Relative categoricity in abelian groups

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Abstract

We consider structures A consisting of an abelian group with a subgroup A^P distinguished by a 1-ary relation symbol P, and complete theories T of such structures. Such a theory T is (κ, λ)-categorical if T has models A of cardinality λ with $|A^P| = \kappa$, and given any two such models A, B with $A^P = B^P$, there is an isomorphism from A to B which is the identity on A^P. We state all true theorems of the form: If T is (κ, λ)-categorical then T is (κ', λ')-categorical. We classify the A of finite order λ with A^P of order κ which are (κ, λ)-categorical.

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The paper falls into four parts. Part I introduces some definitions and sets up the needed machinery. Part II shows that (κ, λ)-categorical theories of pairs of abelian groups must satisfy certain conditions which depend on κ and λ. In Part III we show that the conditions derived in Part II are not only necessary but also sufficient for the relevant kind of relative categoricity, and we draw some corollaries. Part IV gives a complete classification of the finite relatively categorical p-group pairs where p is an odd prime, and also when p is 2 under the further assumption that A^P is a characteristic subgroup of A.

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Part I
Introductory

1 Relative categoricity

A complete theory is a consistent first-order theory whose models are all elementarily equivalent. We write $A \equiv B$ when A is elementarily equivalent to B, and $A \preccurlyeq B$ when A is an elementary substructure of B.

Throughout this paper, T is a complete theory in a countable first-order language $L(P)$, one of whose symbols is a 1-ary relation symbol P; L is the language got by dropping P from $L(P)$; and for every model A of T, the set P_A of elements of A which satisfy the formula $P(x)$ is the domain of a substructure A^P of the reduct $A \restriction L$. We call A^P the *P-part* of A. Since a sentence is true in A^P if and only if its relativisation to P is true in A, the complete theory of A^P is determined by T; we write it as T^P.

We write κ, λ etc. for cardinals. By a (κ, λ)-*structure* (or a (κ, λ)-*model* if we are talking about models of T) we mean an $L(P)$-structure A with $|A^P| = \kappa$ and $|A| = \lambda$.

We say that T is (κ, λ)-*categorical* if T has (κ, λ)-models, and whenever A, B are any two such models with $A^P = B^P$, there is an isomorphism from A to B over A^P (i.e. which is the identity on A^P).

We say that T is *relatively categorical* if whenever A, B are any two models of T with $A^P = B^P$, there is an isomorphism from A to B over A^P.

Lemma 1.1 *If T is (κ, λ)-categorical and A is a (κ, λ)-model of T, then every automorphism of A^P extends to an automorphism of A.*

Proof. Let α be an automorphism of A^P. Construct a structure B and an isomorphism $\gamma : A \to B$ so that $A^P = B^P$ and γ extends α^{-1}. By assumption there is an isomorphism $\beta : A \to B$ over A^P. Then $\gamma^{-1}\beta$ is an automorphism of A extending α. □

2 Abelian groups

We use standard abelian group notation, as for example in Fuchs [3]. We write 0 for the trivial group; if B and C are subgroups of the abelian group A, we say that B and C are *disjoint* when $B \cap C = 0$. We write $A[n]$ for the subgroup of A consisting of the elements a such that $na = 0$. An element a of A is m-*divisible* if $a = mb$ for some element b of A. If $b_1, \ldots, b_n$ are elements of a group B, then $\langle b_1, \ldots, b_n \rangle$ means the subgroup of B generated by $b_1, \ldots, b_n$. A group B is *bounded* if for some finite n, $nB = 0$; the least such n is the *exponent* of B. If B is a torsion group, then B is the direct sum of its p-components, $B = \bigoplus_{p \text{ prime}} B_p$, and this decomposition is unique. We write $\mathbb{Q}$ for the additive group of rationals, $\mathbb{J}_p$ for the additive group of p-adic integers, $\mathbb{Z}(p^\infty)$ for the Prüfer p-group and $A^{(\mu)}$ for the direct sum of μ copies of the group A.

We will sometimes describe a group A as having a property when its complete first-order theory Th(A) has the property; for example 'A is relatively categorical' means 'Th(A) is relatively categorical'. We use [5] Appendices A and B for facts on the first-order theories of abelian groups. A key result is that every ω_1-saturated abelian group is pure-injective ([5] Theorem 10.7.3).

The following results are now classical. See Macintyre [8] for (a) and (b), while (c) is immediate from the Ryll-Nardzewski theorem.

Theorem 2.1 *Let T be a complete theory of infinite abelian groups.*

- *(a) T is ω-stable if and only if every model of T is the direct sum of a divisible group and a bounded group.*
- *(b) T is uncountably categorical if and only if one of the following holds:*
  - *(i) Every model of T is a direct sum of a finite group and an infinite homocyclic group $\mathbb{Z}(p^k)^{(\mu)}$;*
  - *(ii) every model of T is a direct sum of a finite group, a divisible torsion-free group (possibly trivial), and divisible p-groups of finite rank for each prime p.*
- *(c) T is ω-categorical if and only if every model of T is bounded.*

In any first-order language, a formula is said to be *positive primitive*, or more briefly *p.p.*, if it has the form $\exists \bar{x} \bigwedge_{i \in I} \phi_i$ where each ϕ_i is atomic. A subgroup B of an abelian group A is *pure* if and only if for every tuple $\bar{b}$ of elements of B and every p.p. formula $\phi(\bar{x})$, $A \models \phi(\bar{b})$ implies $B \models \phi(\bar{b})$ (cf. Hodges [5] p. 56).

Let A be an abelian group and p a prime. If k is a natural number or ∞, we define $p^k A$ by induction on k:

$$p^0 A = A; \;\; p^{k+1} A = p(p^k A).$$

The *p-height* of an element a of A, $\mathrm{ht}^p_A(a)$, is the least $k < \omega$ such that $a \in p^k A \setminus p^{k+1}A$, or ∞ if there is no such ordinal k. We put $\infty + 1 = \infty$. (This follows Eklof and Fisher [1]. Fuchs continues the definitions of p-heights into the transfinite ordinals.)

We will say that an abelian group A is *divisible-plus-bounded* if A is the direct sum of a divisible group and a bounded group. Divisible-plus-bounded groups appeared in Theorem 2.1 and they will play a central role in this paper.

The following lemma gives some group-theoretic characterisations of divisible-plus-bounded groups.

Lemma 2.2 *Let A be an abelian group. The following are equivalent:*

(a) A is divisible-plus-bounded.

(b) For some positive integer m, mA is divisible.

(c) There is a positive integer m such that $mA = mnA$ for all positive integers n.

(d) The number of pairs (p, n), such that p is prime and n is a positive integer such that $|p^n A/p^{n+1}A| > 1$, is finite.

(e) There is a finite k such that for each prime p, the p-heights of elements of A are all either ∞ or $\leqslant k$; and for all but finitely many primes p the p-component of A is divisible.

Proof. (a) $\Rightarrow$ (b) is by taking for m the exponent of the bounded part of A. Then (b) $\Rightarrow$ (c) is immediate.

(c) $\Rightarrow$ (d): If (d) fails for one prime p and infinitely many n, then for every positive integer m we have $mA \neq mpA$, since multiplication by a number relatively prime to p makes no difference to p-heights. If (d) fails for infinitely many primes, then for every positive integer m we have $mA \neq qmA$ for some q relatively prime to m, for the same reason.

(e) is a paraphrase of (d).

(e) $\Rightarrow$ (a): If (e) holds with an integer k, then take m divisible by p^{k+1} for the finitely many exceptional primes p. If $a \in mA$ then a has infinite p-height for every prime p, and by Euclid it follows that a is divisible by every positive integer. In particular for every prime p there is b such that $mpb = a$, and so a is pc for an element $c = mb$ of mA. Therefore mA is divisible, proving (b). To derive (a), choose a subgroup B of A which is maximal disjoint from mA. Then $A = mA \oplus B$, and $mB \subseteq mA$ so that $mB = 0$ and B is bounded. □

If an abelian group satisfies (c) or (d) in the lemma, then clearly so does every group elementarily equivalent to A. So the class of divisible-plus-bounded groups is closed under elementary equivalence. The next lemma lists some properties of the complete theories of divisible-plus-bounded abelian groups.

Lemma 2.3 *Let T be a complete first-order theory of abelian groups. The following are equivalent:*

(a) Some (or all) models of T are divisible-plus-bounded.

(b) T has finite models or is ω-stable.

(c) Every model of T is pure-injective.

(d) For every model B of T, $\mathrm{Ext}(\mathbb{Q}, B) = 0$.

Proof. If (a) holds, then we can verify (b) by counting types. Also both bounded and divisible abelian groups are pure-injective, and a direct sum of two pure-injectives is pure-injective, so that (c) holds too, and hence also (d) by [3] Proposition 54.1.

If (a) fails, then by (c) of the previous lemma there is an infinite increasing sequence $(n_0, n_1, \ldots)$ of positive integers such that for each $i < \omega$, $n_{i+1}A$ is a proper subgroup of n_iA. Thus the cosets of the n_iA ($i \in \omega$) form an infinite branching tree. There are continuum many branches, so that in a countable model B of T not all the branches are realised by elements. Hence (b) fails, and we infer (a) $\Leftrightarrow$ (b).

It remains to complete the cycle (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) by proving (d) $\Rightarrow$ (a). Suppose (a) fails and B is a countable Szmielew model of T (cf. [5] p. 663f). Then B has either infinitely many pairwise non-isomorphic cyclic direct summands, or for some prime p a direct summand of the form $\mathbb{Z}_{(p)}$ (the localisation of $\mathbb{Z}$ at p, which is not a pure-injective group). In either case (d) fails (by [3] Corollaries 54.4,5). □

An abelian group B satisfying the condition of (d) of the lemma is said to be *cotorsion* (Fuchs [3] p. 232).

The next lemma describes how divisible-plus-bounded groups behave in short exact sequences.

Lemma 2.4 *Suppose A is an abelian group and B a subgroup of A.*

(a) If A is divisible-plus-bounded then A/B is divisible-plus-bounded.

(b) If A is divisible-plus-bounded and B is a pure subgroup of A then B is divisible-plus-bounded.

(c) If A/B is bounded, then A is divisible-plus-bounded if and only if B is divisible-plus-bounded.

Proof. (a) Suppose mA is divisible. Then $m(A/B) = (mA + B)/B = mA/(mA \cap B)$, which is divisible since it is a homomorphic image of a divisible group.

(b) Assuming A is divisible-plus-bounded, let m be the exponent of the bounded part of A. Then $mB \subseteq mA \cap B$. For each element a of $mA \cap B$ and each positive n, a is divisible by mn in A and hence also in B by purity; so for some $b \in B$, $n(mb) = a$. But $mb \in mA \cap B$, proving that $mA \cap B$ is divisible.

(c) Assume A/B is bounded, say with exponent m. Suppose first that B is divisible-plus-bounded, and the bounded part has exponent n. Then for every element a of A, mna lies in the divisible part of B, which is also a divisible subgroup of A.

Conversely suppose A is divisible-plus-bounded, $A = D \oplus E$, and suppose its bounded part E has exponent k. We can write every element b of B as $d+e$ where d and e come respectively from D and E. Then $kb \in D \cap B$. So it suffices to show that $D \cap B$ is divisible. Neither $\mathbb{Q}$ nor any $\mathbb{Z}(p^\infty)$ has any proper subgroup with bounded quotient; so $D \cap B$ is divisible as required. □

We give two countexamples to strengthenings of the previous lemma. They motivate several of the constructions in this paper.

Example 2.5 (a) A (and hence also A/B) divisible but B not divisible-plus-bounded:

> A is $\mathbb{Q}$ and B is $\mathbb{Z}$.

(b) A/B divisible torsion, B bounded, A not divisible-plus-bounded:

> Let p be a prime and let G be the group $\bigoplus_{1 \leqslant \omega} H_n$ where each H_n is a copy of $\mathbb{Z}(p^n)$ generated by an element h_n. Let C be $\bigoplus_{1 \leqslant \omega} C_n$ where each C_n is a copy of $\mathbb{Z}(p)$ generated by an element c_n. Let K be the subgroup of $H \oplus C$ generated by the

elements $ph_{n+1} - h_n - c_n$ ($n < \omega$), and let A be the group $(H \oplus C)/K$. Let B be the subgroup of A generated by the elements $c_n + K$ ($n < \omega$). The group B is bounded with exponent p, and $A/B \cong \mathbb{Z}(p^\infty)$ is divisible. But A is not divisible-plus-bounded.

3 Group pairs

Definition 3.1 (a) By a *group pair* we mean an $L(P)$-structure A which is an abelian group with A^P a subgroup.

(b) A homomorphism $h : A \to B$ of group pairs is a homomorphism of abelian groups such that $hA^P \subseteq B^P$. (Then isomorphisms, short exact sequences etc. are defined in the obvious way.)

(c) We say that a group pair A is *bounded over* P if the group A/A^P is bounded. (And likewise for other notions where the meaning is clear.)

We will say that a theory has a property when all its models have the property. Thus every complete theory of a divisible-plus-bounded group is pure-injective. It's clear that if A is a group pair that is bounded over P, then every group pair elementarily equivalent to A is also bounded over P; so we will say that the theory of A is bounded over P.

A good deal of the model theory of abelian groups carries over immediately to group pairs. For example if $\phi(x_0, \ldots, x_{n-1})$ is a p.p. formula with no parameters and A is a group pair, then $\{\bar{a} : A \models \phi(\bar{a})\}$ is a subgroup $\phi(A^n)$ of A^n; subgroups of this form are called *p.p. subgroups* of A. By the *Baur-Monk invariant* $\phi(\bar{x})/\psi(\bar{x})$ of A we mean the cardinality of the quotient group

$$\phi(A^n)/(\phi(A^n) \cap \psi(A^n)),$$

counted as either finite or ∞. Note that by replacing ψ by a p.p. formula logically equivalent to the conjunction $\phi \wedge \psi$, we can assume that every Baur-Monk invariant is the cardinality of a quotient group $\phi(A^n)/\psi(A^n)$ where ϕ and ψ are p.p. formulas such that ψ entails ϕ. An *invariant sentence* is a sentence of $L(P)$ expressing that a certain Baur-Monk invariant has value $\leqslant k$, where k is a positive integer.

The same proof as for modules (e.g. [5] section A1) gives:

Theorem 3.2 *For every formula $\phi(\bar{x})$ of $L(P)$ there is a boolean combination $\phi'(\bar{x})$ of invariant sentences and p.p. formulas of $L(P)$, which is equivalent to $\phi(\bar{x})$ in all group pairs. Every complete theory of group pairs is stable.* □

Corollary 3.3 *If T has a (κ, λ)-model with $\omega \leq \kappa < \lambda$, then T has a (κ', λ')-model whenever $\omega \leq \kappa' \leq \lambda'$.*

Proof. This follows from Shelah [10] Conclusion V.6.14(2) (noting the assumption on his p. 223 that T is stable). □

Corollary 3.4 *Suppose $A \subseteq B \subseteq C$ are group pairs. If $A \equiv C$, A is pure in B and B is pure in C, then the inclusions are elementary embeddings. In particular if $A \equiv C$, $A \subseteq C$ and the inclusion is pure, then $A \preccurlyeq C$.*

Proof. Since $A \equiv C$, A and C have the same Baur-Monk invariants. Suppose $\bar{a}$ is a tuple of elements of A, and $A \models \chi(\bar{a})$. By Theorem 3.2, χ is equivalent in all group pairs to a boolean combination of invariant sentences and p.p. formulas. Since A is pure in B and B is pure in C, $\bar{a}$ satisfies the same p.p. formulas in B and C as it does in A. So $C \models \chi(\bar{a})$, showing that $A \preccurlyeq C$.

To complete the proof it suffices to show that B has the same Baur-Monk invariants as A and C, since then $A \equiv B \equiv C$ by Theorem 3.2 again. Consider the quotient group

$$\phi(B^n)/\psi(B^n)$$

where ϕ, ψ are p.p. formulas such that ψ entails ϕ. Since A is pure in B, $\phi(A^n) = \phi(B^n) \cap A^n$ and $\psi(A^n) = \psi(B^n) \cap A^n$. So by the Second Isomorphism Theorem

$$\begin{aligned} \phi(A^n)/\psi(A^n) &= (\phi(B^n) \cap A^n)/(\phi(B^n) \cap A^n \cap \psi(B^n)) \\ &\cong ((\phi(B^n) \cap A^n) + \psi(B^n))/\psi(B^n) \subseteq \phi(B^n)/\psi(B^n). \end{aligned}$$

It follows that each Baur-Monk invariant of A is $\leqslant$ the corresponding invariant of B. The same argument shows the same for B and C. Since A and C have equal invariants, we can replace the $\leqslant$ by $=$. □

In the class of group pairs, we can form direct sums $A = \bigoplus_{i \in I}^{P} A_i$. The definition is the same as for abelian groups, except that we also require that for any element $a = \sum_{i \in I} a_i$ with a_i in A_i,

$$a \in A^P \;\Leftrightarrow\; \text{for all } i \in I, a_i \in A_i^P.$$

Right to left follows from the fact that P picks out a subgroup. Left to right doesn't; we look for criteria which guarantee that it does hold.

Lemma 3.5 *Let A be an abelian group, and let A_i ($i \in I$) be subgroups of A such that $A = \bigoplus_{i\in I} A_i$ as abelian groups. Suppose also that there are a subset J of I and an element j_0 of J such that $A^P \subseteq \bigoplus_{i\in J} A_i$, and $A_j \subseteq A^P$ for all $j \in J \setminus \{j_0\}$. Then $A = \bigoplus^P_{i\in I} A_i$ as group pairs.*

Proof. Suppose $a = \sum_{i\in I} a_i$, $a \in A^P$. Then $a = (\sum_{j_0\neq i\in J} a_i) + a_{j_0}$. By assumption each a_i ($j_0 \neq i \in J$) is in A^P. So a_{j_0} is in A^P too. Also when $i \notin J$, $a_i = 0 \in A^P$. □

We will often use Lemma 3.5 silently.

Lemma 3.6 *Let A be a group pair, and suppose that A and A^P are both divisible-plus-bounded. Then $A = F \oplus^P \bigoplus^P_{p \text{ prime}} A_p$, where F is torsion-free divisible and for each prime p, A_p is the p-component of A with $(A_p)^P = A_p \cap A^P$.*

Proof. Since A is divisible-plus-bounded, we can write A as an abelian group direct sum $A_Q \oplus t(A)$ where A_Q is the torsion-free divisible part of A and $t(A)$ is the torsion part of A. By assumption A^P is also divisible-plus-bounded, so that we can write $A^P = (A^P)_Q \oplus t(A^P)$ likewise. Then clearly $t(A^P) \subseteq t(A)$, and we can choose A^P so that $(A^P)_Q \subseteq A_Q$. Let the group pair F be A_Q with $F^P = (A^P)_Q$. Since the p-components of an element of an abelian torsion group are uniquely determined, each p-component of $t(A^P)$ lies inside $(A^P)_p$. □

A direct sum $A \oplus^P B$ is in fact a direct product $A \times B$, so that the Feferman-Vaught theorem applies (e.g. [5] section 9.6). Thus:

Lemma 3.7 *(a) If $B_1 \equiv C_1$ and $B_2 \equiv C_2$ then $B_1 \oplus B_2 \equiv C_1 \oplus C_2$.*

(b) If $B_1 \preccurlyeq C_1$ and $B_2 \preccurlyeq C_2$ then $B_1 \oplus B_2 \preccurlyeq C_1 \oplus C_2$.

