A model theoretic Baire category theorem for simple theories and its applications

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Abstract

We prove a model theoretic Baire category theorem for $\tilde{\tau}^f_{low}$-sets in a countable simple theory in which the extension property is first-order and show some of its applications. We also prove a trichotomy for minimal types in countable nfcp theories: either every type that is internal in a minimal type is essentially-1-based by means of the forking topology or T interprets a strongly-minimal formula or T interprets an infinite definable 1-based group of finite D-rank.

1 Introduction

The goal of this paper is to generalize a result from [S1] and to give some applications. In [S1] The first step for proving supersimplicity of countable unidimensional simple theories eliminating hyperimaginaries is to show the existence of an unbounded type-definable τ^f-open set (a set defined in terms of forking by formulas, see definition 2.1) of bounded finite SU_{se}-rank (for definition see section 4). In this paper we develop a general framework for this kind of result. It is the new idea of a model theoretic Baire category theorem, namely, one deals with certain "uniformly-definable" family of generalized closed sets (in complicated "logic"), roughly speaking, given a partition of a complicated open set into countably many sets, each of which is the intersection of a "uniformly definable" family of generalized closed sets, one can find a **nice** (this is the main advantage) open set that is contained in some generalized closed set in one of these families. In particular, it is not just the usual Baire category theorem for a complicated topological space. The proof is quite similar to the proof in [S1] and has some important consequences, e.g. in a countable wnfcp theory if for every non-algebraic element a (even in some fixed non-empty $\tilde{\tau}^f_{low}$-set) there is $a' \in acl(a)\backslash acl(\emptyset)$ of finite SU-rank, then there exists a weakly-minimal formula. We also prove a trichotomy for countable nfcp theories as indicated in the abstract.

2 Preliminaries

The forking topology is introduced in [S0] and is a variant of Hrushovski's and Pillay's topologies from [H0] and [P] respectively. In this section T is assumed to be simple and we work in $\mathcal{C}$.

Definition 2.1 Let $A \subseteq \mathcal{C}$ and let x be a finite tuple of variables.
1) An invariant set $\mathcal{U}$ over A is said to be *a basic τ^f-open set over A* if there is $\phi(x,y) \in L(A)$ such that

$$\mathcal{U} = \{a|\phi(a,y) \text{ forks over } A\}.$$

2) An invariant set $\mathcal{U}$ over A is said to be *a basic τ^f_∞-open set over A* if $\mathcal{U}$ is a type-definable τ^f-open set over A.

Note that the family of basic τ^f-open sets over A is closed under finite intersections, thus form a basis for a unique topology on $S_x(A)$. Likewise, we define the τ^f_∞-topologies.

Recall, the following definition from [S0] whose roots are in [H0].

Definition 2.2 We say that *the τ^f-topologies over A are closed under projections (T is PCFT over A)* if for every τ^f-open set $\mathcal{U}(x,y)$ over A the set $\exists y \mathcal{U}(x,y)$ is a τ^f-open set over A. We say that *the τ^f-topologies are closed under projections (T is PCFT)* if they are over every set A.

Recall that a formula $\phi(x,y) \in L$ is *low in x* if there exists $k < \omega$ such that for every $\emptyset$-indiscernible sequence $(b_i|i<\omega)$, the set $\{\phi(x,b_i)|i<\omega\}$ is inconsistent iff every subset of it of size k is inconsistent. T is low if every $\phi(x,y)$ is low in x.

Remark 2.3 *Assume $\phi(x,t) \in L$ is low in t and $\psi(y,v) \in L$ is low in v ($x \cap y$ or $t \cap v$ may not be $\emptyset$). Then $\theta(xy,tv) \equiv \phi(x,t) \vee \psi(y,v)$ is low in tv.*

Proof: Let $k_1 < \omega$ be a witness that $\phi(x,t)$ is low in t and let $k_2 < \omega$ be a witness that $\psi(y,v)$ is low in v. Let $k = k_1 + k_2 - 1$. By adding dummy variables we may assume $x = y$ and $t = v$ (as tuples of variables). Let $(a_i |\ i < \omega)$ be indiscernible such that $\{\phi(a_i,t) \vee \psi(a_i,t) | i < \omega\}$ is inconsistent. Thus, every subset of $\{\phi(a_i,t) | i < \omega\}$ of size k_1 is inconsistent, and every subset of $\{\psi(a_i,t) | i < \omega\}$ of size k_2 is inconsistent. Thus every subset of size k of $\{\phi(a_i,t) \vee \psi(a_i,t) | i < \omega\}$ is inconsistent.

In [BPV, Proposition 4.5] the authors proved the following equivalence which, for convenience, we will use as a definition (their definition involves extension with respect to pairs of models of T).

