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Positive model theory and amalgamations

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Abstract

We continue the analysis of foundations of positive model theory as introduced by Ben Yaacov and Poizat. The objects of this analysis are h-inductive theories and their models, especially the "positively" existentially closed ones. We analyze topological properties of spaces of types, introduce forms of quantifier elimination and characterize minimal completions of arbitrary h-inductive theories. The main technical tools consist of various forms of amalgamations in special classes of structures.

1. *Introduction*

Positive model theory is the study of h-inductive theories through their models, especially those that are existentially closed, and their type spaces using positive logic. It was initiated by Ben Yaacov in [**1**], [**2**] following the line of research on universal theories carried out by Shelah ([**11**]), Hrushovski ([**5**]), Pillay ([**8**]). In its current form, positive model theory was introduced by Ben Yaacov and Poizat in [**3**]. In [**9**] and [**10**], Poizat analyzed the topology of type spaces and introduced the notion of positive elementary extension.

In this article, our ultimate goal is to refine the analysis of classes of structures following the line of research of Ben Yaacov and Poizat. The principal subjects will be universal extensions, topological properties of type spaces, quantifier elimination and connections of these with classes of structures. A recurrent theme will be amalgamation in various classes of structures. Frequently, these structures will be model companions of an h-inductive theory or non elementary classes. The amalgamation analysis consists frequently in verifying that certain classes of models form amalgamation bases (Definition 7), a notion borrowed from [**4**] and [**8**].

The models of an h-inductive theory that are amalgamation bases are those that represent the best the theory in question. In our context, the positively existentially closed (pec in short) models are typical examples of this property. The analysis of quantifier elimination will show that under additional hypotheses, this representational power is shared by larger classes of models of an h-inductive theory: the h-maximal and the positively existentially closed ones (Section 5). In general, these particular models do not form elementary classes, a property connected to the study abstract elementary classes in the sense of Shelah (section 6).

The article is organized as follows. In the second section, after revising the foundations of the subject, we will introduce the notion of universal extension that will be used in verifying the existence of "large" models, in particular models whose classes allow amalgamation. The third section is devoted entirely to amalgamation in various classes of models. In particular, will be proven a caracterization in terms of universal extensions (Theorem 1). The fourth section contains a first application of the amalgamation techniques developed in the third section, especially of those in subsection 3.2: we will analyze the preservation of topological properties of type spaces in substructures and elementary extensions. In the fifth section, we will analyze various aspects of quantifier elimination. In particular, we will use the notion of a positive Robinson theory, an h-inductive theory that allows a certain kind of quantifier elimination.

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In section six, we will finish by studying completions of h-inductive theories. This will set the foundations for the first steps of a work in preparation on positive stability and simplicity.

2. *Positive model theory*

2.1. *Basics*

Positive logic is a branch of first-order mathematical logic whose specific property is not using negation. This restricts the available set of first-order formulas to the set of the positive ones obtained from atomic formulas using $\vee, \wedge, \exists$ as logical operators and quantifier respectively. Eventually, a positive first-order formula is of the form $\exists \bar{y} f(\bar{x}, \bar{y})$, where $f(\bar{x}, \bar{y})$ is quantifier-free. The special symbol $\perp$ denoting the antilogy needs to be added. The rest of this section is devoted to recalling various definitions and notions of positive logic. For further details, [**3**] is a sufficiently complete reference.

As in the first-order logic with negation, a sentence is a formula without free variables. A sentence is said to be h-universal if it is the negation of a positive sentence, i.e. it is of the form $\neg\exists\bar{x} f(\bar{x})$, or equivalently $\forall\bar{x}\neg f(\bar{x})$ where $f(\bar{x})$ is quantifier-free and positive. The conjonction of two h-universal sentences is equivalent to an h-universal sentence. The same is true for their disjunction.

A sentence is said to be simple h-inductive if it can be written in the form

$$\forall\bar{x}[\exists\bar{y} f(\bar{x}, \bar{y}) \leftarrow \exists z g(\bar{x}, \bar{z})] \ ,$$

where f and g are quantifier-free and positive. In prenex normal form, such a sentence is of the form

$$\forall\bar{u}\exists\bar{v}(\neg\phi(\bar{u}) \vee \psi(\bar{u}, \bar{v})) \ ,$$

where ϕ and ψ are quantifier free and positive. It follows that the disjunction of two simple h-inductive sentences are still simple h-inductive. An h-inductive sentence is a finite conjunction of simple h-inductive sentences. The conjunction and disjunction of two h-inductive sentences is still h-inductive.

A first-order theory is said to be h-inductive if it is formed by h-inductive sentences. In the particular case when they are all h-universal such a theory is called h-universal. The h-inductive theories are the objects of analysis of positive model theory.

Let L be a first-order language and M and N be two L-structures. A mapping from M to N is a homomorphism if for every tuple $\bar{m}$ extracted from M ($\bar{m} \in M$ in short) and for every atomic formula ϕ, $M \models \phi(\bar{m})$ implies $N \models \phi(h(\bar{m}))$. In such a case, N is said to be a continuation of M. A homomorphism is an embedding whenever for every atomic formula ϕ $M \models \phi(\bar{m})$ if and only if $N \models \phi(h(\bar{m}))$; it is an immersion whenever $\bar{m}$ and $h(\bar{m})$ satisfy the same positive formulas.

A positive compactnes theorem was proven by Ben Yaacov and Poizat, and we will refer to its following form as "positive compactness":

Fact 1 [**3**, Corollaire 4]. *An h-inductive theory is consistent if and only if every finite subset of it is consistent.*

A class of structures is said to be inductive if it is closed with respect to inductive limits of homomorphisms. It is easy to verify that the class of models of an h-inductive theory is inductive. Theorem 23 of [**3**] shows that this is indeed a caracterization:

Fact 2 [**3**, Théorème 23]. *The class of models of a first-order theory is inductive if and only if it is axiomatized by h-inductive sentences.*

2.2. *Positively existentially closed models*

The notion of positively existentially closed (pec from now on) model is fundamental in positive model theory:

Definition 1. *Let L be a first-order language. A member M of a class C of L-structures is said to be positively existentially closed in C if every homomorphism from M into an element of C is an immersion.*

The following fact will be used without mention together with Fact 2 to verify that every model of an h-inductive theory has a pec continuation:

Fact 3 [**3**, Théorème 1]. *Every member of an inductive class of models has an existentially closed continuation in the same class.*

Definition 2 [**3**]. *Two h-inductive theories are said to be companions if they have the same h-universal consequences.*

Companionship of models is caracterized using the notion of a pec model.

Fact 4 [**3**, lemme 7]. *Two h-inductive theories are companions if and only if they have the same pec models.*

The analysis of h-inductive theories in [**3**] as well as Fact 2 above show that an h-inductive theory T has a maximal companion, denoted Tk and called the Kaiser enveloppe of T; it is the h-inductive theory of the pec models of T, equivalently Tk is the set of all h-inductive sentences true in the pec models of T. At the opposite extreme, T has a minimal companion, denoted Tu, formed by its h-universal consequences. When parameters from a certain set A are allowed, the notation will be $Tu(A)$ and $Tk(A)$.

An h-inductive theory T is said to be model-complete if all its models are pec, in other words, if the class of pec models is axiomatised by the Kaiser enveloppe Tk. An example of a model-complete theory is that of algebraically closed fields of a fixed characteristic in the language of fields.

Fact 5 [**3**, lemme 5]. *Let T be an h-inductive theory and Tu its h-universal consequences. Then a structure has a continuation that is a model of T if and only if it is a model of Tu.*

It follows from this fact that every structure that has a pec continuation that is a model of T is a model of Tu.

Examples.

- Let $L = \{R\}$ where R is a relation symbol and T the h-inductive theory that states that R

is an equivalence relation. Then T has a unique pec model which is the model with a unique equivalence class consisting of a single element.

- Let L be the language with two relation symbols P and Q and T be the h-universal theory $\{\neg\exists x, y\, P(x) \wedge Q(y)\}$. Then T has exactly two pec models: $\{A\} = \{a\}$ such that $A \models P(a)$ and $B = \{b\}$ such that $B \models Q(b)$.

- Let T be the h-inductive theory of algebraically closed fields of characteristic p in the language $L = \{+, -, ., ^{-1}, 0, 1\}$. A model A of T is pec if and only if it is algebraically closed. Thus, the Kaiser envelope of T is the theory algebraically closed fields of the same characteristic.

Since, in general, every structure that has a continuation which is a model of T is a model of Tu (fact 5), every ring that has a continuation which is a pec model of the theory of fields of characteristic p is a model of the h-universal theory of fields of characteristic p. To illustrate, the ring of integers has this property. Since two theories of fields of distinct characteristics have distinct pec models, the repective h-universal theories determine the characteristics of the fields.

