REPRESENTATIVE DEFINABLE C^r FUNCTIONS ON DEFINABLE C^r GROUPS

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Abstract. Let G be a compact affine definable C^r group and let r be ∞ or ω. We prove that the representative definable C^r functions on G is dense in the space of continuous functions on G.

1. Introduction.

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. Everything is considered in $\mathcal{M}$, every definable map is assumed to be continuous and the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{N}$"unless otherwise stated. We assume that r denotes ∞ or ω.

General references on o-minimal structures are [1], [2], also see [13].

Definable C^rG manifolds and definable G sets in $\mathcal{M}$ are studied in [8], [7], [6].

Let G be a definable C^r group and $Def^r(G)$ denote the space of definable C^r functions. Left translations in G induce an action of G defined by $f : G \to \mathbb{R} \mapsto L(g, f) = f(g^{-1}x) : G \to \mathbb{R}$. A function f on G is *representative* if the functions $\{L(g, f) | g \in G\}$ generate a finite dimensional subspace of $Def^r(G)$.

Theorem 1.1. *Let G be a compact affine definable C^r group. Then the representative definable C^r functions on G is dense in the strong topology in the space of continuous functions on G.*

Let X be a definable C^rG manifold. We say that the action of G on X is *definably C^r linearizable* (resp. *C^r linearizable*) if there exist a definable C^r representation of G whose representation space Ω, a definable C^rG submanifold Y of Ω and a definable C^rG diffeomorphism (resp. C^r diffeomorphism) from X to Y.

Theorem 1.2. *Let G be a compact affine definable C^r group and X a compact definable C^rG manifold. Then the action is C^r linearizable.*

Remark that if $\mathcal{M} = \mathcal{R}$, then for any positive dimensional compact connected $C^\infty G$ manifold with non-transitive action, it admits uncountably many nonaffine definable $C^\infty G$ manifold structures ([10]). In Theorem 1.2, we cannot replace C^r linearizable by definably C^r linearizable.

Locally definable C^r manifolds are defined in [9].

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Theorem 1.3. *Let G be a connected locally definable C^r group and $(\tilde{G}, \pi)$ the universal cover of G. Then $\tilde{G}$ can be equipped uniquely with the structure of a locally definable C^r group such that π is a locally definable C^r group homomorphism.*

A locally Nash case of Theorem 1.3 is proved in [5].

2. Preliminaries and proof of results

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f $(\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m)$ is a definable set.

We say that a group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.

A Hausdorff space X is an *n-dimensional definable C^r manifold* if there exist a finite open cover $\{U_i\}_{i=1}^k$ of X, finite open sets $\{V_i\}_{i=1}^k$ of $\mathbb{R}^n$, and a finite collection of homeomorphisms $\{\phi_i : U_i \to V_i\}_{i=1}^k$ such that for any i, j with $U_i \cap U_j \neq \emptyset$, $\phi_i(U_i \cap U_j)$ is definable and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. A definable C^r manifold X is *affine* if X is definably C^r diffeomorphic to a definable C^r submanifold of some $\mathbb{R}^n$.

A definable C^r manifold (resp. An affine definable C^r manifold) G is a *definable C^r group* (resp. an *affine definable C^r group*) if G is a group and the group operations $G \times G \to G, G \to G$ are definable C^r maps.

A subgroup of a definable C^r group is a *definable subgroup* of it if it is a definable C^r submanifold of it. Note that every definable C^r subgroup of a definable C^r group is closed ([12]) and a closed subgroup of a definable C^r group is not necessarily definable.

Let G be a definable C^r group. A group homomorphism from G to some $O_n(\mathbb{R})$ is a *definable C^r representation* if it is a definable C^r map. A *definable C^r representation space* of G is $\mathbb{R}^n$ with the orthogonal action induced from a definable C^r representation of G. A *definable C^rG submanifold* means a G invariant definable C^r submanifold of some definable C^r representation space of G.