Definition 2.4 *The extension property is first-order in T iff for every formulas $\phi(x,y), \psi(y,z) \in L$ the relation $Q_{\phi,\psi}$ defined by:*

$$Q_{\phi,\psi}(a) \text{ iff } \phi(x,b) \text{ doesn't fork over } a \text{ for every } b \models \psi(y,a)$$

is type-definable (here a can be an infinite tuple from $\mathcal{C}$ whose sorts are fixed). We say that T has wnfcp if T is low and the extension property is first-order in T.

Fact 2.5 [$S1$] *Suppose the extension property is first-order in T. Then T is PCFT.*

We say that an A-invariant set $\mathcal{U}$ has finite SU-rank if $SU(a/A) < \omega$ for all $a \in \mathcal{U}$, and has bounded finite SU-rank if there exists $n < \omega$ such that $SU(a/A) \leq n$ for all $a \in \mathcal{U}$. The existence of a τ^f-open set of bounded finite SU-rank implies the existence of a weakly-minimal formula:

Fact 2.6 *[S0, Proposition 2.13] Let $\mathcal{U}$ be an unbounded τ^f-open set over some set A. Assume $\mathcal{U}$ has bounded finite SU-rank. Then there exists a set $B \supseteq A$ and $\theta(x) \in L(B)$ of SU-rank 1 such that $\theta^{\mathcal{C}} \subseteq \mathcal{U} \cup acl(B)$.*

In [S1] the class of $\tilde{\tau}^f$-sets is introduced, this class is much wider than the class of basic τ^f-open sets. Here, we look at the class of $\tilde{\tau}^f_{low}$-sets, instead of the class of $\tilde{\tau}^f_{st}$-sets from [S1].

Definition 2.7 A relation $V(x, z_1, ...z_l)$ is said to be a *pre-$\tilde{\tau}^f$-set relation* if there are $\theta(\tilde{x}, x, z_1, z_2, ..., z_l) \in L$ and $\phi_i(\tilde{x}, y_i) \in L$ for $0 \leq i \leq l$ such that for all $a, d_1, ..., d_l \in \mathcal{C}$ we have

$$V(a, d_1, ..., d_l) \text{ iff } \exists\tilde{a}\ [\theta(\tilde{a}, a, d_1, d_2, ..., d_l) \wedge \bigwedge_{i=0}^{l}(\phi_i(\tilde{a}, y_i) \text{ forks over } d_1d_2...d_i)]$$

(for $i = 0$ the sequence $d_1d_2...d_i$ is interpreted as $\emptyset$). If each $\phi_i(\tilde{x}, y_i)$ is assumed to be low in y_i , $V(x, z_1, ...z_l)$ is said to be a *pre-$\tilde{\tau}^f_{low}$-set relation.*

Definition 2.8 1) A *$\tilde{\tau}^f$-set (over $\emptyset$)* is a set of the form

$$\mathcal{U} = \{a|\ \exists d_1, d_2, ...d_l\ V(a, d_1, ..., d_l)\}$$

for some *pre-$\tilde{\tau}^f$-set relation* $V(x, z_1, ...z_l)$.
2) A *$\tilde{\tau}^f_{low}$-set (over $\emptyset$)* is a set of the form

$$\mathcal{U} = \{a|\ \exists d_1, d_2, ...d_l\ V(a, d_1, ..., d_l)\}$$

for some *pre-$\tilde{\tau}^f_{low}$-set relation* $V(x, z_1, ...z_l)$.

3 The Theorem

In this section T is assumed to be a simple theory and we work in $\mathcal{C}$ (so, T not necessarily eliminates imaginaries).

Definition 3.1 Let $\Theta = \{\theta_i(x_i, x)\}_{i\in I}$ be a set of L-formulas such that $\exists^{<\infty}x_i\theta_i(x_i, x)$ for all $i \in I$. Let s be the sort of x. For $A \subseteq \mathcal{C}^s$, let $acl_\Theta(A) = \{b|\ \theta_i(b, a)$ for some $\theta_i \in \Theta$ and $a \in A\}$.

Definition 3.2 An invariant set $\mathcal{U}(x, y_1, ...y_r)$ is said to be *a generalized uniform family of $\tilde{\tau}^f_{low}$-sets* if there is a formula $\rho(\tilde{x}, x, y_1, ..., y_r, z_1, z_2, ..., z_k) \in L$ and there are formulas $\psi_i(\tilde{x}, v_i), \mu_j(\tilde{x}, w_j) \in L$ for $0 \leq i \leq r$ and $1 \leq j \leq k$ that are low in v_i and low in w_j respectively, such that for all $a, d_1, ...d_r$ we have $\mathcal{U}(a, d_1, ...d_r)$ iff $\exists\tilde{a}\exists e_1...e_k$

$$\rho(\tilde{a}, a, d_1, ...d_r, e_1, ...e_k) \wedge [\bigwedge_{i=0}^{r} \psi_i(\tilde{a}, v_i) \text{ forks over } d_1...d_i] \wedge [\bigwedge_{j=1}^{k} \mu_j(\tilde{a}, w_j) \text{ forks over } d_1...d_re_1...e_j].$$