(c) If X is a set of elements of B and a, b are elements of B, then a, b realise distinct types over X in $B \oplus C$ if and only if they realise distinct types over X in B.

(d) Suppose B and C both have the property that over every countable set of parameters there are at most countably many (1-)types realised. Then the same holds for $B \oplus C$.

(e) Suppose $\phi(\bar{x})$ is a formula of $L(P)$ and T is a complete theory in $L(P)$. Then there is a formula $\theta(\bar{x})$ such that if A, B are $L(P)$-structures and B is a model of T, then for every $\bar{a}$ in A,

$$A \oplus B \models \phi[\bar{a}] \;\Leftrightarrow\; A \models \theta[\bar{a}].$$

Proof. (a), (b) and left to right in (c) are straightforward from the Feferman-Vaught theorem. For right to left in (c), suppose a and b satisfy different types over X in B. By quantifier elimination there is some p.p. formula $\phi(x)$ which is satisfied in B by a and not by b (say). Since ϕ is existential, a satisfies it also in $B \oplus C$. But b doesn't satisfy it in $B \oplus C$, since there is a projection from $B \oplus C$ to B that fixes B pointwise.

For (d), here is a more direct argument which works in our case. Suppose to the contrary that X is a countable set of elements of $B \oplus C$ over which the elements $b_i + c_i$ ($i < \omega_1$) realise distinct types. Without loss we can assume that $X = Y + Z$ where Y, Z are respectively subgroups of B, C. By assumption at most countably many types are realised by the b_i over Y; so we can assume that all the b_i realise the same type over Y. Since $b_0 + c_0$ and $b_1 + c_1$ realise different types over $Y + Z$, quantifier elimination gives us a p.p. formula ψ and elements $\bar{d}, \bar{e}$ of Y, Z respectively, such that

$$B \oplus C \models \psi(b_0 + c_0, \bar{d}, \bar{e}) \wedge \neg\psi(b_1 + c_1, \bar{d}, \bar{e})$$

(or vice versa). Since ψ is p.p., we infer

$$B \models \psi(b_0, \bar{d}, \bar{0})$$

and hence

$$B \models \psi(b_1, \bar{d}, \bar{0}).$$

These two conditions hold also with $B \oplus C$ in place of B, since ψ is existential. So by subtraction

$$B \oplus C \models \psi(c_0, \bar{0}, \bar{e}) \wedge \neg\psi(c_1, \bar{0}, \bar{e}),$$

whence

$$C \models \psi(c_0, \bar{0}, \bar{e}) \wedge \neg\psi(c_1, \bar{0}, \bar{e}).$$

This argument shows that the c_i realise uncountably many different types over Z in $B \oplus C$ and also in C.

To prove (e), assume $\phi(\bar{x})$ is given, and use the Feferman-Vaught theorem as at [5] Theorem 9.6.1 to find $\theta_0(\bar{x}), \ldots, \theta_{k-1}(\bar{x})$ in $L(P)$ such that for any A, B and any $\bar{a}$ in A, the truth of $\phi(\bar{a})$ in $A \oplus B$ is determined by which of the $\theta_i(\bar{a})$ are true in A and which of the $\theta_i(\bar{0})$ are true in B. If B is a model of T then it is determined which $\theta_i(\bar{0})$ are true in B, and so the truth of $\phi(\bar{a})$ in $A \oplus B$ is determined by whether $\theta(\bar{a})$ is true in A, for some boolean combination θ of the θ_i. □

Definition 3.8 Let T_1 and T_2 be complete theories in $L(P)$, and suppose T_2 is disjoint from P. Let T be a complete theory in $L(P)$. We write

$$T = T_1 \oplus T_2$$

to mean that if A_1 and A_2 are models of T_1 and T_2 respectively, then $A_1 \oplus A_2$ is a model of T; and moreover every model of T is of this form.

Note that by Lemma 3.7(a), if A_1 and A_2 have complete theories T_1 and T_2 respectively, then the theory $T = \mathrm{Th}(A_1 \oplus A_2)$ depends only on T_1 and T_2. So proving the decomposition $T = T_1 \oplus T_2$ is generally a matter of showing that every model of T takes this form.

4 Pushouts

Some of our results will need a construction which is one step more complicated than direct sums, namely pushouts or fibred sums.

Let B be a group and let A_i ($i \in I$) be groups which have B as a subgroup. The *pushout* of the A_i over B is a group C together with homomorphisms $\iota_i : B \to C$ such that:

> If D is any group and $\gamma_i : A_i \to D$ are homomorphisms which agree on B, then there is a unique homomorphism $\alpha : C \to D$ such that $\gamma_i = \alpha.\iota_i$ for each $i \in I$.

By general nonsense the pushout always exists (we will construct it in a moment), and it is unique up to isomorphism over the group B.

Lemma 4.1 *Suppose the groups A_i ($i \in I$) all have B as a subgroup. Then their pushout over B is the group*

$$D = \left(\bigoplus_{i \in I} A_i \right) / E$$

where, if b is an element of B and we write b_i for the copy of b in the i-th direct factor, then E is the group generated by all the elements $b_i - b_j$ as i, j range through I and b ranges through B. The map $\iota_i : A_i \to C$ is the embedding of A_i in the direct sum, followed by the natural map to C; it is an embedding.

Proof. The group C is generated by the images of the maps ι_i, so that uniqueness of α in the definition of pushout is guaranteed. For its existence, if ι'_i is the embedding of A_i in the direct sum, then there is a unique homomorphism α' from the direct sum to D, such that $\gamma_i = \alpha'.\iota_i$ for each i. Since the γ_i agree on B, α' is zero on E, and hence it factors through the natural map as required. To confirm that ι_i is an embedding, it suffices to note that E is disjoint from the factor A_i in the direct sum. □

Since the maps ι_i are embeddings which agree on B, we can identify the A_i with subgroups of C. Hence it makes sense to describe 'internal' pushouts where the ι_i are inclusion maps, just as one has internal direct sums.

Lemma 4.2 *Let A be a group and B a subgroup.*

(a) The group A is the (internal) pushout of subgroups A_i ($i \in I$) if and only if the A_i generate A, and for any distinct $i_0, \ldots, i_n \in I$, $A_{i_0} \cap (A_{i_1} + \ldots + A_{i_n}) = B$.

(b) If A is the pushout over B of its subgroups A_i ($i \in I$), then $A/B = \bigoplus_{i \in I}(A_i/B)$.

(c) Conversely if A/B factors as a direct sum $A/B = \bigoplus_{i \in I} C_i$, then if we put

$$A_i \;=\; \{a \in A : a + B \in C_i\},$$

A is the pushout over B of the A_i.

Proof. (a) is clear from the construction, and then (b) follows immediately. For (c), take an arbitrary element a of A. By the direct sum decomposition

$$a + B = c_1 + \ldots + c_n$$

for some $i_1, \ldots, i_n \in I$ and some $c_i \in C_i$. For each c_i, choose $a_i \in c_i$. Then there is $b \in B$ such that

$$a = a_1 + \ldots + (a_n + b).$$

The i-th term on the right is in A_i; so the A_i generate A. Suppose that this element a is also in A_{i_0} where i_0 is distinct from $i_1, \ldots, i_n$. Then $a + B$ lies in C_{i_0}. But A/B is the direct sum of the C_i, so that $a \in B$ as claimed. □

Let A be a group and B a subgroup of A; we write $t(A/B)$ for the torsion subgroup of A/B. We say that an element a of A is *torsion over B* if $a + B$ is torsion in A/B. The set of all elements of A which are torsion over B is a subgroup $t_B(A)$ of A. Let $t(A/B) = \bigoplus_p C_p$ be the primary decomposition of $t(A/B)$, and for each prime p let $t^p_B(A)$ be the group of elements

a of A such that $a + B$ is in C_p. Then $t_B(A)$ is the pushout over B of the groups $t^p_B(A)$. We call $t_B(A)$, $t^p_B(A)$ respectively the *torsion-over-B* and the *p-torsion-over-B components* of A.

Part II
Obstructions to categoricity

Following Shelah's recipe [11], we start with the available ways of constructing many models of a theory T over the same P-part B.

5 Copies of $\mathbb{Q}$ outside P

Lemma 5.1 *Suppose A is a group pair with A/A^P unbounded. Then for any cardinal $\kappa > 0$, $A \preccurlyeq A \oplus \mathbb{Q}^{(\kappa)}$.*

Proof. For each element a of A, introduce a new constant c_a; for each rational q introduce a new constant c_q. Let T be the following theory:

> The elementary diagram of A (i.e. the set of all first-order sentences true in A using the new constants c_a);
> the diagram of $\mathbb{Q}$ (i.e. the set of all atomic or negated atomic sentences true in $\mathbb{Q}$, written with the new constants c_q);
> for all $q \neq 0$, the sentence $\neg P(c_q)$.

Since A/A^P is unbounded, every finite subset of T is satisfiable in A, and so by compactness T has a model B^+. Write B for the reduct of B^+ to the language of A. Then B is up to isomorphism an elementary extension of A, and B contains a copy Q of $\mathbb{Q}$ which is disjoint from B^P. Let C be a subgroup of B which contains B^P and is maximal disjoint from Q. Then $B = C \oplus \mathbb{Q}$ since $\mathbb{Q}$ is divisible (cf. [3] Theorem 21.2). Since $B^P \subseteq C$, the sum $C \oplus \mathbb{Q}$ is a group pair direct sum (by Lemma 3.5)

By the Szmielew invariants ([5] section A2) or more simply the upward Löwenheim-Skolem theorem, $\mathbb{Q} \preccurlyeq \mathbb{Q} \oplus \mathbb{Q}^{(\kappa)}$ as abelian groups, and so by Feferman-Vaught for group pairs (Lemma 3.7),

$$A \preccurlyeq B = C \oplus \mathbb{Q} \preccurlyeq C \oplus \mathbb{Q} \oplus \mathbb{Q}^{(\kappa)} = B \oplus \mathbb{Q}^{(\kappa)}.$$

But also, by Feferman-Vaught again,

$$A \subseteq A \oplus \mathbb{Q}^{(\kappa)} \preccurlyeq B \oplus \mathbb{Q}^{(\kappa)}.$$

So $A \preccurlyeq A \oplus \mathbb{Q}^{(\kappa)}$. □

Lemma 5.2 *Suppose A, B are group pairs with $A \subseteq B$ and $A^P = B^P$. Suppose $B/A = F$ for some torsion-free group F. Then $B \equiv A \oplus F$ as group pairs.*

Proof. By expressing A, B and F as parts of a single structure, we can form an ω_1-saturated elementary extension

$$A' \longrightarrow B' \longrightarrow F'$$

of the short exact sequence of group pairs

$$A \longrightarrow B \longrightarrow F.$$

Since F is torsion-free, so is F', and hence both sequences are pure exact for abelian groups. The abelian group A' is ω_1-saturated and hence pure-injective, so that the first short exact sequence splits as a sequence of abelian groups. Since $(B')^P \subseteq A'$, we have the group pair direct sum $B' = A' \oplus F'$. Then

$$B \equiv B' = A' \oplus F' \equiv A \oplus F$$

as group pairs, using Feferman-Vaught (Lemma 3.7(b)). □

The main device of the proof of this lemma, namely putting various groups together in a structure and taking an ω_1-saturated elementary extension, will occur so often that it will be helpful to have a name for it. We call it *blowing up*.

Theorem 5.3 *Let A be a group pair with A/A^P unbounded. Then:*

(a) Th(A) is not (κ, κ)-categorical for any infinite κ.

(b) Th(A) is not (κ, ω)-categorical for any finite κ.

Proof. (a) We can assume A is a (κ, κ)-structure. Let B be $A \oplus \mathbb{Q}^{(\omega)}$, which is also a (κ, κ)-model of Th(A). Let D be a subgroup of B which contains $\mathbb{Q}^{(\omega)}$, is divisible torsion-free and disjoint from B^P, and is maximal with these properties. Then (using [3] Theorem 21.2 and Lemma 3.5 again) we can write B as a group pair direct sum $B_1 \oplus D$ with $B^P \subseteq B_1$. Now split into two cases according as B_1/B_1^P is bounded or not. If it is unbounded, then by Lemma 5.1, $B_1 \preccurlyeq B \preccurlyeq B_1 \oplus \mathbb{Q}^{(\kappa)}$, and the first and third of these group pairs are non-isomorphic (κ, κ)-models with the same P-part, contradicting (κ, κ)-categoricity. On the other hand if B_1/B_1^P is bounded, compare $B_2 = B_1 \oplus \mathbb{Q}$ with $B_3 = B_1 \oplus \mathbb{Q}^{(\kappa)}$; they are both (κ, κ)-models with the same P-part, but they are not isomorphic since the dimensions of the $\mathbb{Q}$-vector spaces $\mathbb{Q} \otimes (B_2/B_2^P)$ and $\mathbb{Q} \otimes (B_3/B_3^P)$ are different.

(b) We can assume A is a (κ, ω)-structure, and by Lemma 5.1 there is no loss of generality in supposing that $\mathbb{Q}$ is a direct summand of A disjoint from A^P, but $\mathbb{Q}^{(2)}$ is not. Then A and $A \oplus \mathbb{Q}$ are (κ, ω)-models with the same P-part, but they are not isomorphic over A^P. □

Theorem 5.4 *Let A be a group pair and p a prime such that $\mathbb{Z}(p^\infty)^{(\omega)}$ is a subgroup of A disjoint from A^P. Then Th(A) is not (κ, λ)-categorical for any $\lambda > \kappa + \omega$.*

Proof. For such a group A (or any group elementarily equivalent to it), $A \equiv A \oplus \mathbb{Z}(p^\infty)^{(\lambda)}$ (for example using [5] Corollary 9.6.7 and Lemma A.1.6). But also A/A^P is unbounded, so that $A \equiv A \oplus \mathbb{Q}^{(\lambda)}$. □

Proposition 5.5 *(a) The property 'A/A^P is unbounded' is EC$_\Delta$, expressed by the theory*

$$|p^n A/(P_A \cap p^n A)| > 1 \quad \textit{for all primes } p \textit{ and integers } n \geqslant 1$$

(b) If the abelian group A is divisible-plus-bounded, then $\mathbb{Z}(p^\infty)^{(\omega)}$ is a subgroup of A/A^P if and only if

$$|p^n A[p]/(P_A \cap p^n A[p])| > n \textit{ for all } n < \omega.$$

Proof. These can be extracted from the Szmielew invariants, cf. [5] pp. 666ff. □

6 When groups are not divisible-plus-bounded

In this section we assume that λ is uncountable, and we violate (κ, λ)-categoricity under the assumption that at least one of A, A^P and A/A^P is not divisible-plus-bounded. In fact if A is divisible-plus-bounded, then so is A/A^P, so that we need only consider A and A^P.

We will distinguish models by the number of types realised over countable sets of elements. The P-part is still relevant, because we have to vary the number of types without varying the P-part. In the case of A^P, the trick is to find a non-split extension of A^P by $\mathbb{Q}$, so that as in the previous section we can put the $\mathbb{Q}$ outside A^P. But there are two ways in which a group B can generate many types through a non-split extension by $\mathbb{Q}$. The first is where some model of Th(B) contains the group $\mathbb{J}_p$ of p-adic integers. The second is that the first doesn't hold, but the reduced part of B is unbounded torsion; in this case the reduced p-components of B must be nontrivial for infinitely many primes p. Through the Szmielew invariants, Th(B) distinguishes these cases as follows.

Proposition 6.1 *We consider abelian groups.*

(a) For each prime p, the class of groups A with $\mathbb{J}_p$ as a direct summand is not EC_Δ, but its closure under elementary equivalence is axiomatised by the theory

$$|p^n A/p^{n+1}A| > 1 \text{ for all integers } n \geqslant 1.$$

(b) If A is a group not satisfying the theory of (a) for any prime p, then the reduced part of A is unbounded torsion if and only if A is a model of a theory

$$|A/pA| > 1 \text{ for all primes } p \in X$$

where X is an infinite set of primes.

□

We start our construction by collecting up some lemmas.

Lemma 6.2 *Let A be a group pair with A^P infinite. Then for every infinite κ there is a group pair B which is a (κ, κ) model of Th(A), and is such that the statement "Over any countable set of elements only countably many types are realised." is true for B, B^P and B/B^P.*

Proof. Let B be an Ehrenfeucht-Mostowski model of Th(A) with spine taken inside the P-part; let the spine be well-ordered of cardinality κ. Then in B at most countably many types are realised over any countable set X of elements ([5] Theorem 11.2.9(b)). If X is in B^P and a, b are elements of B^P realising distinct types over X in B^P, then by relativising formulas, a and b realise distinct types over X in B too; so the statement holds for B^P. Finally if X is a set of elements of B/B^P, choose representatives c_x in B so that $x = c_x + B^P$ for each $x \in X$. If $a + B^P$ and $b + B^P$ realise distinct types over X in B/B^P, then this fact can be expressed in the types of a and b over the c_x. So the statement holds for B/B^P too. □

In what follows we refer to the models constructed in Lemma 6.2 simply as *Ehrenfeucht-Mostowski models*.

Lemma 6.3 *Let J be a reduced group elementarily equivalent to $\mathbb{J}_p$ for some prime p, and suppose that in J only countably many types are realised over any countable set of elements of J. Then J is countable, and there is a pure extension J' of J of cardinality ω_1 in which uncountably many types are realised over a single element, and J'/J is torsion-free divisible.*

Proof. Let A be an ω_1-saturated elementary extension of J. Then A is pure-injective and torsion-free, $|A/pA| = p$ and $qA = A$ for all primes $q \neq p$. By the structure theory of pure-injective groups, $A = \mathbb{J}_p \oplus \mathbb{Q}^\lambda$ for some λ. But J is reduced, so that $J \preccurlyeq \mathbb{J}_p$ and $\mathbb{J}_p$ is the pure-injective hull of J. Since $pJ \neq J$, J contains an element c not divisible by p. If a is any element of J, then a is determined by its cosets modulo $p^k c$ for each $k < \omega$. Since at most countably many types over c are realised in J, J is countable. Also $\mathbb{J}_p/J$ is divisible (by [3] Lemma 41.8(ii)) and torsion-free (as the quotient of a torsion-free group by a pure subgroup). Choose J' to be a pure subgroup of $\mathbb{J}_p$ containing J and of cardinality ω_1. □

The next lemma must surely be well known, but we don't know a reference for it.

