Abstract

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We consider the theory of algebraically closed fields of characteristic zero with raising to powers operations. In an earlier paper we have described complete first-order theories of such a structures, provided that a diophantine conjecture CIT does hold. Here we get rid of this assumption. The theory of complex numbers with raising to real powers satisfies the description if Schanuel's conjecture holds. In particular, we have proved that a (weaker) version of Schanuel's conjecture implies that every well-defined system of exponential sums with real exponents has a solution. Recent result by Bays, Kirby and Wilkie states that the required version of Schanuel's conjecture holds for almost every choice of exponents. It follows that for the corresponding choice of real exponents we have an unconditional description of the first order theory of the complex numbers with raising to these powers.

Raising to powers revisited

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1 Introduction

In this paper we return to the two-sorted structures studied in [6], analogues of the field of complex numbers with the multivalued operations $x^k = \exp(k \ln(x))$, for $k \in K \subseteq \mathbb{C}$. This operation can be better presented in the structure corresponding to the diagram

$$\mathbb{C} \xrightarrow{\exp} \mathbb{C},$$

where the complex numbers on the left and on the right are seen as two distinct sorts. The right-hand sort is endowed with the usual field structure, whereas the left-hand one is considered a vector space over a subfield $K \subseteq \mathbb{C}$ of finite transcendence degree, with one function symbol in the language for each transformation $v \mapsto k \cdot v$, for each $k \in K$. Naturally, for x in the right-hand sort, $\ln x$ is defined as $\exp^{-1}(x)$, a subset of the left-hand sort. Now x^k, $k \in K$, means $\exp(kv)$ for some $v \in \ln(x)$, so 'raising to powers' is definable in the two-sorted structure.

The abstract structure of raising to powers consists of two sorts V and $\mathrm{F}^\times$, with $\mathrm{F}^\times$ given a structure of $\mathbb{F} \setminus \{0\}$, $\mathbb{F}$ a field of characteristic zero and V a vector space over a field K of characteristic zero. There is also a map exp which is assumed to be a homomorphism of the additive group V into the multiplicative group of $\mathrm{F}^\times$,

$$\mathrm{V} \xrightarrow{\exp} \mathrm{F}^\times.$$

In [6] we used a version of Hrushovski's amalgamation process to construct, for each K, a class of algebraically closed fields with raising to powers K, denoted $\mathbb{F}^K$, satisfying (a corollary of) Schanuel's conjecture and showed, using CIT, the *Conjecture on Intersecting varieties* *with Tori* (see [7]), that any completion of the theory is near model complete and superstable.

Here we refine this result by proving essentially the same statement without any assumptions. The use of CIT is avoided by using the solution of Mordell problem (Falting's Theorem) about finitely generated subgroups of semi-abelian varieties, in fact for $(\mathbb{C}^{\times})^n$.

Our main theorem gives, for every $\mathbb{F}^K$, a set of axioms that determines a complete first order theory of $\mathbb{F}^K$, which we also prove to be near model complete and superstable.

The present version of the theorem can be used to study the effect of Schanuel's conjecture. Let $\mathbb{C}^K$ denote the field of complex numbers with raising to powers $K \subseteq \mathbb{C}$.

We prove (Theorem 6.2) that in case $K \subseteq \mathbb{R}$, if $\mathbb{C}^K$ satisfies the corresponding version of Schanuel's conjecture, it is *exponentially-algebraically closed*, that is any well-defined system of exponential sums equations with exponents in K has a solution in $\mathbb{C}$. The conjecture that $\mathbb{C}_{\exp}$, the field of complex numbers with exp, is exponentially-algebraically closed was made in [8]. This has been studied by a number of researchers including attempts to refute the conjecture. Theorem 6.2 brings hopes that in general exponential-algebraic closedness follows from Schanuel's conjecture implies . So far we don't know if this is true even for $\mathbb{C}^K$, when K is not a subfield of the reals.

Another important corollary of the main theorem (Theorem 4.16) states that under Schanuel's conjecture for $\mathbb{C}^K$ solutions to an overdetermined system of exponential sums equations lie in a finitely many, modulo $2\pi i$, cosets of proper $\mathbb{Q}$-linear subspaces. Moreover, there is a bound on the number of such cosets, uniform in coefficients of the system (but possibly not on the exponents).

Finitely, we invoke a recent result by M.Bays, J.Kirby and A.Wilkie that implies that for "almost any" tuple λ in $\mathbb{C}$, for $K = \mathbb{Q}(\lambda)$, the structure $\mathbb{C}^K$ satisfies the corresponding version of Schanuel's conjecture. Thus, the above theorems are applicable to such $\mathbb{C}^K$ unconditionally. In particular, when also $\lambda \subseteq \mathbb{R}$, we know the complete theory of $\mathbb{C}^K$.

Of course, Faltings' Theorem has been proven in much greater generality and, as in [6], one can easily replace $\mathrm{F}^{\times}$ by any semiabelian variety $\mathbf{A}$ and carry out the same construction and axiomatisation since also a corresponding analogue of Ax's Theorem and its corollaries is available. More precisely, one needs the following (weak CIT) to hold for $\mathbf{A}$.

1.1 Theorem (J.Kirby, [5]) *Given* $W(e) \subseteq \mathbf{A}^n$, *with* $W \subseteq \mathbf{A}^{n+l}$ *algebraic subvariety defined over* k *and* $e \in \mathbf{A}^l$ *there are finitely many codimension* 1 $\mathrm{End}\mathbf{A}$*-linear subspaces* $\mu(W) = \{M_1, \ldots, M_m\}$ *of* V^n *such that for any* $\mathrm{End}\mathbf{A}$*-linear subspace* $N \subseteq \mathrm{V}^n$, $b \in \mathbf{A}^l$, *and any positive dimensional atypical component* S *of the intersection* $W(e) \cap \exp(N) \cdot b$ *there is* $M \in \mu(W)$ *and* $s \in S$ *such that* $S \subseteq \exp(M) \cdot s$.

Here, *atypical* for an irreducible component S of the intersection of algebraic subvarieties $W(e)$ and $\exp(N) \cdot b$ of $\mathbf{A}^n$ (observe that $\exp(N)$ is an algebraic subgroup of $\mathbf{A}^n$) means that

$$\dim S > \dim W(e) + \dim \exp(N) - \dim \mathbf{A}^n.$$

To prove our main result in full generality, for semi-abelian $\mathbf{A}$, we would need to consider V as an $\mathrm{End}\mathbf{A}$-module which, for $\dim \mathbf{A} = g > 1$, contains divisors of zero and makes the linear structure on V more involved. This would complicate definitions and some argument, without visible advantages for applications. But the proof below goes through practically without changes for any elliptic curve without complex multiplication defined over $\mathbb{Q}$.

2 Definitions and notation

2.1 This section along with definitions and notations discusses basic ingredients of Hrushovski's construction which is standard enough, so the reader can guess the proofs if they seem too short or are absent.

We use here some of the terminology of [7], slightly improved, where we discussed K-linear and affine spaces, tori and their intersections with algebraic varieties.

For technical reasons we find it more convenient to represent the two-sorted structures $(\mathrm{V}, \mathrm{F}^\times)$ in the equivalent way as one sorted structures in the language $\mathcal{L}_K$ which is the extension of the language of vector spaces over $\mathbb{Q}$ by:

- an equivalence relation E,
- n-ary predicates $L(x_1, \ldots, x_n)$ for linear subspaces $L \subseteq \mathrm{V}^n$ given by a set of K-linear equations in $x_1, \ldots, x_n$,
- n-ary predicates EW for algebraic varieties $W \subseteq (\mathrm{F}^\times)^n$ definable and irreducible over $\mathbb{Q}$.

The interpretation can be explained in the above mentioned terms as follows:

- $E(x,y) \equiv [\exp(x) = \exp(y)]$,
- $L(x_1, \dots, x_n) \equiv [\langle x_1, \dots, x_n \rangle \in L]$,
- $EW(x_1, \dots, x_n) \equiv [\langle \exp(x_1), \dots, \exp(x_n) \rangle \in W]$.

2.2 Definition $\mathcal{E}^K$ is the class of structures $\mathbb{F}^K$ in language $\mathcal{L}_K$ with axioms saying that V is an infinite-dimensional vector space over K, E is an equivalence relation on V which is congruent with respect to the relations $EW(x_1, \dots, x_n)$, $\mathrm{F}^\times = \mathrm{V}/E$ can be identified with the group of the multiplicative group $\mathrm{F}^\times$, of a field of characteristic zero, and the predicates EW define its algebraic varieties over $\mathbb{Q}$. The canonical mapping

$$\exp : \mathrm{V} \to \mathrm{F}^\times$$

is a surjective homomorphism of the additive group of V onto the group $\mathrm{F}^\times$.
The underlying set of axioms we denote PK, (*powered field with exponents in* K).

