Simplicity of some automorphism groups

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Abstract: Let M be a countably infinite first order relational structure which is homogeneous in the sense of Fraïssé. We show, under the assumption that the class of finite substructures of M has the *free* amalgamation property, along with the assumption that $\mathrm{Aut}(M)$ is transitive on M but not equal to $\mathrm{Sym}(M)$, that $\mathrm{Aut}(M)$ is a simple group. This generalises results of Truss, Rubin and others. The proof uses the Polish group structure of the automorphism group and generalises to certain other homogeneous structures, with prospects for further application.

1 Introduction

In this paper, by a *homogeneous structure* we mean a countably infinite relational structure such that every isomorphism between finite substructures of M extends to an automorphism of M. Such structures are typically constructed by Fraïssé amalgamation, and their automorphism groups provide a rich supply of groups interesting both as permutation groups and as topological groups. Under small extra assumptions (for example that the language has finitely many

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relation symbols), the automorphism group will be oligomorphic, or, equivalently, M will be *ω-categorical*, that is, any countably infinite L-structure which satisfies the same first order sentences as M will be isomorphic to M.

There are many results on normal subgroup structure of such automorphism groups. For example, the automorphism group of a pure countably infinite set (an *indiscernible set*), that is the symmetric group $\mathrm{Sym}(\mathbb{N})$, has as its proper non-trivial normal subgroups just the group $\mathrm{FSym}(\mathbb{N})$ of all permutations of finite support, and its subgroup of index two consisting of the even permutations; and the group $\mathrm{Aut}(\mathbb{Q}, <)$ has as its proper non-trivial normal subgroups just the group $L(\mathbb{Q})$ consisting of automorphisms g which 'live on the left' (they fix pointwise some interval (a, ∞) for some $a \in \mathbb{Q}$), a corresponding group $R(\mathbb{Q})$ of permutations which 'live on the right', and the intersection $B(\mathbb{Q}) := L(\mathbb{Q}) \cap R(\mathbb{Q})$. Likewise Truss [22] showed that the automorphism group of the random graph is simple, as is that of the random graph with edges coloured randomly from a countable set C. (The *random graph* is the unique countable homogeneous graph which embeds all finite graphs.) The corresponding result was proved for the universal homogeneous partial order in [9], and for the generic k-uniform hypergraphs and some other structures by Lovell [17]. Earlier, Rubin [19], in an unpublished manuscript, gave a proof of simplicity for some binary homogeneous structures including the K_n-free graphs and the universal homogeneous tournament, and the result for the tournament was also proved in unpublished work by Jaligot. For a survey of some of this work, including the unpublished work of Rubin, see [23]. At the other extreme, the 2-homogeneous countably infinite trees considered by Droste [5], which may be viewed as homogeneous structures in an appropriate finite relational language, have automorphism groups with $2^{2^{\aleph_0}}$ distinct normal subgroups [6]. The examples suggest that in general, if $\mathrm{Aut}(M)$ is not simple, then it has some *obvious* proper non-trivial normal subgroups, explicable in terms of its action, but the proofs involved are often very intricate.

In this paper, we give a uniform proof of the simplicity of the automorphism groups in cases (other than a pure set) where the underlying amalgamation is very canonical. Our methods do not work for structures involving orderings, such as the universal homogeneous partial order, but they work for the random graph, the random K_n-free graphs and the 'Henson digraphs' and higher arity analogues, and also (after a small tweak) for the random tournament. Our main theorem is the following.

Theorem 1.1 *Let M be a homogeneous structure which is free in the sense of Definition 2.1, and such that $\mathrm{Aut}(M)$ is transitive on M but is not equal to $\mathrm{Sym}(M)$. Then $\mathrm{Aut}(M)$ is a simple group.*

Transitivity is required, since the structure consisting of a countably infinite set with an infinite coinfinite subset defined by a unary predicate is free homogeneous, but its automorphism group is not simple. In fact, it is isomorphic to a direct product of $\mathrm{Sym}(\mathbb{N}) \times \mathrm{Sym}(\mathbb{N})$ and hence even has proper non-trivial *closed* normal subgroups. The conclusion of Lemma 2.11 below does not hold in this example.

Our proof exploits the Polish group structure of the automorphism group. The proof reduces to two technical steps, namely Lemma 2.11, which eliminates the possibility of a normal subgroup consisting of 'bounded' automorphisms, and Proposition 3.1. It is based on a method of Lascar, from [16] – see in particular Lemma 3.3 below. Lascar used his approach to show that a certain quotient of the group of 'strong' automorphisms of a countable saturated strongly minimal set must be simple. The same method was later exploited by Gardener [8] to describe the normal subgroup structure of certain infinite dimensional classical groups. Note though that in the above work of Truss, Rubin, Lovell, and Jaligot, an explicit small number n is computed such that if g, h are automorphisms of the structure under consideration with $n \neq 1$, then g is a product of at most n conjugates of h and h^{-1}. Our methods appear not to provide such a bound.

We believe our method to have considerable potential for further generalisation, but have not yet achieved this. Possible further examples to consider include the following.

(i) The automorphism groups of the generalised polygons constructed by Tent in [21]. This would be particularly striking, since these groups have a BN-pair, so would provide new examples of non-algebraic simple groups with a (non-split) BN-pair.

(ii) Urysohn space (see [24] or for example [3]), and the countable universal homogeneous metric space with rational distances. Here one should not expect simplicity as there is a normal subgroup consisting of isometries of 'bounded displacement', but the corresponding quotient groups may be simple.

(iii) Certain 'Hrushovski constructions', such as the ω-categorical pseudoplane (not published by Hrushovski, but see [25]), and 'ab initio' structures obtained before 'collapse' in Hrushovski's construction in [14] of a strongly minimal set.

In each case, the amalgamation can be done canonically, and in (i) and (iii) it may be viewed as 'free amalgamation'. In (i) and (iii) we still need an analogue of Lemma 2.11, eliminating any possibility of a normal subgroup of 'bounded' automorphisms. Also, in (i) and (iii) the model-theoretic algebraic closure operator is non-trivial, which causes problems when extending partial automorphisms in the proof of Proposition 3.1. A further problem is that in our proof of 3.1, we appear to need that if A and B are freely amalgamated over C, and $C_0 \subset C$, then $A \setminus C$ and $B \setminus C$ are freely amalgamated over C_0, and this does not hold for the canonical form of amalgamation used in (ii).