The following conclusion will be useful in various constructions that make use of inductive limits:

FACT 6 [**3**, Lemme 12]. *The class of pec models of an h-inductive theory T is inductive.*

A recent result on pec structures has been proven by Almaz Kungozhin:

FACT 7 [**6**]. *Let L be a relational language and T be a finitely axiomatizable h-universal theory. Then the class of pec models of T is elementary.*

2.3. *Type spaces*

As in every model-theoretic analysis, the notion of type is fundamental in positive model theory. The positive context forces the types under analysis to consist of positive formulas et requires a subtler definition that is the following:

DEFINITION 3 [**3**], [**10**]. *Let T be an h-inductive theory in a language L. An n-type is a maximal set of positive formulas in n variables that is consistent with T or with one of its companions.*
An n-type with parameters in M is a maximal set of positive formulas in n variables with parameters in M, that is consistent with $T(M)$, equivalently with $Tk(M)$.

It is worth emphasizing that one can also define a positive type as the set of positive formulas satisfied by an element of a pec model of an h-inductive theory. This allows to caracterize the pec models by the maximality of the sets of positive formulas that tuples of their elements satisfy:

FACT 8 [**3**]. *A model A of T is pec if and only if for every $\bar{a} \in A$ the set of positive formulas satisfied by $\bar{a}$ is a type.*

From this fact, one deduces that if A is pec and $\bar{a} \in A$ such that $A \models \neg\phi(\bar{a})$, where ϕ is a positive formula, then there exists ψ such that $A \models \psi(\bar{a})$ and $T \vdash \neg\exists x(\phi(\bar{x}) \wedge \psi(\bar{x}))$.

Following the preceding line of thought, when A is a model of an h-inductive theory T and $\bar{a} \in A$, we note $F_A(\bar{a})$ the set of formulas satisfied by $\bar{a}$ in A. Thus, if A is not a pec model, $F_A(\bar{a})$ is not necessarily a type (a maximal set).

The usual notation is adopted to denote types. We denote by $S_n(T)$ (resp. $S_n(M)$) the space of n-types of a theory T (resp. of the theory $T(M)$ with parameters in M). An n-type of $S_n(T)$ (resp. of $S_n(M)$) has a realization in a pec model of T (resp. in an elementary extension of M).

One defines on $S_n(A)$ a topology of which the basis of closed sets is the set of F_f, where f ranges over the entire set of positive formulas, and

$$F_f \;=\; \{\; p \in S_n(A) \mid p \vdash f \;\} \;.$$

The space of positive types is compact (quasi-compact in some mathematical cultures) by Fact 1, but it is not necessarily Hausdorff. In [**10**] Poizat analyzed consequences of the lack of the Hausdorff property. In section 4, we will concentrate on this problem in a systematic way.

2.4. *Positive elementary extensions*

The notion of elementary extension in positive model theory was introduced and analyzed in [**9**]:

Definition 4 [**9**]. *Let M and N be two L-structures such that N is a continuation of M. The structure N is an elementary extension of M if N is a pec member of the class of models of the h-universal theory $Tu(M)$ in the language $L(M)$.*

In [**9**], Poizat proves the following caracterization of positive elementary extensions

Fact 9 [**9**, Lemme 1]. *A continuation N of M is an elementary extension of the latter if and only if the following two conditions are satisfied:*
- (i) *M is immersed in N;*
- (ii) *for every $\bar{b} \in N$, and every positive existential L-formula $f(\bar{x})$ not satisfied by $\bar{b}$ in N, there exists a positive existential formula $g(\bar{x}, \bar{a})$, with parameters $\bar{a} \in M$ that is satisfied by $\bar{b}$ et contradictory with $f(\bar{x})$: the sentence $\neg(\exists \bar{x}, \bar{y}, \bar{z})(f(\bar{x}, \bar{y}) \wedge g(\bar{x}, \bar{z}, \bar{a}))$ belongs to $Tu(M)$ where $f(\bar{x}, \bar{y})$ and $g(\bar{x}, \bar{z}, \bar{a})$ quantifier-free.*

2.5. *Universal extensions*

The notion of universal extension is reminiscent of universal objects in category theory. In our context, inductive limits of universal extensions generalize the notion of saturation and are also relevant for relationshisps with abstract elementary classes (see section 6).

In this section, we will analyze properties of this notion, and in the next one, we will obtain a caracterization of structures that admit a universal extension. In an article in preparation, we will use this notion to obtain "monster models" in connection with positive stability and simplicity.

Definition 5. *Let A and B models of an h-inductive theory T, h a homomorphism from A to B. The pair (B, h) is said to be a universal extension of A if for every model C of T of cardinality at most $|A|$ such that there is a homomorphism from A to C, there exists a* *homomorphism g from C to B such that the following diagram commutes:*

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[ur, "f"] \arrow[r, "h"'] & B \end{tikzcd}$$

REMARK. Let (B, h) be a universal extension of A and g be a homomorphism from B to a model C of T. Then $(C, g \circ h)$ is also a universal extension of A. In particular, A admits a universal extension (B_e, h'), where B_e is a pec model of T.

The following lemma is a positive form of the descending Löwenheim-Skolem theorem. It is slightly modified version of lemma 11 in [**3**].

LEMMA 1. *Let T be an h-inductive theory, A a model of T and B a subset of A. Then there exists a model $B^\star$ of T of cardinality at most $\max(|B|, |L|)$ that contains B and that is immersed in A.*

Proof. The proof is the same as that of lemma 11 in [**3**]. It suffices to note that the structure $B^\star$ obtained at the end of the construction in [**3**] is a model of T. In this vein, suppose that

$$T \vdash \forall \bar{x}[\exists \bar{y} \varphi(\bar{x}, \bar{y}) \to \exists \bar{z} \psi(\bar{x}, \bar{z})] \ .$$

If $B^\star \models \exists \bar{y} \varphi(\bar{a}, \bar{y})$ where $\bar{a} \in B^\star$, then $A \models \exists \bar{y} \varphi(\bar{a}, \bar{y})$, and $A \models \exists \bar{z} \psi(\bar{a}, \bar{z})$. One then deduces from the construction of $B^\star$ that $B^\star \models \exists \bar{z} \psi(\bar{a}, \bar{z})$. Hence, $B^\star \models T$. $\square$

DEFINITION 6. *Let T be an h-inductive theory and α an ordinal. A universal chain of length α of T is an inductive family of models $\{A_i : i < \alpha\}$ (resp. $\{A_i : i \leq \alpha\}$ if α is a successor ordinal) of T with a family of homomorphisms $\{f_{ij} : i \leq j < \alpha\}$ (resp. $\{f_{ij} : i \leq j < \alpha\}$ if α is a sucessor ordinal) such that for every ordinal $\beta < \alpha$, $(A_{\beta+1}, f_{\beta,\beta+1})$ is a universal extension of A_β and that if $\beta \leq \alpha$ is a limit ordinal then A_β is the inductive limit of the A_i with $i < \beta$, $f_{i\beta}$ being defined as the canonical mapping from A_i into A_β.*

LEMMA 2. *Let $\{A_i, f_{ij} : i \leq j < \alpha\}$ be a universal chain of an h-inductive theory T. Then for every limit ordinal $i \leq \alpha$, A_i is a pec model of T. In this case, if $j \leq i$ then h_{ji} is the canonical mapping A_j to A_i for inductive limits, and (A_i, h_{ji}) is a universal extension of A_j.*

Proof. Let A_i be a member of the universal chain with i a limit ordinal. We will first show that A_i is a pec model of T. As A_i is an inductive limit of models of T, A_i is itself a model of T (Fact 2). Let now f be a homomorphism from A_i to a model B of T. Let us suppose that $B \models \varphi(f(\bar{a}))$, where φ is positive formula and $\bar{a} \in A_i$. Then there exists $\beta < i$ such that $\bar{a} \in A_\beta$ and $\bar{a} = f_{\beta,i}(\bar{a})$.

By Lemma 1, there exists B', a model of T generated by $f \circ f_{\beta,i}(A_\beta)$ of cardinality at most $|A_\beta|$ such that $B' \models \varphi(f \circ f_{\beta,i}(\bar{a}))$. As $(A_{\beta+1}, f_{\beta,\beta+1})$ is a universal extension of A_β, there exists an homomorphism h from B' into $A_{\beta+1}$ such that the following diagram commutes:

$$\begin{tikzcd}
A_\beta \arrow[r, "f_{\beta,i}"] \arrow[drr, "f_{\beta,\beta+1}"'] & A \arrow[r, "f"] & B' \arrow[r, "i_d"] \arrow[d, "h"] & B \\
& & A_{\beta+1} \arrow[ul, "f_{\beta+1,i}"] &
\end{tikzcd}$$

As $B' \models \varphi(f(f_{\beta,i}(\bar{a})))$ and $h \circ f \circ f_{\beta,i} = f_{\beta,\beta+1}$, we conclude that $A_{\beta+1} \models \varphi(f_{\beta,\beta+1}(\bar{a}))$. By the definition of an inductive limit, $f_{\beta+1,i} \circ f_{\beta,\beta+1} = f_{\beta,i}$, and so $A \models \varphi(\bar{a})$, which implies that f is an immersion. Hence, A is a pec model of T.