Notation For finite $X, X' \subseteq \mathrm{V}$, $Y, Y' \subseteq \mathrm{F}^\times$

$\mathrm{lin.dim}_K(X)$ the dimension of the vector space $\mathrm{span}_K(X)$ generated by X over K;

$\mathrm{lin.dim}_{\mathbb{Q}}(X)$ the dimension of the vector space $\mathrm{span}_{\mathbb{Q}}(X)$ generated by X over $\mathbb{Q}$;

$\mathrm{tr.deg}(Y)$ the transcendence degree of Y;

$\delta^K(X)$ **the predimension** of finite $X \subseteq \mathrm{V}$:

$$\delta^K(X) = \mathrm{lin.dim}_K(X) + \mathrm{tr.deg}(\exp(X)) - \mathrm{lin.dim}_{\mathbb{Q}}(X);$$

$$\delta^K(X/X') = \delta^K(X \cup X') - \delta^K(X');$$

For infinite $Z \subseteq \mathrm{V}$ and $k \in \mathbb{Z}$ $\delta^K(X/Z) \geq k$ by definition means that for any $Y \subseteq_{fin} Z$ there is $Y \subseteq_{fin} Y' \subseteq Z$ such that $\delta^K(X/Y') \geq k$, and $\delta^K(X/Z) = k$ means $\delta^K(X/Z) \geq k$ and not $\delta^K(X/Z) \geq k+1$.

We recall that $A \subset \mathrm{V}$ is said to be **self-sufficient in** $\mathbb{F}^K$ if $\delta^K(X/A) \geq 0$ for all finite $X \subseteq \mathrm{V}$. This is written as

$$A \leq \mathbb{F}^K.$$

2.3 We let also
$\mathrm{lin.dim}_K(X/X') = \mathrm{lin.dim}_K(X \cup X') - \mathrm{lin.dim}_K(X')$;
$\mathrm{tr.deg}(Y/Y') = \mathrm{tr.deg}(Y \cup Y') - \mathrm{tr.deg}(Y')$;
$\mathrm{lin.dim}_{\mathbb{Q}}(X/X') = \mathrm{lin.dim}_{\mathbb{Q}}(X \cup X') - \mathrm{lin.dim}_{\mathbb{Q}}(X')$;
ker is the name of a unary predicate of type EW : $x \in \ker \equiv \exp(x) = 1$. We write $\ker_{|A}$ for the realisation of this predicate in A.

Given $d \in \mathbb{Z}$, let $\mathcal{E}^K_d$ be the subclass of $\mathcal{E}^K$ consisting of all $\mathbb{F}^K$ satisfying the condition:

$$\delta^K(X/\ker) \geq -d \text{ for all finite } X \subseteq \mathrm{V},$$

where $\ker = \ker_{|\mathbb{F}}$.

2.4 Denote sub$\mathcal{E}^K$ the class of the substructures of the structures of $\mathcal{E}^K$ in the language $\mathcal{L}_K$.

Given an integer d, let sub$\mathcal{E}^K_d$ be the subclass of sub$\mathcal{E}^K$ consisting of A which satisfy $\delta^K(X) \geq -d$ for any finite $X \subseteq A$.

Remark For any structure A in sub$\mathcal{E}^K$ and any $X \subseteq \ker_{|A}$ in the structure

$$\delta^K(X) = 0$$

and thus $\mathcal{E}^K_d$ is empty for $d < 0$.

On the other hand, for any K we have by Lemmas 2.7 and 2.8 of [6]

$$\mathcal{E}^K_0 \neq \emptyset.$$

2.5 Assuming $\mathrm{F}^\times = \mathbb{C}^\times$, the algebraic torus, and the Schanuel conjecture we can make a better estimates for the minimal d such that $\mathbb{C}^K$, the complex numbers with raising to powers $K \subseteq \mathbb{C}$, belongs to $\mathcal{E}^K_d$.

By Schanuel's conjecture, for any finite $X \subseteq \mathbb{C}$

$$\mathrm{tr.deg}(X, \exp(X)) \geq \mathrm{lin.dim}_{\mathbb{Q}}(X).$$

Recall that we assumed that $\mathrm{tr.deg}(K)$ is finite. Obviously,

$$\mathrm{lin.dim}_K(X) + \mathrm{tr.deg}(\exp X) + \mathrm{tr.deg}(K) \geq \mathrm{tr.deg}(X, \exp(X)).$$

Hence,

$$\delta^K(X) \geq \mathrm{tr.deg}(X, \exp(X)) - \mathrm{lin.dim}_{\mathbb{Q}}(X) - \mathrm{tr.deg}(K) \geq -\mathrm{tr.deg}(K).$$

Since $\text{lin.dim}_K(\ker) = \text{lin.dim}_{\mathbb{Q}}(\ker) = 1$ for $\ker = 2\pi i\mathbb{Z}$, we have

$$\delta^K(X/\ker) \geq -(\text{tr.deg}(K) + 1).$$

Thus, under the conjecture,

$$\mathbb{C}^K \in \mathcal{E}_d^K, \text{ for } d = \text{tr.deg}(K) + 1.$$

Remark. In case of an elliptic curve one can produce similarly the estimate $d = \text{tr.deg}(K) + 2$ assuming the corresponding analogue of Schanuel's conjecture (see Be).

2.6 Given $\mathbb{F}^K \in \mathcal{E}_d^K \setminus \mathcal{E}_{d-1}^K$ one can find a finite $A \subseteq \mathbb{F}^K$ with

$$\delta^K(A/\ker) = -d. \tag{1}$$

By minimality

$$A \cup \ker \leq \mathbb{F}^K.$$

The equality (1) does not change if we extend A by elements of $\ker$. In case $\delta^K(X)$ is bounded from below by, say $-d'$, for all finite $X \subset \ker$, in particular, if $\ker$ is a finite rank group, then

$$\delta^K(Y) \leq -(d + d') \text{ for all finite } Y \subseteq \mathbb{F}.$$

It follows that there is a finite $A_0 \subseteq \ker$ such that if $A_0 \subseteq A$ and (1) holds then the value of $\delta^K(A)$ reaches minimum and so

$$A \leq \mathbb{F}^K \tag{2}$$

that is

$$\delta^K(X/A) = \delta^K(X/A \cup \ker) \geq 0 \text{ for every } X \subseteq \mathbb{F}.$$

2.7 We will assume throughout that A with the property (1) and (2) does exist, as is the case for $\mathbb{C}^K$ under Schanuel's conjecture.

In fact, this is a form of Schanuel's conjecture for $\mathbb{C}^K$ for a given $K \subseteq \mathbb{C}$.

2.8 Definition A structure $\mathbb{F}^K$ in $\mathcal{E}_d^K$ is said to be **$\mathcal{E}_d^K$-exponentially-algebraically closed** (e.a.c.) if for any $\mathbb{F}_1^K \in \mathcal{E}_d^K$, such that $\mathbb{F}^K \leq \mathbb{F}_1^K$, any finite quantifier-free type over $\mathbb{F}^K$ which is realized in $\mathbb{F}_1^K$ has a realization in $\mathbb{F}^K$.
Denote $\mathcal{EC}_d^K$ the class of $\mathcal{E}_d^K$-exponentially-algebraically closed structures, or, in the shorter form, $\mathcal{EC}^K$.

Using the standard Fraisse construction in the class $\mathcal{E}_d^K$ relative to the strong embedding $\leq$ one can prove:

2.9 Propositon [Proposition 1 of [6]] For any $\mathbb{F}^K$ in $\mathcal{E}_d^K$ there exists an $\mathcal{E}_d^K$-e.a.c. structure containing $\mathbb{F}^K$.

More difficult is the following result, based on Theorem 1.1. Its proof using Ax's Theorem is given in [6].

2.10 Proposition [Corollaries 1 and 2, section 4 of [6]] There exists a set EC of first order $\forall\exists$ axioms such that, for any $\mathbb{F}^K \in \mathcal{E}_d^K$,

$$\mathbb{F}^K \models \mathrm{EC} \text{ iff } \mathbb{F}^K \text{ is exponentially-algebraically closed.}$$

2.11 For $W \subseteq (\mathrm{F}^\times)^{n+l}$ an algebraic variety, $b = \langle b_1, \ldots, b_l \rangle$ denote

$$W(b) = \{\langle x_1, \ldots, x_n \rangle \in (\mathrm{F}^\times)^n : \langle x_1, \ldots, x_n, b_1, \ldots, b_l \rangle \in W\}.$$

A subspace $L \subseteq \mathrm{V}^n$ is said to be K-**linear** if there are $k_{ij} \in K$ $(i \leq r,\ j \leq n)$ such that

$$L = \{\langle x_1, \ldots, x_n \rangle \in \mathrm{V}^n : k_{i1}x_1 + \cdots + k_{in}x_n = 0\}.$$

Define $\dim L = \text{co-rank}(k_{ij})$, the co-rank of the matrix (k_{ij}).

Let $L \subseteq \mathrm{V}^{n+l}$ be a K-linear subspace, $\bar{a} = \langle a_1, \ldots, a_l \rangle$. Let

$$L(a) = \{\langle x_1, \ldots, x_n \rangle \in \mathrm{V}^n : \langle x_1, \ldots, x_n, a_1, \ldots, a_l \rangle \in L\}.$$

We call such an $L(a)$ a K-**affine subspace defined over** a. The same terminology is applied for $\mathbb{Q}$ instead of K.

2.12 Lemma A K-affine subspace $L(a) \subseteq \mathrm{V}^n$ can be represented equivalently and uniformly on a as

$$L(a) = L(0) + r(a), \quad r(a) \in \mathrm{V}^n, \quad r \text{ is a } K\text{-linear map}, \quad \mathrm{V}^l \to \mathrm{V}^n$$

and $L(0)$ is a K-linear subspace.