Notation 1.2 If G is a group of automorphisms of a structure M, then group elements act on the left, i.e. we write $g(x)$ for the image of $x \in M$ under g. Let $g^h := g^{-1}hg$, and $[g, h] = g^{-1}h^{-1}gh$. If the group G acts on M and $D \subset M$, then $G_{(D)}$ denotes the pointwise stabiliser of D; if $\bar{d}$ enumerates D this may also be denoted by $G_{\bar{d}}$.

If M is a homogeneous structure and $\bar{a}, \bar{a}', \bar{b}$ are tuples from M, we write $\bar{a} \equiv_{\bar{b}} \bar{a}'$ if $\bar{a}$ and $\bar{a}'$ lie in the same orbit of $\mathrm{Aut}(M)_{\bar{b}}$, that is, in model-theoretic language, if they have the same *type* over $\bar{b}$. If M is a first-order structure and $n \in \mathbb{N}^{>0}$, we say that a subset D of M^n is *definable* if it is the solution set of a first-order formula, possibly with parameters; D is A-*definable*, where $A \subset M$, if the parameters can be chosen from M. Also, D is A-*invariant* if it is a union of $\mathrm{Aut}(M)_{(A)}$-orbits on M^n, and D is *invariant* if it is A-invariant for some finite A. If M is ω-categorical, for example if M is homogeneous over a *finite* relational language, and $A \subset M$ is finite, then, by the Ryll-Nardzewski Theorem (see [13] or [1]), $D \subset M^n$ is A-definable if and only if it is A-invariant.

If A, B are first order structures, we write $A \leq B$ if A is a substructure of B in the sense of model theory (which corresponds to the graph-theorist's notion of 'induced substructure').

Finally, by a *digraph* we mean a structure in a language with a single binary irreflexive relation R satisfying $\forall x \forall y (Rxy \to \neg Ryx)$; it is a *tournament* if in addition it satsfies $\forall x \forall y (x = y \vee Rxy \vee Ryx)$.

We shall freely use that if M is a countably infinite structure, then $G := \mathrm{Aut}(M)$ has naturally the structure of a Polish group, that is, a topological group such that the topology comes from a Polish space structure (that is, a complete separable metric space). For example, let $M = \{a_n : n \in \omega\}$, and define a metric d on G, putting $d(g, h) = \frac{1}{n+1}$ where n is least such that $g(a_n) \neq h(a_n)$ or $g^{-1}(a_n) \neq h^{-1}(a_n)$. If f is a finite partial isomorphism of M (that is, an isomorphism between finite substructures of M), let $O_f := \{g \in G : g \text{ extends } f\}$. Then the set of such O_f form a basis of neighbourhoods for this topology; in particular, there is a basis of neighbourhoods of the identity consisting of subgroups $G_{(F)}$ (F a finite subset of M).

Recall that a subset A of a Polish space X has the *Baire Property* if there is an open set U such that the symmetric difference $A \Delta U$ is meagre. The following result of Pettis is well-known – see [15, Theorem 9.9].

Proposition 1.3 *Let G be a Polish group and let H be a subgroup of G with the Baire property. Then H is meagre or clopen.*

For general background on homogeneous structures, see [1] or [18]. Another possible source is [4], in particular, the introduction – this monograph gives the classification of countable homogeneous digraphs.

2 Free amalgamation

Let L be a first order language containing no function or constant symbols. As a general assumption for the paper, we assume that for each relation symbol R of L, if $Ra_1 \ldots a_k$ holds in a structure, then $a_1, \ldots, a_k$ are distinct. This assumption is harmless, since the relations in the language can be adjusted to ensure this, without affecting automorphism groups, homogeneity, or the notion of free amalgamation below.

Recall that an *age* over L is a collection of finite L-structures, containing just countably many non-isomorphic structures, which is closed under isomorphism and (induced) substructure, and has the Joint Embedding Property (JEP). If M is a countably infinite L-structure, then its age $\mathrm{Age}(M)$ is the collection of all finite L-structures which embed in M.

Definition 2.1 (i) An age $\mathcal{C}$ is an *amalgamation class* or has the *amalgamation property*, if, whenever $A, B_1, B_2 \in \mathcal{C}$ and $f_i : A \to B_i$ are embeddings ($i = 1, 2$) there is $D \in \mathcal{C}$ and embeddings $g_i : B_i \to D$ such that $g_1 \circ f_1|_A = g_2 \circ f_2|_A$.

(ii) We say the amalgamation class $\mathcal{C}$ has the *disjoint amalgamation property* (DAP) if in (i), the g_i and D can be chosen so that $g_1(B_1) \cap g_2(B_2) = f_1(A)$.

(ii) The amalgamation class $\mathcal{C}$ has *the free amalgamation property* (FAP) if, whenever $B_1, B_2 \in \mathcal{C}$, $A \in \mathcal{C}$, and $f_i : A \to B_i$ are embeddings ($i = 1, 2$) there are $D \in \mathcal{C}$ and embeddings $g_i : B_i \to D$ such that $g_1 \circ f_1|_A = g_2 \circ f_2|_A$ and $g_1(B_1) \cap g_2(B_2) = f_1(A)$, and in addition, for each relation symbol R of L, no tuple of D which satisfies R meets both of $g_1(B_1) \setminus g_1 f_1(A)$ and $g_2(B_2) \setminus g_2 f_2(A)$.

By Fraïssé's Theorem [7, 1], if $\mathcal{C}$ is an amalgamation class, then there is a (unique up to isomorphism) countably infinite homogeneous L-structure M such that $\mathrm{Age}(M) = \mathcal{C}$. We shall refer to M as the *Fraïssé limit* of $\mathcal{C}$. Furthermore, the age of any homogeneous L-structure is an amalgamation class. We shall say that M is a *free* homogeneous L-structure if $\mathrm{Age}\, M$ is a free amalgamation class. If M is a homogeneous L-structure and A, B_1, B_2 are finite substructures of M such that $B_1 \cap B_2 \subseteq A$ and no tuple of $B_1 \cup B_2 \cup A$ satisfying an L-relation meets both $B_1 \setminus A$ and $B_2 \setminus A$, we denote the structure on $B_1 \cup B_2 \cup A$ as $B_1 \oplus_A B_2$, and write $B_1 \downarrow_A B_2$. As a slight abuse of notation, we allow here that $A = \emptyset$, and then write $B_1 \downarrow B_2$. This notation is motivated by non-forking in model-theoretic stability theory.