Definition 3.3 An invariant set $\mathcal{F}(x, y_1, ...y_r)$ is said to be *a generalized uniform family of $\tilde{\tau}^f_{low}$-closed sets* if $\mathcal{F}(x, y_1, ...y_r) = \bigcap_i \neg\mathcal{U}_i(x, y_1, ...y_r)$, where each $\mathcal{U}_i(x, y_1, ...y_r)$ is a generalized uniform family of $\tilde{\tau}^f_{low}$-sets.

Fact 3.4 *Assume the extension property is first-order in T. Then*
1) Let $\mathcal{U}$ be an unbounded $\tilde{\tau}^f$-set over $\emptyset$. Then there exists an unbounded τ^f-open set $\mathcal{U}^*$ over some finite set A^* such that $\mathcal{U}^* \subseteq \mathcal{U}$. In fact, if $V(x, z_1, ..., z_l)$ is a pre-$\tilde{\tau}^f$-set relation such that $\mathcal{U} = \{a | \exists d_1...d_l V(a, d_1, ..., d_l)\}$, and $(d^*_1, ..., d^*_m)$ is any maximal sequence (with respect to extension) such that $\exists d_{m+1}...d_l V(\mathcal{C}, d^*_1, ..., d^*_m, d_{m+1}, ..., d_l)$ is unbounded, then

$$\mathcal{U}^* = \exists d_{m+1}...d_l V(\mathcal{C}, d^*_1, ..., d^*_m, d_{m+1}, ..., d_l)$$

is a τ^f-open set over $d^*_1...d^*_m$.

Theorem 3.5 *Let T be a countable simple theory in which the extension property is first-order. Assume:*
1) $\Theta = \{\theta_i(x'_i, x)\}_{i<\omega}$ is a set of L-formulas such that $\exists^{<\infty} x'_i \theta_i(x'_i, x)$ for all $i < \omega$.
2) $\mathcal{U}_0(x)$ is a non-empty $\tilde{\tau}^f_{low}$-set over $\emptyset$.
3) $\{F_n(x_n)\}_{n<\omega}$ is a family of $\emptyset$-invariant sets such that $F_n(\mathcal{C}) \cap acl(\emptyset) = \emptyset$ for all $n < \omega$.
4) For every $n < \omega$ and every variables $\bar{y} = y_1, ...y_r$, let $\mathcal{F}^{\bar{y}}_n(x_n, \bar{y})$ be a generalized uniform family of $\tilde{\tau}^f_{low}$-closed sets such that $F_n(\mathcal{C}) \subseteq \mathcal{F}^{\bar{y}}_n(\mathcal{C}, \bar{d})$ for all $\bar{d}$.
Now, assume for all $a \in \mathcal{U}_0$ there exists $b \in acl_\Theta(a)$ and $n < \omega$ such that $b \in F_n(\mathcal{C})$. Then there is an unbounded τ^f_∞-open set $\mathcal{U}^*$ over a finite tuple $\bar{d}^*$ and variables $\bar{y}^*$ of the sort of $\bar{d}^*$, and $n^* < \omega$ such that

$$\mathcal{U}^* \subseteq \mathcal{F}^{\bar{y}^*}_{n^*}(\mathcal{C}, \bar{d}^*) \cap acl_\Theta(\mathcal{U}_0).$$

Proof: First, we may assume Θ is closed downwards (i.e. if $\theta \in \Theta$ and $\theta' \vdash \theta$ then $\theta' \in \Theta$). Assume the conclusion of the theorem is false. It will be sufficient to show that for every non-empty $\tilde{\tau}^f_{low}$-set $\mathcal{U} \subseteq \mathcal{U}_0$, every $\theta \in \Theta$, and every $n < \omega$ there exists a non-empty $\tilde{\tau}^f_{low}$-set $\mathcal{U}^* \subseteq \mathcal{U}$ such that either $\neg\exists x'\theta(x', a)$ for all $a \in \mathcal{U}^*$ or for all $a \in \mathcal{U}^*$ there exists $b \models \theta(x', a)$ with $b \notin F_n(\mathcal{C})$. Indeed, by iterating this for every pair $(\theta, n) \in \Theta \times \omega$ we get by compactness a^* such that for all $\theta \in \Theta$ and all $n < \omega$ either $\neg\exists x'\theta(x', a^*)$