Lemma 6.4 *Let T be a torsion group such that for infinitely many primes p the p-component T_p of T is not empty, but every T_p is bounded. Let $\hat{T}$ be the pure-injective hull of T and A a pure subgroup of $\hat{T}$ containing T. Suppose that in A only countably many types are realised over any countable set of elements. Then there is a pure extension C of A in which uncountably many types over T are realised, and C/A is torsion-free divisible.*

Proof. It suffices to show that we can find A' realising at least one more quantifier-free type over T than is realised in A. For then we can iterate to form $A'' = A^{(2)}, A^{(3)} \ldots$, taking unions at limit ordinals; let C be $A^{(\omega_1)}$. The quotients $A^{(i)}/A$ form an increasing pure chain of divisible torsion-free groups; the quotient C/A is the union of the chain, so that it is divisible torsion-free. The quantifier-free type of an element of $A^{(i)}$ over T is the same in C as it is in $A^{(i)}$.

For each nonzero T_p, choose a nonzero cyclic direct summand C_p. List these cyclic direct summands as $(C_{p_n} : n < \omega)$, and for each n choose a generator c_n of C_{p_n}. Put $T_{p_n} = C_{p_n} \oplus C'_{p_n}$ for each n. Then the pure-injective hull $\hat{T}$ of T is $\prod_{n<\omega}(C_{p_n} \oplus C'_{p_n}) = \prod_{n<\omega} C_{p_n} \oplus \prod_{n<\omega} C'_{p_n}$. Write $B = \bigoplus_{n<\omega} C_{p_n}$ and $D = \bigoplus_{n<\omega} C'_{p_n}$, so that $\hat{T} = \hat{B} \oplus \hat{D}$. If $b \in \hat{B}$, we write $b = (b(n) : n < \omega)$ with each $b(n)$ in C_{p_n}.

If the order of c_n is o_n, then for each k ($0 \leqslant k < o_n$) we write $\theta_{k,n}(x)$ for the p.p. formula "$o_n|(x - kc_n)$". Then for any element b of $\prod_{n<\omega} C_{p_n}$, the formulas $\theta_{k,n}$ determine the $\prod_{m \neq n} C_{p_m}$-coset of b.

We have $B \oplus D \subseteq A \subseteq \hat{B} \oplus \hat{D}$; so every element of A has the form $b + d$ with $b \in \hat{B}$ and $d \in \hat{D}$. By assumption only countably many types over $\{c_n : n < \omega\}$ are realised in A. Choose a countable set X of elements of A representing each of these types; list the elements of X as $x_i + d_i$ ($i < \omega$) with $x_i \in \hat{B}$ and $d_i \in \hat{D}$.

We will construct a matrix $(b_{mn} : m, n < \omega, m \geqslant 1)$ so that the following conditions are met:

(a) For each $m, n < \omega$ with $m \geqslant 1$, b_{mn} is a nonzero element of C_{p_n}.

(b) For each $i, j < \omega$ with $j \geqslant 1$ there is $n < \omega$ such that $jb_{1n} \neq x_i(n)$.

(c) For each $m \geqslant 2$ there is $N_m < \omega$ such that for all $n \geqslant N_m$, $b_{1n} = mb_{mn}$.

For each $m \geqslant 1$ we write $b^\star_m$ for the element of $\hat{T}$ defined by

$$b^\star_m(n) = b_{mn}, \quad {b^\star_m}'(n) = 0 \quad \text{for all } n < \omega.$$

The group A' will be the subgroup of $\hat{T}$ generated by A and all the elements $b^\star_m$ with $m \geqslant 1$. The conditions (c) ensure that for every $m \geqslant 2$, $mb^\star_m - b^\star_1 \in T \subseteq A$, so that $b^\star_1 + A$ is a divisible element of A'/A. The conditions (b) ensure that each element $jb^\star_1 + a$ with $j \geqslant 1$ and $a \in A$ realises (in $\prod_{n<\omega} C_{p_n}$) a type over $\{c_n : n < \omega\}$ that is not already realised in A, using the formulas $\theta_{k,n}$. It follows in particular that $jb^\star_1 \notin A$, and hence $b^\star_1 + A$ is a torsion-free element of A'/A. Thus $A'/A \cong \mathbb{Q}$, and since $\mathbb{Q}$ is torsion-free, this implies that A' is a pure extension of A.

It remains to find the elements $b_{m,n}$. We proceed in stages σ_k ($k < \omega$). At stage σ_k we choose a prime number $p_{f(k)}$ so that f is strictly increasing; then we choose the elements $b_{1,n}$ with $f(k-1) < n \leqslant f(k)$ so as to deal with (b) for the cases $i, j \geqslant k$ (so far as they haven't already been dealt with) by our choice of $b_{1,f(k)}$, and for (c) we find N_k and we choose b_{mn} when $m < k$ and $N_m \leqslant n \leqslant f(k)$ (where not already chosen). Thus at stage σ_k we have finitely many tasks to perform.

We begin stage σ_k by examining k (when $k > 1$) and choosing $N_k > f(k-1)$ so that no prime dividing k is among the p_n with $n \geqslant N_k$. Then we turn to (b) and assemble the pairs i, j for which (b) is not already ensured. This is a finite set of pairs. We solve the following problem at the first $n \geqslant N_k$ where it is solvable:

$b_{1,n}$ is chosen to be a nonzero element of C_{p_n} so that for each of the relevant pairs i, j, $jb_{1n} \neq x_i(n)$.

If n is chosen so that p_n is greater than any prime factor of any of the relevant j, then for any choice of nonzero $b_{1,n}$ the elements $jb_{1,n}$ will also be nonzero elements of C_{p_n}, so it's clear that by taking n large enough we can solve the problem. Having found this n, we put $f(k) = n$. It remains to deal with the requirements at (c). These are met whenever $N_m \leqslant n \leqslant f(k)$ by choosing m' such that $m'n \equiv 1$ modulo the order of C_{p_n} (which is possible by the choice of N_m) and putting $b_{mn} = m'b_{1n}$.

Finally we need to check that the types of the added elements $b_1^\star$ over $\{c_n : n < \omega\}$ remain distinct in the final structure C. Since the formulas $\theta_{k,n}$ are p.p., it suffices to check that the type of $b_1^\star$ is new in A'. If $b_{1,n} = kc_n$ then $o_n | (b_{1,n} - kc_n)$; it remains to show that $o_n | (b_1^\star - (b_{1,n} - kc_n))$. But this follows from (c) and the fact that the groups C_i ($i < N_m$), apart from C_n itself, are all of orders prime to p_n. □

Theorem 6.5 *Suppose A is a group pair, A/A^P is unbounded and A is not divisible-plus-bounded. Then $Th(A)$ is not (κ, λ)-categorical when $\omega \leqslant \kappa$ and $\omega_1 \leqslant \lambda$.*

Proof. Consider an ω_1-saturated elementary extension A' of A. Since A' is not divisible-plus-bounded, either (a) it has a direct summand of the form $\mathbb{J}_p$ for some prime p, or (b) it has no such direct summand, and in this case its reduced torsion part contains nonzero p-components for infinitely many primes p, though each reduced p-component is bounded.

In case (a), choose a direct summand of the form $\mathbb{J}_p$, and add relation symbols to A' so as to express that this is a direct summand. Then take a (κ, κ) Ehrenfeucht-Mostowski model of the resulting theory. We can assume without loss that the original group pair A is the reduct of this Ehrenfeucht-Mostowski model to the language of group pairs. Now A has a direct summand which is elementarily equivalent to $\mathbb{J}_p$. Separating the divisible and reduced parts of this summand, we reach a direct summand J of A which is reduced and elementarily equivalent to $\mathbb{J}_p$, and hence is embeddable as a pure (in fact elementary) subgroup in $\mathbb{J}_p$. By the Ehrenfeucht-Mostowski construction and Lemma 3.7(c), in J there are only countably many types realised over any countable set. So by Lemma 6.3 there is an extension J' of J of cardinality ω_1 such that uncountably many types are realised over some countable set in J', and J'/J is divisible torsion-free. Form the abelian group B by replacing the direct summand J by J'. Then $A \subseteq B$ as abelian groups, and $B/A \cong J'/J$. By Lemma 3.7(c) for abelian groups, B realises uncountably many types over some countable set.

Make B into a group pair by putting $B^P = A^P$. Then we are in the situation of Lemma 5.2, so $B \equiv A \oplus \mathbb{Q}^\mu$ for some cardinal $\mu > 0$. Now consider the two group pairs $A_1 = A \oplus \mathbb{Q}^\lambda$ and $B_1 = B \oplus \mathbb{Q}^\lambda$. These are both (κ, λ) models (bearing in mind the assumption that $\lambda \geqslant \omega_1$). Since $B \equiv A \oplus \mathbb{Q}^\mu$, we have $A \equiv A_1 \equiv B_1$. Also $A_1^P = A^P = B^P = B_1^P$. But in A_1 at most countably many types are realised over any countable set of elements (using Lemma 3.7(d) and the fact that this holds for both A and the ω-stable group $\mathbb{Q}^\lambda$). Since we constructed B_1 to realise uncountably many types over some countable set, $A_1 \ncong B_1$ and (κ, λ)-categoricity fails.

In case (b) we proceed similarly but using Lemma 6.4 in place of Lemma 6.3. □

Theorem 6.6 *Suppose A is a group pair, A/A^P is unbounded and A^P is not divisible-plus-bounded. Then Th(A) is not (κ, λ)-categorical when $\omega \leqslant \kappa$ and $\omega_1 \leqslant \lambda$.*

Proof. With obvious adjustments we can proceed as in the proof of Theorem 6.5, to the point where we assume A is a (κ, κ) Ehrenfeucht-Mostowski model where one of the case assumptions (a), (b) of that proof applies to A^P. As in that proof, we construct an abelian group E extending A^P, so that E/A^P is a torsion-free divisible group of cardinality ω_1, and E realises uncountably many types over some countable set.

We need to build a model B of Th(A) so that $A^P = B^P$. For this we build up the commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^P & \longrightarrow & E & \longrightarrow & E/A^P & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & R & \longrightarrow & 0
\end{array}$$

as follows. The lefthand square is a pushout of the inclusions of A^P in E and A. We make the top and bottom rows exact, and then the map from E/A^P to R is determined by exactness of the top row and the fact that the map $A^P \to E \to B \to R$ is zero. The construction of pushouts ensures that that this map on the right is an isomorphism.

Now make B into a group pair by putting $B^P = A^P$. By Lemma 5.2, $A \equiv B$ as group pairs. Put $A_1 = A \oplus \mathbb{Q}^{(\lambda)}$ and $B_1 = B \oplus \mathbb{Q}^{(\lambda)}$. Then both A_1 and B_1 are (κ, λ) models of Th(A), and they have the same P-part A^P. But they are not isomorphic, since in B_1 uncountably many types are realised over a countable set of elements, and this is false in A_1. □

7 The part outside P

Theorem 7.1 *Suppose A is a group pair and $Th(A)$ is (κ, λ)-categorical for some κ and λ. Then:*

(a) A/A^P is divisible-plus-bounded.

(b) If $\kappa < \lambda$ then both A and A^P are divisible-plus-bounded.

Proof. When $\kappa = \lambda \geqslant \omega$, A/A^P is bounded by Theorem 5.3(a). When $\kappa < \omega = \lambda$, A/A^P is bounded by Theorem 5.3(b); also A^P is finite and A is ω-categorical as an abelian group, hence bounded by Theorem 2.1(c). When $\lambda < \omega$, all of A, A^P and A/A^P are finite and hence bounded.

When $\omega \leqslant \kappa < \lambda$, A and A^P are divisible-plus-bounded by Theorems 6.5 and 6.6, so A/A^P is divisible-plus-bounded by Lemma 2.4(a). When $\kappa < \omega < \lambda$, $\mathrm{Th}(A)$ is uncountably categorical; hence A is ω-stable, and so divisible-plus-bounded by Theorem 2.1(a). Also in this case A^P is finite. □

Definition 7.2 Let A be an abelian group and B a subgroup. We say that A is a *tight* extension of B if there is no nontrivial subgroup D of A disjoint from B such that $\langle D + B\rangle/B$ is pure in A/B.

Theorem 7.3 *Let A be a group pair such that A/A^P is divisible-plus-bounded. Then A is a direct sum $A = C \oplus D$ where C is a tight extension of A^P. If also A is divisible-plus-bounded, then in every such decomposition, $|C| \leqslant |A^P| + \omega$.*

Proof. By Zorn's Lemma there is a subgroup D of A which is maximal with the properties (1) D is disjoint from A^P, (2) $\langle D + A^P\rangle/A^P$ is a pure subgroup of A/A^P. As a pure subgroup of a divisible-plus-bounded group, $\langle D+A^P\rangle/A^P$ is a direct summand of A/A^P. Write $A/A^P = C' \oplus D$ where C' is a subgroup of A/A^P. Let C be the pre-image of C' in A. Then $A = C \oplus D$ with $A^P \subseteq C$. If C is not a tight extension of A^P, then C' contains a direct summand disjoint from A^P, contradicting the maximality of D.

For the cardinality calculation, write C (which is divisible-plus-bounded by Lemma 2.4(b)) as a direct sum of finite cyclic groups and groups of the forms $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$. Since C is a tight extension of A^P, each of these direct summands contains a nonzero element of A^P. □

Example 7.4 In general neither of the direct summands C and D are unique, even given the other. For example let A be $\mathbb{Z}(p^2) \oplus \mathbb{Z}(p)$, where the two cyclic summands are generated by c, d respectively, and let A^P be the subgroup generated by pc. Let C be the subgroup of A generated by $c+d$, with

$C^P = A^P$. Let D be the subgroup of A generated by $pc + d$. Then A can also be written as a group pair direct sum with C in place of $\mathbb{Z}(p^2)$, or with D in place of $\mathbb{Z}(p)$, or both. Here Th(A) is clearly relatively categorical.

However, we will establish in Theorems 10.1, 12.1 and 12.2 that when Th(A) is (κ, λ)-categorical for some κ and λ, some features of the decomposition in Theorem 7.3 are determined by T. We assume the relevant facts as a hypothesis in the next theorem. In the terminology of Definition 3.8, it follows that Th$(A) =$ Th$(C) \oplus$ Th(D).

Theorem 7.5 *Suppose A is a (κ, λ) group pair such that A/A^P is divisible-plus-bounded and Th$(A) = T_1 \oplus T_2$. Let κ and λ be cardinals such that every (κ, λ)-model of Th(A) can be written as $B = C' \oplus D'$ where C' is a model of T_1 that is a tight extension of B^P, and D' is a model of T_2 that is disjoint from P; suppose that for each such B the choice of summand C' is unique up to isomorphism over B^P. Let $A = C \oplus D$ be a decomposition of A with this form. Then:*

- *(a) If $\kappa + \omega < \lambda$ and D is not ω_1-categorical, then Th(A) is not (κ, λ)-categorical.*
- *(b) If $\kappa + \omega < \lambda$, A/A^P is unbounded and D is bounded, then Th(A) is not (κ, λ)-categorical.*
- *(c) If $\omega < \kappa$ and D is infinite, then Th(A) is not (κ, κ)-categorical.*
- *(d) If $\kappa \leqslant \omega$ and D is unbounded then Th(A) is not (κ, ω)-categorical.*

Proof. (a) We can assume without loss that A is a (κ, λ)-structure. By Theorems 7.1(b) and 7.3, $|C| = \kappa < \lambda$ so that $|D| = \lambda$. If $D' \equiv D$ with $|D'| = \lambda$, then A is elementarily equivalent to $C \oplus D'$ by Lemma 3.7(a) and has the same P-part, so (κ, λ)-categoricity would make A and $C \oplus D'$ isomorphic. Now D is a direct sum of finite cyclic groups and groups of the forms $\mathbb{Z}(p^\infty)$ or $\mathbb{Q}$. There are only countably many isomorphism types of such summands, and since $\lambda > \kappa$, at least one of these types must appear at least κ^+ times in D. For any type except $\mathbb{Q}$, the presence of these $\geqslant \kappa^+$ summands of the given type can be recognised from the Szmielew invariants if we allow these invariants to have arbitrary cardinal values (cf. [5] p. 666ff). Also if any one of these types appears infinitely often, the number of its summands can be shrunk to κ or expanded to λ without altering Th(D). So (κ, λ)-categoricity requires that only one type of summand appears infinitely many times. If D is unbounded, this one type must be $\mathbb{Q}$ and Th(D) is ω_1-categorical. If D is bounded, the one type can be any $\mathbb{Z}(p^k)$, but the remaining summands form a finite group; again Th(D) is ω_1-categorical.

(b) Again taking A to be a (κ, λ)-structure, if A/A^P is unbounded then $A \equiv A \oplus \mathbb{Q}$ by Lemma 5.1. In the decomposition of $A \oplus \mathbb{Q}$, the $\mathbb{Q}$ must go with the D part, so D is unbounded.

(c) Assume A is a (κ, κ)-structure. By Theorem 5.3(a) we can assume A/A^P is bounded and so D is bounded. If D is infinite, then D has a direct summand that is an infinite homocyclic subgroup $\mathbb{Z}(p^k)^{(\mu)}$. By compactness,

$$\mathbb{Z}(p^k)^{(\mu)} \equiv \mathbb{Z}(p^k)^{(\omega)} \equiv \mathbb{Z}(p^k)^{(\kappa)},$$

so by Lemma 3.7 there are D' and D'' elementarily equivalent to D, where D' has a direct summand $\mathbb{Z}(p^k)^{(\kappa)}$ but D'' has no direct summand $\mathbb{Z}(p^k)^{(\mu)}$ with $\mu > \omega$. So $C \oplus D'$ and $C \oplus D''$ are (κ, κ)-models of $\mathrm{Th}(A)$ whose decompositions have respectively D' and D'' as the models of T_2. The difference between $A = C \oplus D'$ and $B = C \oplus D''$ can be recognised from the dimensions of the $\mathbb{F}_p$ vector spaces

$$(p^{k-1}A[p] + A^P)/(p^k A[p] + A^P), \;\; (p^{k-1}B[p] + B^P)/(p^k B[p] + B^P).$$

(d) Suppose A is a (κ, ω)-structure. If D is unbounded then the elementary equivalence class of D is not altered by adding countably many summands of the form $\mathbb{Q}$, or removing all but at most one such direct summand. We reach a contradiction to (κ, ω)-categoricity as in the previous cases. □

8 Reduction property

We say that T has the *Reduction Property* if for every formula $\phi(\bar{x})$ of $L(P)$ there is a formula $\phi^\star(\bar{x})$ of L such that if A is any model of T and $\bar{a}$ a tuple of elements of A^P, then

$$A \models \phi(\bar{a}) \;\; \Leftrightarrow \;\; A^P \models \phi^\star(\bar{a}).$$

The next result is in some sense a model-theoretic version of Lemma 1.1. The proof adapts Pillay and Shelah [9], who proved it when $\kappa = \lambda$. For any complete theory T and positive integer n, we write S^n_T for the set of complete types of T over the empty set in the variables $v_0, \dots, v_{n-1}$.