Remark 2.2 1. If M is free homogeneous then for any finite sets $A, B, C \subset M$ there is $g \in \mathrm{Aut}(M)_{(A)}$ such that $g(B) \downarrow_A C$. For non-empty A, this holds by amalgamating $B \cup A$ and $C \cup A$ freely over A (as structures in $\mathrm{Age}(M)$), and then using homogeneity. The statement holds even if A is empty. Indeed, if $A = \emptyset$, choose $a \in M \setminus (B \cup C)$ and $g \in \mathrm{Aut}(M)_a$ such that $g(B) \downarrow_a C$; then $g(B) \downarrow C$.

2. If $B_1 \downarrow_{A \cup A'} B_2$ and $A' \cap (B_1 \cup B_2) = \emptyset$, then $B_1 \downarrow_A B_2$. This is not a standard property of model-theoretic independence relations, but is important in the proof of Proposition 3.1 below.

3. The relation $\downarrow$ also satisfies a model-theoretic 'stationarity' property: $A, C \subset M$ are finite, $\bar{b}$ and $\bar{b}'$ lie in the same $\mathrm{Aut}(M)_{(C)}$-orbit, and $\bar{b} \downarrow_C A$ and $\bar{b}' \downarrow_C A$, then $\bar{b}$ and $\bar{b}'$ lie in the same $\mathrm{Aut}(M)_{(C \cup A)}$-orbit. This is an immediate consequence of homogeneity.

4. If A is finite and $\bar{b} \equiv_A \bar{b}'$ with $\bar{b} \downarrow_A \bar{b}'$, then there is $g \in \mathrm{Aut}(M)_{(A)}$ with $g(\bar{b}\bar{b}') = \bar{b}'\bar{b}$.

Example 2.3 We give some examples of free homogeneous relational structures.

(i) The random graph and random digraph, and the generic K_n-free homogeneous graph. The last example is the Fraïssé limit of the class of finite graphs which do not have K_n as an induced subgraph; this class is a free amalgamation class since the free amalgam $B_1 \oplus_A B_2$ is K_n-free provided B_1 and B_2 are.

(ii) The 'Henson digraphs' [10]. Such a digraph is determined by a collection $\mathcal{T}$ of finite tournaments, and consists of all digraphs not embedding any member of $\mathcal{T}$.

(iii) For any $k > 2$, the generic k-hypergraph, and, for each $\ell > k$, the homogeneous k-hypergraph which is universal subject to not embedding an ℓ-*pyramid*, that is, an ℓ-set all of whose k-subsets are hyper-edges.

Some examples of homogeneous structures which are *not* free, for various different reasons, include: the universal homogeneous tournament; any homogeneous structure (M, E) where E is an equivalence relation on M; $(\mathbb{Q}, <)$ (and the countable universal homogeneous poset); the universal homogeneous two-graph [2, Section 7]; and the countable universal homogeneous metric space with rational distances (since amalgamation is constrained by the triangle inequality, and since in a metric space any two points have to have a specified distance).

Remark 2.4 If A is a structure over a relational language L, then an L-structure B is a *weak substructure* of A if its domain is a subset of that of A, and for any $n \in \mathbb{N}$, relation symbol R in L of arity n, and $b_1, \ldots, b_n \in B$, $B \models Rb_1 \ldots b_n \Rightarrow A \models Rb_1 \ldots b_n$. We shall say that the homogeneous structure M is *monotone* if $\mathrm{Age}(M)$ is closed under weak substructure. Then by the main theorem of [11], if M is a monotone free homogeneous structure over a finite relational language and $\mathcal{C} := \mathrm{Age}(M)$, then $\mathcal{C}$ has Herwig's extension property: for any $A \in \mathcal{C}$ there is $B \in \mathcal{C}$ such that $A \leq B$ and every partial isomorphism between substructures of A extends to an automorphism of B. As a consequence, M has the *small index property*: every subgroup of $\mathrm{Aut}(M)$ of index less than the continuum is open.

Many other properties of free, and of monotone free, homogeneous structures are summarised in [18, 6.5.6 and 6.5.7]. For example, if M is free homogeneous over a finite relational language, then if G acts without inversions on a combinatorial tree T then every element of G fixes a vertex of T, so G is not a non-trivial free product with amalgamation. If in addition M is monotone, then G is not the union of a countable chain of proper subgroups, so as G also does not have $(\mathbb{Z}, +)$ as a homomorphic image, G has Serre's property (FA).

We begin with some easy remarks on free homogeneous L-structures. Observe first that if G is *any* closed permutation group on a countably infinite set X, then there is a homogeneous structure M with domain X such that

$\mathrm{Aut}(M) = G$ (as permutation groups): introduce a relation symbol for each G-orbit on k-tuples, for all $k \in \mathbb{N}$. However, our first lemma ensures, for example, that a *free* homogeneous structure cannot have locally compact automorphism group (with respect to the topology defined above).

Lemma 2.5 *Let M be a free homogeneous L-structure. Then for any finite $A \subset M$, $\mathrm{Aut}(M)_{(A)}$ has no finite orbits on $M \setminus A$.*

Proof. This follows from (2.15) of [1], as (FAP) implies (DAP). $\square$

Lemma 2.6 *Let M be a transitive free homogeneous L-structure. Then $G := \mathrm{Aut}(M)$ acts primitively on M.*

Proof. We shall apply the criterion of D.G. Higman [12]: a transitive permutation group H on X is primitive if and only if, for every orbit Ω of H on unordered pairs from X, the 'orbital graph' Γ_Ω with vertex set X and edge set Ω is connected.

Choose distinct $a, b \in M$, and by (FAP) find a' such that $a \downarrow_b a'$. Let Ω be the orbit $\{\{g(a), g(a')\} : g \in G\}$ of G on 2-subsets of M.

Given any distinct $a, b \in M$, there is $c \in M$ such that $\{a, c\}$ and $\{b, c\}$ both lie in Ω. Indeed, choose $c \in M$ so that $\{a, c\} \in \Omega$. We may in addition choose c so that $b \downarrow_a c$, so also $\{b, c\} \in \Omega$. It follows that the orbital graph Γ_Ω with edge set Ω is connected.