or there exists $b_{n,\theta} \models \theta(x', a^*)$ such that $b_{n,\theta} \notin F_n(\mathcal{C})$. Since we assume Θ is closed downwards, we get contradiction to the assumption that for all $a \in \mathcal{U}_0$ there exists $b \in acl_\Theta(a)$ and $n < \omega$ such that $b \in F_n(\mathcal{C})$ (note that $F_n(\mathcal{C})$ is $\emptyset$-invariant). To show this, let $\mathcal{U}$, θ and $n < \omega$ be given. Let $V(x, z_1, ...z_l)$ be a pre-$\tilde{\tau}^f_{low}$-set relation such that

$$\mathcal{U} = \{a | \ \exists d_1, d_2, ...d_l \ V(a, d_1, ..., d_l)\}.$$

where V is defined by:

$$V(a, d_1, ..., d_l) \text{ iff } \exists \tilde{a} \ [\sigma(\tilde{a}, a, d_1, d_2, ..., d_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}, t_i) \text{ forks over } d_1 d_2 ... d_i)]$$

for some $\sigma(\tilde{x}, x, z_1, z_2, ..., z_l) \in L$ and $\phi_i(\tilde{x}, t_i) \in L$ which are low in t_i for $0 \leq i \leq l$. Let V_θ be defined by: for all $b, d_1, ..., d_l \in \mathcal{C}$,

$$V_\theta(b, d_1, ..., d_l) \text{ iff } \exists a(\theta(b, a) \wedge V(a, d_1, ..., d_l)).$$

and let

$$\mathcal{U}_\theta = \{b | \ \exists d_1, d_2, ...d_l \ V_\theta(b, d_1, ..., d_l)\}.$$

Since by the assumption $F_n(\mathcal{C}) \cap acl(\emptyset) = \emptyset$, we may assume $\mathcal{U}_\theta \cap acl(\emptyset) = \emptyset$ and $\mathcal{U}_\theta$ is non-empty. Now, let $\bar{d}^* = (d^*_1, ..., d^*_m)$ be a maximal sequence, with respect to extension ($0 \leq m \leq l$) such that

$$\tilde{V}_\theta(x') \equiv \exists d_{m+1}, d_{m+2}, ...d_l V_\theta(x', d^*_1, ...d^*_m, d_{m+1}, ...d_l)$$

is non-algebraic. We may assume $m < l$ (by choosing V appropriately). By Fact 3.4, $\tilde{V}_\theta(\mathcal{C})$ is an unbounded basic τ^f_∞-open set over $\bar{d}^*$. Since we assume the conclusion of the theorem is false, $\tilde{V}_\theta(\mathcal{C}) \not\subseteq \mathcal{F}^{\bar{y}^*}_n(\mathcal{C}, \bar{d}^*)$ where $\bar{y}^* = y^*_1, ..., y^*_m$ has the same sort as $\bar{d}^*$. Now, let $\mathcal{U}_{s,n}(x_n, \bar{y}^*)$ for $s < \omega$ be each a generalized uniform family of $\tilde{\tau}^f_{low}$-sets such that $\mathcal{F}_n(x_n, \bar{y}^*) = \bigcap_s \neg \mathcal{U}_{s,n}(x_n, \bar{y}^*)$. Let $b^* \in \tilde{V}_\theta(\mathcal{C}) \backslash \mathcal{F}^{\bar{y}^*}_n(\mathcal{C}, \bar{d}^*)$. So, there exists $s^* < \omega$ such that $b^* \in \mathcal{U}_{s^*,n}(\mathcal{C}, \bar{d}^*)$. Let $\rho(\tilde{x}', x_n, y^*_1, ..., y^*_m, z'_1, z'_2, ..., z'_k) \in L$ and let $\psi_i(\tilde{x}', v_i), \mu_j(\tilde{x}', w_j) \in L$ for $0 \leq i \leq m$ and $1 \leq j \leq k$ be low in v_i and low in w_j respectively, such that for all $b, d_1, ...d_m$ we have $\mathcal{U}_{s^*,n}(b, d_1, ...d_m)$ iff $\exists \tilde{b} \exists e_1 ... e_k$

$$\rho(\tilde{b}, b, d_1, ...d_m, e_1, ...e_k) \wedge [\bigwedge_{i=0}^{m} \psi_i(\tilde{b}, v_i) \text{ forks over } d_1 ... d_i] \wedge [\bigwedge_{j=1}^{k} \mu_j(\tilde{b}, w_j) \text{ forks over } d_1 ... d_m e_1 ... e_j].$$

Now, let $d^*_{m+1}, ...d^*_l$ and $a^*, \tilde{a}^*$ and $E^* = (e^*_1, ..., e^*_k)$ and $\tilde{b}^*$ be such that

$$\theta(b^*, a^*) \wedge \sigma(\tilde{a}^*, a^*, d^*_1, d^*_2, ..., d^*_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}^*, y_i) \text{ forks over } d^*_1 d^*_2 ... d^*_i) \quad (*1)$$

and

$$\rho(\tilde{b}^*, b^*, d^*_1, ..d^*_m, e^*_1, ..e^*_k) \ (*2)$$

and

$$[\bigwedge_{i=0}^{m} \psi_i(\tilde{b}^*, v_i) \text{ forks over } d^*_1 ... d^*_i] \wedge [\bigwedge_{j=1}^{k} \mu_j(\tilde{b}^*, w_j) \text{ forks over } d^*_1 ... d^*_m e^*_1 ... e^*_j] \ (*3).$$