Moreover, if r' is any K-linear map such that $r'(a) \in L(a)$ for all $a \in \mathrm{pr}_{n+1\ldots n+l}L$, then also

$$L(a) = L(0) + r'(a).$$

Proof Let L be determined by the system of linear equations

$$\sum_{j=1}^{n} q_{ij}v_j + \sum_{s=1}^{l} k_{is}w_s = 0, \quad q_{ij}, k_{is} \in K. \tag{3}$$

Then the system

$$\sum_{j=1}^{n} q_{ij}v_j + \sum_{s=1}^{l} k_{is}a_s = 0,$$

determines $L(a)$. It follows that

$$\sum_{j=1}^{n} q_{ij}v_j = 0 \tag{4}$$

determines $L(0)$.

By linear algebra there is an n-tuple $\{r_j(w_1, \dots, w_l) : j = 1, \dots, n\}$ of K-linear functions $V^l \to V$ such that, if for a given $(w_1, \dots, w_l)$ the system (3) is consistent, then $v_j = r_j(w_1, \dots, w_l)$ gives a solution to the system. Hence, $v - r(a)$ is a solution to the homogeneous system (4) iff v is a solution of the system (3) with $w = a$.

The 'moreover' statement follows immediately from the fact that $r'(a) - r(a) \in L(0)$. $\square$

2.13 Lemma Given a K-linear $L \subseteq V^{n+l}$ and 0 the zero of V^l

(i) there exists a unique maximal $\mathbb{Q}$-linear subspace $N_L \subseteq L$;

(ii) $N_{L(0)} = N_L(0)$;

(iii) given $a \in \mathrm{pr}_{n+1\dots n+l}L$ and $q(a) \in L(a) \cap \mathrm{span}_Q(a)$ there exists a maximal $\mathbb{Q}$-affine subspace $N_{L,q(a)}(a) \subseteq L(a)$ over $\mathrm{span}_Q(a)$ containing $q(a)$, and in this case $N_{L,q(a)}(a) = N_L(0) + q(a)$.

Proof (i) $N_L(0)$ exists since the sum of two $\mathbb{Q}$-linear subspaces of $L(0)$ is again a $\mathbb{Q}$-linear subspace of $L(0)$.

(ii) Obviously, $N_L(0)$ is a $\mathbb{Q}$-linear subspace of $L(0)$, so $N_{L(0)} \supseteq N_L(0)$.

$N_{L(0)}$ is a $\mathbb{Q}$-linear subspaces of $L(0)$, so $N_{L(0)} \times \{0\}$ is a $\mathbb{Q}$-linear subspaces of L, hence $N_{L(0)} \times \{0\} \subseteq N_L$ and $N_{L(0)} \subseteq N_L(0)$.

(iii) $N_L(0) + q(a)$ is a $\mathbb{Q}$-affine subspace of $L(0) + q(a) = L(a)$. If $M + q(a)$ is another $\mathbb{Q}$-affine subspace of $L(a)$, containing $q(a)$ then $M = (M + q(a)) - q(a) \subseteq L(0)$ and hence $M \subseteq N_L(0)$, $M + q(a) \subseteq N_L(0) + q(a)$. $\square$

3 Intersections in semi-abelian varieties

3.1 Lemma. In the statement of Theorem 1.1 we can assume that $s \in \mathrm{acl}(b, e)$.

Proof S is an irreducible component of the set $W(e) \cap \exp(N) \cdot b$ definable over (e, b) hence it is definable over $\mathrm{acl}(b, e)$. Thus it contains points from the algebraically closed field $\mathrm{acl}(b, e)$.$\square$

3.2 Proposition. Given $W \subseteq (\mathrm{F}^\times)^{n+l}$, with W algebraic subvariety defined over $\mathbb{Q}$ there are finitely many proper $\mathbb{Q}$-linear subspaces $\pi(W) = \{M_1, \ldots, M_m\}$ of V^n such that for any $e, b \in (\mathrm{F}^\times)^l$ and a $\mathbb{Q}$-linear subspaces $N \subseteq V^n$, for any positive dimensional atypical component S of the intersection $W(e) \cap \exp(N) \cdot b$ there is a $M \in \pi(W)$ and $s \in S \cap \mathrm{acl}(e, b)$ such that $S \subseteq \exp(M) \cdot s$ and S is a typical component of $\exp(N) \cdot b \cap \exp(M) \cdot s \cap W(e)$ with respect to the group variety $\exp(M) \cdot s$.

Proof Notice first that by obvious transformations of W we can assume that the family $\{W(e) : e \in (\mathrm{F}^\times)^l\}$ is invariant with respect to shifts by elements of $(\mathrm{F}^\times)^l$, that is, for every $b, e \in (\mathrm{F}^\times)^l$ there is $e' \in (\mathrm{F}^\times)^l$ such that

$$W(e) \cdot b = W(e').$$

By induction on the dimension of a proper algebraic subgroup P of $(\mathrm{F}^\times)^n$, for any algebraic subvariety W_P of $P \subseteq (\mathrm{F}^\times)^l$ over $\mathbb{Q}$ we construct a collection of proper algebraic subgroups $\pi_P(W_P)$ of P such that the statement of the lemma holds for $\exp(N) \cdot b \cap W_P(e) \subseteq P$.

For $\dim P = 1$ the statement is trivially true for there is no atypical components in any intersection.

Consider the general case. Assume by induction that $\pi_P(W_P)$ has been constructed for $\dim P < \dim(\mathrm{F}^\times)^n$. Notice that by invariance $\pi_P(W_P)$ will be the same if we replace P by $P \cdot b$, for $b \in (\mathrm{F}^\times)^l$ a parameter.

Assume that for all $P \subset (\mathrm{F}^\times)^n$ proper, $\pi_P(W_P)$ exists.

Given $W \subseteq (\mathrm{F}^\times)^{n+l}$, we let

$$\pi(W) = \bigcup_{Q=\exp(M),\ M \in \mu(W)} \pi_Q(W_Q) \cup \{M\},$$

where W_Q is $W \cap (Q \times (\mathrm{F}^\times)^l)$.

Now, if $S \subseteq \exp(N) \cdot b \cap W(e)$ is atypical, then by Theorem 1.1 $S \subseteq \exp(M) \cdot s$ for some $M \in \mu(W)$. Hence $S \subseteq \exp(N) \cdot b \cap W_Q(e)$,

for $Q = \exp(M) \cdot s$, and is a component of the intersection. In other words, S is a component of the intersection $\exp(N \cap M) \cdot b' \cap W_Q(e)$ for some $b' \in (\mathrm{F}^\times)^l$. Either S is typical in this intersection with respect to $Q = \exp(M)$, and hence the statement of the proposition holds for the chosen M belonging to $\pi_Q(W_Q)$ by definitions, or S is atypical but we can find by induction $M' \in \pi_Q(W_Q) \subseteq \pi(W)$ such that $S \subseteq \exp(M') \cdot s$ and is a typical component in the intersection with respect to $\exp(M')$.
□

We want to show now that under certain conditions we can factor out M in the previous proposition.

3.3 Let $M \subseteq \mathrm{V}^n$ be a $\mathbb{Q}$-linear subspace. We see V^n as a subspace of V^{n+l}, equivalently $\mathrm{V}^{n+l} = \mathrm{V}^n \dot{+} \mathrm{V}^l$, with $a \in \mathrm{V}^l$, $e = \exp(a) \in (\mathrm{F}^\times)^l$.

By definitions M is definable by $c = \operatorname{codim} M$ independent Q-linear equations

$$m_{i1}v_1 + \cdots + m_{in}v_n = 0, \quad i = 1, \ldots c,$$

where $(v_1, \ldots, v_n) \in \mathrm{V}^n$. The same in matrix notation

$$\bar{m}\bar{v} = \bar{0}.$$

We now choose $\bar{m}^\perp$, a $(n-c) \times n$-matrix consisting of vectors $(m_{j1}, \ldots, m_{jn}) \in \mathbb{Q}^n$, $j = c+1, \ldots, n$, which extend $\bar{m}$ to the basis of the $\mathbb{Q}$-vector space $\mathbb{Q}^n$. We let $M^\perp$ to be the set of solutions to the system

$$\bar{m}^\perp \bar{v} = \bar{0}.$$

This determines the definable decomposition

$$\mathrm{V}^n = M \dot{+} M^\perp \cong M \dot{\times} \mathrm{V}^n / M.$$

Applying exp we correspondingly have the decomposition

$$(\mathrm{F}^\times)^n = Q \cdot Q^\perp \cong Q \times (\mathrm{F}^\times)^n / Q,$$

where $Q = \exp(M)$ and $Q^\perp = \exp(M^\perp)$.

Note that the structure $(M^\perp, \exp, Q^\perp)$ is by construction isomorphic to $(V^c, \exp, (\mathrm{F}^\times)^c)$ in language $\mathcal{L}_K$.

We denote the natural mappings

$$\mathrm{V}^n \to M^\perp \text{ and } (\mathrm{F}^\times)^n \to Q^\perp$$

associated with the above decomposition as

$$v \mapsto v + M \text{ and } x \mapsto x \cdot Q,$$

correspondingly.