It remains to check that any other orbital graph with edge set Δ is connected. By the last paragraph, it suffices to show that if $\{a_1, a_2\} \in \Omega$ then a_1, a_2 are at distance two in the orbital graph Γ_Δ. To see this, let $\{a, b\} \in \Delta$. Choose $g \in G_a$ so that if $b' := g(b)$ then $b \downarrow_a b'$. Then $\{b, b'\} \in \Omega$ and there is a path of length two in Γ_Δ from b to b', as required. $\square$

The next lemma yields that ω-categorical free homogeneous L-structures satisfy the model-theoretic condition 'weak elimination of imaginaries'. See [13, p.161] for details on this.

Lemma 2.7 *Let M be a free homogeneous L-structure, and $G := \mathrm{Aut}(M)$.*

(i) If $A, B \subset M$ are finite, then $G_{(A \cap B)} = \langle G_{(A)}, G_{(B)} \rangle$

(ii) If $X \subseteq M^n$ is invariant, there is a unique smallest set $D \subset M$ such that X is D-invariant.

Proof. (i) The containment $\supseteq$ is clear. For $\subseteq$, suppose $g \in G_{(A \cap B)}$. Using Remark 2.2(1) choose $h_1 \in G_{(A)}$ with $h_1 g(A) \downarrow_A B$. Then $h_1 g(A) \cap B = A \cap B$. There is $h_2 \in G_{(B)}$ with $h_2 h_1 g(A) \downarrow_B A$. Then as $h_2 h_1 g(A) \cap B = A \cap B$, Remark 2.2(2) yields $h_2 h_1 g(A) \downarrow_{A \cap B} A$. Likewise choose $h_3 \in G_{(B)}$ so that $h_3(A) \downarrow_B A$, so $h_3(A) \downarrow_{A \cap B} A$. Now if $\bar{a}$ enumerates A, then by Remark 2.2(3), $h_3(\bar{a}) \equiv_A h_2 h_1 g(\bar{a})$. Hence there is $h_4 \in G_{(A)}$ so that $h_4 h_2 h_1 g|_A = h_3|_A$. Thus, $h_3^{-1} h_4 h_2 h_1 g \in G_{(A)}$, so $g \in \langle G_{(A)}, G_{(B)} \rangle$.

(ii) This is a standard consequence of (i). Suppose that X is invariant over the finite sets D_1 and D_2. Put $D = D_1 \cap D_2$. Then if $g \in G_{(D)}$ then $g \in \langle G_{(D_1)}, G_{(D_2)} \rangle$ by (i), so g fixes X setwise. Hence X is D-invariant. $\square$

Lemma 2.8 *Assume M is a free homogeneous L-structure and $G = \mathrm{Aut}(M)$. Then G has no proper non-trivial open normal subgroups.*

Proof. Suppose for a contradiction that K is a proper non-trivial open normal subgroup of G. Then there is finite $F \subset M$ such that $G_{(F)} \leq K$. Also, $G \setminus K$ is open, as it is a union of (open) cosets of K. Thus, there is an open set $U = O_f \subseteq G \setminus K$. Here, f is a finite partial automorphism of M so U consists of all extensions of f in G. Let $A := \mathrm{dom}(f)$ and $B := \mathrm{ran}(f)$. By (FAP) there is $F' \equiv F$ with $F' \downarrow A \cup B$. Let $g \in G$ with $g(F) = F'$. Then as K is normal in G, $G_{(F')} = G_{(F^g)} \leq K$. Now $\mathrm{id}_{F'} \cup f$ is a partial isomorphism, so by homogeneity extends to some $h \in G$. However $h \in K$ (as h fixes F'), and $h \in G \setminus K$, as $h \in U_f$. This is a contradiction. $\square$

We aim next to show that if M is a free homogeneous structure which is not a 'pure set', then any non-trivial normal subgroup of $\mathrm{Aut}(M)$ contains fixed-point-free elements. This (slightly strengthened) is then combined with Proposition 3.1 to prove Theorem 1.1.

Lemma 2.9 *Let M be a transitive free homogeneous L-structure, $G = \mathrm{Aut}(M)$, and $\bar{d}$ be a finite tuple from M. Let $g \in G_{\bar{d}} \setminus \{1\}$. Suppose that $\{\Omega_i : i \in I\}$ is the set of infinite orbits of $G_{\bar{d}}$ on M, and that $I_1 := \{i \in I : \mathrm{supp}(g) \cap \Omega_i \neq \emptyset\}$. Then $I_1 = I$.*

Proof. Suppose this is false, so $I_1 \neq I$, and let $I_2 = I \setminus I_1$. For $j = 1, 2$, let G_j be the group induced by $G_{\bar{d}}$ on $\bigcup_{i \in I_j} \Omega_i$, and let M_j be the substructure of M induced on $\bigcup_{i \in I_j} \Omega_i$.

First, we suppose that $G_{\bar{d}}$ induces $G_1 \times G_2$. By free amalgamation, it follows that $M_1 \downarrow_{\bar{d}} M_2$. In particular,
(*) for each $i \in I_1$ there is $h \in G_{\bar{d}}$ fixing M_2 pointwise with $\mathrm{supp}(h) \cap \Omega_i \neq \emptyset$ (put $h = g$) and for each $i \in I_2$ there is $h \in G_{\bar{d}}$ fixing M_1 pointwise with $\mathrm{supp}(h) \cap \Omega_i \neq \emptyset$.
By free amalgamation, there is some (unique) $r \in I$ such that for $x \in \Omega_r$, $x \downarrow \bar{d}$. By symmetry (since this part of the argument uses (*), not the definition of I_1), we may suppose that $r \in I_2$. It follows easily, by free amalgamation over $\bar{d}$, that any finite substructure of M embeds in Ω_r.