By maximality of $\bar{d}^*$, we know $b^* \in acl(\bar{d}^* d^*_{m+1})$. Thus, by taking a non-forking extension of $tp(\tilde{b}^* E^* / acl(\bar{d}^* d^*_{m+1}))$ over $acl(d^*_1 ... d^*_l a^* \tilde{a}^*)$ we may assume E^* is independent from $d^*_1 ... d^*_l a^* \tilde{a}^*$ over $\bar{d}^* d^*_{m+1}$ and $(*1)$, $(*2)$ and $(*3)$ still hold. We conclude that

$$\bigwedge_{i=m+1}^{l} (\phi_i(\tilde{a}^*, t_i) \text{ forks over } d^*_1 d^*_2 ... d^*_i E^*).$$

Now, we define the $\tilde{\tau}^f_{low}$-set $\mathcal{U}^*$. First, define a relation V^* by:

$$V^*(a, d_1, ...d_m, e_1, ...e_k, d_{m+1}, ..d_l) \text{ iff } \exists \tilde{a}, b, \tilde{b} (\theta^* \wedge V^*_0 \wedge V^*_1 \wedge V^*_2),$$

where θ^* is defined by: $\theta^*(\tilde{a}, b, \tilde{b}, a, d_1, ..d_m, e_1, ...e_k, d_{m+1}, ..d_l)$ iff

$$\theta(b, a) \wedge \sigma(\tilde{a}, a, d_1, d_2, ..., d_l) \wedge \rho(\tilde{b}, b, d_1, ...d_m, e_1, ..., e_k),$$

V^*_0 is defined by: $V^*_0(\tilde{a}, \tilde{b}, d_1, ...d_m)$ iff

$$\bigwedge_{i=0}^{m} (\phi_i(\tilde{a}, t_i) \vee \psi_i(\tilde{b}, v_i) \text{ forks over } d_1 d_2 ... d_i),$$

V^*_1 is defined by $V_1(\tilde{b}, d_1, ..d_m, e_1, ...e_k)$ iff

$$\bigwedge_{j=1}^{k} (\mu_j(\tilde{b}, w_j) \text{ forks over } d_1 ... d_m e_1 ... e_j), \text{ and}$$

V_2^* is defined by $V_2(\tilde{a}, d_1, ...d_m, e_1, ...e_k, d_{m+1}, ..d_l)$ iff

$$\bigwedge_{i=m+1}^{l} (\phi_i(\tilde{a}, t_i) \text{ forks over } d_1 d_2 ... d_i e_1 ... e_k).$$

Note that V^* is a pre-$\tilde{\tau}^f_{low}$-set. Let

$$\mathcal{U}^* = \{a | \exists d_1, ..d_m, e_1, ...e_k, d_{m+1}, ..d_l \ V^*(a, d_1, ..d_m, e_1, ...e_k, d_{m+1}, ...d_l)\}.$$

By the definition of $\mathcal{U}^*$, $\mathcal{U}^* \subseteq \mathcal{U}$. $\mathcal{U}^*$ is a $\tilde{\tau}^f_{low}$-set using Remark 2.3. By the construction, $\mathcal{U}^* \neq \emptyset$. Now, let $a \in \mathcal{U}^*$. By the definition of $\mathcal{U}^*$, there are $\tilde{b}, b, d_1, ...d_m, e_1, ...e_k$ such that $\theta(b, a)$, $\rho(\tilde{b}, b, d_1, ...d_m, e_1, ..., e_k)$,

$\bigwedge_{i=0}^{m} (\psi_i(\tilde{b}, v_i)$ forks over $d_1, ...d_i)$ and $\bigwedge_{j=1}^{k} (\mu_j(\tilde{b}, w_j)$ forks over $d_1, ...d_m e_1 ... e_j)$.

Thus $\mathcal{U}_{s^*,n}(b, d_1...d_m)$ and therefore $\neg \mathcal{F}_n^{\bar{y}^*}(b, d_1...d_m)$. Hence $b \notin F_n$ as required.

4 Applications

In this section we show some applications of Theorem 3.5. In fact, we will show several instances of this theorem that apparently new even for stable theories. In this section T is assumed to be a simple theory and we work in $\mathcal{C}$.

We start by pointing out that theorem 3.5 generalizes [S1, Theorem 9.4] that is one of the essential steps towards the proof of supersimplicity of countable simple unidimensional theories with elimination of hyperimaginaries. First recall the following definitions from [S1].

