DEFINABLE MORSE FUNCTIONS IN A REAL CLOSED FIELD

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ABSTRACT. Let X be a definably compact definable C^r manifold and $2 \leq r < \infty$. We prove that the set of definable Morse functions is open and dense in the set of definable C^r functions on X with respect to the definable C^2 topology.

1. INTRODUCTION

In Morse theory the topological data of a given space can be described by Morse functions defined on the space. We refer the reader to the book by J. Milnor [10] for Morse theory on compact C^∞ manifolds.

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Everything is considered in $\mathcal{N}$, the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{N}$", each definable map is assumed to be continuous and $2 \leq r < \infty$.

General references on o-minimal structures are [2], [3], also see [13].

Definable C^r Morse functions in an o-minimal expansion of the standard structure of a real closed field are considered in [11].

In this paper we consider a definable C^r version of Morse theory in a real closed field R when $2 \leq r < \infty$.

Definable C^r manifolds are studied in [11], [1], and definable C^rG manifolds are studied in [4]. If R is the field $\mathbb{R}$ of real numbers, then definable C^rG manifolds are considered in [8], [7], [6] [5].

Let $Def^r(R^n)$ denote the set of definable C^r functions on R^n. For each $f \in Def^r(R^n)$ and for each positive definable function $\epsilon : R^n \to R$, the ϵ-neighborhood $N(f;\epsilon)$ of f in $Def^r(R^n)$ is defined by $\{h \in Def^r(R^n) | |\partial^\alpha(h-f)| < \epsilon, \forall \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \leq r\}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n, |\alpha| = \alpha_1 + \dots + \alpha_n, \partial^\alpha F = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. We call the topology defined by these ϵ-neighborhoods the *definable C^r topology*.

Theorem 1.1 (10.7 [1]). *Every definably compact definable C^r manifold X is definably C^r diffeomorphic to a definable C^r submanifold of some R^n.*

By Theorem 1.1, we can consider the set $Def^r(X)$ of definable C^r functions on X as a subspace of $Def^r(R^n)$.

Theorem 1.2. *Let X be a definably compact definable C^r manifold. Then the set of definable Morse functions $Def^r_{Morse}(X)$ is open and dense in the set $Def^r(X)$ of definable C^r functions on X with respect to the definable C^2 topology.*

Theorem 1.1 is a generalization of [9].

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2. Preliminaries.

Let $W_1 \subset R^n, W_2 \subset R^m$ be definable open sets and $f : W_1 \to W_2$ a definable map. We say that f is a *definable* C^r *map* if f is of class C^r. A definable C^r map is a *definable* C^r *diffeomorphism* if f is a C^r diffeomorphism.

Definition 2.1. *A Hausdorff space X is an n-dimensional definable C^r manifold if there exist a finite open cover $\{U_i\}_{i=1}^k$ of X, finite open sets $\{V_i\}_{i=1}^k$ of R^n, and a finite collection of homeomorphisms $\{\phi_i : U_i \to V_i\}_{i=1}^k$ such that for any i, j with $U_i \cap U_j \neq \emptyset$, $\phi_i(U_i \cap U_j)$ is definable and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. This pair $(\{U_i\}_{i=1}^k, \{\phi_i : U_i \to V_i\}_{i=1}^k)$ of sets and homeomorphisms is called a definable C^r coordinate system.*

A definable C^r manifold X is *definably compact* if for every $a, b \in R \cup \{\infty\} \cup \{-\infty\}$ with $a < b$ and for every definable map $f : (a, b) \to X$, $\lim_{x\to a+0} f(x)$ and $\lim_{x\to b-0} f(x)$ exist in X.

If $R = \mathbb{R}$, then for any definable C^r manifold X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \le x \le 1\}$ is definably compact but not compact.

Let X be an m-dimensional definable C^r manifold and $f : X \to R$ a definable C^r function. A point $p \in X$ is a *critical point* of f if the differential of f at p is zero. If p is a critical point of f, then $f(p)$ is called a *critical value* of f. Let p be a critical point of f and $(U, \phi : (U, p) \to (V, 0))$ a definable C^r neighborhood around p. The critical point p is *nondegenerate* if the Hessian of $f \circ \phi^{-1}$ at 0 is nonsingular. Direct computations show that the notion of nondegeniricity does not depend on the choice of a local coordinate neighborhood. We say that f is a *definable Morse function* if every critical point of f is nondegenerate.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following results.