We will now show that for every $\beta < i$, $(A_i, f_{\beta,i})$ is a universal extension of A_β. Let C be a model of T, g a homomorphism from A_β into C and assume that $|C| \leq |A_\beta|$. As $(A_{\beta+1}, f_{\beta,\beta+1})$ is a universal extension of A_β, there exist f and H such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "f"] & A_{\beta+1} \arrow[d, "h"] \\
A_\beta \arrow[u, "g"] \arrow[ur, "f_{\beta,\beta+1}"] \arrow[r, "f_{\beta,i}"'] & A_i
\end{tikzcd}$$

We deduce from the commutativity of the diagram $f_{\beta,i} = h \circ f_{\beta,\beta+1} = h \circ f \circ g$. The conclusion follows. $\square$

3. *Amalgamations*

The possibility of amalgamating the structures in a given class allows a finer study of it. This section continues the analysis of amalgamation techniques initiated in [**3**]. In the first subsection, we will introduce and caracterize the amalgamation bases following [**4**] et [**8**].

The ability to amalgamate, being a property of "maximal" structures, is strongly connected to the analysis of "maximal" h-inductive theories. The second section is devoted to the analysis of amalgamation of models of the Kaiser envelope of an h-inductive theory.

3.1. *Amalgamation bases*

Definition 7. *Let T be an h-inductive theory. A model A of T is said to be an amalgamation basis if for every pair of models B and C of T such that there exist homomorphisms f and g from A to B and C respectively, there exist a model D of T, and f', g' homomorphisms such that the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\
C \arrow[r, "f'"'] & D
\end{tikzcd}$$

A theory is said to have the amalgamation property if each model of T is an amalgamation basis.

We remind that for a structure A, $F_A(\bar{a})$ is the set of positive formulas satisfied by $\bar{a}$ in A.

Lemma 3. *Let T be an h-inductive theory and A a model of T. Then the following properties are equivalent:*

(i) *A is an amalgamation basis;*
(ii) *for every $\bar{a} \in A$, there exists a unique type in $S(T)$ that contains $F_A(\bar{a})$.*

Proof. $((i) \Rightarrow (ii))$ Let $\bar{a} \in A$. We suppose that there exist two distinct types $p \neq q$ in $S(T)$ that contain $F_A(\bar{a})$.

We first show that since $p \vdash F_A(\bar{a})$, there exist B a pec model of T and f a homomorphism from A to B that maps $\bar{a}$ to $\bar{b}$ that realizes p. In this vein, it suffices to show that the family $\Gamma = T \cup \mathrm{Diag}^+(A) \cup p(\bar{a})$ is consistent. Let A' be a model of T that realizes p with $\bar{a}'$ and $\varphi(\bar{a}, \bar{m}) \in \mathrm{Diag}^+(A)$. Then $\exists \bar{y} \varphi(\bar{x}, \bar{y}) \in F_A(\bar{a})$, so $p \vdash \exists \bar{y} \varphi(\bar{x}, \bar{y})$. Hence there exists $\bar{c}'$ in A' such that $A' \models \varphi(\bar{a}', \bar{c}')$, from which follows that the family Γ is consistent. Let B' be a model of Γ and B a pec model of T that is a continuation of B'. Then there is a homomorphism from A into B and $\bar{b}$, which is the image of $\bar{a}$ in B realizes p.

Similarly, there exists C a pec model of T and a homomorphism g from A to C which maps $\bar{a}$ to $\bar{c}$, a realization of q. Since A has the amalgamation property, there exists D a model of T such that the following diagramm commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "f'"] \\ C \arrow[r, "g'"'] & D \end{tikzcd}$$

Thus $f'(\bar{b}) = g'(\bar{c})$, and hence $p = q$, a contradiction.

$((ii) \Rightarrow (i))$ Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

with A a model of T. Let B, C be pec models of T, f, g homomorphisms as in the diagram. Suppose one cannot amalgamate f and g. This would mean that there exists $\bar{a} \in A$ such that $F_B(f(\bar{a}))$ and $F_C(g(\bar{a}))$ (which are types as B and C are pec models of T) are contradictory. This contradicts hypothesis (ii). $\square$

Theorem 1. *Let A be a model of an h-inductive theory T. Then A has a universal extension if and only if A is an amalgamation basis.*

Proof. Let us suppose that A has a universal extension (B, h) and show that A is an amalgamation basis. Let $A_i \models T$, $i = 1, 2$, be two continuations of A by the homomorphisms f_i. In order to verify the amalgamation property, it suffices to show that the following family is consistent:

$$\Gamma \ = \ T \cup \mathrm{Diag}^+(A_1) \cup \mathrm{Diag}^+(A_2) \ ,$$

by interpreting the parameters from A by the same symbols in A_1 and A_2.

We fix a subset $T \cup \Gamma_1 \cup \Gamma_2$ where the Γ_i are finite subsets of $\mathrm{Diag}^+(A_i)$ for $i = 1, 2$. Let $\bar{a}_i$ be the parameters from A_i that are used in Γ_i. By Lemma 1, there exists a model B_i of

$T \cup \Gamma_i$ that contains $A \cup \{\bar{a}_i\}$ and that has the same cardinality as A. We will denote by g_i the homomorphism from A into B_i defined by $g_i(a) = f_i(a)$ for every $a \in A$. Then by definition of universal extensions, one has the following diagram

$$\begin{tikzcd} & B_1 \arrow[d, "h_1"] \\ A \arrow[ur, "g_1"] \arrow[r, "h"'] \arrow[dr, "g_2"'] & B \\ & B_2 \arrow[u, "h_2"'] \end{tikzcd}$$

This implies that B is a model of the set $T \cup \Gamma_1 \cup \Gamma_2$. By positive compactness, Γ is consistent. This proves the existence of the amalgamation being sought for.

In order to prove the other implication, we assume that A is an amalgamation basis. We will show that it has a universal extension. Let Δ be the family of all pairs (M, f) with M a model of T of cardinality at most $|A|$ such that there exists a homomorphism f from A to M. By the Axiom of Choice, we may suppose that Δ is well-ordered. Its order type will be denoted by α.

We will construct an inductive family $\{A_\beta : \beta \leq \alpha\}$ of models of T with a coherent family of homomorphisms $\{h_{i,j} : i \leq j \leq \alpha\}$. The homomorphisms will be indexed by pairs of ordinals up to α. The last member of the sequence, A_α, will be the universal extension that is being sought for.

To start the construction, we set $A_0 = A$, and $h_{0,0}$ is defined as the identity mapping. Since A is an amalgamation basis, there exist A_1, a model of T, and two homomorphisms $h_{0,1}$ and g_0 from A_0 to A_1 and from M_0 to A_1 respectively such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "h_{0,0}"] \arrow[d, "f_0"'] & A_0 \arrow[d, "h_{0,1}"] \\ M_0 \arrow[r, "g_0"'] & A_1 \end{tikzcd}$$

For the inductive step, we assume that the family $\{A_\beta : \beta < \gamma \leq \alpha\}$ with the corresponding homomorphisms has been constructed. If γ is a successor of the form $\beta + 1$, there exist a model A_γ of T, homomorphisms $h_{\beta,\beta+1}$ and g_β such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "h_{0,\beta}"] \arrow[d, "f_\beta"'] & A_\beta \arrow[d, "h_{\beta,\beta+1}"] \\ M_\beta \arrow[r, "g_\beta"'] & A_{\beta+1} \end{tikzcd}$$

For every $i \leq \beta$, we set $h_{i,\beta+1} = h_{\beta,\beta+1} \circ h_{i,\beta}$. The coherence of the homomorphisms already constructed inductively implies that the new family is still coherent. In other words, we continue to have an inductive family of models of T.

If γ is a limit ordinal, then one defines A_γ as the inductive limit of the already constructed inductive family. As for the new homomorphisms, for every $i < \gamma$, $h_{i,\gamma}$ is the natural mapping from A_i to A_γ. The new family of models and homomorphisms is also inductive.

The construction ends when α is reached. By construction, either α is a successor and thus A_α is constructed as in the inductive step for successors, or α is limit and A_α is the inductive limit of the family $\{A_i : i < \alpha\}$.

To finish the proof, we show that $(A_\alpha, h_{0,\alpha})$ is a universal extension of A. Let M be a model of T of cardinality at most $|A|$ such that there exists a homomorphism f from A into M. By the definition of the family Δ, there exists $\beta \leq \alpha$ such that $(M, f) = (M_\beta, f_\beta)$. If $\beta = \alpha$, then the identity mapping on A_α suffices. Otherwise, $\beta < \alpha$ and by construction, the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "h_{0,\beta}"] \arrow[d, "f_\beta"'] & A_\beta \arrow[d, "h_{\beta,\beta+1}"] & \\ M_\beta \arrow[r, "g_\beta"'] & A_{\beta+1} \arrow[r, "h_{\beta+1,\alpha}"'] & A_\alpha \end{tikzcd}$$

The equalities $h_{0,\alpha} = h_{\beta+1,\alpha} \circ h_{\beta,\beta+1} \circ h_{0,\beta} = h_{\beta+1,\alpha} \circ g_\beta \circ f_\beta$ that follow from this diagram yield the desired conclusion. □

3.2. *Amalgamations in models of Kaiser envelopes*

In earlier works on positive model theory, the existence of amalgamations is frequently analyzed in the context of h-universal theories. Here, it will be necessary to extend the context to Kaiser envelopes. To start, we will prove a slightly modified version of the so-called "asymmetric amalgamation" of Ben Yaacov and Poizat (Lemma 8 in [**3**]).