Definition 4.1 *For $a \in \mathcal{C}$, $A \subseteq B \subseteq \mathcal{C}$, $a \not\downarrow^s_A B$ if for some stable $\phi(x, y) \in L$, there is b over B and $a' \models \phi(x, b)$ for some $a' \in dcl(Aa)$ such that $\phi(x, b)$ forks over A.*

Definition 4.2 *The SU_{se}-rank of $tp(a/A)$ is defined by induction on α: if $\alpha = \beta + 1$, $SU_{se}(a/A) \geq \alpha$ if there exist $B_1 \supseteq B_0 \supseteq A$ such that $a \not\downarrow^s_{B_0} B_1$ and $SU_{se}(a/B_1) \geq \beta$. For limit α, $SU_{se}(a/A) \geq \alpha$ if $SU_{se}(a/A) \geq \beta$ for all $\beta < \alpha$.*

Remark 4.3 First, recall that in a simple theory in which $Lstp = stp$ over sets $\downarrow^{s}$ is symmetric [Lemma 6.7, S1]. Thus for any finite tuples of sorts s_0 and s_1 and $n < \omega$ the set $\mathcal{F}_n^{s_0,s_1}$ defined by

$$\mathcal{F}_n^{s_0,s_1} = \{(a, A) \in \mathcal{C}^{s_0} \times \mathcal{C}^{s_1} |\ SU_{se}(a/A) < n\}.$$

is a generalized uniform family of $\tilde{\tau}^f_{low}$-closed sets.

For an A-invariant set V, let $acl_1(V) = \{a'|\ a' \in acl(a)$ for some $a \in V\}$. The following corollary generalizes [S1, Theorem 9.4].

Corollary 4.4 *Let T be a countable simple theory in which the extension property is first-order and assume $Lstp = stp$ over sets. Let $\mathcal{U}_0$ be a non-empty $\tilde{\tau}^f_{low}$-set. Assume for every $a \in \mathcal{U}_0$ there exists $a' \in acl(a) \backslash acl(\emptyset)$ such that $SU_{se}(a') < \omega$. Then there exists an unbounded τ^f_∞-open set $\mathcal{U} \subseteq acl_1(\mathcal{U}_0)$ over a finite set such that $\mathcal{U}$ has bounded finite SU_{se}-rank.*

Proof: Let x be the variable of $\mathcal{U}_0$, so $\mathcal{U}_0 = \mathcal{U}_0(x)$. Let

$$\Theta = \{\theta(x', x)|\ \exists^{<\infty} x' \theta(x', x), x' \text{ any variable}\}.$$

Let $\mathcal{S}$ be the set of sorts. Let $I : \omega \to \mathcal{S} \times \omega$ be a bijection, I_1, I_2 the projections of I to the first and second coordinate respectively. Now, for each $n < \omega$ let $F_n = \{a \in \mathcal{C}^{I_1(n)} \backslash acl(\emptyset)\ |SU_{se}(a) < I_2(n)\}$. Now, for every finite tuple of variables Y and $n < \omega$ let $s(Y)$ be the finite sequence of sorts of Y and let

$$\mathcal{F}_n^Y = \{(a, A) \in \mathcal{C}^{I_1(n)} \times \mathcal{C}^{s(Y)} |\ SU_{se}(a/A) < I_2(n)\}.$$

Now, by the definition of the SU_{se}-rank, $\mathcal{F}_n(\mathcal{C}) \subseteq \mathcal{F}_n^Y(\mathcal{C}, A)$ for every $n < \omega$ and every Y, A. By Remark 4.3, $\mathcal{F}_n^Y$ is a generalized uniform family of $\tilde{\tau}^f_{low}$-closed sets for all Y, n. By our assumptions, we see that the assumptions of Theorem 3.5 hold for $\mathcal{U}_0(x)$, Θ ,$\{F_n\}_n$ and $\{\mathcal{F}_n^Y\}_{Y,n}$ and thus by its corollary we are done.

Corollary 4.5 *Let T be a countable theory with wnfcp. Let $\mathcal{U}_0$ be an unbounded $\tilde{\tau}^f$-set over $\emptyset$ of finite SU-rank. Then there exists a finite set A and a SU-rank 1 formula $\theta \in L(A)$ such that $\theta^{\mathcal{C}} \subseteq \mathcal{U}_0 \cup acl(A)$.*