Theorem 8.1 *Suppose T doesn't have the Reduction Property. Then T is not (κ, λ)-categorical for any κ and λ.*

Proof. We first claim that

For each $n < \omega$ there is a function $\sigma : S^n_{T^P} \to S^n_T$ such that for every model B of T and every n-tuple $\bar{a}$ of elements of B^P, if $\bar{a}$ realises $p \in S^n_{T^P}$ then $\bar{a}$ realises $\sigma(p)$ in B.

Suppose first that κ is infinite. Let B, C be models of T and $\bar{b}, \bar{c}$ finite sequences in B^P, C^P respectively which realise p in B^P, C^P. We have to show that $\bar{b}, \bar{c}$ realise the same type in B, C respectively. Since $B \equiv C$, we can elementarily embed both in a single model and thus assume $B = C$. If $\kappa = \lambda$ then by Löwenheim-Skolem we can assume that B is a (κ, λ)-model. On the other hand if $\kappa < \lambda$, then we can choose a (κ, κ)-model and (by the classification above) blow it up to a (κ, λ)-model. By Theorems 7.1(b) and 7.3, $B = C \oplus D$ where $B^P \subseteq C$ and $|C| = |B^P| = \kappa$. Now put $C_0 = C$. By assumption $(C^P_0, \bar{b}) \equiv (C^P_0, \bar{c})$. Let a be any element of C^P_0 and let $q(\bar{b}, x)$ be the type of a over $\bar{b}$ in C^P_0. Then $q(\bar{c}, x)$ relativised to P is consistent with the elementary diagram of C_0; so there exists an elementary extension C_1 of C_0 with an element d such that $(C^P_1, \bar{b}, a) \equiv (C^P_1, \bar{c}, d)$. We can choose C_1 to be of cardinality κ. Now we repeat this move back and forth, so as to build up an elementary chain $C_0 \preccurlyeq C_1 \preccurlyeq \dots$ of length κ. It can be arranged that the elements of C^P_κ are listed as $\bar{a}$ and as $\bar{d}$ so that $(C^P_\kappa, \bar{b}, \bar{a}) \equiv (C^P_\kappa, \bar{c}, \bar{d})$. Then the map $\bar{a} \mapsto \bar{d}$ defines an automorphism of C^P_κ which takes $\bar{b}$ to $\bar{c}$. By Lemma 1.1 and (κ, λ)-categoricity, this automorphism extends to the whole of $C_\kappa \oplus D$, and hence $\bar{b}$ and $\bar{c}$ have the same type in $C_\kappa \oplus D$ and therefore also in B. This proves the claim.

If κ is finite then B^P is determined up to isomorphism by T, and it already has the property that for any two tuples which realise the same type in B^P there is an automorphism of B^P taking one to the other. So a much shorter version of the previous argument applies, and again we have the claim.

We infer the Reduction Property as follows. Consider a formula $\phi(\bar{x})$ of $L(P)$. If $\phi \in \sigma(p)$ then T implies this; so by compactness there is a formula $\theta_p \in p$ such that $\phi \in \sigma(q)$ whenever $\theta_p \in q$. Then modulo T, ϕ implies the infinite disjunction $\bigvee\{\theta_p : \phi \in \sigma(p)\}$. By compactness again, ϕ implies a finite disjunction, which will serve as $\phi^\star$. □

Theorem 8.2 *Suppose there are complete theories T_1, T_2 such that every model of T has the form $A = C \oplus D$ where $A^P \subseteq C$, $C \models T_1$ and $D \models T_2$. If T_1 has the Reduction Property then so does T.*

Proof. Let $\phi(\bar{x})$ be a formula of $L(P)$. If $\theta(\bar{x})$ is as in Lemma 3.7(e), and the Reduction Property finds $\theta^\star(\bar{a})$, then for every tuple $\bar{a}$ in A^P,

$$A \oplus B \models \phi(\bar{a}) \ \Leftrightarrow\ A \models \theta(\bar{a}) \ \Leftrightarrow\ A^P \models \theta^\star(\bar{a}).$$

□

Part III
Proving categoricity

In this part we show that the conditions for categoricity described in the previous subsection are not just necessary; they are also sufficient.

In all cases we have group pairs A, B and a pure embedding $i : A^P \to B^P$, and we lift i to a pure embedding $j : A \to B$. When i is an isomorphism, j will also be an isomorphism. The statement 'i preserves finite p-heights in A and B' means that the p-height of each element a of A^P in A is equal to the p-height of $i(a)$ in B, where p-heights are reckoned as either finite or ∞.

9 Invariants

We will need some invariants that are not necessarily first-order expressible.

Definition 9.1 Let A be a group and B a subgroup. Let p be a prime and $n < \omega$. Then the *Ulm-Kaplansky invariant*, in symbols $UK_{p,n}(A, B)$, is the rank of

$$p^n A[p]/((p^{n+1}A + B) \cap p^n A[p])$$

as vector space over $\mathbb{F}_p$. (Cf. [4] p. 61.)

Lemma 9.2 *For each prime p the Ulm-Kaplansky invariants $UK_{p,n}$ are additive in group pair direct sums, and zero in divisible groups and q-groups with $q \neq p$.*

□

Definition 9.3 When p is a prime and A is an abelian p-group with a subgroup C, we say that an element a of A is *proper* over C if $\mathrm{ht}^p_A(a) \geqslant \mathrm{ht}^p_A(a+c)$ for every $c \in C$.

Lemma 9.4 *Suppose p is a prime, A is a p-group, C is a subgroup of A and $a \in A$. Then a is proper over C if and only if for every element c of C,*

$$ht^p_A(a + c) = \min\{ht^p_A(a), ht^p_A(c)\}.$$

Hence if a is proper over C and c is an element of C with $ht^p_A(a) = ht^p_A(c)$, then $a + c$ is also proper over C.

Proof. Straightforward, [4] p. 61. □

Lemma 9.5 *Let A be an abelian group and B a subgroup of A. Assume that either (i) A/B is a bounded p-group or (ii) A is a divisible-plus-bounded p-group. Then if A is a tight extension of B, the following hold:*

(a) For every $k < \omega$, $p^k A[p] \subseteq p^{k+1}A + B$.

(b) $p^\infty A[p] \subseteq B$.

(c) (Villemaire [13]) For every prime p and every $n < \omega$, the Ulm-Kaplansky invariant $UK_{p,n}(A, B)$ is zero.

(d) If $a \in A \setminus B$, $pa \in B$ and a is proper over B, then $ht^p_A(pa) = ht^p_A(a) + 1$.

Proof. (a) Assume that $a \in A \setminus B$ has p-height k in A, and $pa = 0$. Suppose that k is finite, and choose d such that $p^k d = a$. Consider the cyclic group $D = \langle d \rangle$. This group is disjoint from B, so by tightness $(D + B)/B$ is not pure in A/B. Hence $a + B$ has height $> k$ modulo B, in other words $a \in p^{k+1}A + B$.

(b) Suppose next that k is infinite. Then (ii) applies, so A (and hence also A/B by Lemma 2.4(a)) is divisible-by-bounded. For contradiction assume $a \in p^\infty A[p] \setminus B$. Let p^m be the exponent of the bounded part of A. Put $a_1 = a$, and inductively choose a_i of infinite p-height so that $pa_{i+1} = a_i$ as follows. When a_i has been chosen, let c be an element of A such that $p^{m+1}c = a_i$. Then $p^m c$ has p-height $\geqslant m$ and hence is in the divisible part of A. Put $a_{i+1} = p^m c$. Let D be the subgroup of A generated by $\{0, a_1, a_2, \ldots\}$. Then D is divisible and disjoint from B, so that A can be split as $C \oplus D$ where $B \subseteq C$. So D is a pure subgroup of $A/B = C/B \oplus D$, contradicting tightness.

(c) is immediate from (a) by the definition of the invariants.

(d) When k is infinite the statement is trivial (recalling that we put $\infty + 1 = \infty$). When k is finite, for contradiction let a be an element of A such that $p^k a \in A \setminus B$, $p^{k+1}a \in B$ and $k = \mathrm{ht}^p_A(p^k a)$. Suppose $p^{k+1}a$ has p-height $> k + 1$; then there is $b \in A$ such that $p^{k+2}b = p^{k+1}a$. So by (a), $p^{k+1}b - p^k a \in p^k A[p] \subseteq p^{k+1}A + B$, hence the coset of $p^k a$ over B contains an element of p-height $\geqslant k + 1$, contradicting the properness of $p^k a$ over B. □

Corollary 9.6 *Suppose A is a group pair such that A/A^P is divisible-plus-bounded, A is a tight extension of A^P, and B is a group pair $\equiv A$. Assume also that if A/A^P is unbounded then both A and A^P are divisible-plus-bounded. Then $B = C \oplus \mathbb{Q}^{(\mu)}$ where $B^P \subseteq C$, C is tight over B^P and $\mu \geqslant 0$. When A/A^P is bounded, $\mu = 0$.*

Proof. Since $B \equiv A$, $B/B^P \equiv A/A^P$ and hence B/B^P is also divisible-plus-bounded. Suppose that B is not tight over B^P. Then there is a nonzero subgroup D of B disjoint from B^P, such that $\langle D+B^P \rangle/B^P$ is pure in B/B^P. Taking direct summands in B/B^P, we can suppose that $\langle D + B^P \rangle/B^P$ has one of the following forms:

(a) A cyclic p-group for some prime p.

(b) $\mathbb{Z}(p^\infty)$ for some prime p.

(c) $\mathbb{Q}$.

We show first that (a) is impossible.

Assuming (a), let d be an element of D of order p^{k+1}, such that $d + B^P$ generates $\langle D + B^P \rangle/B^P$. Then $p^k d \in p^k B[p]$. We claim that $p^k d$ is not in $p^{k+1}B + B^P$. For suppose to the contrary that $p^k d = p^{k+1} b + a$ with $a \in B^P$. Then $p^k d + B^P$ has p-height $\leqslant k + 1$ in B/B^P, contradicting that $\langle D + B^P \rangle/B^P$ is pure in B/B^P. It follows that $p^k B[p] \nsubseteq p^{k+1}B + B^P$. But this is a first-order property of B and hence of A, contradicting Lemma 9.5(a). Hence case (a) is impossible.

Now if A/A^P is bounded, then so is B/B^P. In this case (b) and (c) are ruled out too, so that B is tight over B^P.

Suppose then that A/A^P and hence B/B^P are unbounded; in this case we assume A^P is divisible-plus-bounded, so the same holds for B^P. Then nothing prevents (c), since $B \equiv B \oplus \mathbb{Q}^{(\mu)}$ for any cardinal μ, by Lemma 5.1. But now the short exact sequence

$$B^P \longrightarrow D + B^P \longrightarrow \mathbb{Q}$$

splits by Lemma 2.3(d), making $\mathbb{Q}$ a direct summand disjoint from B^P. The same applies with $\mathbb{Q}^{(\mu)}$ in place of $\mathbb{Q}$.

It remains to show that even when A/A^P is unbounded, case (b) never occurs. Recall our assumption that when A/A^P is unbounded, A is divisible-plus-bounded. Assuming case (b), let m be the exponent of the bounded part of A (and hence also of B). Choose $d \in D$ so that $md + B^P$ has order p in $\langle D + B^P \rangle/B^P$. Then $md \in p^\infty B$. Also $pmd \in D \cap B^P = \{0\}$, so that $md \in p^\infty B[p] \subseteq B^P$ by Lemma 9.5(b) (which holds also in B since $A \equiv B$). This contradicts the choice of $md + B^P$ as nonzero. □

Lemma 9.7 *Let A be an abelian p-group and B a subgroup of A with A/B bounded. Then the following are equivalent:*

(a) A is a tight extension of B.

(b) For every $k < \omega$, $p^k A[p] \subseteq p^{k+1}A + B$.

(c) If $A = C \oplus D$ with $B \subseteq C$ then $D = 0$.

In particular if A/B is bounded, B is finite and A is a tight extension of B then A is finite.

Proof. Certainly (a) implies (b).

(b) $\Rightarrow$ (c): Suppose $A = C \oplus D$ with $B \subseteq C$, and $D \neq 0$. Let d be a nonzero element of $D[p]$; let k be the p-height of d. Then $p \in p^k A[p] \setminus (p^{k+1}A + B)$, contradicting (b).

(c) $\Rightarrow$ (a): Suppose A is not a tight extension of B. Then there is a nontrivial subgroup D of A disjoint from B such that $\langle D + B\rangle/B$ is pure in A/B. Since pure subgroups of bounded groups are direct summands, we can put $A/B = C' \oplus D$ (noting that $D \cong \langle D + B/B\rangle$ since D is disjoint from B). Put $C = \{c : c + B \in C'\}$. We claim $A = C \oplus D$. First, if $a \in A$ then $a + B = (c + B) + d$ with $c \in C$ and $d \in D$, so $a = (c + b) + d$ for some $b \in B$. Then $a \in C + D$ since $B \subseteq C$. Second, suppose $d \in C \cap D$. Then in A/B, $d + B \in C' \cap D$, so $d \in B$, and hence $d = 0$ since D is disjoint from B.

For the final clause, if A/B and B are bounded then A is bounded, so A can be written as a direct sum of cyclic groups. Since B is finite, all but finitely many of the direct summands of A are disjoint from B; so by (c), A has only finitely many direct summands. □

10 Pinning down the part outside P

Theorem 10.1 *Suppose A is a group pair of the form $C \oplus D$ where $A^P \subseteq C$ and C is a tight extension of A^P. Suppose also that A/A^P is divisible-plus-bounded, and that if A/A^P is unbounded then so is D. Under these conditions Th(D) can be read off from Th(A).*

Proof. We refer to the Szmielew invariants (cf. [5] p. 666ff). The theory of the bounded direct summand of D is determined by the invariants

$$U(p, k; D) = |p^k D[p]/p^{k+1} D[p]| \in \omega \cup \{\infty\}$$

for each prime p and each $k < \omega$. To compute these invariants, we introduce an Ulm-Kaplansky invariant $UK(p, k; A)$ for group pairs by writing

$$UK(p, k; A) = p^{UK_{p,k}(A,A^P)} = |p^n A[p]/((p^{n+1}A + A^P) \cap p^n A[p])|.$$

(Cf. Definition 9.1.) Then $UK(p, k; -)$ is a Baur-Monk invariant, so it is multiplicative in direct sums (cf. [5] Lemma A.1.9). Hence

$$UK(p, k; A) = UK(p, k; C) \cdot UK(p, k; D) = UK(p, k; D)$$

since $UK(p, k; C) = 1$ by Lemma 9.5(b). But $U(p, k; D) = UK(p, k; D)$ since $D^P = \{0\}$. So we can recover $U(p, k; D)$ from Th(A).

Next we consider the divisible p-torsion part D_p of D; the relevant Szmielew invariants are

$$D(p, k; D) = |p^k D[p]| \in \omega \cup \{\infty\}$$

with $k < \omega$. For determining Th(D) we need only the values of $D(p, k; D)$ for large enough k. Now $D(p, k; D)$ is not necessarily equal to either of $D(p, k; A)$ or $D(p, k; A/A^P)$, since for example there may be divisible p-torsion groups in A whose socle lies inside A^P. It suffices to use instead the Baur-Monk invariant $D^\star(p, k; -)$ where for any group pair B,

$$D^\star(p, k; B) = |p^k B[p]/(p^k B[p] \cap B^P)|.$$

Since $D^P = \{0\}$, $D^\star(p, k, D) = D(p, k; D)$ for all p and k. By Lemma 9.5(a), $D^\star(p, k; C) = 1$ when we take k so that p^k is greater than the exponent of the bounded part of A, so that $p^k C = p^\infty C$. Hence $D(p, k; D)$ for all large enough k is equal to $D^\star(p, k; A)$ and hence is determined by Th(A).

The Szmielew invariants Tf$(p, k; -)$ are not needed for determining the theory Th(D), since D is divisible-plus-bounded. The only other piece of information that we need is whether D is bounded or not. If A/A^P is bounded then clearly D is bounded; by assumption if A/A^P is unbounded then D is unbounded. □

This theorem does half the work of showing that our necessary conditions for (κ, λ)-categoricity are also sufficient.

Corollary 10.2 *Let A and B be two elementarily equivalent (κ, λ) group pairs, where $A = C \oplus D$, $B = C' \oplus D'$, C is a tight extension of A^P and C' is a tight extension of B^P. Suppose also that A/A^P (and hence also B/B^P) is divisible-plus-bounded. Then:*

(a) *If $\kappa + \omega < \lambda$, D is ω_1-categorical, and either A/A^P is bounded or D is unbounded, then $D \cong D'$.*

(b) *If $\kappa \leqslant \lambda \leqslant \omega$ and D is bounded, then $D \cong D'$.*

(c) *If D is finite then $D \cong D'$.*

Proof. (a) By Theorems 7.1(b) and 7.3, $|C| = |C'| = \kappa$. If A/A^P is bounded then so are D and D', so by Theorem 10.1, D and D' are elementarily equivalent λ-categorical groups of cardinality λ. If A/A^P is unbounded then so is D by our assumption, and Theorem 10.1 applies again.

(b) In this case $D \equiv D'$ by Theorem 10.1 since both are bounded. Since $\lambda \leqslant \omega$, both D and D' must have the same cardinality (finite or ω), so by Theorem 2.1(c) they are isomorphic.

(c) This follows at once from Theorem 10.1. □

So the main task still to do is to show that if A and B are elementarily equivalent tight extensions of $A^P = B^P$, then they are isomorphic over A^P.

11 The lifting

The main result of this section has the following form. Given group pairs A, B and an embedding $i : A^P \to B^P$, we extend i to a group pair embedding from A into B. This result is needed for showing that under certain conditions a theory of group pairs is (κ, λ)-categorical. But curiously we need it in the other direction too, for showing that the hypothesis of Theorem 7.5 holds.

Our main tool is the Kaplansky-Mackey extension lemma, [4] Lemma 77.1, suitably adapted. In that lemma the ground model is a p-group. We can assume this when A and A^P are both divisible-by-bounded, because we can separate off the p-components in both A and A^P and argue at each prime separately. However, A/A^P may have a nontrivial divisible p-torsion part. The following example makes it unlikely that we can handle the bounded part and the divisible part separately; so we need to adjust Kaplansky-Mackey to handle both simultaneously.