LEMMA 4. *Let A, B, C be L-structures, g an immersion from A into B, and h a homomorphism from A to C. Then, there exist a model D of $Tk(C)$, a homomorphism g' from B to D and an immersion h' from C into D, such that $g' \circ g = h' \circ h$.*

Proof. We use the same symbols to name the elements of A in B and C. The proof consists in showing that the set

$$Tk(C) \cup \mathrm{Diag}^+(B)$$

of sentences is consistent. In this vein, let $f(\bar{a}, \bar{b})$ be in $\mathrm{Diag}^+(B)$ with $\bar{a} \in A$ and $\bar{b} \in B$. Then $A \models \exists \bar{y} f(\bar{a}, \bar{y})$ since A is immersed in B. Hence, one can interpret $\bar{b}$ by an element of A. The final formula belongs to $Tk(C)$. □

This lemma has the following corollary mentioned in [**3**] in a different form.

COROLLARY 1. *The pec models of an h-inductive theory are amalgamation bases.*

We deduce the following connection with universal extensions.

COROLLARY 2. *Let A_e be a pec model of an h-inductive theory. Then A_e has a universal extension (B_e, i), where B_e is another pec model and i an immersion from A_e into B_e.*

Proof. Since every pec model is an amalgamation basis, the corollary follows from Theorem 1. □

The following lemma and its corollary, fundamental for section 4, are also of independant interest.

LEMMA 5. *Let A be an L-structure, B a model of $Tk(A)$ and C an L-structure in which A is immersed. Then there exist a model D of $Tk(C)$ and two immersions φ and ψ such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "im"] \arrow[d, "im"'] & B \arrow[d, "\varphi"] \\ C \arrow[r, "\psi"'] & D \end{tikzcd}$$

Proof. We name the elements of A in B and C by the same symbols and note $L^\star$ the enlarged language. The proof of the theorem consists in showing that the following set of h-inductive sentences is consistent:

$$\Gamma \ = \ Tk(C) \cup Tu(B) \cup \mathrm{Diag}^+(B) \ .$$

Let $F = \{\chi, f(\bar{\beta}, \bar{b}), \neg\exists\bar{y}g(\bar{y}, \bar{b})\}$ be a finite subset of Γ where $\chi \in Tk(C)$, $f(\bar{\beta}, \bar{b}) \in diag^+(B)$ and $\neg\exists\bar{y}g(\bar{y}, \bar{b}) \in Tu(B)$.

As $B \models \neg\exists\bar{y}g(\bar{y}, \bar{b})$ and $B \models \exists\bar{x}f(\bar{x}, \bar{b})$, we conclude that the h-inductive sentence

$$\forall\bar{z}[\exists\bar{x}f(\bar{x}, \bar{z}) \to \exists\bar{y}g(\bar{y}, \bar{z})]$$

does not belong to $Tk(B)$, and thus nor to $Tk(A)$. This implies that one can find $\bar{a} \in A$ such that $A \models \neg\exists\bar{y}g(\bar{y}, \bar{a})$, and $A \models \exists\bar{x}f(\bar{x}, \bar{a})$. It follows that we can interpret the sentences in A and thus in C, and hence conclude that Γ is consistent. □

Corollary 3. *Let A be an L-structure and B a model of $Tk(A)$. Then every model of $Tk(A)$ is immersed in a model of $Tk(B)$, and every model of $Tk(B)$ is immersed in a model of $Tk(A)$.*

4. *Hausdorff type spaces and elementary extensions*

This section is devoted to the analysis of topological properties of spaces of positive types of an h-inductive theory. The main theorem, that answers a question of Poizat, is concerned with the Hausdorff property of type spaces. Its proof depends heavily on amalgamation techniques developed in earlier sections.

Definition 8 **[10]**. *An h-inductive theory T (resp. a structure M), is said to be Hausdorff if and only if for every natural number n, the space $S_n(T)$ (resp. $S_n(M)$) is Hausdorff.*

Such a definition would be useless if negation were in the language. But, the exclusion of negation, which makes the topology of S_n closer to the Zariski topology in algebraic geometry, yields rapidly examples of h-inductive theories whose type spaces are not Hausdorff (see the example after Lemma 6).

A natural question is the connection between the Hausdorff property of an h-inductive theory and those of its individual models. This necessitates the analysis of the preservation of the Hausdorff property when one goes to elementary extensions or restrictions. An affirmative answer concerning the passage to elementary extensions was proven by Poizat in **[10]**. The main result of this section gives an affirmative answer for passage to elementary substructures.

We start with a technical notion introduced in **[4]**, (Section 8.5).

Definition 9. *Let T be an h-inductive theory and φ a positive formula. The resultant of φ, denoted by $Res_T(\varphi)$, is the set of positive formulas ψ sucht that $T \vdash \neg\exists x(\varphi(x) \wedge \psi(x))$.*

Lemma 6. *Let T be an h-inductive theory and $S_n(T)$ be its space of n-types. Then $S_n(T)$ is Hausdorff if and only if for every pair of distinct types $p, q \in S_n(T)$, there exist two positive formulas f and g such that $p \vdash f$, $q \vdash g$, and every formula in $Res_T(f)$ is contradictory to every formula in $Res_T(g)$.*

Proof. Let O_f and O_g be two basic open sets in $S_n(T)$, in other words, $O_f = \{r \in S_n(T) : r \nvdash f\}$, and similarly $O_g = \{r \in S_n(T) : r \nvdash g\}$. Equivalently, $O_f = \cup_{h \in Res_T(f)} F_h$ where F_h is nothing but the closed set defined by h, and similarly for O_g. The topology on $S_n(T)$ is Hausdorff if and only if there exist f, g such that $p \vdash f$, $q \vdash g$ and $O_f \cap O_g = \emptyset$. This is equivalent to the conclusion of the lemma. □

Before going any further, we will use this lemma to illustrate an example of a non-Hausdorff theory. A slightly different version of this example was given at the end of [**3**]. Let $L = \{R_i : i < \omega\}$ be relational language. The h-inductive theory T assures that for every $i < \omega$, $Res_T(R_i)$ contains all but finitely many of the R_j, $j \neq i$. Then, by Lemma 6, T is not Hausdorff.

In [**3**], the following caracterization of the Hausdorff property of type spaces was shown:

Fact 10 [**3**, Théorème 20]. *The spaces of type of an h-inductive theory are Hausdorff if and only if one can amalgamate the homomorphisms between models of the Kaiser envelope Tk; i.e. for any three models M_1, M_2 and M_3 of Tk, such that there is a homomorphism f from M_1 to M_2 and a homomorphism g from M_1 to M_3, there exist M_4 a model of Tk and s, h homomorphisms such that the following diagram commutes:*

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "g"'] & M_2 \arrow[d, "s"] \\ M_3 \arrow[r, "h"'] & M_3 \end{tikzcd}$$

The following corollaries offer example of Hausdorff h-inductive theories.

Corollary 4. *Every model-complete h-inductive theory is Hausdorff.*

Proof. As T is model-complete, by definition its class of pec models is elementary and axiomatized by Tk. Consequently, every model of Tk is pec. The conclusion follows from Corollary 1 and Fact 10. □

Corollary 5. *An h-inductive theory that has the amalgamation property is Hausdorff.*

Corollary 6. *Let L be a relational language and T a finitely axiomatizable h-universal theory. Then T is Hausdorff.*

Proof. By Fact 7, Tk is model-complete. The conclusion follows from Corollary 4. □

We use this corollary to verify that an example in [**6**] is a Hausdorff theory. Let L be a language that contains a single relational predicate R and let T be the h-universal theory $\{\neg\exists xy\ R(x, y) \wedge R(y, x)\}$. By Corollary 6, T is Hausdorff.

We will now attack the question of preservation of Hausdorffness. In [**10**], Poizat makes the following remark:

Fact 11 [**10**]. *An elementary extension of a Hausdorff structure is also Hausdorff.*

The reverse implication was left as an open problem in [**10**]. Theorem 2 below answers affirmatively this question. Amalgamation in classes of models of Kaiser enveloppes will be a major tool in the proof (Lemma 5 and Corollary 3).

Theorem 2. *An elementary substructure of a Hausdorff structure is Hausdorff.*

Proof. The main point of the proof will be to replace models of $Tk(M)$ with models of $Tk(N)$ in order to be able to use the amalgamation property of the latter and Fact 10.