Proof: First, by modifying $\mathcal{U}_0$, we may assume $\mathcal{U}_0 \cap acl(\emptyset) = \emptyset$. Let $\Theta = \{x' = x\}$, $\mathcal{U}_0(x) = \mathcal{U}_0$. Let $s(x)$ be the sort of x. Now, for each $n < \omega$ let

$$F_n = \{a \in \mathcal{C}^{s(x)} \backslash acl(\emptyset) \ | SU(a) < n\}.$$

For every finite tuple of variables Y and $n < \omega$ let $s(Y)$ be the finite sequence of sorts of Y and let

$$\mathcal{F}^Y_n = \{(a, A) \in \mathcal{C}^{s(x)} \times \mathcal{C}^{s(Y)} | \ SU(a/A) < n\}.$$

By symmetry of forking and the assumption that T is low, each $\mathcal{F}^Y_n$ is a generalized uniform family of $\tilde{\tau}^f_{low}$-closed sets. Clearly, $F_n(\mathcal{C}) \subseteq \mathcal{F}^Y_n(\mathcal{C}, A)$ for every $n < \omega$ and every Y, A. By our assumption, the assumptions of Theorem 3.5 are satisfied for $\mathcal{U}_0$, Θ, $\{F_n\}_n$ and $\{\mathcal{F}^Y_n\}_{Y,n}$ and thus by its corollary there exists an unbounded τ^f_∞-open set $\mathcal{U}^* \subseteq \mathcal{U}_0$ over a finite set A_0 and $\mathcal{U}^*$ has bounded finite SU-rank. By Fact 2.6, there exists a finite set $A \supseteq A_0$ and there exists a SU-rank 1 formula $\theta \in L(A)$ such that $\theta^{\mathcal{C}} \subseteq \mathcal{U}^* \cup acl(A)$.

Corollary 4.6 *Let T be a countable theory with wnfcp. Let $\mathcal{U}_0$ be a non-empty $\tilde{\tau}^f$-set over $\emptyset$. Assume for every $a \in \mathcal{U}_0$ there exists $a' \in acl(a) \backslash acl(\emptyset)$ such that $SU(a') < \omega$. Then there exists a finite set A and a SU-rank 1 formula $\theta \in L(A)$ such that $\theta^{\mathcal{C}} \subseteq acl_1(\mathcal{U}_0) \cup acl(A)$.*

Proof: Just like the proof of Corollary 4.5.

5 Dichotomies for countable theories with the wnfcp

In this section we show the dichotomy [S1, Theorem 5.5] implies a strong dichotomy between essential 1-basedness and supersimplicity in the case T is a countable wnfcp theory that eliminates hyperimaginaries. Before we state the above dichotomy for the special case of the τ^f-topologies (simplified version), let us recall the basic definitions. In this section T is assumed to be simple.

Definition 5.1 A type $p \in S(A)$ is said to be *essentially 1-based by means of the τ^f-topologies* if for every finite tuple $\bar{c}$ from p and for every type-definable τ^f-open set $\mathcal{U}$ over $A\bar{c}$, the set $\{a \in \mathcal{U} | \ Cb(a/A\bar{c}) \not\in bdd(aA)\}$ is nowhere dense in the Stone-topology of $\mathcal{U}$.

Fact 5.2 *Let T be a countable simple theory with PCFT that eliminates hyperimaginaries. Let p_0 be a partial type over $\emptyset$ of SU-rank 1. Then, either there exists an unbounded finite-SU-rank τ^f-open set over some countable set, or every type $p \in S(A)$, with A countable, that is internal in p_0 is essentially 1-based by means of the τ^f-topologies.*

Let V, W be invariant sets (over some small set). We say that V almost contain W if for some A over which W is invariant $V \supseteq W \backslash acl(A)$. We say that V is generated over W if there exists a small set B such that $V \subseteq dcl(W \cup B)$.

Theorem 5.3 *Let T be a countable theory with wnfcp that eliminates hyperimaginaries. Let p be a partial type over $\emptyset$ of SU-rank 1 . Then, either*
1) every type $q \in S(A)$, with A countable, that is internal in p is essentially 1-based by means of the τ^f-topologies, or
2) there exists $n < \omega$ and $g(x, y) \in L$ such that $g(x, a)$ is a definable function for all a, and there exists a small sequence $(a_i | i < \alpha)$ such that $\bigcup_{i<\alpha} g(p^n, a_i)$ almost contains a weakly-minimal definable set. In particular, there exists a weakly-minimal definable set that is generated over $p(\mathcal{C})$.

Proof: Assume 1) is false. By the proof of Fact 5.2, there exists an unbounded basic τ^f-open set $\mathcal{U}$ over some countable set A such that $tp(a/A)$ is almost p-internal for every $a \in \mathcal{U}$.