Lemma 3.1 (6.3.6 [2]). *Let $A \subset R^n$ be a definable set which is the union of definable open subsets $U_1, \dots, U_n$ of A. Then A is the union of definable open subsets $W_1, \dots, W_n$ of A with $cl_A(W_i) \subset U_i$ for $i = 1, \dots, n$, where $cl_A(W_i)$ denotes the closure of W_i in A.*

Theorem 3.2 ([12]). *For a definable subset of R^n, it is definably compact if and only if it is closed and bounded.*

Theorem 3.3 (5.8 [1]). *Let $X \subset R^l$ be a definable C^r manifold. Given two disjoint definable sets $F_0, F_1 \subset X$ closed in X, there exists a definable C^p function $\delta : X \to R$ which is 0 exactly on F_0, 1 exactly on F_2 and $0 \le \delta \le 1$.*

The following result is a definable version of Sard's Theorem.

Theorem 3.4 (3.5 [1]). *Let $X_1 \subset R^s$ and $X_2 \subset R^t$ be definable C^r manifolds of dimension m and n, respectively. Let $f : X_1 \to X_2$ be a definable C^r map. Then the set of critical values of f has dimension less than n.*

By Theorem 3.4, we have the following lemma.

Lemma 3.5. *Let U be a definable open subset of R^m and $f : U \to R$ a definable C^r function. There exist $a_1, \dots, a_m \in R$ such that $F(x_1, \dots, x_m) = f(x_1, \dots, x_m) - (a_1x_1 + \dots + a_mx_m)$ is a definable Morse function on U and $|a_1|, \dots, |a_m|$ are sufficiently small.*

Let $\{\phi_i : U_i \to V_i\}_{i=1}^k$ be a definable C^r coordinate system of X. By Lemma 3.1, Theorem 3.2, Theorem 1.1 and X is definably compact, shrinking $\{U_i\}_{i=1}^k$, if necessary, there exists a finite collection $\{K_i\}_{i=i}^k$ of definably compact subsets with $K_i \subset U_i$ such that $X = \cup_{i=1}^k K_i$. From now on we fix $\{U_i\}_{i=1}^k$ and $\{K_i\}_{i=1}^k$.

Let $f, g : X \to R$ be definable C^r functions and $\epsilon > 0$. We say that g is a (C^2, ϵ) *approximation of* f on a definably compact subset K of X if the following three inequalities hold for any point $p \in K$.

$$\begin{cases} |f(p) - g(p)| < \epsilon, \\ |\frac{\partial f}{\partial x_i}(p) - \frac{\partial g}{\partial x_i}(p)| < \epsilon, 1 \le i \le n, \\ |\frac{\partial^2 f}{\partial x_i \partial x_j}(p) - \frac{\partial^2 g}{\partial x_i \partial x_j}(p)| < \epsilon, 1 \le i, j \le n. \end{cases}$$

Definition 3.6. *Let $f : X \to R$ be a definable C^r function and $\epsilon > 0$. A definable C^r function $g : X \to R$ is a (C^2, ϵ) approximation of f if g is a (C^2, ϵ) approximation of f on any K_i.*

Proposition 3.7. *Let C be a definably compact subset of X, $h : X \to R$ a definable C^r function and $\epsilon > 0$ is sufficiently small. If there are no degenerate critical points of h in C, then for every definable C^r function $h' : X \to R$ which is a (C^2, ϵ) approximation of h, C does not contain a degenerate critical point of h'. In particular $Def^r_{Morse}(X)$ is open in $Def^r(X)$ with respect to the definable C^2 topology.*

Proof. We consider in a definable C^r coordinate neighborhood $(U_l, (x_1, \dots, x_m))$. Let the Hessian of h with respect to $(U_l, (x_1, \dots, x_m))$ be $(\frac{\partial^2 h}{\partial x_i \partial x_j})$. Then h has no degenerate critical points in $C \cap K_l$ if and only if $|\frac{\partial h}{\partial x_1}| + \dots + |\frac{\partial h}{\partial x_n}| + |\det(\frac{\partial^2 h}{\partial x_i \partial x_j})| > 0$ holds in $C \cap K_l$. If $\epsilon > 0$ is sufficiently small, then for any h' which is a (C^2, ϵ) approximation of h, $|\frac{\partial h'}{\partial x_1}| + \dots + |\frac{\partial h'}{\partial x_n}| + |\det(\frac{\partial^2 h'}{\partial x_i \partial x_j})| > 0$ holds in $C \cap K_l$. Thus h' has no degenerate critical points in $C \cap K_l$. By a similar argument, h' has no degenerate critical points in $C = \cup_{i=1}^k C \cap K_l$. □

Proof of Theorem 1.2. Proposition 3.7 proves that $Def^r_{Morse}(X)$ is open in $Def^r(X)$.