Let M_1, M_2, M_3 three models of $Tk(M)$, φ_2 (resp. φ_3) a homomorphism from M_1 to M_2 (resp. from M_1 to M_3). By Corollary 3, there exists N_1, a model of $Tk(N)$ such that M_1 is immersed in N_1 and that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "im"] \arrow[d, "im"'] & M_1 \arrow[d, "i_1"] \\ N \arrow[r, "im"'] & N_1 \end{tikzcd}$$

As M_1 is immersed in N_1, an application of the asymmetric amalgamation yields the following commutative diagram:

$$\begin{tikzcd} M_1 \arrow[r, "\varphi_2"] \arrow[d, "i_1"'] & M_2 \arrow[d, "i_2"] \\ N_1 \arrow[r, "{\varphi'}_2"'] & M' \end{tikzcd}$$

with M' a model of $Tk(M_2)$, and thus a model of $Tk(M)$. The following commutative diagram illustrates this.

$$\begin{tikzcd} M \arrow[r, "im"] \arrow[d, "im"'] & M_1 \arrow[r, "\varphi_2"] \arrow[d, "i_1"] & M_2 \arrow[d, "i_2"] \\ N \arrow[r, "i"'] & N_1 \arrow[r, "{\varphi'}_2"'] & M' \end{tikzcd}$$

On this diagram, one remarks that the mapping ${\varphi'}_2 \circ i$ defined from N to M' is an immersion because N is a pec model of $Tk(M)$ and $M' \models Tk(M)$, which implies that M' is a model of $Tu(N)$. This allows us to find a continuation N_2 of M' that is a pec model of $Tu(N)$ (Fact 3). Since it is pec, it is also a model of $Tk(N)$. We thus obtain the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "im"] \arrow[d, "im"'] & M_1 \arrow[r, "\varphi_2"] \arrow[d, "i_1"] & M_2 \arrow[d, "i_2"] \arrow[dr, "f_2 \circ i_2"] & \\ N \arrow[r, "i"'] & N_1 \arrow[r, "{\varphi'}_2"'] & M' \arrow[r, "f_2"'] & N_2 \end{tikzcd}$$

We repeat the same construction for M_3 and obtain the following commutative diagram

$$\begin{tikzcd}
M \arrow[r, "im"] \arrow[d, "im"'] & M_1 \arrow[r, "\varphi_3"] \arrow[d, "i_1"] & M_3 \arrow[d, "i_3"] \arrow[dr, "f_3 \circ i_3"] & \\
N \arrow[r, "i"'] & N_1 \arrow[r, "{\varphi'}_3"'] & M'' \arrow[r, "f_3"'] & N_3
\end{tikzcd}$$

where M'' is a model of $Tk(M)$, N_3 a model of $Tk(N)$ and f_3 a homomorphism. We then have the following diagram:

$$\begin{tikzcd}
& M_2 \arrow[r, "f_2 \circ i_2"] & N_2 \\
M_1 \arrow[ur, "\varphi_2"] \arrow[r, "i_1"] \arrow[dr, "\varphi_3"'] & N_1 \arrow[ur, "f_2 \circ \varphi_2'"'] \arrow[dr, "f_3 \circ \varphi_3'"] & \\
& M_3 \arrow[r, "f_3 \circ i_3"'] & N_3
\end{tikzcd}$$

The amalgamation in models of $Tk(N)$ (Fact 10) yields the following commutative diagram

$$\begin{tikzcd}
N_1 \arrow[r, "f_2 \circ \varphi_2'"] \arrow[d, "f_3 \circ \varphi_3'"'] & N_2 \arrow[d, "\psi_2"] \\
N_3 \arrow[r, "\psi_3"'] & N'
\end{tikzcd}$$

where N' is a model of $Tk(N)$, and thus of $Tk(M)$ as well. It follows from this that

$$\psi_2 \circ f_2 \circ \varphi_2' \circ i_1 = \psi_3 \circ f_3 \circ \varphi_3' \circ i_1 \ .$$

This implies

$$\psi_2 \circ f_2 \circ i_2 \circ \varphi_2 = \psi_3 \circ f_3 \circ i_3 \circ \varphi_3 \ .$$

The following commutative diagram illustrates this construction:

$$\begin{tikzcd}
& M_2 \arrow[r, "f_2 \circ i_2"] & N_2 \arrow[dr, "\psi_2"] & \\
M_1 \arrow[ur, "\varphi_2"] \arrow[r, "i_1"] \arrow[dr, "\varphi_3"'] & N_1 \arrow[ur, "f_2 \circ \varphi_2'"'] \arrow[dr, "f_3 \circ \varphi_3'"] & & N' \\
& M_3 \arrow[r, "f_3 \circ i_3"'] & N_3 \arrow[ur, "\psi_3"'] &
\end{tikzcd}$$

Finally, we conclude the following commutative diagram in the class of models of $Tk(M)$:

$$\begin{tikzcd}
M_1 \arrow[r, "\varphi_2"] \arrow[d, "\varphi_3"'] & M_2 \arrow[d, "\psi_2 \circ f_2 \circ i_2"] \\
M_3 \arrow[r, "\psi_3 \circ f_3 \circ i_3"'] & N'
\end{tikzcd}$$

The theorem follows from Fact 10. $\square$

We end this section with an example that shows that the topology of a space of types is too weak to determine all properties of an h-inductive theory. The theory in question will be Hausdorff but the class of its pec models will not be elementary.

EXAMPLE. Let L be the relational language $\{P_i, R_i : i < \omega\}$, T be the h-universal theory $\{\neg\exists x P_i(x) \wedge P_j(x), \forall x P_i(x) \vee R_i(x), \neg\exists x P_i(x) \wedge R_i(x) \mid i \neq j,\, i, j < \omega\}$. A pec model A of T has the following properties:

1. For every $i < \omega$ the h-inductive sentences $\forall x, y(P_i(x) \wedge P_i(y) \rightarrow x = y)$ belong to Tk, because otherwise one can continue A into a model of T using a homomorphism that maps x and y to the same image. Such a homomorphism would not be an immersion and this would keep A from being pec.

2. For every $i < \omega$, $A \models \exists x P_i(x)$. Indeed, one can continue A in a model that satisfies $\exists x P_i(x)$. But, A is pec, and thus this sentence is also true in A. In particular, since the P_i are mutually incompatible, A is necessarily infinite.

There exist exactly two pec models of T: either $A = \{a_i, \mid i < \omega\}$ such that for every $i < \omega$, one has $A \models P_i(a_i)$, or $B = A \cup \{x\}$ where x satisfies R_i for every $i < \omega$, and A is the first pec model.

It follows from the classical Lǿwenheim-Skolem theorem that the class of pec models is not elementary. Equivalently, Tk is not model-complete. But T is Hausdorff by Fact 10. Indeed, as an arbitrary model of Tk is a pec model of T augmented by a possibly empty set of points that satisfy no P_i but all the R_i, the amalgamation in models of Tk is equivalent to compressing these points.

5. *Positive Robinson theories and quantifier elimination*

In this section, we will discuss quantifier elimination in the positive context. The determination of positive types by their quantifier-free parts will play an important role. More generally, the "density" of the quantifier-free positive formulas within the set of positive formulas satisfied by an element in an arbitrary model of an h-inductive theory characterizes the general notion of elimination (Definition 13).

The characterizations of quantifier elimination vary according to classes of models and companion theories in question. In the case where the analysis is done within the class of pec models, one deals with a positive Robinson theory, notion of which precursors are in [**5**] and [**1**] (see in particular Lemmas 8 and 9 below). In the general case, a similar analysis is carried out on all models of an h-inductive theory (Definition 13), and the final characterization for the h-universal theories is obtained using Theorem 3.

By definition, an embedding is an equivalence of quantifier-free types. Hence, in the case of a theory that assigns heavier weight to its quantifier-free formulas, it is natural that embeddings are closer to immersions than in general. This aspect of elimination is described by the notions of an h-maximal model (Definition 10) and of a weakly pec model (Definition 12).

DEFINITION 10 [**6**]. *Let T be an h-inductive theory. A model A of T is said to be h-maximal if every homomorphism from A to another model of T is an embedding.*

LEMMA 7. *The class of h-maximal models of an h-inductive theory is inductive.*

Proof. Let T be an h-inductive theory, α a limit ordinal and $\{M_i; f_{ij} \mid i \geq j, i, j < \alpha\}$ an inductive family of h-maximal models of T of which the inductive limit will be denoted M.

We will show that M is h-maximal. Since M is an inductive limit of models of T, M is model of T (Fact 2). Let $N \models T$ and f a homomorphism from M to N. For every $\bar{m} \in M$, there exists $i < \alpha$ such that $\bar{m} \in M_i$. Let h_i be the canonical homomorphism from M_i into M, suppose that $N \models \varphi(f(\bar{m}))$, where φ is a free positive formula. As M_i is h-maximal, the homomorphism $f \circ h_i$ is an embedding. But φ is a quantifier-free formula, thus $M_i \models \varphi(\bar{m})$ and $M \models \varphi(\bar{m})$. Hence, f is an embedding. □

We will denote by Tm the h-inductive theory of the h-maximal models of an h-inductive theory T. Note that $Tu \subset Tm \subset Tk$.

Corollary 7. *Let T be an h-inductive theory. The class of h-maximal models of T is elementary if and only if it is axiomatized by Tm.*

Proof. Let T' denote the theory axiomatizing the h-maximal models of T. Then $M \models T'$ if and only if M is h-maximal, and so $M \models Tm$. Hence, $T' \vdash Tm$. As for the reverse implication, by Fact 2, T' is an h-inductive theory. Since Tm is the set of h-inductive sentences that are true in all h-maximal models of T, one concludes that $Tm \vdash T'$. □

Let A be a structure and $\bar{a} \in A$. We will denote by $tpsq(\bar{a})$ the set of positive quantifier-free formulas satisfied by $\bar{a}$ in A.