It is easy to see that $x \mapsto x \cdot Q$ is a proper mapping on $(\mathrm{F}^\times)^n$ (and $(\mathrm{F}^\times)^{n+l}$) hence the images $W/\exp M$ (that is W/Q) and $W(e)/\exp M$ are algebraic subvarieties of $\exp M^\perp \times (\mathrm{F}^\times)^l$ and $\exp M^\perp$, correspondingly.

The same algebraicity holds for $S/\exp M$ and $b \cdot \exp(N)/\exp M$).

3.4 On the other hand, exp can be naturally extended to the quotient spaces

$$\exp : \ \mathrm{V}^n/M \to (\mathrm{F}^\times)^n/Q.$$

So, $(\mathrm{V}^n/M, \exp, (\mathrm{F}^\times)^n/Q)$ and $(\mathrm{V}^{n+l}/M, \exp, (\mathrm{F}^\times)^{n+\ell}/Q)$ are canonically isomorphic to $(\mathrm{V}^c, \exp, (\mathrm{F}^\times)^c)$ and $(\mathrm{V}^{c+l}, \exp, (\mathrm{F}^\times)^{c+l})$ correspondingly and hence, for $(\mathrm{V}, \exp, \mathrm{F}^\times) \in \mathcal{E}_d^K$, $u \in \mathrm{V}^n$ and $\overrightarrow{A} \in \mathrm{V}^l$, $A \leq \mathbb{F}^K$,

$$\delta^K(u+M) = \mathrm{lin.dim}_K(u+M) + \mathrm{tr.deg}(\exp(u+M)) - \mathrm{lin.dim}_{\mathbb{Q}}(u+M) \geq d,$$

and

$$\delta^K(u+M/A) = \mathrm{lin.dim}_K(u+M/A) + \mathrm{tr.deg}(\exp(u+M)/\exp\ A) - \mathrm{lin.dim}_{\mathbb{Q}}(u+M/A) \geq 0.$$

3.5 We have also the decomposition

$$L(a) = L(\bar{0}) \cap M \dot{+} L/M(a) \tag{5}$$

where

$$L/M = L/(L \cap (M \times \{\bar{0}\})) \subseteq V^{n+\ell}/(M \times \{\bar{0}\}),\ \bar{0} \in \mathrm{V}^\ell,\ L/M(a) \subseteq M^\perp.$$

Thus we can naturally identify $L(a)/M$ with $L/M(a)$.

4 Axiomatizing $\mathcal{E}_d^K$.

Fix $A \subseteq \mathrm{V}$ and consider pairs $(L(a), W(\exp\ a))$, where L is a K-affine subspace of V^n over $a = \overrightarrow{A} \in \mathrm{V}^l$ and W an algebraic subvariety of $(\mathrm{F}^\times)^n$ over $\mathbb{Q}$.

Definition A pair $(L(a), W(\exp\ a)$ is said to be **special** if L is not contained in any proper $\mathbb{Q}$-linear subspace of $V^{n+\ell}$ and

$$\dim L(a) + \dim W(\exp\ a) < n. \tag{6}$$

(This corresponds to 0-*special* in the terminology of [6].)

4.1 Let, for a $\mathbb{Q}$-subspace M of V^n

$$d(W(\exp\ a), \exp(M)) = \min\{\dim(\bar{W}(\exp\ a) \cap w \cdot \overline{\exp\ M} : w \in W(\exp\ a)\},$$

where $\bar{W}$ and $\overline{\exp\ M}$ is the closure in the ambient projective space.

Let

$$W^{\exp\ M}(\exp\ a) = \{w \in W(\exp\ a) : \dim(W(\exp\ a) \cap w \cdot \exp\ M) > d(W(\exp\ a), \exp\ M)\}.$$

Since minimal dimension fibers are located over an open subset
$W^{\exp\ M}(\exp\ a)$ *is a proper closed subset of* $W(\exp\ a)$,
maybe empty, if $d(W(\exp\ a), \exp\ M) = \dim(W(\exp\ a) \cap \exp(x) \exp\ M)$.

4.2 Suppose that $(L(a), W \exp\ a))$ is special. Suppose $M \subseteq L(0)$. Consider the quotients $M^\perp = \mathrm{V}^n/M$, $\exp(M)^\perp = (\mathrm{F}^\times)^n/\exp(M)$ and subsets $L(a)/M$ and $W(\exp\ a)/\exp(M)$.

Then, $\dim W(\exp\ a)/\exp(M) < n - \dim M$, that is

$W(\exp, a)/\exp(M)$ *is a proper subvariety of* $(\mathrm{F}^\times)^n/\exp(M)$.

Indeed, by addition formula,

$$\dim W(\exp\ a)/\exp(M) = \dim W(\exp\ a) - d(W(\exp\ a), \exp\ M)$$

and

$$d(W(\exp\ a), \exp\ M) \geq 0 > \dim W(\exp\ a) + \dim M - n,$$

since $W(\exp\ a), L(a)$ is special.

4.3 *Assume $A \leq \mathbb{F}^K$ is finite and let $x \in \mathrm{V}^n$. We analyse first order consequences of this assumption. We aim to show also that the analysis yields the same conclusions and formulas when we replace A by B with* $\mathrm{qftp}(A) = \mathrm{qftp}(B)$.

Suppose $(L(a), W \exp\ a))$ is **special**, Suppose $x \in L(a)$ and $\exp(x) \in W(\exp\ a)$, in $\mathbb{F}^K$ and $A \leq \mathbb{F}^K$.

Since

$$\text{lin.dim}_K(x/A)+\text{tr.deg}(\exp(x)/\exp(A))-n \leq \dim L(a)+\dim W(\exp\ a)-n < 0$$

and, by $A \leq \mathbb{F}^K$,

$$\delta^K(x/A) \geq 0,$$

we have $\text{lin.dim}_{\mathbb{Q}}(x/A) < n$, so $x^\frown a \in N$ for some proper $\mathbb{Q}$-linear subspace of V^{n+l}, We assume N is minimal for $x^\frown a$.

We have $\exp(x) \in S_x \subseteq \exp(N(a)) \cap W(\exp\ a)$, where S_x is a component of $\exp(N(a)) \cap W(\exp\ a)$.

Case 1. The component S_x is of dimension 0.

Then $\text{tr.deg}(\exp(x)/\exp(A)) = 0$, which implies $\text{lin.dim}_K(x/a) = \text{lin.dim}_{\mathbb{Q}}(x/a)$, that is $\dim N(a) = \dim N(a) \cap L(a)$ and so $N(a) \subseteq L(a)$ is a $\mathbb{Q}$-affine subspace over a, thus $N(a) = N_L(0) + q(a)$, for some $q(a) \in L(a) \cap \text{span}_Q(a)$ (Lemma 2.13).

So

$$x \in N_L(0) + q(a) \text{ for some } q(a) \in L(a) \cap \text{span}_Q(a).$$

Subcase 1.1 $d(W(\exp\ a), \exp(N_L(0))) < \dim(W(\exp\ a) \cap \exp(x + N_L(0)))$.

Under this assumption $W^{\exp(N_L(0))}(\exp\ a)$ is a proper closed subset of $W(\exp\ a)$ containing $\exp(x)$, by 4.1.

Otherwise we have

Subcase 1.2. $d(W(\exp\ a), \exp(N_L(0))) = \dim(W(\exp\ a) \cap \exp(x + N_L(0)))$.

We have $W^{\exp(N_L(0))}(\exp\ a) = \emptyset$ in this case.

Consider the quotients $N_L^{\perp}(0) = \mathrm{V}^n/N_L(0)$, $\exp(N_L(0))^{\perp} = (\mathrm{F}^{\times})^n/\exp(N_L(0))$ and subsets $L/N_L(a)$ and $W(\exp, a)/\exp(N_L(0))$. By 4.2 $W(\exp\ a)/\exp(N_L(0)) \subsetneq (\mathrm{F}^{\times})^n/\exp(N_L(0))$.

Obviously, for the x above, $\exp(x/N_L(0))$ is a singleton in $W(\exp\ a)/\exp(N_L(0)$ and is also equal to $\exp(q(a)/N_L(0))$.

Let

$$\Gamma_a = \{\exp(q(a)^\frown a) : q(a) \in \text{span}_Q(a)\}$$

This is a coset $\Gamma^0 \cdot s(\exp a)$ of a finite rank subgroup Γ^0 of $(\mathrm{F}^\times)^{n+l}$ (depending on the choice of $\exp a$).

By Faltings' Theorem there are finitely many, say k_a, cosets $T_i(\exp a) \subseteq (\mathrm{F}^\times)^{n+l}$ of group subvarieties (tori) $T_i \subseteq (\mathrm{F}^\times)^{n+l}$, $T_i(\exp a) \subseteq W(\exp a)$ (note the notation, $T_i(\exp a)$ as defined in 2.11) such that

$$\Gamma_a \cap W(\exp a) = \cup_{i\le k_a}\Gamma_a \cap T_i(\exp a). \tag{7}$$

Hence, in this case

$$\exp(x/N_L(0)) \in \bigcup_{i\le k_a} T_i(\exp a) \tag{8}$$

and $T_i(\exp a) \subseteq W(\exp\ a)/\exp N_L(0) \subsetneq (\mathrm{F}^\times)^n/\exp(N_L(0)$.