Example 11.1 Let p be a prime. Put $A = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^2)$. In A let a be an element of $\mathbb{Z}(p^\infty)$ of order p and b an element of $\mathbb{Z}(p^2)$ of order p. Make A into a group pair by taking for A^P the subgroup generated by $a + b$. Then A is divisible-plus-bounded, and A is tight over A^P. But we can't split A into two group pair summands, one divisible and one bounded. Also note that $\mathbb{Z}(p^\infty) + A^P$ is not tight over A^P; the whole divisible part becomes a separate summand. So fibring over A^P won't help either.

When A/A^P is bounded, this problem vanishes but another one appears. In this case we have no constraints at all on A^P, so we don't know that the p-components can be split off. The device that works in this case is to analyse A as a pushout of the separate p-torsion extensions of A^P. We pay for this liberty by having to check that the Kaplansky-Mackey lemma still works when the ground group A^P is arbitrary.

Theorem 11.2 *Suppose A and B are groups and C, D are subgroups of A, B. Assume A is a tight extension of C and B is a tight extension of D. Let $i : C \to D$ be an embedding which preserves p-heights in A and B for some prime number p. Assume either*

(a) A/C and B/D are bounded p-groups and i preserves finite q-heights for all primes q, or

(b) A and B are divisible-plus-bounded p-groups.

Then there is a pure embedding $j : A \to B$ which extends i. When i is an isomorphism, j can be taken to be an isomorphism.

Proof. We define, by induction on α, an increasing chain of embeddings $i_\alpha : A_\alpha \to B_\alpha$ ($\alpha \leqslant \xi$) such that

- $A_0 = C$, $B_0 = D$ and $A_\xi = A$.
- Each i_α preserves q-heights in A and B, for all primes q. (Under assumption (b) all q-heights in A and B are ∞ when $q \neq p$, so that on this assumption we need only consider p.)

Since $i : C \to D$ is an embedding which preserves p-heights from A to B, we can put $i_0 = i$. At limit ordinals we take unions.

We define $i_{\alpha+1} : A_{\alpha+1} \to B_{\alpha+1}$, assuming that $i_\alpha : A_\alpha \to B_\alpha$ has been defined and $A_\alpha \neq A$. Since A/A_α is a p-group, there is some element $a \in A \setminus A_\alpha$ with $pa \in A_\alpha$. By the assumption that A/C is (divisible-plus-)bounded, we can assume that a is proper over A_α. Put $k = \mathrm{ht}^p_A(a)$.

Now there are two cases, according as k is finite or infinite.

If k is finite, then since A is tight over A_α, Lemma 9.5(c) tells us that $\mathrm{ht}^p_A(pa) = k + 1$. Since i_α preserves finite p-heights in A and B, $i_\alpha(pa)$ has p-height $k + 1$ in B. Choose b in B so that $pb = i_\alpha(pa)$ and $\mathrm{ht}^p_B(b) = k$.

If k is infinite, then we are in assumption (b). Certainly pa has infinite p-height in A. Using the facts that i_α preserves finite p-heights in A and B, and that elements of large enough p-height have infinite p-height, we can choose b in B so that $pb = i_\alpha(pa)$ and $\mathrm{ht}^p_B(b) = \infty = k$.

Either way, we have chosen b in B so that $pb = i_\alpha(pa)$ and $\mathrm{ht}^p_B(b) = \mathrm{ht}^p_A(a)$.

Claim A. The element b is proper over B_α. This is trivial if k is infinite. If k is finite and the claim fails, then there is $b' \in B_\alpha$ such that $b + b'$ has p-height $> k$, where necessarily b' has p-height k and so does $a' = i_\alpha^{-1}(b')$. Since a was proper over A_α, the same holds for $a + a'$ by Lemma 9.4, and hence by tightness and Lemma 9.5 (a), $p(a + a')$ has p-height $k + 1$. But $p(b+b')$ has p-height $> k+1$; since $i_\alpha(p(a+a')) = p(b+b')$, this contradicts the induction assumption that i_α preserves finite p-heights in A and B. The claim is proved.

Claim B. For every $a' \in A_\alpha$ and every positive integer m,

$$\mathrm{ht}^q_A(ma + a') = \mathrm{ht}^q_B(mb + i_\alpha a') \text{ for all primes } q.$$

First we consider the case where $q = p$. Since A/A_α and B/B_α are p-groups and pa, pb are in A_α, B_α, it suffices to prove the claim when $m = 1$. But in this case the claim follows at once from Lemma 9.4 and the fact that a and b are proper.

Next we suppose $q \neq p$. Then the q-heights of $ma + a'$ and $mb + i_\alpha a'$ are the same as those of $p(ma + a')$ and $p(mb + i_\alpha a') = i_\alpha(p(ma + a'))$ respectively, so the result follows by the induction assumption on i_α. The claim is proved.

Claim C. For every $a' \in A_\alpha$ and every positive integer m,

$$ma + a' = 0 \text{ if and only if } mb + i_\alpha(a') = 0.$$

If $ma + a' = 0$ then $ma \in A_\alpha$, so that $m = pn$ for some n. Then $a' = -ma = -npa$ and $i_\alpha(a') = -ni_\alpha(pa) = -mb$. So the left equation implies the right, and vice versa by symmetry, proving the claim.

We define $A_{\alpha+1}$ to be the subgroup of A generated by A_α and a, and $B_{\alpha+1}$ likewise in B with b. We define $i_{\alpha+1}$ by

$$i_{\alpha+1}(ma + a') = mb + i_\alpha a'$$

for all integers m and all $a' \in A_\alpha$. By Claim C this defines an embedding from $A_{\alpha+1}$ to $B_{\alpha+1}$ which extends i_α. By Claim B, $i_{\alpha+1}$ preserves finite q-heights in A and B for all primes q.

We put $j = i_\xi$ when $A_\xi = A$. By construction j preserves finite q-heights in A and B for all primes q, and hence $j : A \to B$ is a pure embedding.

Suppose $i : C \to D$ is an isomorphism. If there is an element b of $B \setminus j(A)$ with $pb \in j(A)$, then we can continue the construction using j^{-1} to extend the domain of j. This is absurd, and hence there is no such element. Thus j is an isomorphism. □

Theorem 11.2 handled each prime p separately. The next two theorems amalgamate the results, using pushouts in case (a) and a direct sum decomposition in case (b).

Theorem 11.3 *Suppose*

(a) A and B are group pairs,

(b) A/A^P and B/B^P are both bounded,

(c) A and B are tight extensions of A^P and B^P respectively, and

(d) $i : A^P \to B^P$ is an embedding which preserves finite q-heights in A and B, and also in A^P and B^P, for all primes q.

Then there is a pure embedding $j : A \to B$ which extends i. When i is an isomorphism, j can be taken to be an isomorphism.

Proof. We can write the bounded group A/A^P as a direct sum of its p-components for each prime p. This direct sum makes A the pushout of subgroups A_p (p prime) over A^P. Likewise the decomposition of B/B^P makes B the pushout of subgroups B_p (p prime) over B^P. Each A_p is uniquely determined as the set of elements a of A such that $p^k a \in A^P$ for some $k < \omega$; and likewise with the B_p's.

Consider a prime p for which $A_p \neq A^P$. Since A/A^P is bounded, A_p/A^P is a bounded p-group. Now finite p-heights in A_p are the same as in A, and finite q-heights in A_p are the same as in A^P for every other prime q, and likewise for B_p. So the assumption (d) implies that i preserves q-heights in A_p, for all primes q. Hence by Theorem 11.2 (case (a)) there is a pure embedding $j_p : A_p \to B_p$ extending i. The pushout property (section 3) amalgamates these maps j_p into a single pure embedding $j : A \to B$ extending i. When i was an isomorphism, each of the j_p is an isomorphism (again by Theorem 11.2), and so j is an isomorphism. □

Theorem 11.4 *Suppose*

(a) A and B are group pairs which are divisible-plus-bounded groups,

(b) A^P and B^P are both divisible-plus-bounded,

(c) A and B are tight extensions of A^P and B^P respectively, and

(d) $i : A^P \to B^P$ is an embedding which preserves finite q-heights in A and B for every prime q.

Then there is a pure embedding $j : A \to B$ which extends i. When i is an isomorphism, j can be taken to be an isomorphism.

Proof. Let $A = F \oplus^P \bigoplus^P_{p \text{ prime}} A_p$ be the decomposition given by Lemma 3.6, and $B = G \oplus^P \bigoplus^P_{p \text{ prime}} B_p$ the corresponding decomposition of B. Since A is a tight extension of A^P, F must equal F^P; likewise $G = G^P$. Therefore $i \restriction F$ is already an embedding from F to G; it is pure because the groups are divisible. When i is an isomorphism, so is $i \restriction F$.

For each prime p, $i \restriction A_p$ is an embedding from $(A^P)_p$ to $(B^P)_p$. The p-height in A of an element of $(A^P)_p$ is the same as its p-height in A^P, and likewise in B. So $i \restriction (A^P)_p$ preserves finite p-heights in A and B, and thus by Theorem 11.2 case (b) it extends to a pure embedding $j_p : A_p \to B_p$. If i was an isomorphism then so is $i \restriction (A^P)_p$ by Theorem 11.2 again. Amalgamating the maps $i \restriction F$ and j_p yields a pure embedding $j : A \to B$, which is an isomorphism when i was an isomorphism. □

12 Uniqueness of the decomposition

Theorem 12.1 *Let T be a complete theory of group pairs which has the Reduction Property. Let A be a model of T, and suppose $A = C \oplus D = C' \oplus D'$ where C, C' are tight extensions of A and D, D' are disjoint from A^P. Suppose also that if A/A^P is unbounded then both D and D' are unbounded, and suppose either*

(a) A/A^P is bounded, or

(b) A and A^P are divisible-plus-bounded.

Then the identity map $i : A^P \to A^P$ extends to an isomorphism from C to C'.

Proof. Clearly i preserves finite q-heights in A and in A^P, for all primes q. It follows that i extends to an isomorphism from C to C', by Theorem 11.3 in case (a) and by Theorem 11.4 in case (b). □

Theorem 12.2 *Let T be a complete theory of group pairs which has the Reduction Property, and let A be a divisible-plus-bounded model of T such that $A = C \oplus D$ where C is a tight extension of A^P and either*

(a) A/A^P is bounded, or

(b) A and A^P are divisible-plus-bounded.

Suppose also that if A/A^P is unbounded then so is D. Then $T = Th(C) \oplus Th(D)$.

Proof. Let B be any other model of T. Under (a) or (b), B/B^P is divisible-plus-bounded, so that by Theorem 7.3 B has a decomposition as $B = C' \oplus D'$ where C' is a tight extension of B^P. By saturation arguments we can blow up A and B to isomorphic elementary extensions $\tilde{A}$ and $\tilde{B}$. We can arrange that $\tilde{A} = \tilde{C} \oplus \tilde{D}$ where $\tilde{C}$, $\tilde{D}$ are elementary extensions of the group pairs C, D respectively; but in general there is no guarantee that $\tilde{A}$ is a tight extension of $\tilde{A}^P$. By Corollary 9.6 the blowup $\tilde{C}$ of C has the form $C_0 \oplus \mathbb{Q}^{(\mu)}$ where C_0 is a tight extension of $\tilde{C}^P$, and $\mu = 0$ unless A/A^P is unbounded. If A/A^P is unbounded, then by assumption D and hence also $\tilde{D}$ will be unbounded, so $\mathrm{Th}(\tilde{C})$ and $\mathrm{Th}(\tilde{D})$ are not affected by transferring the $\mathbb{Q}^{(\mu)}$ to $\tilde{D}$. Assume this done, so that again $\tilde{C}$ is a tight extension of $\tilde{A}^P$. Do likewise with $\tilde{B}$. Now by Theorem 12.1 any isomorphism from $\tilde{A}^P$ to $\tilde{B}^P$ extends to an isomorphism from $\tilde{C}$ to $\tilde{C}'$. Hence

$$C \equiv \tilde{C} \equiv \tilde{C}' \equiv C'.$$

Also

$$D \equiv \tilde{D} \equiv \tilde{D}' \equiv D'$$

where the middle equivalence is by Theorem 10.1. □

Theorem 12.3 *Let $A = C \oplus D$ be a group pair with C a tight extension of A^P. Then $Th(A)$ is (κ, λ)-categorical if and only if the following three conditions hold:*

(a) $Th(A)$ has a (κ, λ)-model,

(b) $Th(A)$ has the Reduction Property, and

(c) the cardinals κ, λ obey a condition in the following diagram which corresponds to conditions that hold on C and D:

	D finite	D bounded	D bounded ω_1-categorical	D unbounded ω_1-categorical
C/A^P bounded	$\omega < \kappa = \lambda$	$\kappa \leqslant \lambda \leqslant \omega$	—	—
C/A^P bounded, C div+bdd	—	—	$\kappa + \omega < \lambda$	$\kappa + \omega < \lambda$
C/A^P unbounded, C, A^P div+bdd	—	—	—	$\kappa + \omega < \lambda$

Proof. First we prove necessity. Clause (a) is by the definition of (κ, λ)-categoricity. Clause (b) is by Theorem 8.1.

Suppose $\kappa + \omega < \lambda$. By Theorem 6.6, A and A^P are divisible-plus-bounded. By Theorem 7.3, $A = C \oplus D$ where C is tight over A^P. By Theorem 12.2 the hypotheses of Theorem 7.5 hold, so that we can read off from (a) of that theorem that D must be ω_1-categorical, and from (b) that if C/A^P is unbounded then D is unbounded.

Next suppose $\omega < \kappa = \lambda$. Then by Theorem 5.3(a), A/A^P is bounded. Again the hypotheses of Theorem 7.5 are confirmed, and we read off from (c) of that theorem that D must be finite.

Next suppose $\kappa \leqslant \lambda = \omega$. Then by Theorem 5.3(b), A/A^P is bounded and again we can invoke Theorem 7.5. By (b) of that theorem, D is bounded. If $\kappa \leqslant \lambda < \omega$ then trivially D is bounded.

Secondly we prove sufficiency. We assume throughout that A and B are (κ, λ)-models of T and $i : A^P \to B^P$ is an isomorphism; and that $A = C \oplus D$ and $B = C' \oplus D'$ where C is a tight extension of A^P and C' is a tight extension of B^P. We assume also that T has the Reduction Property. We note that by the Reduction Property, i preserves finite q-heights in A and B for all primes q.

Again suppose first that $\kappa + \omega < \lambda$, C/A^P is bounded, C is divisible-plus-bounded and D is ω_1-categorical. Then by Theorem 11.3 the isomorphism i extends to an isomorphism i^+ from C to C'. By Theorem 10.1 we can infer $D \equiv D'$. Since D and D' are elementarily equivalent models of a λ-categorical theory and they both have cardinality λ by Theorem 7.3, they are isomorphic. Combining this isomorphism with i^+ gives the required isomorphism $i : A \to B$.

Next suppose that $\kappa + \omega < \lambda$, C/A^P is unbounded but C and A^P are both divisible-plus-bounded, and D is ω_1-categorical. By Lemma 3.6 we can decompose C as $F \oplus^P \bigoplus^P_{p \text{ prime}} A_p$, where F is torsion-free divisible and for each prime p, A_p is the p-component of A with $(A_p)^P = A_p \cap A^P$. Let

$\mathbb{Q}$ be a direct summand of F. By tightness, $\mathbb{Q}$ has some nonzero element in A^P; since A^P is divisible-plus-bounded, this implies that the whole of $\mathbb{Q}$ lies in A^P. So $F \subseteq A^P$ and C/A^P is torsion. The same decomposition applies to B. Now for each prime p, Theorem 11.2(b) extends i to an isomorphism from A_p to B_p. Since C/A^P is torsion, these extensions fit together to give an isomorphism from C to C' as required. By Theorem 10.1 we can infer $D \equiv D'$, provided that we are not in the case where A/A^P is unbounded and D is bounded—but this is exactly the case missing in the chart. The rest is as in the previous case.

Next suppose that $\omega < \kappa = \lambda$, C/A^P is bounded and D is finite. We note that since $i : A^P \to B^P$ is an isomorphism, it preserves finite q-heights in A^P and B^P. So the conditions for applying Theorem 11.3 are met, and we infer that i extends to an isomorphism i^+ from C to C'. Also D and D' are elementarily equivalent finite groups by Theorem 10.1, so they are isomorphic and we conclude as in the previous case.

Next suppose that $\kappa \leqslant \lambda = \omega$, C/A^P is bounded and D is bounded. The isomorphism i extends to an isomorphism $i^+ : C \to C'$ by the previous case. Since D and D' are (by Theorem 10.1) elementarily equivalent bounded groups of cardinality at most ω, they must be isomorphic.

The final case is where $\kappa \leqslant \lambda < \omega$. In this case $A \cong B$ and the Reduction Property says that every automorphism of A^P extends to an automorphism of A, which establishes (κ, λ)-categoricity. □

The conditions in Theorem 12.3 are stated in terms of C and D rather than their theories. The next theorem restates the results in terms of theories.

Definition 12.4 We say that a theory T of group pairs is *tight-bounded* if every model A of T is a tight extension of its P-part. We say that T is *tight-unbounded* if every model A of T is of the form $C \oplus \mathbb{Q}^{(\mu)}$ where C is a tight extension of A^P, C/A^P is unbounded and $\mu \geqslant 0$. (Cf. Corollary 9.6.)

Theorem 12.5 *Let T be a complete theory of group pairs. Then T is (κ, λ)-categorical if and only if the following three conditions hold:*

- *(a) T has (κ, λ)-models,*
- *(b) $T = T_1 \oplus T_2$ where T_1 has the Reduction Property and is tight-bounded or tight-unbounded, and T_2 is disjoint from P, and*
- *(c) the cardinals κ, λ obey a condition in the following diagram which corresponds to conditions that hold on T_1 and T_2:*

	T_2 *finite*	T_2 *bounded*	T_2 *bounded* ω_1*-categorical*	T_2 *unbounded* ω_1*-categorical*
T_1 *tight-bounded*	$\omega < \kappa = \lambda$	$\kappa \leqslant \lambda \leqslant \omega$	—	—
T_1 *tight-bounded, div+bdd*	—	—	$\kappa + \omega < \lambda$	$\kappa + \omega < \lambda$
T_1 *tight-unbounded, div+bdd,* T^P *div+bdd*	—	—	—	$\kappa + \omega < \lambda$

Proof. Most of this follows directly from what precedes.

We need to check that assuming the Reduction Property on T is equivalent to assuming it on T_1. If T has the Reduction Property, then so does T_1 by Theorem 8.2. In the other direction, if T_1 has the Reduction Property then by Theorem 12.1, if C and C' are models of T_1 and there is an isomorphism $i : C^P \to C'^P$, then the isomorphism extends to C and C' apart from some possible direct summands $\mathbb{Q}$ lying outside the P-part. These extra direct summands appear only in the tight-unbounded case (the bottom row of the diagram), and in this case they can be transferred to the model of T_2. This argument also explains how in this theorem we can allow T_1 to be tight-unbounded while in Theorem 12.3 we required C to be tight over its P-part. □

Corollary 12.6 *Let T be a complete theory of group pairs.*

(a) If T is (κ, λ)-categorical for some pair (κ, λ) of infinite cardinals with $\kappa < \lambda$, then it is (κ, λ)-categorical for all such pairs.

