we can define Morley sequences $\langle a_i b_i \rangle$ of ab over M and $\langle a_i^\pi b_i^\pi \rangle$ of $a^\pi b^\pi$ over M^π such that a_i^π, b_i^π are the projection of a_i, b_i. This contradicts Lemma A.2. □

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