Note that (7) and (8) continue to hold with the same T_i and k_a if we replace a by b with $\exp \frac{a}{n} \equiv \exp \frac{b}{n}$, all n, in the field language, that is Galois conjugated.

Case 2. $\dim S_x > 0$. Then, by 3.2, $S_x \subseteq c \cdot \exp(M)$, for some $\mathbb{Q}$-linear subspace $M \in \pi_W$ of $\mathrm{V}^{n+\ell}$, $c \in \mathrm{acl}(\exp a)$, and S_x is typical in the intersection
$W(\exp a) \cap c \cdot \exp(M)$ with respect to $c \cdot \exp(M)$. The latter gives us, for $a' = \ln c$,

$$\dim S_x = \dim(\exp(N(a))\cap\exp(M+a'))+\dim(W(\exp a)\cap\exp(M+a'))-\dim\exp(M+a').$$

It is easy to see that $\delta^K(x/Aa') \ge 0$ and hence we obtain

$$\dim(L(a) \cap N(a) \cap (M + a')) + \dim S_x - \dim(N(a) \cap (M + a')) \ge 0.$$

Combining with the above we get

$$\dim(L(a)\cap N(a)\cap(M+a'))+\dim(W(\exp a)\cap\exp(M+a'))-\dim\exp(M+a') \ge 0.$$

And so

$$\dim(L(0) \cap M) + \dim(W(\exp a) \cap \exp(M + a')) - \dim M \ge 0. \tag{9}$$

Subcase 2.1 $d(W(\exp a), \exp(M)) < \dim(W(\exp a)\cap c\cdot\exp(M))$.
Under this assumption $W^{\exp(M)}(\exp a)$ is a proper closed subset of $W(\exp a)$ containing $\exp(x)$.

Otherwise, we have

Subcase 2.2. $d(W(\exp\ a), \exp(M)) = \dim(W(\exp\ a) \cap c{\cdot}\exp(M))$.

So,

$$W^{\exp(M)}(\exp\ a) = \emptyset. \tag{10}$$

We now apply the factorisation of 3.3-3.4.

Obviously, for our x,

$$\exp(x/M) \in S_x/\exp(M) \cap \exp(L(a)/M)$$

and $S_x/\exp(M)$ is a singleton in $(\mathrm{F}^\times)^{n-\dim M}$ defined over the same parameters as S_x, that is over $\mathrm{acl}(\exp(A))$.

Now we notice that

$$\dim L/M(a) + \dim W(\exp\ a)/\exp\ M =$$

$$= \dim L(a) - \dim L(0) \cap M + \dim W(\exp\ a) - d(W(\exp\ a), \exp(M)) =$$

$$= [\dim L(a) + \dim W(\exp\ a)] - [\dim L(0) \cap M + d(W(\exp\ a), \exp(M))].$$

The sum in the first bracket is less than n by assumptions, and the sum in the second bracket is not less than $\dim M$ by (9). Hence

$$\dim L/M(a) + \dim W(\exp\ a)/\exp\ M < n - \dim M,$$

that is *the pair is special.*

This means that after factorisation by M we are in case 1 again.

Hence either, as in subcase 1.1,

4.4 $W^{\exp(N_{L/M}(0))}(\exp\ a)$ is a proper closed subset of $W(\exp\ a)/\exp\ M$ containing $\exp(x + M)$.

or, as in subcase 1.2,

4.5

$$\exp(x + N_{L/M}(0) + M) \in \bigcup_{i \leq k_a} T_i(\exp\ a),$$

for group subvarieties T_i,

$$T_i(\exp\ a) \subseteq W(\exp\ a)/\exp(N_{L/M}(0)+M) \subsetneq (\mathrm{F}^\times)^n/\exp(N_{L/M}(0)+M).$$

Note again as in subcase 1.2 that the latter continues to hold with the same T_i if we replace a by b with $\exp\ \frac{a}{n} \equiv \exp\ \frac{b}{n}$, all n, in the field language, that is Galois conjugate.

4.6 For L and W as above let, for a of the length equal to the size of A,

$$\Phi_{L,W,A}(a,x) := x^\frown a \in L \ \& \ \exp(x^\frown a) \in W \to \bigvee_{M \in \pi_W \vee M = N_L(0) \vee M = \{0\}}$$
$$\Big[\exp(x) \in W^{\exp\ M}(\exp\ a) \ \vee \ \exp(x+M) \in W^{\exp\ N_{L/M}}(\exp\ a) \ \vee \ \exp(x+M) \in \bigcup_{i \le k_a} \exp(N$$

Ths is a quantifier-free formula and, by the analysis above

$$A \le \mathbb{F}^K \Rightarrow \mathbb{F}^K \models \forall x \Phi_{L,W,A}(\overrightarrow{A}, x).$$

4.7 Consequence of the analysis. Given $A \le \mathbb{F}^K$ and a special pair L, W, the ingredients of formula $\Phi_{L,W,A}(a,x)$, except for $\exp(N_{i,a,M})$, depend on A as parameters only. And $\exp(N_{i,a,M})$, as noted in cases 1.2 and 2.2 where these have been introduced, are of the form $T_i(\exp \frac{a}{m})$, $m \in \mathbb{N}$, T_i are group subvarieties (subtori) of $(\mathrm{F}^\times)^{n+\ell}$ and T_i depend on the Galois type of $\exp(\mathrm{span}_Q A) \subseteq \mathrm{F}^\times$ only.

In particular, if

$$B \le \mathbb{F}^K \ \text{ and } \exp(\mathrm{span}_Q A) \equiv_{\text{fields}} \exp(\mathrm{span}_Q B)$$

then

$$\mathbb{F}^K \models \forall x \Phi_{L,W,A}(\overrightarrow{B}, x).$$

4.8 We define a **strong embedding type** with variables Y, $|Y| = |A|$, $\overrightarrow{Y} = y$,

$$\mathrm{sttp}_A(y) := \mathrm{qftp}_A(y) \cup \{\forall x \ \Phi_{L,W,A}(y,x) : \ (L(y), W(\exp\ y)) \ \text{special}\},$$

where $\mathrm{qftp}_A(y)$ denotes the quantifier-free type of A in variables Y.

This is a type consisting of universal formulas.

Now we can reformulate 4.7

4.9 $B \le \mathbb{F}^K$ and $\mathbb{F}^K \models \mathrm{qftp}_A(B) \Rightarrow \mathbb{F}^K \models \mathrm{sttp}_A(B)$.

We assume below that $\mathbb{F}^K \in \mathcal{E}^K_d$ and A has been chosen so that $A \le \mathbb{F}^K$ as well as $A \cup \ker \le \mathbb{F}^K$ as discussed in 2.7.

4.10 Lemma.

$$\mathbb{F}^K \models \mathrm{sttp}_A(B) \Rightarrow B \le \mathbb{F}^K.$$

Proof Assume w.l.o.g. that $x \in V^n$ is $\mathbb{Q}$-linearly independent over B, $a = \overrightarrow{B}$, $L(a)$ is the minimal K-affine subspace over A containing x, and $W(\exp\, a)$ the minimal algebraic variety over $\exp(B)$ containing $\exp(x)$. Notice that under this choice $\exp\, x$ is multiplicatively independent over $\exp\, B$.

We show that $(L(a), W(\exp\, a)$ can not be special, thus proving $\delta^K(x/B) \geq 0$.

Indeed, if the pair were special, $\forall x \Phi_{L,W,A}(a, x)$ would imply that x satisifes one of the conditions on the second or third line of the definition of $\Phi_{L,W,A}(a, x)$.

Any of the conditions on the second line contradicts the assumptions that $W(\exp\, a)$ is the algebraic locus of $\exp\, x$ over $\exp\, B$, since $W^Q(\exp\, a)$, for Q a group subvariety, is a proper subvariety of $W(\exp\, a)$ by 4.1 and 4.4.

The condition on the third line can not hold because by 4.5 it would contradict the fact that $\exp\, x$ is multiplicatively independent over $\exp\, B$. □

4.11 Proposition. The following two conditions are equivalent:

$$\mathbb{F}^K \models \mathrm{sttp}_A(B) \tag{11}$$

and

$$B \leq \mathbb{F}^K \ \ \& \ \ \mathbb{F}^K \models \mathrm{qftp}_A(B) \tag{12}$$

Proof Lemma 4.9 proves (12) ⇒ (11). The converse follows from Lemma 4.10 and the definition of sttp.□

4.12 Let $\mathbb{F}^K$ be a member of $\mathcal{E}_d^K$ and $A \subseteq \mathbb{F}^K$ be a finite subset containing generators of ker(exp) such that

$$\delta^K(A) = -d.$$

It follows that $A \leq \mathbb{F}^K$.

Let

$$\mathrm{SCH}_A = \{\exists X \bigwedge S(X) : \ S \subset \mathrm{sttp}_A(X), \ \ \text{finite}, |X| = n\},$$

be the set of ∃∀-sentences stating the consistency of type sttp_A.

Recall the notation PK for the axioms of powered fields with powers in K.