Subclaim 5.4 *There exists an unbounded basic τ^f-open set $\mathcal{U}^* \subseteq \mathcal{U}$ over A that is generated over $p(\mathcal{C})$.*

Proof: By [WB] or [S2, Corollary 4.9], for every $a \in \mathcal{U} \backslash acl(A)$ there exists $a' \in dcl(aA) \backslash acl(A)$ such that $tp(a'/A)$ has fundamental system of solutions over $p(\mathcal{C})$, (i.e. $tp(a'/A)$ is generated over $p(\mathcal{C})$ by a set of realizations p.) In particular, there exists a (finite) set A' of realizations of $tp(a'/A)$ that is independent from A' and tuple $\bar{c}$ of realizations of p such that $a' \in dcl(A'\bar{c})$. For every A-definable functions f, g let

$$F_{f,g} = \{a \in \mathcal{U} | f(a) = g(\bar{b}, \bar{c}) \notin acl(A) \text{ for some } \bar{b}, \bar{c} \text{ with } f(a) \perp \bar{b} \text{ , where } \bar{c} \text{ is a tuple of realizations of } p, \text{ and } \bar{b} \text{ is a tuple of realizations of } tp(f(a)/A)\}.$$

Note that each $F_{f,g}$ is τ^f-closed over A. Thus, by Baire category theorem for the τ^f-topology and Fact 2.5 we conclude that there exists an unbounded τ^f-open set $\mathcal{U}^*$ over A and A-definable function g^* such that for every $a \in \mathcal{U}^*$ there exists a tuple $\bar{b}$ of realizations of $tp(a)$ that is independent from a such that $a = g^*(\bar{b}, \bar{c})$ for tuple $\bar{c}$ of realizations of p. Now, the subclaim follows directly from fact [S2, Theorem 3.7]:

Fact 5.5 *Let $p \in S(\emptyset)$ and let $\mathcal{R}$ be $\emptyset$-invariant. Suppose the internality of p in $\mathcal{R}$ is witnessed by a generic parameter whose type q is $\mathcal{R}$-internal. Then p is generated over $\mathcal{R}$ by a set of realizations of q.*

Moreover, we may assume $A = acl(A)$, so it is not hard to see that there exists a definable family of functions $g^{**}(x, y) \in L$ (constructed from g^*) such that for every $a \in \mathcal{U}^*$ there exists a' such that if $n = l(x)$, then $g^{**}(p^n, a')$ contains $tp(a)^{\mathcal{C}}$ (see the method in [S3, Theorem 5.6]). Now, as $\mathcal{U}^*$ has bounded finite SU-rank (the bound is determined by g^*), by Fact 2.6, there exists a SU-rank 1 formula $\theta(x, b)$ such that $\theta(\mathcal{C}, b) \subseteq \mathcal{U}^* \cup acl(Ab)$. Thus 2) follows.

5.1 A trichotomy for countable theories with the nfcp

Work in $\mathcal{C} = \mathcal{C}^{eq}$.

Theorem 5.6 *Let T be a countable theory with nfcp. Let $p \in S(\emptyset)$ be minimal. Then, either*
1) every type $q \in S(A)$, with A countable, that is internal in p is essentially 1-based by means of the τ^f-topologies, or
2) there exists a strongly minimal definable set that is p-internal, or
3) there is an infinite definable 1-based group of finite D-rank that is p-internal.

Proof: Assume 1) is false. By Theorem 5.3, there exists a weakly-minimal formula $\theta(x, b)$ that is p-internal. First, assume $\theta(\mathcal{C}, b) \subseteq acl(p(\mathcal{C}) \cup b)$. By Baire category theorem, there exists a b-definable $\theta^*(x, b) \vdash \theta(x, b)$ and b-definable functions f, g and $n < \omega$ such that $g[p^n(\mathcal{C})] \supseteq f[\theta^*] \equiv f[\theta^*(\mathcal{C}, b)]$ and $f[\theta^*]$ is non-algebraic. Since p is minimal, $f[\theta^*]$ must have Morley rank and thus contains a strongly minimal formula. Thus, we may assume

$\theta(\mathcal{C}, b) \not\subseteq acl(p^{\mathcal{C}} \cup b)$. Let $a \in \theta(\mathcal{C}, b) \backslash acl(p^{\mathcal{C}} \cup b)$. Let $q = tp(a/acl(b))$ and let $\Gamma = Aut(q^{\mathcal{C}}/p^{\mathcal{C}} \cup acl(b))$. Then there is an infinite type-definable group G over $acl(b)$ that is isomorphic to Γ. Now, G can be realized as $acl(b)$-definable projections of certain type-definable set over $acl(b)$ of finite tuples from q. As $q \vdash \theta(x, b)$, and G is intersection of definable groups over $acl(b)$, we conclude that there an infinite $acl(b)$-definable group G_0 that is p-internal and has finite D-rank. By Buechler's dichotomy, every minimal type $r \vdash G_0$ is either 1-based or of Morley rank 1. Thus if 2) fails, then any such r is 1-based. As G_0 has finite SU-rank, every non-algebraic type of G_0 is non-orthogonal to a minimal type in G_0 and therefore analyzable in it. By [W], G_0 is 1-based.

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