Definition 11. *Let T be an h-inductive theory. The theory T is said to be positive Robinson if it satisfies the following condition:*
for any two pec models A and B of T, if $\bar{a} \in A$, $\bar{b} \in B$ and $tpsq(\bar{a}) \subset tpsq(\bar{b})$, then $tp(\bar{a}) = tp(\bar{b})$.

Remarks. 1. This definition is equivalent to saying that in the pec models of a positive Robinson theory, the types are entirely determined by their quantifier-free parts.

2. An h-inductive theory is positive Robinson if and only if it has a companion that has this property.

Example. The theory of fields of a given characteristic is a positive Robinson theory because its maximal h-inductive companions has as models the algebraically closed fields of the same characteristic.

Lemma 8. *An h-inductive theory T is positive Robinson if and only if it satisfies the following condition:*
for every pec model A of T, for every quantifier-free positive formula $\varphi(\bar{x})$ and $\bar{a} \in A$, $A \models \neg\varphi(\bar{a})$ if and only if there exists a quantifier-free positive formula $\psi(\bar{x})$ such that $A \models \psi(\bar{a})$ and $T \vdash \neg\exists\bar{x}\psi(\bar{x}) \wedge \varphi(\bar{x})$.

Proof. Let A be a pec model of T and $\bar{a} \in A$. Let us suppose that $A \models \neg\varphi(\bar{a})$. This implies $\varphi(\bar{x})$ does not belong to the type of $\bar{a}$. We first show that $T \cup tpsq(\bar{a}) \cup \{\varphi(\bar{a})\}$ is inconsistent. If not, then there exists a pec model B and $\bar{b} \in B$ such that $B \models \varphi(\bar{b})$ and $tpsq(\bar{a}) \subset tp(b)$. Since T is positive Robinson, $tp(\bar{a}) = tp(\bar{b})$, a contradiction. It follows that there exists a finite subset $\psi(\bar{x})$ of $tpsq(\bar{a})$ such that $T \vdash \neg\exists\bar{x}\psi(\bar{x}) \wedge \varphi(\bar{x})$.

For the reverse direction, we assume that for every positive formula φ, Res_φ is equivalent modulo T to a set of quantifier-free positive formulas. Let A and B two pec models of T, $\bar{a} \in A$ and $\bar{b} \in B$ such that $tpsq(\bar{a}) \subset tpsq(\bar{b})$. Let φ be a positive formula such that $A \models \neg\varphi(\bar{a})$, then by hypothesis, there exists a positive quantifier-free formula $\psi(\bar{x})$ such that $A \models \psi(\bar{a})$ and $T \vdash \neg\exists\bar{x}\psi(\bar{x}) \wedge \varphi(\bar{x})$. Since $tpsq(\bar{a}) \subset tpsq(\bar{b})$, $B \models \psi(\bar{b})$. This implies that $B \models \neg\varphi(\bar{b})$. It follows that $tp(\bar{b}) \subset tp(\bar{a})$. By the maximality of positive types, we deduce that $tp(\bar{a}) = tp(\bar{b})$. $\square$

Corollary 8. *Let T be a positive Robinson theory and A a model of T. Then A is h-maximal if and only if it satisfies the following condition:*
for every quantifier-free positive formula $\varphi(\bar{x})$ and $\bar{a} \in A$, $A \models \neg\varphi(\bar{a})$ if and only if there exists a quantifier-free positive formula $\psi(\bar{x})$ such that $A \models \psi(\bar{a})$ and $T \vdash \neg\exists\bar{x}\psi(\bar{x}) \wedge \varphi(\bar{x})$.

Proof. Let A be an h-maximal and B a pec model of T such that A embeds in B. We assume that $A \models \neg\varphi(\bar{a})$. Thus, $B \models \neg\varphi(\bar{a})$. Since B is a pec model, by Lemma 8 there exists $\psi(\bar{x})$ a quantifier-free positive formula such that $T \vdash \neg\exists\bar{x}\psi(\bar{x}) \wedge \varphi(\bar{x})$ et $B \models \psi(\bar{a})$. This implies that $A \models \psi(\bar{a})$. In the reverse direction, every model of T that satisfies the condition above is h-maximal. $\square$

Lemma 9. *Let T be a positive Robinson theory. Then the following conditions are satisfied:*
1. Every model of T that embeds in a pec model of T is h-maximal.
2. The h-maximal models of T have the amalgamation property.
Moreover, if T is h-universal then these two conditions are sufficient to conclude that T is a positive Robinson theory.

It is worth noting that the second condition of Lemma 9 shows that the h-maximal models of a positive Robinson theory are amalgamation bases.

Proof. Let A, B, C be three models of T such that there is an embedding i from A into B and a homomorphism f from A into C, and B is a pec model. Let Ce be a pec model of T such that there is a homomorphism j from C to Ce:

$$\begin{tikzcd}
 & B & \\
A \arrow[ur, "i"] \arrow[dr, "f"'] & & \\
 & C \arrow[r, "j"'] & Ce
\end{tikzcd}$$

For any $\bar{a} \in A$, we have that $tpsq(\bar{a}) = tpsq(i(\bar{a}))$, and $tpsq(\bar{a}) \subset tpsq(f(\bar{a})) \subset tpsq(j \circ f(\bar{a}))$. Since T is positive Robinson, B and Ce are pec models of T and $tpsq(i(\bar{a})) \subset tpsq(j \circ f(\bar{a}))$, $tpsq(i(\bar{a})) = tpsq(j \circ f(\bar{a}))$. Hence, $tpsq(\bar{a}) = tpsq(f(\bar{a}))$, which implies that f is an embedding.

We will now show the amalgamation property for h-maximal models. Let A, B and C be h-maximal models of T with i and j embeddings from A into B and C respectively. Let Be and Ce be pec models of T that are continuations of B and C respectively. We then have the following diagram:

$$\begin{tikzcd} & B \arrow[r, "f"] & Be \\ A \arrow[ur, "i"] \arrow[dr, "j"'] & & \\ & C \arrow[r, "g"'] & Ce \end{tikzcd}$$

For every $\bar{a} \in A$, $tpsq(\bar{a}) = tpsq(f \circ i(\bar{a})) = tpsq(g \circ j(\bar{a}))$. Thus, $\bar{a}$ has the same type p in Be and Ce.

In order to complete the amalgamation argument, we will show using positive compactness that $T \cup D^+(Be) \cup D^+(Ce)$ is consistent. Let $\varphi(\bar{a}, \bar{b}) \in D^+(Be)$ and $\psi(\bar{a}, \bar{c}) \in D^+(Ce)$, where $\bar{a}$ is the tuple of parameters that belong to A. Then, $\exists y \varphi(x, y)$ and $\exists z \psi(x, z)$ belong to p. As a result, there exists $\bar{c}' \in Ce$ such that $\varphi(\bar{a}, \bar{c}') \wedge \psi(\bar{a}, \bar{c}) \in D^+(Ce)$, and the consistency follows. It follows that $T \cup D^+(Be) \cup D^+(Ce)$ has a model D that one can continue to an h-maximal model of T (a pec model of T for example). The amalgamation property for the h-maximal models follows.

Now, we assume that T is h-universal and the two conditions in the statement hold. We will prove that T is positive Robinson. Let A be a pec model of T, $\bar{a}, \bar{b} \in A$ such that $tpsq(\bar{a}) \subset tpsq(\bar{b})$. Let $\langle \bar{a} \rangle$ be the substructure of A generated by $\bar{a}$. As T is h-universal, $\langle \bar{a} \rangle \models T$. Since the inclusion $\langle \bar{a} \rangle$ in A is an embedding, $\langle \bar{a} \rangle$ is an h-maximal model of T by condition 1. Hence, the homomorphism f from $\langle \bar{a} \rangle$ into A that maps $\bar{a}$ to $\bar{b}$ is an embedding. The amalgamation property of the h-maximal models (condition 2) shows that there exists B that may be chosen to be a pec model such that the following diagram commutes:

$$\begin{tikzcd} \langle \bar{a} \rangle \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[d, "h"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

Thus $h(\bar{a}) = g(\bar{b})$. Note that, since A is a pec model, g, h are immersions. Hence, $\bar{a}$ and $\bar{b}$ have the same type. $\square$

Remark. If every model of T that embeds in a pec model is an h-maximal, then every model of T that embeds in an h-maximal is an h-maximal.

Corollary 9. *If T is a positive Robinson theory of which the class of h-maximal models is elementary, then T is Hausdorff.*

Proof. As T is positive Robinson, its h-maximal models have the amalgamation property, i.e. they satisfy condition 2 of Lemma 9. Since the class of h-maximal models is elementary, it is axiomatized by the h-inductive theory Tm (Corollary 7). Let M_1, M_2, M_3 be models of Tk with f and g homomorphisms from M_1 to M_2 and M_3 respectively. Since $Tm \subset Tk$, these three models are h-maximal. As a result, there exists a model N of Tm and a pec continuation

M of N such that the following diagram commutes

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "g"'] & M_2 \arrow[d] & \\ M_3 \arrow[r] & N \arrow[r] & M \end{tikzcd}$$

Since $M \models Tk$, the Hausdorff property of T follows from Fact 10. □

In the rest of this section, we will extend the preceding discussion to all models of an h-inductive theory. The notion of a weakly pec model and the property EQ will be crucial.