4.13 Lemma. Let $\mathbb{F}^K$ be a model of $\mathrm{PK} + \mathrm{SCH}_A$ which realises the type sttp_A. Then $\mathbb{F}^K \in \mathcal{E}_d^K \setminus \mathcal{E}_{d-1}^K$.

Proof By assumption we have a finite subset $B \subseteq \mathbb{F}$ such that $\mathbb{F}^K \models \mathrm{sttp}_A(B)$. By Proposition 4.11

$$B \leq \mathbb{F}^K.$$

As a consequence of qftp_A we have $\delta^K(B) = -d$. It follows that $\mathbb{F}^K \notin \mathcal{E}_{d-1}^K$.

To see that $\mathbb{F}^K \in \mathcal{E}_d^K$ we need to prove that $\delta^K(Z) \geq -d$ for any finite $Z \subseteq \mathbb{F}^K$.

Let Y be a $\mathbb{Q}$-linear basis of $\mathrm{span}_Q(Z) \cap \mathrm{span}_Q(B)$. We have then $\mathrm{lin.dim}_{\mathbb{Q}}(Z/B) = \mathrm{lin.dim}_{\mathbb{Q}}(Z/Y)$ and thus $\delta^K(Z/Y) \geq \delta^K(Z/B) \geq 0$. But $\delta^K(Z) = \delta^K(Z/Y) + \delta^K(Y)$, so $\delta^K(Z) \geq \delta^K(Y) \geq -d$.□

4.14 Theorem. *Assume* $\delta^K(A) = -d$. *The following two conditions are equivalent for a structure* $\mathbb{F}^K$:

(i) $\mathbb{F}^K \models \mathrm{PK}{+}\mathrm{SCH}_A$;

(ii) $\mathbb{F}^K \in \mathcal{E}_d^K \setminus \mathcal{E}_{d-1}^K$ *and* qftp_A *is realised in some* ${}^*\mathbb{F}^K \succ \mathbb{F}^K$.

If in (i) $\mathbb{F}^K$ *also satisfies* EC *then in (ii)* $\mathcal{E}_d^K$ *should be replaced by* $\mathcal{EC}_d^K$.

Proof Assume (i). By the definition of SCH_A there is ${}^*\mathbb{F}^K \succ \mathbb{F}^K$ which realises sttp_A, say by B. By Lemma 4.13, ${}^*\mathbb{F}^K \in \mathcal{E}_d^K \setminus \mathcal{E}_{d-1}^K$, so ${}^*\mathbb{F}^K \in \mathcal{E}_d^K \setminus \mathcal{E}_{d-1}^K$.

It follows that $\mathbb{F}^K \in \mathcal{E}_d^K$, since $\delta^K(X/\ker) \geq -d$ for all $X \subseteq {}^*\mathrm{V}$.

It remains to see that $\mathbb{F}^K \notin \mathcal{E}_{d-1}^K$. Indeed, if it were in $\mathcal{E}_{d-1}^K$, we would have $A' \leq \mathbb{F}^K$ with $\delta^K(A') = -d'$, $d' \leq d-1$, and by the analysis in 4.3 arrive at the fact that $\mathbb{F}^K$ and ${}^*\mathbb{F}^K$ realise $\mathrm{sttp}_{A'}$, hence using again 4.13, ${}^*\mathbb{F}^K \in \mathcal{E}_{d'}^K$, a contradiction. This proves (ii).

Now, conversely, assume (ii).

We claim that ${}^*\mathbb{F}^K \in \mathcal{E}_d^K$. Indeed, since $\mathbb{F}^K \in \mathcal{E}_d^K$ by 4.3 we find a B with $\delta^K(B/\ker) = -d$, so $B \leq \mathbb{F}^K$ and $\mathbb{F}^K \models \mathrm{SCH}_B$. So ${}^*\mathbb{F}^K \models \mathrm{SCH}_B$ and as shown in the first part of the proof, it follows ${}^*\mathbb{F}^K \in \mathcal{E}_d^K$.

By assumptions, up to isomorphism, $A \subseteq {}^*\mathbb{F}^K$. Since $\delta^K(A) = -d$, we have $A \leq {}^*\mathbb{F}^K$. It follows ${}^*\mathbb{F}^K \models \mathrm{SCH}_A$, so $\mathbb{F}^K \models \mathrm{SCH}_A$ and (i) proved.

The last statement of the theorem follows from Proposition 2.10. □

4.15 Let $L \subseteq \mathrm{V}^n$ be a K-linear subspace of dimension l and $W \subseteq (\mathbb{F}^{\times})^{n+m}$ be a variety over $\mathbb{Q}$. Define the set of special parameters

$$P_l(W) = \{p \in \mathbb{F}^m : \dim W(p) < n - l\}.$$

This set is constructible (quantifier-free definable in the field language).

Theorem. Assume $\mathbb{F}^K \models \mathrm{PK}+\mathrm{SCH}_A$. There are a number N and codimension 1 $\mathbb{Q}$-linear subspaces $M_1, \ldots, M_N \subsetneq \mathrm{V}^n$ depending on L and W such that for every $p \in P_l(W)$ for some $a_1, \ldots a_N \in \mathrm{V}^n$

$$\{z \in \mathrm{V}^n : z \in L \ \&\ \exp(z) \in W(p)\} \subseteq \bigcup_{i \leq N} (M_i + a_i + \ker^n).$$

Proof First we consider the case of a single $p \in P_l$. Choose a finite set $B \subset \mathrm{V}$ so that $p \subseteq \exp(B)$ and $B \cup \ker \leq \mathbb{F}^K$. Let $z = \langle z_1, \ldots, z_n \rangle \in L$ such that $\exp(z) \in W(p)$. By the choice of B

$$\delta^K(z/B \cup \ker) \geq 0.$$

But $\mathrm{lin.dim}_K(z/B \cup \ker) + \mathrm{tr.deg}(\exp(z)/\exp(B) < n$. It follows, $\mathrm{lin.dim}_{\mathbb{Q}}(z/B \cup \ker) < n$. In other words,

$$m_1 z_1 + \ldots m_n z_n - b \in \ker,$$

for some $m_1, \ldots, m_n \in \mathbb{Z}$, not all zero, and $b \in \mathrm{span}_{\mathbb{Q}}(B)$. Denote

$$M = \{\langle x_1, \ldots, x_n \rangle \in \mathrm{V}^n : \ m_1 x_1 + \ldots m_n x_n = 0\}.$$

We have proved that

$$z \in L \ \&\ \exp(z) \in W(p) \ \Rightarrow\ z \in M + a + \ker^n \text{ for some } a \in \mathrm{span}_{\mathbb{Q}}(B)^n \text{ and } M \tag{13}$$

Claim. For a given p there is finitely many M and a such that (13).

Indeed, if not then the type saying that $z \in L \ \&\ \exp(z) \in W(p)$ and $z \notin M + a + \ker^n$, for M running through all $\mathbb{Q}$-linear subspaces of codimension 1 and $a \in \mathrm{span}_{\mathbb{Q}}(B)^n$ is consistent. This type would be realised in some ${}^*\mathbb{F}^K \succ \mathbb{F}^K$ contradicting (13).

The proved claim implies the existence of the bound N_p on the number of cosets $M + a + \ker^n$ satisfying (13). We need to show that there is an N that bounds all the N_p. Assuming such a bound does not exist we can find a $p \in P_l(W)$ in some ${}^*\mathbb{F}^K \succ \mathbb{F}^K$ for which no finite bound N_p does exist, contradicting the Claim.$\square$

4.16 Corollary. *Suppose $\mathbb{C}^K$ the structure on complex numbers, for some $K \subseteq \mathbb{C}$ satifies the assumptions 2.7 for some $A \subseteq \mathbb{C}$, that is*

$$\mathbb{C}^K \models \mathrm{PK}+\mathrm{SCH}_A.$$

Then there are codimension-1 $\mathbb{Q}$-linear subspaces $M_1, \ldots, M_N \subsetneq \mathbb{C}^n$ such that for every $p \in P_l(W)$ there are $a_1, \ldots a_N \in \mathbb{C}^n$ with the property that for every irreducible component S of the analytic set $\{z \in \mathbb{C}^n : z \in L \ \& \ \exp(z) \in W(p)\}$ there is $j \in \{1, \ldots, N\}$ and $k \in \mathbb{Z}^n$ such that $S \subseteq M_j + a_j + 2\pi i k$.

The corollary is immediate from the theorem once one takes into account that irreducible components of the analytic set $M_i + a_i + \ker^n$ are of the form $M_i + a_i + 2\pi i k$.

5 Completeness, near model completeness and superstability

5.1 Definition The extension of the initial language $\mathcal{L}_K$ by existential predicates

$$E_P(\bar{x}) \equiv \exists \bar{y} P(\bar{x}, \bar{y}),$$

where P is a quantifier-free formula, is denoted $\mathcal{L}_K^E$.

We assume throughout that A is a finite subset of sub$\mathcal{E}_d^K$, $\delta^K(A) = -d$

5.2 Lemma. *Assuming $\mathbb{F}_1^K \subseteq \mathbb{F}^K$ as $\mathcal{L}_K^E$-structures and $\mathbb{F}^K \models$ PK+SCH$_A$, we have $\mathbb{F}_1^K \models$ PK+SCH$_A$ and $\mathbb{F}_1^K \leq \mathbb{F}^K$.*

Proof $\mathbb{F}_1^K \in \mathcal{E}_d^K$ for every $\mathcal{L}_K$-substructure of $\mathbb{F}^K$, since facts of the form $\delta^K(X) = m$ are fixed by quantifier-free types.