For each $i \in I$, we may write Ω_i as $\Omega_i(\bar{d})$, so $\Omega_i(\bar{d}')$ is the image of Ω_i under any automorphism f such that $f(\bar{d}) = \bar{d}'$. Pick $s \in I_1$. If $x \in \Omega_s$ then by free amalgamation there is $\bar{d}' \equiv_x \bar{d}$ with $\bar{d} \downarrow_x \bar{d}'$ (so $\bar{d} \downarrow \bar{d}'$). Clearly such $\bar{d}'$ lies in Ω_r, so for any $y \in \Omega_s$, there is $f \in G_{\bar{d},\bar{d}'}$ with $f(x) = y$ (we use here that $G_{\bar{d}} = G_1 \times G_2$). That is, $\Omega_s(\bar{d}) \subseteq \Omega_s(\bar{d}')$ for any $\bar{d}' \equiv \bar{d}$ with $\bar{d} \downarrow \bar{d}'$. For such $\bar{d}'$, by Remark 2.2(4) there is $h \in G$ interchanging $\bar{d}$ and $\bar{d}'$. Hence in fact $\Omega_s(\bar{d}) = \Omega_s(\bar{d}')$. However, for $x \in \Omega_s$, we may choose $\bar{d}' \equiv \bar{d}$ with $\bar{d}' \downarrow \bar{d}x$. Then $x \in \Omega_s(\bar{d}) \cap \Omega_r(\bar{d}')$, with $\bar{d} \downarrow_x \bar{d}'$, contradicting that $\Omega_s(\bar{d}) = \Omega_i(\bar{d}')$.

Suppose now that $G_{\bar{d}}$ does not induce $G_1 \times G_2$. Then there is an L-relation R, a subtuple $\bar{d}'$ of $\bar{d}$, and $\bar{a}$, $\bar{a}'$ in M_1 and $\bar{b}, \bar{b}'$ in M_2 such that $\bar{a} \equiv_{\bar{d}} \bar{a}'$, $\bar{b} \equiv_{\bar{d}} \bar{b}'$, and $R\bar{a}\bar{b}\bar{d}' \wedge \neg R\bar{a}'\bar{b}'\bar{d}'$ (after reordering variables in R if necessary). Using an automorphism, we may suppose $\bar{b} = \bar{b}'$. By free amalgamation, we may suppose $\bar{a}' \downarrow_{\bar{d}} \bar{b}$. Let $\bar{a}$ enumerate the set A.

Claim. We may suppose $g(A) \neq A$.

Proof. By definition of I_1, and replacing g by a $G_{\bar{d}}$-conjugate if necessary, we may suppose $g|_A \neq \mathrm{id}_A$, but that $g(A) = A$. Hence, we may also suppose $|A| > 1$. Let $\bar{a} = (a_1, ..., a_n)$ and assume $g(a_1) \neq a_1$. There is $\bar{a}^* \equiv_{\bar{d}a_1} \bar{a}$ with $\bar{a}^* \downarrow_{\bar{d}a_1} \bar{a}$. Choose $h \in G_{\bar{d}}$ with $h(\bar{a}) = \bar{a}^*$. Then $h^{-1}gh(A) \neq A$, so we may replace g by $h^{-1}gh$ to obtain the claim.

Put $A^* := A \cap g(A)$. By free amalgamation, there is $\bar{a}'' \equiv_{\bar{d}\bar{a}} g(\bar{a})$ with $\bar{a}'' \downarrow_{\bar{d}\bar{a}} \bar{b}$. Then $\bar{a}'' \downarrow_{\bar{d}A^*} \bar{b}$ by Remark 2.2(2), so in particular, $\neg R\bar{a}''\bar{b}\bar{d}'$. Now find $h \in G_{\bar{d}\bar{a}}$ with $h^{-1}(g(\bar{a})) = \bar{a}''$. Then g^h maps $\bar{a}$ to $\bar{a}''$ and fixes $\overline{db}$ (as g fixes $\bar{b}$). This is impossible, as $R\bar{a}\bar{b}\bar{d}' \wedge \neg R\bar{a}''\bar{b}\bar{d}'$. □

Corollary 2.10 *Let M be a transitive free homogeneous L-structure, let $G = \mathrm{Aut}(M)$ and assume $G \neq \mathrm{Sym}(M)$. Let $g \in G \setminus \{1\}$, let $D \subset M$ be finite, and let U be an infinite $G_{(D)}$-orbit on M. Then*

(i) g does not fix U pointwise, and in fact

(ii) $\mathrm{supp}(g) \cap U$ is infinite.

Proof. (i) Suppose for a contradiction that g fixes U pointwise. We may suppose that D is minimal such that U is D-invariant. Let $\bar{d}$ be an enumeration of D. By Lemma 2.9, $g(\bar{d}) \neq \bar{d}$. Put $\bar{d}' := g(\bar{d})$.

We first claim that $g(D) = D$. Indeed, by the minimality of $|D|$ and Lemma 2.7(ii), D is the unique smallest set over which U is invariant. Hence, as $g(U) = U$, $g(D) = D$.

Next, by a theorem of Wielandt (Exercise 3 on p. 38 of [1]), as G is primitive (Lemma 2.6), $\mathrm{supp}(g)$ is infinite.

It follows easily that there is $h \in G_{\bar{d}} \setminus C_G(g)$. Indeed, find distinct $a_1, a_2 \in \mathrm{supp}(g) \setminus D$ with $g(a_1) = a_2$, and use (DAP) to find $a_1' \neq a_1$ with $a_1'a_2\bar{d} \equiv a_1a_2\bar{d}$. There is $h \in G$ with $h(a_1a_2\bar{d}) = a_1'a_2\bar{d}$, and it follows that $g^h(a_1) \neq g(a_1)$. Now $[g, h]$ is a non-identity element of G which fixes $\bar{d}$ and satisfies $\mathrm{supp}([g, h]) \cap U = \emptyset$. This is impossible, by Lemma 2.9.

(ii) This follows immediately from (i). For if $F := \mathrm{supp}(g) \cap U$ is finite, then there is an infinite $G_{(D \cup F)}$-orbit contained in U which is fixed pointwise by g. □

Lemma 2.11 *Assume M is a homogeneous L-structure whose age has (DAP), and assume $g \in G := \mathrm{Aut}(M)$ that for each finite $D \subset M$, g does not fix pointwise any infinite $G_{(D)}$-orbit. Then there is $h \in G$ such that $[g, h]$ is fixed-point-free and has no 2-cycles.*

Proof. We build h by a 'back-and-forth' construction as the union of a chain of finite partial automorphisms, so we must show how to add elements to its domain and range. Suppose that h_n has been defined, $a \notin \mathrm{dom}(h_n)$, and our task is to extend h_n to h_{n+1} so that $h_{n+1}(a)$ is defined. Since the age of M has (DAP), there is an infinite set of points $b \in M$ such that $h_n \cup \{(a,b)\}$ is a partial automorphism, and this is an orbit of $G_{(\mathrm{ran}(h_n))}$; hence by our assumption we may choose such b also to lie in $\mathrm{supp}(g)$. In particular, we may choose b so that in addition $h_n^{-1}(g(b))$ is undefined. Then define $h_{n+1}(a) = b$ and build in, for further extensions h_m of h_{n+1}, the requirement that $h_m^{-1}(g(b)) \neq g(a)$. This ensures that $[g,h](a) \neq a$, and we may arrange also that $g^{-1}h_{n+1}^{-1}gh_{n+1}$ does not fix any other point by choosing b outside the finite set $\mathrm{ran}(gh_n)$. We also ensure, when choosing b, that there is no point c such that $[g,h_{n+1}]^2(c) = c$. This again eliminates only finitely many possibilities for b. Thus, at any given stage there will be finitely many 'commitments', i.e. finitely many points to avoid when making a one-point extension.