Definition 12. *Let T be an h-inductive theory. A model A of T is said to be weakly pec if every embedding from A into a model of T is an immersion.*

We first refine a notation already introduced. For an h-inductive theory T, a model M of T and $\bar{a} \in M$, we will denote by $f_M(\bar{a})$ the set of quantifier-free positive formulas satisfied by $\bar{a}$ in M.

Definition 13. *An h-inductive theory T is said to have the property EQ if it satisfies the following hypothesis:*
for every pair of models A and B of T, $\bar{a} \in A$ and $\bar{b} \in B$, $f_A(\bar{a}) = f_B(\bar{b})$ if and only if $F_A(\bar{a}) = F_B(\bar{b})$.

The property EQ will allow us to characterize the elimination of quantifiers in h-universal theories. We start with a general lemma:

Lemma 10. *If an h-inductive theory T has the property EQ, then every embedding between models of T is an immersion. In particular, every model of T is weakly pec.*

Proof. We assume that T has the property EQ. Let A and B be two models of T, i an embedding of from A into B, and $\bar{a} \in A$. Then $\bar{a}$ and $i(\bar{a})$ satisfy the same quantifier free positive formulas. They satisfy the same positive formulas since T has the property EQ. It follows that i is an immersion, and one concludes that every model of T is weakly pec. □

Corollary 10. *If T is a theory having the property EQ, then every h-maximal model of T is pec.*

Corollary 11. *If T is an h-universal theory having the property EQ, then T is a positive Robinson theory.*

Proof. Since T has the property EQ, Lemma 10 shows that every embedding between models of T is an immersion. Subsequently, every model A of T that embeds in a pec model of T is a pec model; it is in particular h-maximal.
By Corollary 10, every h-maximal model of T is pec. The amalgamation property for h-maximal models follows from Corollary 1.
The conclusion of the corollary follows from Lemma 9. □

THEOREM 3. *Let T be an h-universal theory. Then the following conditions are equivalent:*

1. T has the property EQ;

2. every model of T is weakly pec;

3. every positive formula is equivalent modulo T to a quantifier-free positive formula.

Proof. $(1 \Rightarrow 2)$ This is Lemma 10.

$(2 \Rightarrow 1)$ Let $\bar{a} \in A$, $\bar{b} \in B$ such that $f_A(\bar{a}) = f_B(\bar{b})$ and $\langle \bar{a} \rangle$ the substructure of A generated by $\bar{a}$. Since T is h-universal, $\langle \bar{a} \rangle$ is a model of T. It embeds in A through the inclusion mapping that we will denote by i and in B through the embedding that maps $\bar{a}$ onto $\bar{b}$ and that we will denote by j. By hypothesis 2, the embeddings i and j are immersions. By the asymmetric amalgamation (Lemma 4), there exist a model D of T together with an immersion f and a homomorphism g that make the following diagram commute:

$$\begin{tikzcd} \langle \bar{a} \rangle \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

If $B \models \exists \bar{y} \varphi(\bar{b}, \bar{y})$, where $\bar{b} = j(\bar{a})$, then $B \models \exists \bar{y} \varphi(j(\bar{a}), \bar{y})$, and $D \models \exists \bar{y} \varphi(g \circ j(\bar{a}), \bar{y})$. Since f is an immersion, $A \models \exists \bar{y} \varphi(a, \bar{y})$, thus $F_B(\bar{b}) \subset F_A(\bar{a})$.

In order to show that $F_A(\bar{a}) \subset F_B(\bar{b})$ one redoes the same argument on the commutative diagram

$$\begin{tikzcd} \langle \bar{a} \rangle \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

this time with g as an immersion.

$(3 \Rightarrow 2)$ We assume 3. Then every embedding is an immersion, thus every model of T is weakly pec.

The idea of the proof of $(1 \Rightarrow 3)$ is from [**4**] (Lemma 8.4.8).

$(1 \Rightarrow 3)$ Let φ be a positive formula, Δ be the set of quantifier-free positive formulas ψ such that $T \vdash \varphi \to \psi$. We will show that $T \cup \Delta(x) \vdash \varphi(x)$ in the language $L \cup \{x\}$. Let B be a model of $T \cup \Delta(x)$. Let Γ be the set of quantifier-free positive formulas χ such that $B \models \neg\chi(x)$ and T' be the theory $T \cup \{\varphi(x)\} \cup \{\neg\chi(x) \mid \chi \in \Gamma\}$.

Suppose towards a contradiction that T' is not consistent. Then there exists a quantifier-free positive formula χ such that $T \vdash \varphi(x) \to \chi(x)$. By the definition of Δ, one concludes $\chi \in \Delta$. Since $B \models \neg\chi(x)$, we reach a contradiction with the fact that B is a model of Δ. Hence, T' is consistent.

Let A be a model of T', C the substructure of A generated by the constants in the language $L \cup \{x\}$. So C embeds in A, and as T is h-universal, $C \models T$. By Lemma 10, this embedding is an immersion.

On the other hand, the mapping j from C to B that maps every constant of L onto itself and x onto x is a homomorphism. Indeed, suppose $C \models \alpha(x, a)$ and $B \models \neg\alpha(x, a)$, with x, a constants of the language $L \cup \{x\}$, and α a quantifier-free positive formula. The fact that $B \models \neg\alpha(x, a)$ implies that $\alpha(x, a) \in \Gamma$. As a result, $C \models \neg\alpha(x, a)$, a contradiction. Hence j is a homomorphism. We obtain the following diagram:

$$\begin{tikzcd} & A \\ C \arrow[ur, "im"] \arrow[dr, "j"'] & \\ & B \end{tikzcd}$$

where im is an immersion and j a homomorphism. Since $A \vdash \varphi(x)$, $C \vdash \varphi(x)$, and thus $B \vdash \varphi(x)$. Hence, $T \cup \Delta(x) \vdash \varphi(x)$.

The conclusion of the preceding paragraph et positive compactness imply that there exists $\psi \in \Delta$ such that $T \vdash \psi \to \varphi$. By the definition of Δ, we conclude that $T \vdash \varphi \leftrightarrow \psi$. □

6. *Complete theories, abstract elementary classes*

In this final section, we will discuss general aspects of h-inductive theories and connections with classes of models of such theories. In the first subsection, we will extend the analysis of **[3]** on complete theories, and in the second one, we will investigate connections with abstract elementary classes in the sense of Shelah.

6.1. *Complete theories*

In **[3]** Ben Yaacov and Poizat introduced the notion of a complete theory as the h-universal theory of a structure and showed that in the case of an h-universal theory this notion is equivalent to the joint continuation property defined below. In this section, we will pursue their approach and analyze completions of an arbitrary h-inductive theory.

Definition 14. *An h-inductive theory T is said to be complete if it has the joint continuation property:*
for any two models of T, there exists a third model C of T that is a continuation of both A and B.

Fact 12 **[3]**. *An h-inductive theory is complete if and only if it has a companion that is complete.*

Lemma 11. *An h-inductive theory that has a unique pec model is complete.*

Proof. Every model of T has a pec continuation. Since there is only one such, the joint continuation condition is satisfied. □

Examples.

- Let L consist of the sole relational predicate R, and T be the h-universal theory $\{\neg\exists x, yR(x, y) \wedge R(y, x)\}$. Let A and B be models of T. We set $C = A \cup B$. Then C is a continuation of A and B which is also a model of T. Hence, T is complete.

- Let L be the relational language $\{Q, R\}$ and T be the h-universal theory $\{\neg\exists x, yQ(x) \wedge R(y)\}$. The theory T is not complete since if A and B are models of T such that $A \models R(a)$ and $B \models Q(b)$, then there does not exist a model of T that is a continuation of A and B.

We will introduce a method to obtain and caracterize minimal completions (Definition 15) of an arbitrary h-inductive theory. The fundamental ingredient is an equivalence relation on the pec models of the theory in question. In this vein, let T be an h-inductive theory, Ae and Be pec models of T. Let $\Re$ be the binary relation defined on the class of pec models of T by: $Ae\Re Be$ if and only if there exists a model C of T that is a common continuation of Ae and Be. Note that $Ae\Re Be$ is also equivalent to saying that there exists a pec model Ce that is common continuation of Ae and Be.