(b) If T is (κ, κ)-categorical for some uncountable cardinal κ, then T is relatively categorical.

□

Corollary 12.6(b) is not true for arbitrary theories of pairs of structures; see [12]. It's reasonable to conjecture that Corollary 12.6(a) has counterexamples too in the general case.

Corollary 12.7 *Let κ, λ be cardinals such that T is (κ, λ)-categorical. Let A, B be (κ, λ) models of T and $f : A^P \to B^P$ an elementary embedding. Then f extends to an elementary embedding of A into B.*

Proof. The proofs of the relevant results in sections 11 and 12 imply that f extends to a pure embedding of A into B. But $A \equiv B$, so a pure embedding from A to B is elementary by Corollary 3.4. □

13 Comparison with Shelah [11]

Like us, Shelah considers $L(P)$-structures A where the 1-ary relation symbol P picks out an L-substructure A^P, the *P-part* of A. He assumes we consider models of a complete first-order theory T in $L(P)$. Our L is countable, but Shelah doesn't assume this.

Shelah's Hypothesis 1.0 is that $|A| = |A^P|$. He doesn't assume this in general; he marks with * those results that do assume it.

Shelah's Hypothesis 1.1, alias Hypothesis A, is the Reduction Property. This property holds for all (κ, κ)-categorical theories for reasons of general model theory [9]. Our proof used mostly general theory, but also a direct sum decomposition. Does the Reduction Property hold for (κ, λ)-categorical pairs in general?

Shelah's Hypothesis 1.2, alias Hypothesis B, is the definability of types. More precisely he assumes that for every model A of T, every tuple $\bar{a}$ in A and every formula $\phi(\bar{x}, \bar{y})$ of $L(P)$ there are a formula $\psi(\bar{y})$ of L and a tuple $\bar{c}$ in A^P such that for every tuple $\bar{b}$ in A^P,

$$A \models \phi(\bar{a}, \bar{b}) \;\Leftrightarrow\; A^P \models \psi(\bar{b}, \bar{c}).$$

In our case Hypothesis B is guaranteed by the stability of T (e.g. [5] Theorem 6.7.8).

Shelah defines: a subset X of a model A of T is *complete* if whenever $A \models (\exists \bar{x} \subseteq P)\phi(\bar{x}, \bar{a})$ with $\bar{a}$ in X, there is $\bar{c}$ in $X \cap A^P$ such that $A \models \psi(\bar{c}, \bar{a})$. Hypothesis C is that for every model A of T and every complete set $X \subseteq A$, the cardinality of

$$\{\text{tp}(\bar{b}/X) : \bar{b} \cap \mathbb{M}^P = \emptyset, X \cup \{\bar{b}\} \text{ is complete }\}$$

is at most $|X|^{|L|}$. Hypothesis C holds in our case because A/A^P is finite or ω-stable under any relative categoricity assumption.

Shelah defines a notion of a set being 'stable'. He shows that if all sets in models of T are stable then Hypothesis C holds, and he proves (using diamonds) that in general relative categoricity assumptions imply that every model of T is stable.

Shelah's Question D is whether for every pair of models $A \preccurlyeq B$, the set $A \cup B^P$ is stable. When the answer is Yes, he shows that every suitably saturated model E of T^P is the P-part of a model A of T which is prime over E.

Shelah is also concerned with the number of models not isomorphic over a given P-part. Write $I(T, \kappa, \lambda)$ for the supremum, over families of

(κ, λ)-models of T with the same P-part, of the number of isomorphism types of models over the P-part. Then our calculations show that when T has (κ, λ)-models,

- If $\kappa = \lambda = \omega_\alpha$ with $\alpha > 0$, and T is not bounded over its P-part, then $I(T, \kappa, \lambda) \geqslant |\omega + \alpha|$. (Theorem 5.3(a))
- If $\kappa < \omega = \lambda$ and T is not bounded over its P-part, then $I(T, \kappa, \lambda) \geqslant \omega$. (Theorem 5.3(b))
- If $\kappa = \omega_\alpha < \lambda = \omega_\beta$ and T has models with $\mathbb{Z}(p^\infty)^{(\omega)}$ a direct summand disjoint from P, then $I(T, \kappa, \lambda) \geqslant |\beta - \alpha|$. (Theorem 5.4)
- If $\kappa < \lambda$ and some model of T contains in its P-part μ direct summands of the form either $\mathbb{J}_p$ or an unbounded torsion group in which infinitely many primes are represented, then $I(T, \kappa, \lambda) \geqslant 2^\mu$. (Theorem 6.6)
- If $T = T_1 \oplus T_2$ where T_1 is tight-bounded or tight-unbounded, and T_2 is disjoint from P, then $I(T, \kappa, \lambda)$ is at least the number of isomorphism types of models of T_2 of cardinality λ.

Part IV

Finite groups

Using Lemma 3.6, a finite group pair is relatively categorical if and only if its p-components are relatively categorical for each prime p. When $p \neq 2$ we will describe the relatively categorical finite p-group pairs. The case where $p = 2$ is more complicated, and our description in this case will be complete only when the P-part A^P is a characteristic subgroup of A.

14 Preliminaries

If the group pair A is finite, then $\mathrm{Th}(A)$ determines A up to isomorphism. So in particular the relative categoricity of $\mathrm{Th}(A)$ is the same thing as the relative categoricity of A.

Lemma 14.1 *Let p be a prime and A a finite p-group pair. Then the following are equivalent:*

(a) *A is relatively categorical.*

(b) *Every automorphism of A^P extends to an automorphism of A.*

(c) *Every automorphism of A^P extends to an endomorphism of A.*

(d) *A has the Reduction Property.*

(e) *$A = C \oplus D$ where $A^P \subseteq C$, and C is relatively categorical and a tight extension of A^P.*

Proof. (a) implies (b) by Lemma 1.1 and (d) by Theorem 8.1. By Theorem 12.5, (a) implies that A has a decomposition $A = C \oplus D$ where $A^P \subseteq C$, C is a tight extension of A^P and C has the Reduction Property.

Since A is finite, (b) and (d) are equivalent; clearly (b) implies (a). This shows that (a), (b) and (d) are equivalent. The implication (d) $\Rightarrow$ (a), applied to C in the previous paragraph, proves that C is relatively categorical and hence that (a) implies (e). The implication from (e) to (a) is clear.

Clearly (b) implies (c). For the converse, suppose β is an automorphism of A^P which extends to an endomorphism α of A. By Fitting's Lemma (e.g. Jacobson [7] p. 113), there is an abelian group direct sum decomposition $A = A_1 \oplus A_2$ such that α is an automorphism on A_1 and nilpotent on A_2. Then $A^P \subseteq A_1$. The automorphism which agrees with α on A_1 and with the identity on A_2 is an automorphism of A extending β. □

In view of Lemma 14.1 we can use 'relatively categorical' henceforth as meaning the purely group-theoretic statement that every automorphism of A^P extends to an automorphism of A.

Lemma 14.2 *Let A be a finite p-group pair where $B = A^P$ is cyclic, generated by an element $b \neq 0$. Then A is a tight extension of B if and only if A can be written as $A_1 \oplus \ldots \oplus A_n$ where*

(a) *each A_i is cyclic, generated by a nonzero element a_i of order p^{r_i}, and $b = p^{s_1}a_1 + \ldots + p^{s_n}a_n$ where $0 \leqslant s_i < r_i$ for each i;*

(b) *$s_i < s_j$ whenever $1 \leqslant i < j \leqslant n$;*

(c) *$1 \leqslant r_i - s_i < r_j - s_j$ whenever $1 \leqslant i < j \leqslant n$.*

When A is a tight extension of B, the sequence $(s_1, \ldots, s_n; r_1, \ldots, r_n)$ is uniquely determined by A.

Proof. Suppose first that A is a tight extension of B. Write A as a direct sum of nonzero cyclic groups. We can arrange that b is of the form $p^{s_1}a_1 + \ldots + p^{s_n}a_n$ by multiplying each generator a_i by a suitable integer prime to p. Then tightness guarantees that $s_i < r_i$ for each i. Order the summands so that $r_1 \leqslant \ldots \leqslant r_n$.

Suppose $i < j$ and $s_i \geqslant s_j$. Then we can replace a_j by $a_j + p^{s_i - s_j}a_i$; this reduces $A_i \cap B$ to 0 and contradicts tightness (by Lemma 9.7). So (b) is proved.

We turn to (c). Let $i < j$ and for contradiction suppose $r_i - s_i \geqslant r_j - s_j$. Then we can replace a_i by $a_i + p^{s_j - s_i}a_j$, since by supposition $r_j - (s_j - s_i) \leqslant r_i$. This reduces $A_j \cap B$ to 0 and again contradicts tightness. So (c) is proved. Note that (b) and (c) together imply that $r_j \geqslant r_i + 2$ whenever $i < j$.

Conversely suppose A is a direct sum of cyclic groups with (a)–(c) holding. Write $c_1, \ldots, c_n$ for the generators of the cyclic summands of A. We show that for every $k \leqslant r_n$, $p^k A[p] \subseteq p^{k+1}A + B$; by Lemma 9.5 this implies that A is a tight extension of B.

Suppose there is some element a of $p^k A[p] \setminus p^{k+1}A[p]$. Then $k = r_i - 1$ for some unique i, and by subtracting suitable elements of $p^{k+1}A$ we can suppose that a generates the socle $A_i[p]$. Consider the element $d = p^{r_i - s_i - 1}b$ of B, (which exists since $r_i - s_i - 1 \geqslant 0$ by (a)). For each $j < i$, $r_j - s_j - 1 \geqslant r_i - s_i$ by (c) again, and hence the element d lies in $A_i \oplus \ldots \oplus A_n$. Then for some suitable m prime to p, $ma = d + a'$ where a' lies in $A_{i+1} \oplus \ldots \oplus A_n$. Now if $i < j$ then by (b), $s_i < s_j$. It follows that a' has p-height at least $r_i - s_i - 1 + s_{i+1} \geqslant r_i \geqslant k + 1$, so that $a' \in p^{k+1}A$ as required.

The numbers r_i are recoverable from the Szmielew invariants U of the group A in the usual way. The number s_1 is the minimum p-height in A of elements of A^P. Then $s_2 - s_1$ is the minimum p-height in $p^{s_1}A$ of elements of $p^{s_1}A \cap A^P$; and so on. Hence the sequences of r_i's and s_i's are recoverable from the group pair A. □

Definition 14.3 We refer to the sequence $(s_1, \ldots, s_n; r_1, \ldots, r_n)$ in Lemma 14.2 as the *ticket* of the group pair A.

Lemma 14.4 *Let A be a finite p-group pair.*

(a) Suppose A is a group pair direct sum, $A_1 \oplus \ldots \oplus A_n$. If A is relatively categorical then so is each A_i.

(b) Suppose $A = C \oplus D$ where $A^P \subseteq C$ and C is a tight extension of A^P. Then A is relatively categorical if and only if C is relatively categorical.

Proof. (a) Assume A is relatively categorical and let α be an automorphism of A_i^P. Extend α to the whole of A^P by taking it to be the identity on each A_j^P with $j \neq i$. By assumption α extends to an automorphism β of the whole of A. Let γ be $g\beta \restriction A_i$ followed by projection onto A_i along the remaining direct summands. Then γ is an endomorphism of A_i which extends α, so A_i is relatively categorical by Lemma 14.1(c).

(b) is then clear. Note that by Theorem 7.3, A can always be decomposed in this form. □

15 Automorphisms

Definition 15.1 Fixing a prime p, we consider an abelian p-group $A = A_1 \oplus \ldots \oplus A_n$ where each A_i has generator a_i of order p^{r_i} with $r_1 \leqslant \ldots \leqslant r_n$. By an *elementary automorphism* of A (with respect to the given decomposition) we mean an automorphism α of A with one of the following two forms:

(a) For some i and some m prime to p, $\alpha a_i = m a_i$, and $\alpha a_k = a_k$ for all $k \neq i$.

(b) For some i and j with $i \neq j$, $\alpha a_i = a_i + p^h m a_j$ where m is prime to p and $r_j - h \leqslant r_i$; and $\alpha a_k = a_k$ for all $k \neq i$.

(We use 'elementary' here in the sense of linear algebra; of course all automorphisms are elementary embeddings in the model-theoretic sense.)

Lemma 15.2 *If A is as in Definition 15.1, then every automorphism of A is a product of elementary automorphisms. Also every elementary automorphism of the form (b) is a power of the elementary automorphism where $m = 1$ and $h =$* $\max(0, r_j - r_i)$.

Proof. Use Gaussian elimination with the obvious adjustments. If α is an automorphism of A, we write $\alpha a_i = \sum_j m_{ij} a_j$ for each i; here each m_{ij} is a unique integer modulo p^{r_j}. If M is the matrix (m_{ij}), we write M^{-1} for the matrix of the inverse automorphism α^{-1}. Since αa_n has order p^{r_n}, there is at least one j with a_j of order p^{r_n} and m_{nj} prime to p. So elementary row operations on the matrix (m_{ij}) bring the final column to the form $(0, \ldots, 0, 1)^T$, and then elementary column operations bring the last row to the form $(0, \ldots, 0, 1)$. Applying the same argument to $n-1$, $n-2 \ldots$ in place of n, there are matrices P, Q of elementary automorphisms such that $P(m_{ij})Q$ is the unit matrix, and hence $(m_{ij}) = P^{-1}Q^{-1}$. The righthand side of this equation is the matrix of a product of elementary automorphisms; hence so is the lefthand. □

Definition 15.3 Let A be a group pair with $A^P = B$. We say that A is *separated* if A has a group pair direct sum decomposition $A = A_1 \oplus \ldots \oplus A_k$ such that each A_i^P is cyclic.

Lemma 15.4 *Suppose the finite group pair A is separated as in Definition 15.3, and each A_i^P is generated by an element b_i of order p^{s_i}. Then A is relatively categorical if and only if :*

> *for all $i \neq j$ $(1 \leqslant i, j \leqslant k)$, there is a group homomorphism $g : A_i \to A_j$ taking b_i to $p^h b_j$ where $h = \max(0, s_j - s_i)$.*

Proof. Write B for A^P. First we show that if A is relatively categorical then the condition holds. Let α be the automorphism of B which takes b_i to $b_i + p^h b_j$ and fixes each $b_{i'}$ $(i' \neq i)$. Then α extends to an automorphism α^+ of A. The automorphism $\alpha^+ \restriction A_i$ takes b_i to $b_i + p^h b_j$, and so $\alpha^+ \restriction A_i$ followed by projection onto A_j takes b_i to $p^h b_j$ as required.

Second, suppose the condition holds. To show that A is relatively categorical, it suffices to show that each elementary automorphism of B lifts to A. The one-dimensional automorphisms ((a) in Definition 15.1) lift immediately; multiply A by the same scalar as B. Suppose next that $i \neq j$ and α is an automorphism of B which takes b_i to $b_i + p^h m b_j$ where $s_j - h \leqslant s_i$ and m is prime to p. By assumption there is a group homomorphism β from A_i to A_j which takes b_i to $p^{\max(0, s_j - s_i)} b_j$. Suppose $p^h m = m' p^{\max(0, s_j - s_i)}$. (It must have this form, since $h \geqslant a_j - s_i$ and $h \geqslant 0$.) Then $m'\beta$ is a homomorphism from A_i to A_j taking b_i to $p^h m b_j$. Counting $m'\beta$ as zero on all elements outside A_i, the endomorphism $1_A + m'\beta$ of A is an automorphism extending α. □

Now when A is a separated group pair, we can combine the lemmas above and read off necessary and sufficient conditions for A to be relatively categorical, in terms of the tickets of the direct summands of A. By Lemma 14.4(b) there is no loss in assuming that A is a tight extension of A^P, so that all the direct summands of A contain nontrivial cyclic subgroups of A^P and hence are tight extensions of their P-parts.

Theorem 15.5 *Let A be a separated p-group pair and $\bigoplus_{i \in I} A_i$ a decomposition of A as a direct sum of indecomposable group pairs. Assume A is a tight extension of A^P, and for each $i \in I$ let $\tau_i = (s_{i,1}, \ldots, s_{i,n_i}; r_{i,1}, \ldots, r_{i,n_i})$ be the ticket of A_i. Then A is relatively categorical if and only if for all $i \neq j$ in I, and for each k' $(1 \leqslant k' \leqslant n_j)$, either $r_{j,k'} - s_{j,k'} \leqslant h$ or there is k $(1 \leqslant k \leqslant n_i)$ such that*

(a) $s_{i,k} \leqslant s_{j,k'} + h$ *and*

(b) $r_{i,k} - s_{i,k} \geqslant r_{j,k'} - s_{j,k'} - h$

where $h = \max(0, (r_{j,n_j} - s_{j,n_j}) - (r_{i,n_i} - s_{i,n_i}))$.

Proof. For each i let b_i be a generator of A_i^P as in Lemma 14.2; note that the order of b_i is $p^{r_{i,n_i} - s_{i,n_i}}$. By Lemma 15.4 it suffices to show that the condition above is equivalent to: There is an abelian group homomorphism $g : A_i \to A_j$ taking b_i to $p^h b_j$.

The condition is sufficient. Suppose the condition holds. If k' is such that $r_{j,k'} - s_{j,k'} \leqslant h$, then $p^h b_j$ has zero k'-th component and we can ignore it. For each k' where this inequality fails, choose a k as in the condition, and define a homomorphism $\alpha_{k'}$ from the subgroup $\mathbb{Z}a_{i,k}$ to A_j by putting

$$\alpha_{k'}(a_{i,k}) = p^{s_{j,k'} + h - s_{i,k}} a_{j,k'}.$$

The element of A_j is well-defined by (a). To ensure that $\alpha_{k'}$ is a homomorphism we need to know that the order of $\alpha_{k'}(a_{i,k})$ is at most $p^{r_{i,k}}$, in other words that

$$r_{j,k'} - s_{j,k'} - h + s_{i,k} \leqslant r_{i,k}.$$

But (b) guarantees precisely this. So the homomorphisms $\alpha_{k'}$ are well-defined. Now define $\beta : A_i \to A_j$ to be the sum $\sum_{1 \le k' \le n_j} \alpha_{k'}$. One can check that $\beta(b_i) = p^h b_j$.