The other case is when h_n has been defined, with $b \notin \mathrm{ran}(h_n)$, and we must find a such that $h_{n+1} := h_n \cup \{(a,b)\}$ is a partial isomorphism. If $b \notin \mathrm{supp}(g)$, then we may choose a to be any point in $\mathrm{supp}(g)$ such that $h_{n+1} := h_n \cup \{(a,b)\}$ is a partial isomorphism; then $g^{-1}h_{n+1}^{-1}gh_{n+1}(a) = g^{-1}(a) \neq a$. Such a exists by our assumption, and if we choose a such that in addition $g^{-1}(a) \notin \mathrm{dom}(h_n)$, then $g^{-1}h_{n+1}^{-1}gh_{n+1}$ is fixed-point-free. So suppose $b \in \mathrm{supp}(g)$. If $h_n^{-1}(g(b))$ is defined, and equals c, say, choose a so that $h_n \cup \{(a,b)\}$ is a partial isomorphism and $a \neq g^{-1}(c)$. On the other hand, if $h_n^{-1}(g(b))$ is undefined, choose a to be any point in $\mathrm{supp}(g)$ so that $h_n \cup \{(a,b)\}$ is a partial isomorphism, and build in for the future the commitment $h_n^{-1}(g(b)) \neq g(a)$; we also ensure a is chosen with $g^{-1}(a) \notin \mathrm{dom}(h_n)$, to ensure $g^{-1}h_{n+1}^{-1}gh_{n+1}$ is fixed-point-free. In both cases in this paragraph, when choosing a, we also avoid finitely many points, to ensure that eventually $[g,h]$ has no 2-cycle. $\square$

Remark 2.12 The conclusion of Lemma 2.11 holds for many homogeneous structures for which the amalgamation is not free. Note though that it fails for $\mathrm{Aut}(\mathbb{Q},<)$, since an automorphism may have support within a bounded interval, in which case all elements of its normal closure have support within a bounded interval. It also fails for many treelike structures.

3 Proof of the Theorem 1.1.

The heart of the proof of Theorem 1.1 is the following proposition (an analogue of Lemme 9 of [16]). A small adaptation of the proof, with an appropriate notion of $\downarrow$, yields simplicity also for the automorphism group of the universal homogeneous tournament (see Remark 3.2 below).

Proposition 3.1 *Let M be a free homogeneous L-structure, let $G := \mathrm{Aut}(M)$, and suppose that $g \in G$ is fixed-point-free and has no 2-cycles. Define $\alpha : G^6 \to$*

G by $\alpha(h_1,h_2,h_3,h_4,h_5,h_6) := g^{h_1}g^{h_2}g^{h_3}g^{h_4}g^{h_5}g^{h_6}$. Let U_1,U_2,U_3,U_4,U_5,U_6 be non-empty open subsets of G. Then there is non-empty open $Y \subset G$ such that $\alpha(U_1 \times \ldots \times U_6)$ is dense in Y.

Proof of Proposition 3.1. We may suppose that there are finite partial automorphisms u,v,w,x,y,z of M such that $U_1 = O_u$, $U_2 = O_v$, $U_3 = O_w$, $U_4 = O_x, U_5 = O_y$ and $U_6 = O_z$. It suffices to prove the proposition with u,v,w,x,y,z replaced by finite extensions. Let $\alpha(u,v,w,x,y,z)$ denote the partial function

$$u^{-1}guv^{-1}gvw^{-1}gwx^{-1}gxy^{-1}gyz^{-1}gz.$$

For $a \in M$, we write $\alpha(a)$ for $\alpha(u,v,w,x,y,z)(a)$. In the claim below, part (1) and (2) ensure that if a sufficiently big final segment of α is defined on $a \in M$, then $\alpha(a)$ is defined, and (2) ensures the same for α^{-1}.

Claim 1. We may extend u,v,w,x,y,z finitely to ensure the following.

(1) For any $a \in M$, if $yz^{-1}gz(a)$ is defined, then $\alpha(a)$ is defined.

(2) For any $a \in M$, if $vu^{-1}g^{-1}u(a)$ is defined, then $\alpha^{-1}(a)$ is defined.

(3) If $\bar{a} := \mathrm{dom}(w) \cap \mathrm{dom}(x)$, $\bar{b} := \mathrm{dom}(w) \setminus \mathrm{dom}(x)$, and $\bar{c} := \mathrm{dom}(x) \setminus \mathrm{dom}(w)$, then $\bar{b} \downarrow_{\bar{a}} \bar{c}$.

(4) If $a \in \mathrm{dom}(w) \cap \mathrm{dom}(x)$ then $(g^u \circ g^v \circ g^w)(a))$ is defined and lies in $\mathrm{ran}(\alpha(u,v,w,x,y,z))$.

(5) $\mathrm{dom}(v) \cap (\mathrm{dom}(x) \setminus \mathrm{dom}(w)) = \emptyset$ and $\mathrm{dom}(y) \cap (\mathrm{dom}(w) \setminus \mathrm{dom}(x)) = \emptyset$.

Proof of Claim 1. We first indicate the basic idea. We define finitely many one-point extensions of $u,v,\ldots,z$. To avoid proliferation of notation, we keep the same symbols $u,\ldots,z$, i.e., we avoid writing u_n, v_n, etc. Thus, the construction is dynamic in the sense that the meaning of the symbols $u,\ldots,z$ changes as the construction proceeds, but they always denote finite partial isomorphisms. We remind the reader that unlike $u,\ldots,z$, the element g is already completely defined as an automorphism of M.