Let G be a definable C^r group. A *defin-able C^rG manifold* is a pair (X, ϕ) consisting of a definable C^r manifold X and a definable C^r action $\phi : G \times X \to X$ on X of G. For abbreviation, we write X instead of (X, ϕ). A definable C^rG manifold is *affine* if it is definably C^rG diffeomorphic to a definable C^rG submanifold of some definable C^r representation space of G.

Proof of Theorem 1.1. Since G is compact and affine, there exists a definable C^rG diffeomorphism f from G to a definable C^rG submanifold G' of some definable C^r representation space Ω of G.

Let $r : G \to \mathbb{R}$ be a continuous function. Applying Polynomial Approximation Theorem to $r \circ f^{-1} : G' \to \mathbb{R}$, we have a polynomial function $q : G' \to \mathbb{R}$ approximating $r \circ f^{-1}$. Since f is equivariant and G acts orthogonally on Ω and by P107 [11], $q \circ f : G \to \mathbb{R}$ is a representative on G which is a definable C^r function approximating r. □

By a way similar to the proof of results of [10], we have the following result.

Theorem 2.1. *Let G be a compact affine definable C^r group and X a compact $C^\infty G$ manifold. Then X is $C^\infty G$ diffeomorphic to a definable C^rG submanifold Y of some representation space of G.*

Proof of Theorem 1.2. We only have to prove the case where $r = \omega$. By Theorem 2.1, there exist a representation space Ω of a definable C^r representation of G, a definable C^rG submanifold Y of Ω and a $C^\infty G$ diffeomorphism $f : X \to Y$. By [P 233 [4]], any Whitney neighborhood of a $C^\infty G$ map to a representation space contains a $C^\omega G$ map. Thus we can approximate f by a $C^\omega G$ map $h : X \to \Omega$. Therefore we have a required $C^\omega G$ imbedding. □

A Hausdorff space X is an *n-dimensional locally definable C^r manifold* if there exist a countable open cover $\{U_i\}_{i=1}^\infty$ of X, countably many open sets $\{V_i\}_{i=1}^\infty$ of $\mathbb{R}^n$, and a countable collection of homeomorphisms $\{\phi_i : U_i \to V_i\}_{i=1}^\infty$ such that for any i, j with $U_i \cap U_j \neq \emptyset$, $\phi_i(U_i \cap U_j)$ is definable and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. We call the $(U_i, \phi_i)'s$ the *definable charts* of X.

Note that locally definable (C^0) manifolds are considered in [3].

Let X, Y be locally definable C^r manifolds with definable charts $(U_i, \phi_i)_{i\in I}, (W_j, \psi_j)_{j\in J}$ respectively. A continuous map $f : X \to Y$ is a *locally definable C^r map* if for every finite subset I' of I, there exists a finite subset J' of J such that $f(\cup_{i\in I'} U_i) \subset \cup_{j\in J'} V_j$ and that $f| \cup_{i\in I'} U_i : \cup_{i\in I'} U_i \to \cup_{j\in J'} V_j$ is a definable C^r map.

A bijective locally definable C^r map f between locally definable C^r manifolds is a *locally definable C^r diffeomorphism* if f^{-1} is a locally definable C^r map.

A locally definable C^r manifold X is *affine* if X is locally definably C^r diffeomorphic to a locally definable C^r submanifold of some $\mathbb{R}^n$. Note that for any positive integer s, a locally definable C^r manifold is locally definably C^s imbeddable into some $\mathbb{R}^l$ (1.3 [9]).

A locally definable C^r manifold (resp. An affine locally definable C^r manifold) G is a *locally definable C^r group* (resp. an *affine locally definable C^r group*) if G is a group and the group operations $G \times G \to G, G \to G$ are locally definable C^r maps.

Proof of Theorem 1.3. By the construction of the universal cover $\tilde{G}$ of G, $\tilde{G}$ is a C^r group whose charts are countable and π is a C^r map. Since G is a locally definable C^r group, every transition function is definable. □

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