To see that $\mathbb{F}_1^K \leq \mathbb{F}^K$ it is enough to show that for a finite B

$$B \leq \mathbb{F}_1^K \Rightarrow B \leq \mathbb{F}^K.$$

This follows from Proposition 4.11 if we take into account that sttp_B is $\mathcal{L}_K^E$-quantifier-free.

It remains to see that an elementary extension ${}^*\mathbb{F}_1^K$ of $\mathbb{F}_1^K$ contains a copy of A. This is immediate by the fact that the condition on consistency of qftp_A is given by existential $\mathcal{L}_K$-formulas, so that is by $\mathcal{L}_K^E$-quantifier-free ones. $\square$

5.3 Lemma. *Assume* $\mathbb{F}_1^K, \mathbb{F}_2^K \in \mathcal{E}_d^K$ *and* $\mathbb{F}_1^K \models \mathrm{EC}$. *Suppose* $\mathbb{F}_2^K \leq \mathbb{F}_1^K$. *Then* $\mathbb{F}_2^K \subseteq \mathbb{F}_1^K$ *in the language* $\mathcal{L}_K^E$.

Proof Recall that by Proposition 2.10 $\mathbb{F}_1^K \in \mathcal{EC}^K$. Let $a \subseteq \mathbb{F}_1^K$ be finite and suppose $\mathbb{F}_2^K \models \exists y\ P(a, y)$, where $P(x, y)$ is quantifier-free. By the definition of $\mathcal{EC}^K$ we get then $\mathbb{F}_1^K \models \exists y\ P(a, y)$. $\square$

5.4 Corollary. *For* $\mathbb{F}_1^K, \mathbb{F}_2^K \in \mathcal{EC}_d^K$

$$\mathbb{F}_1^K \subseteq \mathbb{F}_2^K \text{ as } \mathcal{L}_K^E\text{-structures} \quad \text{iff} \quad \mathbb{F}_1^K \leq \mathbb{F}_2^K.$$

5.5 We can always extend $\mathbb{F}^K$ to contain also an infinite cl-independent over A subset I, so suppose $A \cup I \leq \mathbb{F}^K \in \mathcal{E}_d^K$.

We will denote $\mathrm{SCH}_A + \mathrm{ID}$ the set of axioms stating that *the type* $\mathrm{sttp}_{A \cup I}$ *is consistent.*

Notice that given a finite $I_0 \subseteq I$ we have an embedding $A \cup I_0 \leq \mathbb{F}^K$ for every ω-saturated model $\mathbb{F}^K$ of $\mathrm{SCH}_A + \mathrm{ID}$.

We then have by Proposition 4.11.

5.6 Corollary. $\mathrm{PK} + \mathrm{SCH}_A + \mathrm{EC} + \mathrm{ID}$ *axiomatises the subclass of* $\mathcal{EC}_A^K$ *whose* ω*-saturated structures are infinite dimensional.*

We say that a (partial) map $\varphi : \mathbb{F}_1^K \to \mathbb{F}_2^K$ is an $\mathcal{L}_K^E$**-monomorphism**, if it is injective and for any k-ary $\mathcal{L}_K^E$-predicate S and any k-tuple a from the domain of φ

$$\mathbb{F}_1^K \models S(a) \text{ iff } \mathbb{F}_2^K \models S(\varphi(a)).$$

5.7 Lemma. *Let* $\mathbb{F}_1^K$ *and* $\mathbb{F}_2^K$ *satisfy* $\mathrm{PK} + \mathrm{SCH}_A + \mathrm{EC} + \mathrm{ID}$, *and* $B_1 \leq \mathbb{F}_1^K$, $B_2 \leq \mathbb{F}_2^K$ *such that there is an* $\mathcal{L}_K$*-monomorphism*

$$\varphi : B_1 \to B_2.$$

Let $\mathbb{F}_{B_1}^K$ *and* $\mathbb{F}_{B_2}^K$ *be the expansions of* $\mathbb{F}_1^K$, $\mathbb{F}_2^K$ *by constants naming elements of* B_1 *and* B_2 *in correspondence with* φ. *Then*

$$\mathbb{F}_{B_1}^K \equiv \mathbb{F}_{B_2}^K.$$

Proof We prove that given ω-saturated elementary extensions ${}^*\mathbb{F}_1^K$ of $\mathbb{F}_1^K$ and ${}^*\mathbb{F}_2^K$ of $\mathbb{F}_2^K$, given finite $C \subseteq {}^*\mathbb{F}_1^K$, $c \in {}^*\mathbb{F}_1^K$ and a $\mathcal{L}_K^E$-monomorphism φ of $B_1 \cup C$ into ${}^*\mathbb{F}_2^K$ one can extend the monomorphism to c. By symmetry, this yields a winning strategy for the Ehrenfeucht-Fraisse game, and we are done.

We may assume that φ is the identity and $B_1 \cup C = B = \varphi(B)$. It is enough to show that under the assumption for any $c \in {}^*\mathbb{F}_1^K$ we can extend φ to some $B' \supseteq Bc$ as an $\mathcal{L}_K$-monomorphism and $B' \leq {}^*\mathbb{F}_1^K$, $\varphi(B') \leq {}^*\mathbb{F}_2^K$.

If $\partial(c/B) = 1$ then define $B' = Bc$ and $\varphi(c)$ to be any element from ${}^*\mathbb{F}_2^K$ which is not in the ∂-closure of A in ${}^*\mathbb{F}_2^K$ (use ID). Then B' and $\varphi(B')$ are as required.

If $\partial(c/B) = 0$ then extend c to a finite string $\bar{c}$ from ${}^*\mathbb{F}_1^K$ so that $\delta^K(\bar{c}/B) = 0$. The quantifier free type of $\bar{c}$ over B is consistent with $\mathbb{F}_1^K$, by 5.3, and so is realised in ${}^*\mathbb{F}_1^K$, by $\bar{b}$ say. Since $\delta^K(\bar{b}/B) = 0$, we have $A\bar{b} \leq {}^*\mathbb{F}_2^K$. So, we can define $B' = B\bar{c}$ and $\varphi(\bar{c}) = \bar{b}$. □

5.8 Lemma. *Let $\mathbb{F}_1^K, \mathbb{F}_2^K$ be ω-saturated models of* $\mathrm{PK} + \mathrm{SCH}_A + \mathrm{ID}$, *$B_1, B_2$ finite subsets of $\mathbb{F}_1^K$, $\mathbb{F}_2^K$, correspondingly, and $\varphi : B_1 \to B_2$ is a $\mathcal{L}_K^E$-monomorphism. Then, there exists a finite subset $\tilde{B}_1$ such that $\tilde{B}_1 \leq \mathbb{F}_1^K$ and φ can be extended to $\tilde{B}_1$ in such a way that*

$$\varphi(\tilde{B}_1) = \tilde{B}_2 \leq \mathbb{F}_2^K.$$

Proof Let $\bar{b}_1$ be the string of all elements of B_1 and $\bar{c}$ in $\mathbb{F}_1^K$ such that $\delta^K(\bar{b}_1\bar{c}) = \partial(\bar{b}_1)$. It follows $B_1\bar{c} \leq \mathbb{F}_1^K$. Let $m = \partial(\bar{b}_1)$.

Let $q^0(\bar{x}\bar{y})$ be the $\mathcal{L}_K$-quantifier-free type of $\bar{b}_1\bar{c}$. Let $\bar{b}_2$ be a string in $\mathbb{F}_2^K$ which corresponds to $\bar{b}_1$. Then the $\mathcal{L}_K^E$-monomorphism guarantees that $q^0(\bar{b}_2\bar{y})$ is consistent and thus there is $\bar{d}$ in $\mathbb{F}_2^K$ realising the type, in particular $\partial(\bar{b}_2) \leq \delta^K(\bar{b}_1\bar{c}) = m = \partial(\bar{b}_1)$. By symmetry $\partial(\bar{b}_2) = m = \partial(\bar{b}_1)$. Since $\delta^K(\bar{b}_2\bar{d}) = \partial(\bar{b}_2)$, we have $\bar{b}_2\bar{d} \leq \mathbb{F}_2^K$. Now Lemma 5.3 says that $\bar{b}_2\bar{d}$ is of the same $\mathcal{L}_K^E$-quantifier-free type as $\bar{b}_1\bar{c}$.
□

5.9 Main Theorem. *Given $\mathbb{F}^K \in \mathcal{EC}_d^K$, let finite $A \leq \mathbb{F}^K$. Then the following hold:*

(i) The axioms $\mathrm{PK} + \mathrm{SCH}_A + \mathrm{ID} + \mathrm{EC}$ *determine the complete theory* $\mathrm{Th}(\mathbb{F}^K)$ *of* $\mathbb{F}^K$.

(ii) The theory $\mathrm{Th}(\mathbb{F}^K)$ *has quantifier elimination in language* $\mathcal{L}_K^E$.