LEMMA 12. *The relation $\Re$ is an equivalence relation.*

Proof. It is easy to see that $\Re$ is reflexive and symmetric. It remains therefore to check the transitivity property. Let Ae, Be, Ce be pec models of T such that $Ae\Re Be$ and $Be\Re Ce$. Then we have the following diagram

$$\begin{tikzcd}
A_e \arrow[d] & & B_e \arrow[dll, "f"'] \arrow[drr, "g"] & & C_e \arrow[d] \\
D_1 & & & & D_2
\end{tikzcd}$$

where D_1 and D_2 are models of T. Since Be is pec, f and g are immersions. By asymmetric amalgamation,

$$\begin{tikzcd}
A \arrow[d] & & B_e \arrow[dll, "f"'] \arrow[drr, "g"] & & C \arrow[d] \\
D_1 \arrow[drr] & & & & D_2 \arrow[dll] \\
& & D & &
\end{tikzcd}$$

with $D \models T$. Thus, $Ae\Re Ce$, and it follows that $\Re$ is an equivalence relation. $\square$

Let $\mathcal{E}$ denote an equivalence class of $\Re$. We define a subclass of models of T denoted $\Gamma_{\mathcal{E}}$:

$$\Gamma_{\mathcal{E}} \;=\; \{\; A \models T \;:\; A \text{ has a continuation that is a member of } \mathcal{E} \;\} \;.$$

LEMMA 13. *The members of $\mathcal{E}$ have the same h-universal theory.*

Proof. Let Ae and Be in $\mathcal{E}$. By definition, there exists a model C of T in which Ae and Be are immersed (they are pec models of T). Hence, if $Be \models \neg\exists\bar{x}\varphi(\bar{x})$, then $C \models \neg\exists\bar{x}\varphi(\bar{x})$, and similarly for Ae. This implies that Ae and Be have the same h-universal theory. $\square$

We will denote by $Tu(\mathcal{E})$ the h-universal theory found in the preceding lemma.

LEMMA 14. *The class $\Gamma_{\mathcal{E}}$ is axiomatized by the theory $T_{\mathcal{E}} = T \cup Tu(\mathcal{E})$.*

Proof. We will first show that all models of $T_{\mathcal{E}}$ belong to $\Gamma_{\mathcal{E}}$. Let A be a model of $T_{\mathcal{E}}$ and Be in $\mathcal{E}$. Then the h-inductive family $T'_{\mathcal{E}} = T_{\mathcal{E}} \cup \mathrm{Diag}^{+}(A) \cup \mathrm{Diag}^{+}(Be)$ is consistent. For every formula $\exists\bar{x}\phi(\bar{x},\bar{a}) \in \mathrm{Diag}^{+}(A)$, $\neg\exists xy\phi(x,y)$ does not belong to $Tu(\mathcal{E})$ because $A \models T_{\mathcal{E}}$. Moreover, since Be belongs to $\mathcal{E}$, $Tu(\mathcal{E})$ is the h-universal theory of Be by Lemma 13. One can thus find $\bar{b} \in Be$ such that $Be \models \exists x\phi(x,\bar{b})$. Hence, $T'_{\mathcal{E}}$ is consistent, and one deduces from this that A and Be have a joint continuation C that we may continue in a pec model Ce of T. It then follows that $Be\Re Ce$, and thus $Ce \in \mathcal{E}$. Hence, $A \in \Gamma_{\mathcal{E}}$.

As for the reverse implication, if $A \in \Gamma_{\mathcal{E}}$, then there exists $Ae \in \mathcal{E}$, a continuation of A, which forces that $A \models T_{\mathcal{E}}$. One concludes from this that $\Gamma_{\mathcal{E}}$ is an elementary class axiomatized by $T_{\mathcal{E}}$. □

Corollary 12. *The theory $T_{\mathcal{E}}$ of Lemma 14 is complete.*

Proof. Let A and B two models of $T_{\mathcal{E}}$. By Lemma 14, A, $B \in \Gamma_{\mathcal{E}}$. By the definition of $\Gamma_{\mathcal{E}}$, there exist two pec models Ae and Be of T in $\mathcal{E}$ that are continuations of A and B respectively. By the definition of $\mathcal{E}$, Ae and Be have a common continuation in $\mathcal{E}$. The conclusion follows. □

Corollary 13. *The class $\Gamma_{\mathcal{E}}$ is inductive.*

Proof. The conclusion follows from Fact 2 and Lemma 14. □

Definition 15. *Let T be an h-inductive theory. A theory T' is said to be a minimal completion of T if T' is a complete theory that contains T and has as model a pec model of T, and it is minimal with respect to these properties.*

Corollary 14. *The theory $T_{\mathcal{E}}$ is a minimal completion of T. Moreover, there exists a bijective correspondence between the equivalence classes of $\Re$ and the minimal completions of T.*

Proof. We start by verifying the first assertion. By Corollary 12, $T_{\mathcal{E}}$ is complete. Its pec models include the members of $\mathcal{E}$. Finally, $T_{\mathcal{E}}$ is minimal since it is exactly $T \cup Tu(\mathcal{E})$.

Now, we prove the second assertion. We first define the correspondence. We associate to each class $\mathcal{E}$ of $\Re$ the theory $T_{\mathcal{E}}$. Clearly this is well-defined, and surjective by the very definition of a minimal completion of an h-inductive theory (Definition 15).

We will next verify the injective property. The main step in the proof is to prove that the pec members of $\Gamma_{\mathcal{E}}$ are exactly the members of $\mathcal{E}$. By definition, every element of $\mathcal{E}$ is a pec model of T, thus it is a pec member of $\Gamma_{\mathcal{E}}$. As for the other inclusion, let A be a pec member of $\Gamma_{\mathcal{E}}$ that has as continuation a model B of T, and f be the witnessing homomorphism from A to B. By definition of $\Gamma_{\mathcal{E}}$, A has a continuation Ae that belongs to $\mathcal{E}$. As $\mathcal{E}$ is contained in $\Gamma_{\mathcal{E}}$, the homomorphism from A into Ae, say g, is in fact an immersion. By asymmetric amalgamation, there is a model C of T such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Ae & \xrightarrow[f']{} & C \end{array}$$

Since g and f' are immersions, it follows that f is an immersion. Thus $A \in \mathcal{E}$. This finishes the proof of the main step, from which the injectivity follows rapidly. Indeed, if $T_{\mathcal{E}_1} = T_{\mathcal{E}_2}$, then $\Gamma_{\mathcal{E}_1} = \Gamma_{\mathcal{E}_2}$, and the main step shows that $\mathcal{E}_1 = \mathcal{E}_2$. □

6.2. *Abstract elementary classes*

When an h-inductive theory is unbounded, the class of its pec models is in fact an abstract elementary class in the sense of Shelah. This is what we will verify in this section.

DEFINITION 16 [**11**]. *Let L be a first-order language and Γ a class of L-structures together with a binary relation that will be denoted by $\prec$. The pair $(\Gamma, \prec)$ is said to be an abstract elementary class if it verifies the following properties:*
N1. The class Γ is closed under isomorphisms.
N2. The relation $\prec$ is a partial ordering, preserved under isomorphisms, $A \prec B$ implies $A \subset B$.
N3. If α is an ordinal and $\{A_t : t < \alpha\}$ is a continuous $\prec$-chain then
1. $\bigcup_{t<\alpha} A_t \in \Gamma$;
2. for every $j < \alpha$, $A_j \prec \bigcup_{t<\alpha} A_t$;
3. if for every $t < \alpha$, one has $A_t \prec A \in \Gamma$, then $\bigcup_{t<\alpha} A_t \prec A$.
N4. If $A, B, C \in \Gamma$, $A \prec C$, $B \prec C$ and $A \subset B$, then $A \prec B$.
N5. There exists a Löwenheim-Skolem number $LS(\Gamma)$, such that if $A \subset B \in \Gamma$, then there exists $A' \in \Gamma$ such that $A \subset A' \prec B$ and $|A'| \leq |A| + LS(\Gamma)$.

DEFINITION 17. *An h-inductive theory is said to be unbounded if the class of its pec models contains members of arbitrarily large cardinality. An L-structure A is said to be unbounded if $Tk(A)$ is unbounded in the language $L(A)$.*

REMARK. A structure is unbounded if and only if it has positive elementary extensions of arbitrarily large cardinalities.

EXAMPLES. 1. Every model-complete theory is unbounded.
2. An h-inductive theory of which the h-maximal models form an elementary class is unbounded.
3. If A is an L-structure such that $Tk(A)$ defines uniformly positively the formula $x \neq y$, then A is an unbounded structure.
4. We fix $L = \{\leq\}$. The structure $(\mathbb{Q}, \leq)$, where $\leq$ is the usual order relation of rational numbers is bounded. Indeed, its maximal elementary extension is the set of real numbers together with the usual order relation and with a point at each of the positive and negative infinities.
5. Every theory with a countable number of pec models is a bounded theory.

Lemma 15. *Let T be an unbounded h-inductive theory. Then for every cardinality $\lambda \geq |L|$, there exists a pec model of T of cardinality λ.*

Proof. As the theory T is unbounded, there exists a pec model B of size $\gamma \geq \lambda$. Let A be a subset of B of size λ. By Lemma 1, there exists A', a substructure of cardinality $\leq max(\lambda, |L|)$, which is immersed in B and which contains A. Since B is pec, A' is also a pec of T. □

Let T be an h-inductive theory, Γ the class of its pec models and $A,\ B \in \Gamma$. We define the binary relation by $A \prec B$ if and only if A is immersed in B. Now, it is easy to verify the following statement.

Proposition 1. *Let T be an unbounded h-inductive theory. Then the class of its pec models together with the relation $\prec$ is an abstract elementary class.*

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