At any stage, with given defined $u,\ldots,z$, we say that a point $a \in M$ is *old* if $a \in A' := \bigcup_{-2\leq i\leq 2} g^i(A)$, where

$$A = \mathrm{dom}(u) \cup \ldots \cup \mathrm{dom}(z) \cup \mathrm{ran}(u) \cup \ldots \cup \mathrm{ran}(z).$$

Suppose we wish to make a one-point extension of u, by defining $u(a)$. Let A' be set of old points at this stage. If $\bar{b}$ is an enumeration of $\mathrm{dom}(u)$ and $u(\bar{b}) = \bar{c}$, choose d such that $\bar{b}a \equiv \bar{c}d$ and $d \downarrow_{\bar{c}} A$, and define $u(a) = d$. Such d exists by free amalgamation, and the extension of u is a partial isomorphism. The same applies to one-point extensions of u^{-1}, and likewise for v,w,x,y,z and their inverses. Call such one-point extensions *good extensions*. All our extensions below are good.

Initially, we aim for the following strengthenings of (1), (2), namely:

$(1)'$ For any $a \in M$, if $z^{-1}gz(a)$ is defined, then $\alpha(a)$ is defined.

$(2)'$ For any $a \in M$, if $u^{-1}g^{-1}u(a)$ is defined, then $\alpha^{-1}(a)$ is defined.

Suppose for example $(z^{-1}gz)(a)$ is defined, and equals b, but $y(b)$ is undefined (a violation of $(1)'$). To ensure that $\alpha(a)$ is defined, we first make a good extension of y to define $y(b)$. Then, after defining $y(b)$, as g is fixed-point-free and as $y(b)$ was chosen not among the old points at that stage, $y^{-1}(g(y(b)))$ is undefined. Make a good extension of y^{-1} to define this, and proceed right-to-left along $\alpha(u,v,w,x,y,z)$, always making good extensions, to ensure $a \in \mathrm{dom}(\alpha(u,\ldots,z))$. If we do this successively for all points a such that $(z^{-1}gz)(a)$ is defined but $\alpha(a)$ is undefined, then $(1)'$ is achieved. (Of course it could be that for the above pair a,b, the element $y(b)$ was already defined, but $\alpha(u,\ldots,z)(a)$ was not – in that case start the process further to the left along α.) Call this 'Step A'. After Step A, $(1)'$ holds, but $(2)'$ may not.

Next, as Step B, we repeat this process, with α^{-1} in place of α, to ensure that $(2)'$ holds. That is, we work from left to right along α (or right-to-left along α^{-1}), always making good extensions. In the process, for certain points a, we may define $z(a)=b$ and then $z^{-1}(g^{-1}(b))=c$ (or it may be that $z(a)=b$ was already defined, after Step A or in an earlier stage of Step B, but $z^{-1}(g^{-1}(b))=c$ is now defined). In particular, it could happen that we *create* a violation of $(1)'$; that is, for some d, after Step A $z^{-1}gz(d)$ was undefined, but now after Step B it is defined but $\alpha(d)$ is not. So after Step B, $(2)'$ holds but $(1)'$ may fail.

We claim that after Step B, if $z^{-1}gz(d)$ is defined, but $\alpha(d)$ is not defined, then $y(z^{-1}gz(d))$ is undefined. To see this, suppose first that $z^{-1}gz(d)$ became defined when we put $z^{-1}(g^{-1}(b))=c$; that is, there was some c' such that $(u^{-1}g^{-1}u)(c')$ was defined but $\alpha^{-1}(c')$ was undefined, and as part of Step B we ensured that $\alpha^{-1}(c')$ is defined and equals c. If $c=d$, then after Step B, $\alpha(u,\ldots,z)(d)$ is indeed defined, contrary to hypothesis. The other possibility is that $z^{-1}gz(d)$ becomes defined because $g^{-1}(b)=gz(d)$, but then, because c was chosen to witness a good extension of z^{-1}, $y(z^{-1}gz(d))=y(c)$ is undefined. Alternatively, suppose $z^{-1}gz(d)$ became defined when at Step B we put $z(a)=b$, so before putting $z^{-1}(g^{-1}(b))=c$. If $a=d$ then as b was chosen witnessing a good extension of z, and $g(b)\neq b$, $g(b)\notin \mathrm{dom}(z^{-1})$, contradicting that $z^{-1}gz(d)$ becomes defined at this stage. The other possibility is that $z(d)$ was previously defined and $z^{-1}gz(d)$ becomes defined when we put $z(a)=b$, because $b=gz(d)$. This too could not occur, for as b witnesses a good extension of z, $z(d)$ cannot previously have been defined.

As Step C, apply Step A *again*, to ensure that $(1)'$ holds. By the last paragraph, when dealing at Step C with some d such that $z^{-1}gz(d)$ is defined, but $\alpha(d)$ is not, we will make a good extension of u and then a good extension of u^{-1}, i.e. *two* good extensions. Of course, Step C may be applied to *several* such points d, but each such d will involve a good extension of u followed by a good extension of u^{-1}.

It can be checked that now $(2)'$ also holds. For suppose it fails, that is, $u^{-1}g^{-1}u(a)$ is defined, but $\alpha^{-1}(a)$ is not defined. Then this failure was caused at Step C. That is, at Step C, to ensure that $\alpha(u,\ldots,z)$ was defined at some point d, we defined $u(e)=c$ for some e and c, and then $u^{-1}(g(c))=b$. As a result, $u^{-1}g^{-1}u(a)$ became defined. We emphasise that by the last paragraph both c and b are new points witnessing good extensions – there was a good extension of u followed by a good extension of u^{-1}. There are four possibilities.

(i) $a = b$. In this case, $\alpha^{-1}(a)$ is defined and equals d, contrary to hypothesis.

(ii) $a = e$. Then c was chosen outside the previous set $g(\mathrm{dom}(u^{-1}))$ (as it was chosen when making a good extension), and as g has no 2-cycles, $g^{-1}(c) \neq g(c)$, so u^{-1} is still (after putting $u(e) = c$ and $u^{-1}(g(c)) = b$) not defined on $g^{-1}(c)$. (This is the reason for the requirement in Lemma 2.11 that $[g, h]$ have no 2-cycles.) Likewise, by the choice of c, it could not be that earlier in Step C when dealing with another violation of $(1)'$, we defined $u^{-1}(g^{-1}(c))$. Hence $u^{-1}g^{-1}u(a)$ is undefined, a contradiction.