The condition is necessary. Suppose there is a homomorphism $\alpha : A_i \to A_j$ such that $\alpha(b_i) = p^h b_j$, and consider some k' ($1 \leqslant k' \leqslant n_j$). Let β be α followed by projection onto $\mathbb{Z}a_{j,k'}$. If $r_{j,k'} - sj, k' \leqslant h$ then $\beta(b_i) = 0$;. If not then $\beta(b_i)$ has p-height $s_{j,k'} + h$. Write $\beta(a_{i,k}) = p^{\ell_k} m_k a_{j,k'}$ for each k ($1 \leqslant k \leqslant n_i$), where each m_k is prime to p. Partition $\{1, \ldots, n_i\}$ into $K_1 \cup K_2$ where $k \in K_2$ if and only if $s_{i,k} + \ell_k > s_{j,k'} + h$. Let $\gamma : A_i \to A_j$ be the homomorphism that acts on each $a_{i,k}$ like β if $k \in K_1$ and as zero if $k \in K_2$. Then $\beta(b_i) - \gamma(b_i)$ has p-height $> s_{j,k'} + h$; it follows that K_1 is not empty and $\beta(b_i)$ has the same p-height as $\gamma(b_i)$, so $\beta(b_i) = m\gamma(b_i)$ for some integer m prime to p. Finally let $\delta : A_i \to A_j$ be $m\gamma$. Then $s_{i,k} + \ell_k \leqslant s_{j,k'} + h$ for each $k \in K_1$, and in particular (a) holds for each such k.

Since δ is a homomorphism, the order of some $a_{i,k}$ ($k \in K_1$) is at least that of $p^{\ell_k} m_k a_{j,k'}$, in other words $r_{i,k} \geqslant r_{j,k'} - \ell_k$. Then

$$r_{i,k} \leqslant r_{j,k'} - \ell_k \leqslant r_{j,k'} + s_{i,k} - (s_{j,k'} + h)$$

which immediately gives (b). □

Corollary 15.6 *Under the hypotheses of the theorem, suppose that each A_i is cyclic. Then for each i the numbers $r_{i,k}$ and $s_{i,k}$ reduce to single numbers r_i and s_i, and the conditions (a), (b) reduce to*

(a) $s_i \leqslant s_j + h$ and

(b) $r_i - s_i \geqslant r_j - s_j + h$

where $h = \max(0, (r_j - s_j) - (r_i - s_i))$. □

A small part of the information in Theorem 15.5 extends to the non-separated case.

Theorem 15.7 *Let A be a relatively categorical finite p-group pair, and suppose that $A^P = B_1 \oplus B_2$ where B_1 is a cyclic group of order p^r, generated by an element b of p-height s in A. Then every element of B_2 of order $\leqslant p^r$ has p-height $\geqslant s$ in A.*

Proof. Suppose c is an element of B_2 of order $\leqslant p^r$. Then there is an automorphism β of B which takes b to $b + c$. By assumption β extends to an automorphism α of A. Since b and $\alpha(b) = b + c$ both have p-height s, c has p-height $\geqslant s$. □

When is a relatively categorical finite p-group pair separated?

Theorem 15.8 *Let A be a relatively categorical finite p-group pair.*

(a) If $p \neq 2$ then A is separated.

(b) If $p = 2$ then A splits as a group pair direct sum of relatively categorical group pairs $A_1 \oplus A_2$ where A_1 is separated and A_2^P is a direct sum of cyclic groups of pairwise distinct orders.

Proof. Write $B = A^P$ as a direct sum of cyclic groups, $B_1 \oplus \ldots \oplus B_n$ with generators $b_1, \ldots, b_n$.

(a) We assume $p \neq 2$. We will show that A is a group pair direct sum $A_1 \oplus A_2$ with $B_1 \subseteq A_1$ and $\bigoplus_{j \neq 1} B_j \subseteq A_2$, where each of A_1 and A_2 are relatively categorical. Induction completes the argument.

We first show that projection onto B_1 along the other direct summands of B is a linear combination of automorphisms of B. Let β be the automorphism taking b_1 to $2b_1$ and each other generator b_j to b_j. Then the required projection is $\beta - 1_B$. By relative categoricity, β extends to an automorphism α of A. Then $\gamma = \alpha - 1_A$ is an endomorphism of A extending the projection. By Fitting's Lemma the group A has a direct sum decomposition $A = A_1 \oplus A_2$ such that γ is the sum of an automorphism of A_1 and a nilpotent endomorphism of A_2. Clearly then $B_1 \subseteq A_1$ and $\bigoplus_{j \neq 1} B_j \subseteq A_2$, so

$A = A_1 \oplus A_2$ is a group pair direct sum. Both A_1 and A_2 are relatively categorical by Lemma 14.4(a).

(b) Suppose now that $p = 2$. We show that the construction of (a) allows us to separate off one cyclic summand B_i provided that there is $j \neq i$ such that B_i and B_j have the same order. Let α transpose b_i and b_j, fixing the other generators of B pointwise. Let β take b_i to $b_i + b_j$ and fix all generators b_k ($k \neq i$) pointwise. Then $\alpha(\beta - 1_B)$ is the projection of B onto B_i along the other direct summands, and the argument proceeds as before. □

Example 15.9 The following example shows that a relatively categorical finite 2-group pair need not be separated. Let A be $\mathbb{Z}(4) \oplus \mathbb{Z}(16)$, and a_1, a_2 generators of the two summands. Let $B = A^P$ be the subgroup generated by the two elements $a_1 + 2a_2$ and $2a_1$; then B is $B_1 \oplus B_2$ where $a_1 + 2a_2$ generates B_1 and $2a_1$ generates B_2. If C is a direct summand of A containing $a_1 + 2a_2$, then C contains $8a_2$ and so must contain an element c of A with $8c = 8a_2$. All such elements c have the form $m_1a_1 + m_2a_2$ where m_1, m_2 are odd. Hence C contains $2m'a_1 + 2a_2$ for some odd m', and thus also $(2m' - 1)a_1$. It follows quickly that $C = A$.

We confirm that A is relatively categorical. It suffices to check that each elementary automorphism of B extends to one of A. The only automorphisms of B of the form (a) are scalar multiplication by m, where m is an odd integer. Each such automorphism extends to the automorphism multiplying each element of A by m. Next there is the automorphism

$$a_1 + 2a_2 \mapsto 3a_1 + 2a_2, \quad 2a_1 \mapsto 2a_1.$$

This extends to the automorphism

$$a_1 \mapsto 3a_1, \quad a_2 \mapsto a_2.$$

Finally there is the automorphism

$$a_1 + 2a_2 \mapsto a_1 + 2a_2, \quad 2a_1 \mapsto 2a_1 + 8a_2.$$

This extends to the automorphism

$$a_1 \mapsto a_1 + 4a_2, \quad a_2 \mapsto 15a_2$$

of A.

16 Characteristic subgroups

Definition 16.1 Recall that a subgroup B of a group A is *characteristic* if for every automorphism α of A, $\alpha \restriction B$ is an automorphism of B. We call a group pair A *characteristic* if A^P is a characteristic subgroup of the group A.

Definition 16.2 We fix some conventions for this section. The p-group pair A is a group direct sum $A_1 \oplus \ldots \oplus A_n$ where each A_i is a nonzero direct sum of cyclic groups of order p^{r_i}, and $r_1 < \ldots < r_n$. Decomposing each A_k into a direct sum of cyclic groups, we can choose a generator for each of these cyclic groups; the set of these generators for all A_k will be called the *chosen basis*. The chosen basis is not fixed; we can re-choose it. We write $B = A^P$. For each i ($1 \leqslant i \leqslant n$) we define two integers s_i, t_i in the following way. If $B \cap A_i = 0$, set $s_i = r_i$; otherwise s_i is the least nonnegative integer such that $A_i \cap B$ contains an element of p-height s_i. If $B \subseteq A_1 \oplus \ldots \oplus A_{i-1} \oplus A_{i+1} \oplus \ldots \oplus A_n$, set $t_i = r_i$; otherwise, t_i is the least nonnegative integer such that there exists an element $c_1 + \ldots + c_i + \ldots + c_n$ of the group B, with c_j in A_j for each j, such that c_i is an element of p-height t_i. It is clear that $0 \leqslant t_i \leqslant s_i \leqslant r_i$.

Lemma 16.3 *Suppose B is a characteristic subgroup of A. Then:*

(a) For each i, $B \cap A_i = p^{s_i} A_i$.

(b) If $j < i$, then $t_j \leqslant s_j \leqslant t_i \leqslant s_i$.

(c) If $i < j$, then $r_i - t_i \leqslant r_j - s_j$.

(d) For each i, $s_i \leqslant t_i + 1$; if $p \neq 2$ or A_i has rank > 1 then $t_i = s_i$.

Proof. Fix i and j. By the definition of t_i, there exists an element $c = c_1 + \ldots + c_n$ of the group B, with $c_j \in A_j$ for each j, such that the p-height of c_i is equal to t_i. We can arrange the chosen basis so that it contains an element $a_i \in A_i$ for which $p^{t_i} a_i = c_i$. Let $a' \in A_j$, and let $h = \max(0, r_j - r_i)$. In each of the cases

1. $i \neq j$, a' is an element of the chosen basis;
2. $i = j$, $a' = a_i$ and $p \neq 2$;
3. $i = j$, A_i has rank > 1 and a' is an element of the chosen basis distinct from a_i;
4. $i = j$, $a' = pa_i$

there is an automorphism α of A which takes a_i to $a_i + p^h a'$ and fixes each other element of the chosen basis. Since B is a characteristic subgroup of A, the element $b = p^{t_i+h}a' = \alpha(c) - c$ is contained in $B \cap A_j$.

In the first three cases put $b = p^{t_i+h}a' \in B \cap A_j$. Then either b is an element of p-height $t_i + h$ and then $s_j \leqslant t_i + h$, or else $b = p^{t_i+h}a' = 0$ so that $t_i + h \geqslant r_j \geqslant s_j$. Similarly, in the fourth case $b = p^{t_i+1}a_i \in B \cap A_i$ is an element of p-height $t_i + 1$, and then $s_i \leqslant t_i + 1$, or $b = p^{t_i+1}a_i = 0$, which means that $t_i + 1 \geqslant r_i \geqslant s_i$. To complete the proof of (b), (c), (d) of this Lemma it suffices to recall that $t_i \leqslant s_i$ for each i.

To prove (a), arrange the chosen basis so that it contains an element a_i in A_i such that $p^{s_i}a_i \in B$. Let a'_i be another element of the chosen basis that lies in A_i. Then there is an automorphism β of A that transposes a_i and a'_i, and fixes each other element of the chosen basis. Then $\beta(p^{s_i}a_i) = p^{s_i}a'_i$, and this element is in B since B is a characteristic subgroup. Since $A_i \cap B$ is a group, we infer that $p^{s_i}A_i \subseteq B$. The converse inclusion $B \cap A_i \subseteq p^{s_i}A_i$ is an immediate corollary of the definition of s_i. □

Corollary 16.4 *The order of each element of B is at most $p^{r_n - t_n}$. If $b = b_1 + \ldots + b_n$, with $b_j \in A_j$ for each j, is an element of B, and its component b_n in A_n has the least possible p-height t_n, then the order of b is equal to $p^{r_n - t_n}$. Hence the cyclic group generated by b is a direct summand of B.*

Proof. Let $c = c_1 + \ldots c_n$ be an element of the group B, with c_i belonging to A_i for each i. By the definition of t_i, $c_i \in p^{t_i}A_i$, hence $p^{r_i - t_i}c_i = 0$. By (c) of Lemma 16.3, $r_i - t_i \leqslant r_n - s_n \leqslant r_n - t_n$ for each $i < n$ and consequently $p^{r_n - t_n}c_i = 0$. Thus $p^{r_n - t_n}c = 0$ for every element $c \in B$. On the other hand, $p^{r_n - t_n - 1}b_n \in A_n$ is an element of p-height $r_n - t_n - 1 + t_n = r_n - 1$, which implies that $p^{r_n - t_n - 1}b_n \neq 0$ and consequently $p^{r_n - t_n - 1}b \neq 0$. □

From the preceding lemmas we can read off a characterisation of the finite relatively categorical characteristic p-group pairs that are separated (bearing in mind that when $p \neq 2$, all finite relatively categorical p-group pairs are separated).

Theorem 16.5 *The following are equivalent, for any finite p-group pair:*

(a) A is a relatively categorical characteristic p-group pair which is a group pair direct sum of cyclic group pairs.

(b) For some s, $A^P = p^s A$.

(c) In the notation of Definition 16.2, $s_n = t_n$.

(d) *Either $B = 0$, or for some s, $0 \neq p^s A \subseteq B$ and $pB \subseteq p^{s+1}A$.*

If $p \neq 2$, we can leave out the condition that A is a group pair direct sum of cyclics.

Remark 16.6 The condition (d) is technical; we shall need it below.

Proof. (a) $\Rightarrow$ (c): Assume (a) and recall Definition 16.2. Since A is a group pair direct sum of cyclic group pairs, we can choose the decomposition $A = A_1 \oplus \ldots \oplus A_n$ so that each A_i is a group pair direct sum of cyclic group pairs. Then $B = (B \cap A_1) \oplus \ldots \oplus (B_n \cap A_n)$, and it follows that the component $b_n \in A_n$ of any element $b = b_1 + \ldots + b_n \in B$ is contained in the group $B \cap A_n$, which, by Lemma 16.3(a), coincides with $p^{s_n}A$. Therefore $t_n = s_n$.

(c) $\Rightarrow$ (b). Let $s_n = t_n$ and let $a_n \in A_n$ be an element of the chosen basis. Then $b = p^{s_n}a_n \in B$, and the p-height of its component $p^{s_n}a_n$ in A_n is equal to $s_n = t_n$. By Corollary 16.4, the cyclic group D generated by b is a direct summand of B; let B' be any complementary direct summand, so that $B = D \oplus B'$. Again by Corollary 16.4, the order of any element $b' \in B' \subseteq B$ is not greater than the order of $b = p^{s_n}a_n$. Therefore there is an automorphism β of the group B which takes b to $b + b' = p^{s_n}a_n + b'$. By relative categoricity, there exists an automorphism α of A such that $p^{s_n}\alpha(a_n) = \alpha(b) = \beta(b) = p^{s_n}a_n + b'$; therefore, $b' \in p^{s_n}A$. Thus B' is contained in $p^{s_n}A$, and, since $p^{s_n}a_n$ also belongs to $p^{s_n}A$, we obtain that $B \subseteq p^{s_n}A$.

On the other hand, by (b) of Lemma 16.3, $s_i \leqslant s_n$ for each $i \leqslant n$, and consequently $p^{s_n}A_i \subseteq p^{s_i}A_i \subseteq B$. Hence, $p^{s_n}A = p^{s_n}A_1 \oplus \ldots \oplus p^{s_n}A_n \subseteq B$.

(b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (d) are immediate.

(d) $\Rightarrow$ (c). If $B = 0$ then $t_n = s_n = r_n$. Suppose $0 \neq p^s A \subseteq B$, $pB \subseteq p^{s+1}A$. Then $s < r_n$ and $0 \neq p^s A_n \subseteq B \cap A_n = p^{s_n}A_n$; it follows that $s_n \leqslant s < r_n$. If $t_n \neq s_n$, then $t_n = s_n - 1$. There is an element $c_1 + \ldots + c_n$ of the group B, with c_j in A_j for each j, such that c_n has p-height t_n. Then pc_n has p-height $t_n + 1 = s_n < r_n$; on the other hand, $pc_n \in p^{s+1}A_n$, and we obtain that $s + 1 \leqslant s_n \leqslant s$. This contradiction proves that the assumption $t_n \neq s_n$ was erroneous.

Finally if $p \neq 2$, then by Lemma 16.3 (d), each A_i is a group pair direct summand of A with $A_i^P = p^{s_i}A_i$, so a group decomposition of A_i into a direct sum of cyclics is in fact a group pair decomposition. $\square$

We remark that the finite group pairs satisfying Theorem 16.5(b) are exactly those investigated in Evans, Hodges and Hodkinson [2], which characterised those group pairs A of this form for which A is coordinatisable over A^P.

Let us turn now to the case $t_n \neq s_n$; by Lemma 16.3 (d), this is possible only if $p = 2$ and A_n has rank 1. To the end of the section A is a finite relatively categorical characteristic 2-group pair and $B = A^P$. We denote by A' the direct sum $A_1 \oplus \ldots \oplus A_{n-1}$ and by B' the intersection $B \cap A'$. We always assume that $t_n \neq s_n$. Since the integers s_{n-1} and t_n play an essential role in the following argument, we write them s and t for brevity; by Lemma 16.3(b), $s \leqslant t$.

Lemma 16.7 *There exists a subset Γ of the set $\{1, \ldots, n-1\}$, such that:*

(a) if $i \in \Gamma$, then $t_i \neq s_i$ and consequently A_i has rank 1;

(b) generators a_i of the cyclic groups A_i $(i \in \Gamma \cup \{n\})$ can be chosen so that the element $b = 2^t a_n + \sum_{i \in \Gamma} 2^{t_i} a_i$ belongs to B.

Proof. By the definition of $t = t_n$, there exists an element $b' = b_1 + \ldots + b_n \in B$, with $b_i \in A_i$ for each i, such that b_n has 2-height t. Denote by Γ the set of all indices $i < n$, such that $b_i \notin 2^{s_i} A_i \subset B$; observe that $t_i \neq s_i$ for every $i \in \Gamma \cup \{n\}$. Then the element $b = b_n + \sum_{i \in \Gamma} b_i$ also belongs to B, and it is clear that for each $i \in \Gamma \cup \{n\}$ the element b_i has 2-height $< s_i$; since the 2-height of the (nonzero) component in A_i of an element of B cannot be smaller than t_i, and $t_i \geqslant s_i - 1$ by (d) of Lemma 16.3, we find that $t_i = s_i - 1$ and that b_i is an element of 2-height t_i. If $i \in \Gamma \cup \{n\}$, then by (d) of Lemma 16.3 the group A_i has rank 1, and we can choose a generator a_i of A_i so that $b_i = 2^{t_i} a_i$; in particular, $b_n = 2^t a_n$. □

Lemma 16.8 *The group B decomposes into the direct sum of the group $B' = B \cap A'$ and the cyclic group D generated by the element b which was defined in Lemma* 16.7.