(iii) Further on, $\mathrm{Th}(\mathbb{F}^K)$ *is superstable.*

(iv) Moreover, the group structure on ker *is embedded in* $\mathbb{F}^K$ *conservatively, that is no new relations are induced (using parameters) on* ker *from* $\mathbb{F}^K$.

Proof (i) and (ii) It follows from Lemmas 5.7 (with $B_1 \cong A \cong B_2$) and 5.8 that the theory is complete and submodel complete. The latter implies elimination of quantifiers (see e.g. Theorem 13.1 of S).

(iii) To prove superstability consider $\mathbb{F}^K \in \mathcal{EC}_A^K$ of cardinality λ. We want to establish the cardinality of the set $S(\mathbb{F}^K)$ of complete 1-types over $\mathbb{F}^K$. Let ${}^*\mathbb{F}^K$ be an elementary extension of $\mathbb{F}^K$ which realises all n-types over $\mathbb{F}^K$ for all n. Let $S^\#(\mathbb{F}^K)$ the set of all complete n-types over $\mathbb{F}^K$ which are realised in ${}^*\mathbb{F}^K$ by n-tuples $\bar{b} = \langle b_1, \dots, b_n\rangle$ such that $\delta^K(\bar{b}/\mathbb{F}^K) = \partial(b_1/\mathbb{F}^K)$. It follows that card $S(\mathbb{F}^K) \leq$ card $S^\#(\mathbb{F}^K)$.

From general properties of $\leq$ we get $\mathbb{F}\bar{b} \leq {}^*\mathbb{F}^K$, and by Lemma 5.3 the $\mathcal{L}_K^E$-quantifier-free type of $\bar{b}$ over $\mathbb{F}^K$ is determined by the $\mathcal{L}_K$-quantifier-free type of that. Thus card $S(\mathbb{F}^K)$ is less or equal to the cardinality of $QFS(\mathbb{F}^K)$, the set of all $\mathcal{L}_K$-quantifier-free complete types over $\mathbb{F}^K$.

We claim that $QFS(\mathbb{F}^K) \leq \lambda + 2^\omega$. Indeed, each quantifier-free $\mathcal{L}_K$-type of $\bar{b}$ over $\mathbb{F}^K$ is uniquely determined by the minimal K-affine subspace L over $\mathbb{F}^K$ containing $\bar{b}$ and, for each $l \in \mathbb{N}$, the minimal algebraic variety $W^{\frac{1}{l}}$ containing $\exp(\frac{\bar{b}}{l})$. Notice that, once $W = W^1$ is known, for each l there is at most l^n choices of $W^{\frac{1}{l}}$ ($n = |\bar{b}|$), all conjugated by torsion elements of $(\mathbb{F}^\times)^n$ of order l. This branches into at most 2^ω types for each of λ-many varieties W.

(iv) Consider again a saturated model $\mathbb{F}^K$ of the theory, let $C \leq \mathbb{F}^K$ be an arbitrary finite self-sufficient set and let $B = \ker \cup C$. Clearly, $B \leq \mathbb{F}^K$. We claim first that for every finite tuple $\bar{b}$ in B the complete $\mathcal{L}_K$-type of $C \cup \bar{b}$ is determined by the quantifier-free $\mathcal{L}_K$-type of the tuple. This is again a direct consequence of $C \cup \bar{b} \leq \mathbb{F}^K$, by Lemma 5.7. Now, since any type of a tuple in the definable B is equivalent to a $\mathcal{L}_K$-quantifier-free type, any definable subset of B^n is quantifier-free definable, by compactness. We deduce that any C-definable subset of $\ker^n$ is $\mathcal{L}_K(C)$-quantifier-free definable, hence any subset of $\ker^n$ definable with parameters is $\mathcal{L}_K$-quantifier-free definable.

More specifically, let $\bar{b}$ be $\mathbb{Q}$-linearly independent over C. We claim that then it is K-linearly independent over C, which follows from the assumption that $\delta^K(\bar{b}/C) \geq 0$.

It follows that quantifier-free $\mathcal{L}_K(C)$-formulas without parameters restricted to ker are Boolean combinations of formulas of the form $m_1x_1 + \ldots + m_nx_n = k_1c_1 + \ldots + k_pc_p$, for some $m_1, \ldots, m_n \in \mathbb{Z}$, $k_1, \ldots, k_p \in K$, and of the form $\frac{x}{m} \in \ker$ (equivalently, $\exp(\frac{x}{m}) = 1$). The latter can be equivalently rewritten as $\exists y \in \ker\ x = my$. This is the standard form for core formulas in the theory of the $\mathbb{Z}$-group $(\ker, +, 0)$. Which proves that the subsets of $\ker^n$ definable in $\mathbb{F}^K$ are the same as ones definable in $(\ker, +, 0)$. $\Box$

6 Raising to powers in the complex numbers

6.1 Consider the structure $\mathbb{C}^K$ for $K \subseteq \mathbb{C}$. Assume Schanuel's conjecture or, more specifically, its form derived in 2.5:

$$\mathbb{C}^K \in \mathcal{E}_d^K \text{ and } A \leq \mathbb{C}^K.$$

6.2 Theorem. *Assume the corollary of Schanuel's conjecture in the form 6.1. Suppose $K \subseteq \mathbb{R}$. Then*

$$\mathbb{C}^K \models \mathrm{PK}{+}\mathrm{SCH}_A{+}\mathrm{EC} + \mathrm{ID}.$$

In particular, the axioms define the complete theory of the structure which also has the properties described in the Main Theorem 5.9 .

Proof PK and SCH_A are immediate by assumptions. The axiom ID is a corollary of the Countable Closure property that has been proved for $\mathbb{C}_{\exp}$ in [8] under Schanuel's conjecture. The same proof works for $\mathbb{C}^K$ under the version 6.1 of Schanuel's conjecture. It remains to establish EC, the exponential-algebraic closedness.

This property was proved in [7], Theorem 3, under the extra assumption that Schanuel's conjecture holds *uniformly*. The latter is used in the proof just ones, as a condition for the statement of Theorem 2. But the statement of Theorem 2 is exactly Corollary 4.16, proved here using only assumptions of the present theorem.$\Box$

Note that the theorem states in particular that the corresponding form of Schanuel's conjecture (for $K \subseteq \mathbb{R}$) implies EC, exponential-algebraic closedness.

6.3 We recall the following result by A.Wilkie, J.Kirby and M.Bays.

Theorem. ([4] 1.3) *Let* $\mathbb{F}_{\exp}$ *be any exponential field, let* ker *be the kernel of its exponential map, let* C *be an ecl-closed subfield of* $\mathbb{F}_{\exp}$, *and let* λ *be an* m*-tuple which is exponentially algebraically independent over* C, $K = \mathbb{Q}(\lambda)$. *Then for any tuple* z *from* $\mathbb{F}$:

$$\mathrm{tr.deg}(\exp(z)/C(\lambda)) + \mathrm{lin.dim}_K(z/\ker) - \mathrm{lin.dim}_{\mathbb{Q}}(z/\ker) \geq 0. \quad (14)$$

In particular, this holds for the exponential field $\mathbb{C}_{\exp}$ *of complex numbers and* $C = \mathrm{ecl}(\emptyset)$.

Here, an **exponential field** $\mathbb{F}_{\exp}$ is a $(\mathbb{F}, +, \cdot, \exp)$ a field structure with a homomorphism $\exp : \mathbb{F} \to \mathbb{F}^{\times}$. An **ecl-closed subfield** is an exponential subfield $C \subseteq \mathbb{F}$ that is exponentially-algebraically closed inside $\mathbb{F}_{\exp}$ (see [4] for details). In the exponential field $\mathbb{C}_{\exp}$ the ecl-closure $\mathrm{ecl}(X)$ of a countable subset X is countable, by Lemma 5.12 of [8]. In particular, all but countably many complex numbers are exponentially algebraically independent over $\mathrm{ecl}(\emptyset)$.

6.4 Corollary of (14).

Since

$$\mathrm{tr.deg}(\exp(z)/C, \lambda) = \mathrm{tr.deg}(\exp(z), \lambda/C) - \mathrm{tr.deg}(\lambda/C) \leq \mathrm{tr.deg}(\exp z/C)$$

we have as a corollary

$$\mathrm{tr.deg}(\exp(z)/C) + \mathrm{lin.dim}_K(z/\ker) - \mathrm{lin.dim}_{\mathbb{Q}}(z/\ker) \geq 0,$$

and a weaker version, which is of interest to us here,

$$\delta^K(z/\ker) = \mathrm{tr.deg}(\exp z) + \mathrm{lin.dim}_K(z/\ker) - \mathrm{lin.dim}_{\mathbb{Q}}(z/\ker) \geq 0,$$

which amounts to say that

$$\mathbb{F}^K \in \mathcal{E}_0.$$

6.5 Corollary. *Let a finite subset* $\lambda \subseteq \mathbb{C}$ *be exponentially-algebraically independent over* $\mathrm{ecl}(\emptyset)$ *and let* $K = \mathbb{Q}(\lambda)$. *Then* $\mathbb{C}^K$ *satisfies* PK+SCH_0+ID, *where* SCH_0 *denotes* SCH_A *with* $A = \{2\pi i\}$.

In particular, the statement of Corollary 4.16 holds for $\mathbb{C}^K$.

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