DEFINABLY CONNECTED NONCONNECTED SETS

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Abstract. We give an example of a structure R on the real line, and a manifold M definable in R, such that M is definably connected but is not connected.

1. Introduction

Let $\mathcal{R}$ be an expansion of the real field $\bar{\mathbb{R}} := \langle \mathbb{R}, +, \cdot, < \rangle$. Let M be a definable subset of $\mathcal{R}$. We remind that M is called **definably connected** if there is no clopen definable subset Y of M such that $\emptyset \neq Y \neq M$.

Main Theorem. *There exists a structure $\mathcal{R}$ expanding $\bar{\mathbb{R}}$ and set M definable in $\mathcal{R}$, such that:*
(1) *M is a 1-dimensional embedded $\mathcal{C}^1$ submanifold of $\mathbb{R}^3$,*
(2) *M has 2 connected components,*
(3) *M is definably connected.*

We know that such $\mathcal{R}$ cannot be o-minimal, because in an o-minimal expansion of $\mathbb{R}$ every definable and definably connected set is arc-connected (and hence connected). However, we can find $\mathcal{R}$ as above which is also d-minimal (see [Mil05] for the definition and main properties of d-minimal structures).

The main ingredient is the following result, which follows easily from the proof of [MT06, Theorem 1]; we will give some details of the proof in §3.

Lemma 1.1. *There exists a sequence $P = \langle c_n : n \in \mathbb{N} \rangle$ of real numbers, such that*
(1) *P is strictly increasing and unbounded;*
(2) *the set $Q := \{c_n : n \text{ even}\}$ (as a set) is not definable in $\mathcal{R}$, the expansion of $\bar{\mathbb{R}}$ with a new predicate for P (where P is also regarded as a set).*
Moreover, we can find P as above such that $\mathcal{R}$ is also d-minimal.

We will show that $\mathcal{R}$ as in the above lemma satisfies the conclusion of the Main Theorem.

1.1. Application to Pfaffian functions.

Let $\mathcal{R}$ and M be as in the proof of the Main Theorem. In [Fra06], S. Fratarcangeli introduced the relative Pfaffian closure of $\bar{\mathbb{R}}$ inside $\mathcal{R}$. Consider the following 1-form on $\mathbb{R}^3$ $\omega(x, y, z) := dz$. Let L be the xy-coordinate plane. Notice that L is a Rolle Leaf with data $\langle \mathbb{R}^2, \omega \rangle$, according to the definition in [Spe99]. Let X_0 be the translate of M along the z-axis, such that the endpoints of M are on L, X_1 be its mirror image along the xy-plane, and $X := X_0 \cup X_1$. Then, X is a 1-dimensional $\mathcal{C}^1$ manifold which is definable in $\mathcal{R}$ and definably connected, which intersects L in 2 points, but which is never orthogonal to ω. Thus, L is not a $\mathcal{R}$-Rolle Leaf, according to [Fra06, Definition 5.2]: therefore, it is not always the case that a Rolle Leaf (à la Speissegger) definable in $\mathcal{R}$ and with data definable in $\bar{\mathbb{R}}$ is a $\mathcal{R}$-Rolle Leaf à la Fratarcangeli.

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2. Proof of the Main Theorem

$\mathcal{R}$ is the structure introduced in Lemma 1.1. We have to produce the manifold M. Let C_0 be the double helix in $\mathbb{R}^3$ obtained as union of the helices

$$\begin{cases} x = \sin(\theta) \\ y = \cos(\theta) \\ z = \theta/\pi, \end{cases} \quad \text{and} \quad \begin{cases} x = -\sin(\theta) \\ y = -\cos(\theta) \\ z = \theta/\pi, \end{cases}$$

with $0 \leq \theta \leq 1$. Let C be an embedded 1-dimensional manifold definable in $\bar{\mathbb{R}}$, of the "same shape" as C_0, such that the endpoints of C and C_0 coincide. What we mean is the following:

(1) C is the union of 2 disjoint $\mathcal{C}^1$ arcs, C_1 and C_2, where C_1 is the graph of a $\mathcal{C}^1$ function $g_1 : [0,1] \to [-1,1]^2$ (with domain the z-axis), and the same for C_2;
(2) $g_1(0) = \langle 0,1\rangle$, $g_1(1) = \langle 0,-1\rangle$, $g_2(0) = \langle 0,-1\rangle$, $g_1(1) = \langle 0,1\rangle$;
(3) $g_i(t) \in (-1,1)^2$ for $i = 1,2$ and $t \in (0,1)$;
(4) We also ask that $\langle 0,0\rangle = g_1'(0) = g_1'(1) = g_2'(0) = g_2'(1)$ (notice that this latter condition is not satisfied by C_0).

For every $n \in N$, let D_n be the double helix obtained from C by translation and dilation along the z-axis, such that the lower endpoints of D_n are $\langle 0, \pm 1, c_n\rangle$ and the upper endpoints are $\langle 0, \pm 1, c_{n+1}\rangle$. Finally, let $M \coloneqq \bigcup_n D_n$. We claim that M satisfies the conclusion of the Main Theorem. The fact that M is definable in $\mathcal{R}$ and has 2 connected components M_1 and M_2 is clear (where M_1 is the component containing the point $\langle 0, 1, d_0\rangle$). The fact that M is $\mathcal{C}^1$ is clear from (4). It remains to show that M is definably connected. If not, then M_1 would be definable in $\mathcal{R}$. However, $\langle 0, 1, z\rangle \in M_1$ iff $z \in Q$: hence, Q would be definable in $\mathcal{R}$, contradiction.

3. Proof of Lemma 1.1

In this section we sketch the proof of Lemma 1.1. Let $1 < c \in \mathbb{R}$ and P be the sequence $\{c^{2^n} : n \in \mathbb{N}\}$, and $\mathcal{R} \coloneqq \bar{\mathbb{R}}(P)$. By [MT06, Corollary 7], $\mathcal{R}$ is d-minimal. In the proof of [MT06, Theorem 1], Miller and Tyne establish the following result.

Fact 3.1. *Let $\bar{\mathcal{L}} \coloneqq \langle 0, 1, +, \cdot, <\rangle$ be the language of ordered fields, and $\bar{\mathcal{L}}(P)$ be its extension by the new unary predicate P (thus, $\mathcal{R}$ is naturally an $\bar{\mathcal{L}}(P)$-structure). Let $\langle \mathbb{F}, P^*\rangle$ be an elementary extension of $\langle \bar{\mathbb{R}}, P\rangle$. Let b and b' be in P^*, such that b and b' be in Q such that satisfy the same $\bar{\mathcal{L}}$-type over $\mathbb{R}$. Then, b and b' satisfy the same $\bar{\mathcal{L}}(P)$-type over $\mathbb{R}$.*

Assume now, by contradiction, that Q, the subset of P given by the elements with even index, is definable in $\mathcal{R}$, by an $\bar{\mathcal{L}}(P)$-formula (with parameters from $\mathbb{R}$) $\phi(x)$. Let $\langle \mathbb{F}, P^*\rangle$ be an ω-saturated elementary extension of $\mathcal{R}$. Let b be in P^*, such that $b > \mathbb{R}$ and $\phi(b)$ holds. Let b' be the successor of b in P^*. Then, $\neg\phi(b')$ holds. However, b and b' satisfy the same $\bar{\mathcal{L}}$-type over $\mathbb{R}$, contradicting Fact 3.1.

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