Proof. Let $c = c_1 + \ldots + c_n \in B$, where $c_i \in A_i$ for each i. Then the 2-height of the element $c_n \in A_n$ is not greater than the 2-height t of the element b_n, and, since the group A_n is cyclic, there is an integer m such that $c_n = m b_n$. Then obviously $c - mb \in B \cap ((A_1 \oplus \ldots \oplus A_{n-1})) = B'$. Thus, $B = B' + D$, and this sum is direct because the component b_n of the element b in A_n has the same order as the element b itself. □

Lemma 16.9 *The group pair (A', B') is a relatively categorical group pair.*

Proof. It is obvious that the group B' is a characteristic subgroup of the group A'. Further, any automorphism β' of the group B' extends to an automorphism β of the direct sum $B = B' \oplus D$, which in its turn extends to an automorphism α of the group A, because B is relatively categorical in A. The composition of α with the projection $A = A' \oplus A_n \to A'$ is an endomorphism of A', and its restriction to B' coincides with β'; by Lemma 14.1, A' is a relatively categorical group pair. □

Lemma 16.10 *The group $2B'$ is contained in the group $2^{t+1}A'$.*

Proof. Let c be an arbitrary element of $B' = A' \cap B$; there exists an automorphism β of the group $B = B' \oplus D$ which takes b to $b + c$ and which is the identity on B'. Since A is a relatively categorical group pair, there is an automorphism α of the group A which extends β. Unfortunately, we do not know the images $\alpha(2^{t_i}a_i)$, $i \in \Gamma$, because $2^{t_i}a_i \notin B'$. But $2^{t_i+1}a_i = 2^{s_i}a_i \in B'$, and it follows that $\alpha(2^{t_i+1}a_i) = \beta(2^{t_i+1}a_i) = 2^{t_i+1}a_i$ for every $i \in \Gamma$. Therefore

$$
\begin{aligned}
&2c = \beta(2b) - 2b = \alpha(2b) - 2b = (\alpha(2^{t+1}a_n) + \textstyle\sum_{i\in\Gamma} \alpha(2^{t_i+1}a_i)) - \\
&-(2^{t+1}a_n + \textstyle\sum_{i\in\Gamma} 2^{t_i+1}a_i) = 2^{t+1}(\alpha(a_n) - a_n) \in A' \cap 2^{t+1}A = 2^{t+1}A'.
\end{aligned}
$$

Lemma 16.11 *The group B' is equal to the group $2^sA'$.*

Proof. First suppose $s = s_{n-1} < r_{n-1}$. Then $0 \neq 2^sA_{n-1} \subseteq 2^sA'$. Further, $2^sA' \subseteq B'$ because, by Lemma 16.3, $s_i \leqslant s = s_{n-1}$ for each $i < n$, and $2^sA_i \subseteq 2^{s_i}A_i = B \cap A_i$. By Lemma 16.10, $2B' \subseteq 2^{t+1}A'$; but $s \leqslant t$, therefore $2B' \subseteq 2^{s+1}A'$. Now it follows from (d) ⇒ (b) of Theorem 16.5 that $B' = 2^sA'$.

Next suppose $s = r_{n-1}$. Then for each $i < n-1$, $r_i - t_i \leqslant r_{n-1} - s = 0$. This means that if $c = c_1 + \ldots + c_{n-1}$ is an element of B', with $c_j \in A_j$ for each j, then $c_1 = \ldots = c_{n-2} = 0$ and $c = c_{n-1} \in A_{n-1} \cap B = p^{s_{n-1}}A_{n-1} = 0$. Therefore $B' = 0$, and we have again $B' = 0 = 2^sA'$. □

We can now calculate s_i, t_i and obtain information about the set Γ.

Lemma 16.12 *(a) If $i < n$ then $s_i = \min(r_i, s)$. If $i \notin \Gamma$ then $t_i = s_i$. If $i \in \Gamma$ then $t_i = s_i - 1$. Thus Γ consists of all the indices i such that $t_i \neq s_i$.*

(b) If $t < r_{n-1}$ then $s = t$. If $t \geqslant r_{n-1}$ then $s = r_{n-1}$ or $s = r_{n-1} - 1$.

(c) If $i \in \Gamma$ and $r_i \geqslant s$, then $r_{i-1} < s$ or $i = 1$.

(d) If $i \in \Gamma$ and $r_i < s < r_{n-1}$, then $r_{i+1} > s$.

(e) If $n - 1 \in \Gamma$ and $s < r_{n-1}$, then $t < r_n - 2$.

Proof. (a) By Lemma 16.11, $B \cap A' = 2^s A'$ and consequently $B \cap A_i = 2^s A_i$ for each $i < n$; therefore, $s_i = \min(r_i, s)$. If $i \in \Gamma$, then $s_i \neq t_i$ and so $t_i = s_i - 1$. We have seen that B is generated by the group $2^s A'$ and the element $b = 2^t a_n + \sum_{i\in\Gamma} 2^{t_i} a_i$. Hence every element of B is the sum $2^t a' + q(2^t a_n + \sum_{i\in\Gamma} 2^{t_i} a_i)$, with $a' \in A'$, $q \in \mathbb{Z}$, and for $j \notin \Gamma \cup \{n\}$ its component in A_j is contained in $2^s A_j = 2^{s_j} A_j$, which means that $t_j = s_j$.

(b) If $t < r_{n-1}$, then $0 \neq 2^t A' \subseteq 2^s A'$ and $2B' \subseteq 2^{t+1} A'$; using once more Theorem 16.5, we obtain that $B' = 2^t A'$. Thus $0 \neq 2^t A' = B' = 2^s A'$, which implies that $s = t$. If $t \geqslant r_{n-1}$, then $2^{s+1} A_{n-1} = 2 \cdot 2^s A_{n-1} = 2(A_{n-1} \cap B') \subseteq 2B' \subseteq 2^{t+1} A' = 0$, which means that $s + 1 \geqslant r_{n-1} \geqslant s$.

(c), (d), (e) Let $i \in \Gamma$, $i < n$. If both r_{i-1} and r_i are $\geqslant s$, we have $s = s_{i-1} \leqslant t_i = s_i - 1 = s - 1$; contradiction. If both r_i and r_{i+1} are $\leqslant s$, we have $1 = s_i - t_i = r_i - t_i \leqslant r_{i+1} - s_{i+1} = 0$; contradiction. If $n - 1 \in \Gamma$ and $s < r_{n-1}$, then $2 \leqslant r_{n-1} - (s-1) = r_{n-1} - t_{n-1} \leqslant r_n - s_n < r_n - t_n = r_n - t$. □

The following result is now obvious.

Lemma 16.13 *There are only the following variants for the set* $\Gamma \subseteq \{1, \ldots, n-1\}$ *and the integers* s, t, t_i $(i \in \Gamma)$*:*

(a) $\Gamma = \{m-1, m\},\ r_{m-1} < s < r_m,\ t_{m-1} = r_{m-1} - 1,\ t_m = s - 1,\ t = s.$

(b) $\Gamma = \{m-1\},\ r_{m-1} < s < r_m,\ t_{m-1} = r_{m-1} - 1,\ t = s.$

(c) $\Gamma = \{m\},\ s \leqslant r_m,\ r_{m-1} < s$ *or* $m = 1,\ t_m = s - 1,\ t = s.$

(d) $\Gamma = \{n-2, n-1\},\ r_{n-2} < s = r_{n-1} - 1,\ t_{n-1} = r_{n-1} - 2,\ t_{n-2} = r_{n-2} - 1,\ r_{n-1} \leqslant t < r_n - 2.$

(e) $\Gamma = \{n-1\},\ r_{n-2} < s = r_{n-1} - 1,\ t_{n-1} = r_{n-1} - 2,\ r_{n-1} \leqslant t < r_n - 2.$

Bringing together all the preceding results, we obtain the complete description of those finite relatively categorical characteristic p-group pairs which are not group pair direct sums of cyclics.

Theorem 16.14 *Let A be a* 2*-group pair which is a group direct sum* $A_1 \oplus \ldots \oplus A_n$ *where each A_i is a nonzero direct sum of cyclic groups of order* 2^{r_i}*, and* $r_1 < \ldots < r_n$*. Further, let* $s \leqslant r_{n-1}$*,* $\Gamma \subseteq \{1, \ldots, n-1\}$*,* t*,* t_i *satisfy the requirements of one of the items of Lemma* 16.13*. Assume that for each* $i \in \Gamma \cup \{n\}$ *the group* A_i *is cyclic; let* a_i *be a generator of this group. If* $B = A^P$ *is the direct sum of the group* $2^s(A_1 \oplus \ldots \oplus A_{n-1})$ *and the cyclic group generated by the element*

$b = 2^t a_n + \sum_{i\in\Gamma} 2^{t_i} a_i$, *then A is a relatively categorical characteristic group pair. Conversely, any finite relatively categorical characteristic p-group pair which is not a group pair direct sum of cyclics can be obtained in this way.*

Proof. The converse statement is in fact already proved: by Theorem 16.5 if a finite relatively categorical characteristic p-group pair is not a group pair direct sum of cyclics, then $p = 2$, $t_n \neq s_n$, and Lemmas 16.7 – 16.13 show that this group pair has the structure described in the first part of Theorem. Therefore, it remains to check that in all cases B is a characteristic subgroup of A and that every automorphism of B can be extended to an automorphism of A. We shall consider only the most complicated cases (a), (d), because the same argument (and even a part of it) works in the three other cases.

Case (a). B is the direct sum of the group $2^t A'$ and the cyclic group generated by the element $b = 2^{r_{m-1}-1}a_{m-1} + 2^{t-1}a_m + 2^t a_n$, where $1 < m < n$, $r_{m-1} < t < r_m$. Note that $r_n - r_m \geqslant 2$ because $r_m - (t-1) = r_m - t_m \leqslant r_n - s_n = r_n - (t+1)$.

First we show that B is a characteristic subgroup of A. Since the group $C = 2^t A' + 2^{t+1}A_n$ is contained in B and is characteristic in A, it is sufficient to check that every elementary automorphism α of the group A takes the element b into the coset $b + C$.

We can assume that the chosen basis of A contains the elements a_{m-1}, a_m, a_n. If an automorphism of A does not move a_{m-1}, a_m, a_n, then it does not move b. Any elementary automorphism of A which moves one of the elements a_{m-1}, a_m, a_n is a specialisation of one of the automorphisms α, β, γ, such that

$$\begin{aligned}
&\alpha(a_{m-1}) = a_{m-1} + c_i, \quad \alpha(a_m) = a_m + c_j, \quad \alpha(a_n) = a_n + c_k;\\
&\beta(a_{m-1}) = a_{m-1} + 2^{h_1}c_u, \quad \beta(a_m) = a_m + 2^{h_2}c_v, \quad \beta(a_n) = a_n;\\
&\gamma(a_{m-1}) = (2x+1)a_{m-1}, \quad \gamma(a_m) = (2y+1)a_m, \quad \gamma(a_n) = (2z+1)a_n,
\end{aligned}$$

where $i < m-1,\ j < m,\ k < n,\ u > m-1,\ v > m,\ c_i \in A_i,\ c_j \in A_j,\ c_k \in A_k,\ c_u \in A_u,\ c_v \in A_v,\ h_1 = r_u - r_{m-1},\ h_2 = r_v - r_m,\ x, y, z \in \mathbb{Z}$. We have:

$$\begin{aligned}
&\alpha(b) - b = 2^{r_{m-1}-1}c_i + 2^{t-1}c_j + 2^t c_k = 2^t c_k \in 2^t A' \subseteq C,\\
&\beta(b) - b = 2^{h_1 + r_{m-1}-1}c_u + 2^{h_2+t-1}c_v = 2^{r_u-1}c_u + 2^{r_v - r_m + t-1}c_v \in C,\\
&\gamma(b) - b = x\cdot 2^{r_{m-1}}a_{m-1} + y\cdot 2^t a_m + z\cdot 2^{t+1}a_n \in C,
\end{aligned}$$

because $r_{m-1} - 1 \geqslant r_i$, $t-1 \geqslant r_{m-1} \geqslant r_j$, $r_u - 1 \geqslant r_m - 1 \geqslant t$, $r_v - r_m + t - 1 \geqslant 1 + t - 1 = t$ for $v < n$ and $r_n - r_m + t - 1 \geqslant 2 + t - 1 = t + 1$. Thus B is a characteristic subgroup of A.

Now we prove that A is a relatively categorical group pair. The elements b, $2^t a_m$, $2^t a$, where a runs through all elements of the chosen basis in $A_{m+1} \oplus \ldots \oplus A_{n-1}$, constitute a basis of B, which we shall call the chosen basis of B. We must prove that each elementary automorphism of B can be extended to A. Fix an element $a \in A_q$ of the chosen basis of A, where $m < q < n$, and consider the automorphisms φ, χ, ψ of B, which do not move any elements of the chosen basis of B except $b,\ 2^t a_m,\ 2^t a$, and which act on $b,\ 2^t a_m,\ 2^t a$ in the following way:

$$\begin{aligned}
&\varphi(b) = b + 2^t c_i, \quad \varphi(2^t a_m) = 2^t a_m, \quad \varphi(2^t a) = 2^t a + 2^t c_j;\\
&\psi(b) = b, \quad \psi(2^t a_m) = 2^t a_m + 2^{h_1} 2^t c_u + x \cdot 2^{h_2} b,\\
&\qquad\qquad\qquad\qquad \psi(2^t a) = 2^t a + 2^{h_3} 2^t c_v + y \cdot 2^{h_4} b;\\
&\chi(b) = (2x+1)b, \quad \chi(2^t a_m) = (2y+1)2^t a_m, \quad \chi(2^t a) = (2z+1)2^t a,
\end{aligned}$$

where $m \leqslant i < n,\ m \leqslant j \leqslant q,\ m < u < n,\ q < v < n,\ c_i \in A_i,$ $c_j \in A_j,\ c_u \in A_u,\ c_v \in A_v,\ h_1 = r_u - r_m,\ h_2 = r_n - r_m,\ h_3 = r_v - r_q,$ $h_4 = r_n - r_q,\ x, y, z \in \mathbb{Z}$. Besides, we require that if $j = q$, then c_j is not contained in the cyclic group generated by a (otherwise φ is not necessarily an automorphism). Each elementary automorphism of B can be obtained as a specialisation of one of these automorphisms for an appropriate choice of parameters $q,\ a,\ c_i$ etc. Therefore it is sufficient to observe that the automorphisms of A which fix all elements of the chosen basis of A except $a_m,\ a,\ a_n$ and which act on $a_m,\ a,\ a_n$ by the following rules:

$$\begin{aligned}
&a_m \to a_m, \quad a \to a + c_j, \quad a_n \to a_n + c_i;\\
&a_m \to (1 + 2^{h_2-1}x)a_m + 2^{h_1} c_u + 2^{h_2} x a_n,\\
&\qquad\qquad a \to a + 2^{h_3} c_v + y(2^{h_4-1} a_m + 2^{h_4} a_n);\\
&\qquad\qquad\qquad\qquad a_n \to (1 - 2^{h_2-1}x)a_n - 2^{h_1-1} c_u - 2^{h_2-2} x a_m,\\
&a_m \to (2y+1)a_m, \quad a \to (2z+1)a, \quad a_n \to (2x+1)a_n + (x-y)a_m,
\end{aligned}$$

extend respectively $\varphi,\ \psi,\ \chi$ (note that obviously $h_1, h_4 \geqslant 1,\ h_2 \geqslant 2$).

Case (d). B is the group generated by the elements $b_1 = 2^s a_{n-1}$ and $b = 2^{t_{n-2}} a_{n-2} + 2^{t_{n-1}} a_{n-1} + 2^t a_n$, where $t_{n-2} = r_{n-2} - 1 < r_{n-1} - 2 = t_{n-1} = s - 1$, $r_{n-1} \leqslant t < r_n - 2$. We check that B is a characteristic subgroup of A, i.e., that each elementary automorphism of A takes b and b_1 into B. Note first of all that $2^{t+1} a_n = 2b - b_1 \in B$.

Assume that the chosen basis of A contains the elements a_{n-2}, a_{n-1}, a_n. If an automorphism of A does not move a_{n-2}, a_{n-1}, a_n, then it does not move any element of B. Any elementary automorphism of A which moves one of the elements a_{n-2}, a_{n-1}, a_n is a specialisation of one of the automorphisms α, β, γ, such that

$$\begin{aligned}
&\alpha(a_{n-2}) = a_{n-2} + c_i, \quad \alpha(a_{n-1}) = a_{n-1} + c_j, \quad \alpha(a_n) = a_n + c_k;\\
&\beta(a_{n-2}) = a_{n-2} + 2^{h_1} x a_{n-1} + 2^{h_2} y a_n,\\
&\qquad\qquad\qquad\qquad \beta(a_{n-1}) = a_{n-1} + 2^{h_3} z a_n, \quad \beta(a_n) = a_n;\\
&\gamma(a_{n-2}) = (2x+1)a_{n-2}, \quad \gamma(a_{n-1}) = (2y+1)a_{n-1}, \quad \gamma(a_n) = (2z+1)a_n,
\end{aligned}$$

where $x, y, z \in \mathbb{Z},\ h_1 = r_{n-1} - r_{n-2},\ h_2 = r_n - r_{n-2},\ h_3 = r_n - r_{n-1},$ $c_i \in A_i,\ c_j \in A_j,\ c_k \in A_k,\ i < n-2,\ j < n-1,\ k < n$. We have:

$$\alpha(b) = b + 2^{t_{n-2}} c_i + 2^{t_{n-1}} c_j + 2^t c_k = b, \quad \alpha(b_1) = b_1 + 2^s c_j = b_1,$$

because $t_{n-2} \geqslant r_{n-3} \geqslant r_i,\ t_{n-1} \geqslant r_{n-2} \geqslant r_j,\ t \geqslant r_{n-1} \geqslant r_k$;

$$\begin{aligned}
\beta(b) &= b + 2^{t_{n-2}+h_1} x a_{n-1} + (2^{t_{n-2}+h_2} y + 2^{t_{n-1}+h_3} z) a_n =\\
&\qquad = b + x b_1 + (2^{r_n-2-t} y + 2^{r_n-3-t} z) \cdot 2^{t+1} a_n \in B,\\
\beta(b_1) &= b_1 + 2^{s+h_3} z a_n = b_1 + 2^{r_n-t-2} z \cdot 2^{t+1} a_n \in B,
\end{aligned}$$

because $r_n - t - 2 > 0$; finally,

$$\begin{aligned}
&\gamma(b) = b + 2^{t_{n-2}+1} x a_{n-2} + 2^s y a_{n-1} + 2^{t+1} z a_n = b + (y-z) b_1 + 2zb,\\
&\gamma(b_1) = b_1 + 2y \cdot 2^s a_{n-1} = b_1,
\end{aligned}$$

because $t_{n-2} + 1 = r_{n-2},\ s + 1 = r_{n-1}$. Thus B is a characteristic subgroup of A.

To show that A is a relatively categorical group pair, we must show that the elementary automorphisms

$$b \to b + b_1,\ b_1 \to b_1; \quad b \to (2z+1)b,\ b_1 \to b_1; \quad b \to b,\ b_1 \to b_1 + 2^{r_n-t-1} b$$

of the group B can be extended to A. But we have just seen that the first two of them are restrictions of γ respectively for $y = 1,\ z = 0$ and for $y = z$. The endomorphism of A which takes a_{n-1} to $a_{n-1} + 2^{r_n - r_{n-1}} a_n$, a_n to $(1 - 2^{r_n-t-2}) a_n$ and fixes all other elements of the chosen basis of A extends the third elementary automorphism of B; since $r_n - t - 2 > 0$, the integer $(1 - 2^{r_n-t-2})$ is odd, which means that the above endomorphism of A is an automorphism.

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