To prove density of $Def^r_{Morse}(X)$, we proceed by induction on l. Let $g : X \to R$ be a definable C^r function and $\epsilon > 0$. Assume that we have a definable C^r function $f_{l-1} : X \to R$ such that f_{l-1} has no degenerate critical points in $C_{l-1} := \cup_{i=1}^{l-1} K_i$ and it is a (C^2, δ_{l-1}) approximation of g, where $\delta_{l-1} > 0$ is sufficiently smaller than ϵ.

We consider a definable C^r coordinate neighborhood $(U_l, (x_1, \dots, x_m))$. By Lemma 3.5, there exist $a_1, \dots, a_m \in R$ such that $f(x_1, \dots, x_m) - (a_1x_1 + \dots + a_mx_m)$ is a definable Morse function on U_l and $|a_1|, \dots, |a_m|$ are sufficiently small. By Theorem 3.3, we have a definable C^r function $h_l : X \to R$ such that h_l is identically 1 on some definable open neighborhood V_l of K_l in U_l, h_l is identically 0 outside of some definably compact set L_l with $V_l \subset L_l \subset U_l$ and $0 \le h_l \le 1$. We define $f_l : X \to R, f_l = f_{l-1}(x_1, \dots, x_m) - (a_1x_1 + \dots + a_mx_m)h_l(x_1, \dots, x_m)$ on U_l and $f_l = f_{l-1}(x_1, \dots, x_m)$ outside of L_l. By the definition of f_l, f_l is a definable C^r function on X.

Calculating on U_l, $|f_{l-1}(p) - f_l(p)| = |a_1x_1 + \cdots + a_mx_m|h_l(p)$, $|\frac{\partial f_{l-1}}{\partial x_i}(p) - \frac{\partial f_l}{\partial x_i}(p)| = |a_ih_l(p) + (a_1x_1 + \cdots + a_mx_m)\frac{\partial h_l}{\partial x_i}(p)|, 1 \leq i \leq m$, $|\frac{\partial^2 f_{l-1}}{\partial x_i \partial x_j}(p) - \frac{\partial^2 f_l}{\partial x_i \partial x_j}(p)| = |a_i\frac{\partial h_l}{\partial x_j}(p) + a_j\frac{\partial h_l}{\partial x_i}(p) + (a_1x_1 + \cdots + a_mx_m)\frac{\partial^2 h_l}{\partial x_i \partial x_j}(p)|, 1 \leq i, j \leq m$, where $p = (x_1, \dots, x_m)$.

By the construction of h_l and since X is definably compact, $|h_l|, |\frac{\partial h_l}{\partial x_i}|, |\frac{\partial^2 h_l}{\partial x_i \partial x_j}|$ are bounded. Thus f_l is a (C^2, δ_l') approximation of f_{l-1} on K_l if $|a_1|, \dots, |a_m| > 0$ are sufficiently small.

We now consider on K_j when $j \neq l$. Since $f_{l-1} = f_l$ outside of L_l, we only have to evaluate them on $K_j \cap L_l$. Since $K_j \cap L_l \subset U_j \cap U_l$, they are evaluated by the Jacobian of $(U_j, (y_1, \dots, y_m))$ between $(U_l, (x_1, \dots, x_m))$. It is bounded on $K_j \cap L_l$ because $K_j \cap L_l$ is definably compact. Thus they are sufficiently small if $|a_1|, \dots, |a_m| > 0$ are sufficiently small. Hence f_l is a (C^2, δ_l) approximation of f_{l-1}. By Proposition 3.7, f_l has no degenerate critical points in C_{l-1}. By the construction of f_l, f_l has no degenerate critical points in K_l. Thus there are no degenerate critical points of f_l in $C_l := \cup_{i=1}^l K_i$. Therefore $f_k : X \to R$ is the required definable Morse function on X. □

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