(iii) $u(a)$ was defined before Step C or at an earlier part of Step C when handling another violation of $(1)'$, and $g^{-1}(u(a)) = g(c)$, so $u^{-1}g^{-1}u(a) = b$. In this case, $c = g^{-2}(u(a))$, contrary to the choice of c at Step C.

(iv) $u(a)$ was defined before Step C or at an early part of Step C, and $g^{-1}(u(a)) = c$, so $u^{-1}g^{-1}u(a) = e$. In this case, again, $\alpha^{-1}(a)$ is defined and equals d, a contradiction.

Thus, after Steps A-C, conditions $(1)'$ and $(2)'$ hold, and it remains to ensure (3) and (4). At Step D we ensure (3). For this, for any $b \in \mathrm{dom}(w) \setminus \mathrm{dom}(x)$, make a good extension to ensure $x(b)$ is defined, and for any $c \in \mathrm{dom}(x) \setminus \mathrm{dom}(w)$, define $w(c)$ by a good extension. This ensures that $\mathrm{dom}(w) = \mathrm{dom}(x)$, so (3) (and also (5)) hold. It is easily seen that $(1)'$ and $(2)'$ are preserved, since only the 'middle' elements w, x of α are extended.

As Step E, we ensure (4). If $a \in \mathrm{dom}(w) \cap \mathrm{dom}(x)$, make good extensions to ensure that $(g^u \circ g^v \circ g^w)(a))$ is defined, and $((g^{-1})^z \circ (g^{-1})^y \circ (g^{-1})^x)(a)$ is defined. At Step E, we might have to extend w to w', by defining $w'^{-1}(c) = d$, where $c = gw(a)$. Such d will be chosen with $d \downarrow_{\mathrm{dom}(w)} A$, where A is the set of old points at this stage. Likewise, we might have to extend x to x' by defining $x'^{-1}(c) = d$ where $c = g^{-1}x(a)$, and for such d we will have $A \downarrow_{\mathrm{dom}(x)} d$. It follows that Step E cannot create a violation of (3). Likewise no violation of (5) is created.

Step E could create a violation of $(1)'$ or $(2)'$. For example, possibly at Step E we define $u^{-1}(e) = f$, where f is a new point, and possibly $e = g^{-1}u(e')$, so $u^{-1}g^{-1}u(e')$ is defined but $\alpha^{-1}(e')$ is not, violating $(2)'$. However, since f is chosen new, at the end of Step E the element $v(f)$ will not have been defined, so $vu^{-1}g^{-1}u(e')$ is not defined. Thus, (2) holds, and likewise (1) holds.

Given the claim, define y to be the partial map $\alpha(u, v, w, x, y, z)$, and put $Y := O_y$. We show that $\alpha(U_1 \times \ldots \times U_6)$ is dense in Y. That is, we show that for any finite extension y' of y induced by an element of G, there are finite extensions u', v', w', x', y', z' of u, v, w, x, y, z respectively such that $\alpha(u', \ldots, z')$ extends y'. We may suppose that y is the map $\bar{a} \mapsto \bar{b}$, and y' is the map $\bar{a}\bar{e} \mapsto \bar{b}\bar{f}$, with $\bar{e}$ disjoint from $\bar{a}$. Applying $\alpha(u, \ldots, z)$, read from right to left, we may put

$$\bar{a}_0 := \bar{a},\ \bar{a}_1 := z(\bar{a}),\ \bar{a}_2 := gz(\bar{a}),\ \bar{a}_3 = z^{-1}gz(\bar{a}),\ \ \ldots,\ \bar{a}_{18} := \bar{b}.$$

Also put $\bar{e}_0 := \bar{e}$, $\bar{e}_{18} := \bar{f}$. Making good extensions of z, u if necessary, we may suppose $\bar{e}_1 := z(\bar{e}_0)$, $\bar{e}_2 := gz(\bar{e}_0)$, $\bar{e}_3 = g^z(\bar{e}_0)$, and also $\bar{e}_{17} := u(\bar{e}_{18})$, $\bar{e}_{16} := g^{-1}u(\bar{e}_{18})$, and $\bar{e}_{15} = (g^{-1})^u(\bar{e}_{18})$ are defined, but that Claim 1 above still holds.

At this stage, we extend u, v, w, x, y, z to $u^*, v^*, w^*, x^*, y^*, z^*$, successively choosing appropriate $\bar{e}_4, \ldots, \bar{e}_7$, $\bar{e}_{14}, \ldots, \bar{e}_{11}$ and putting $z^* := z$, $y^* = y \cup \{(\bar{e}_3, \bar{e}_4), (\bar{e}_6, \bar{e}_5)\}$ (where $g(\bar{e}_4) = \bar{e}_5$), $x^* = x \cup \{(\bar{e}_6, \bar{e}_7)\}$, and also $u^* := u$, $v^* = v \cup \{(\bar{e}_{15}, \bar{e}_{14}), (\bar{e}_{12}, \bar{e}_{13})\}$ (where $g^{-1}(\bar{e}_{14}) = \bar{e}_{13}$), and $w^* = w \cup \{(\bar{e}_{12}, \bar{e}_{11})\}$. Here, if $\bar{a} := (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$, then $(\bar{a}, \bar{b})$ is a slight abuse of notation for the partial map $\{(a_1, b_1), \ldots, (a_n, b_n)\}$. All such extensions are chosen to be good. Now define $\bar{e}_8 := g(\bar{e}_7)$ and $\bar{e}_{10} := g^{-1}(\bar{e}_{11})$. At this stage conditions (1) and (2) of Claim 1 may be violated, but (3) and (4) hold. The fact that (3) still holds uses part (5) of Claim 1. Indeed, $\bar{e}_{12}$ is in $\mathrm{dom}(w^*)$ but was chosen to have a certain type over $\mathrm{dom}(v)$, and as $\mathrm{dom}(v) \cap (\mathrm{dom}(x) \setminus \mathrm{dom}(w)) = \emptyset$, no violation of (3) was forced, so as $\bar{e}_{12}$ was chosen successively to realise good extensions, there is no such violation. Likewise, $\bar{e}_6$ is in $\mathrm{dom}(x^*)$, but realises a certain type over $\mathrm{dom}(y)$, which is disjoint from $\mathrm{dom}(w) \setminus \mathrm{dom}(x)$, so again causes no violation of (3).

The remaining task is to choose $\bar{e}_9$ so that $x' := x^* \cup (\bar{e}_9, \bar{e}_8)$ and $w' := w^* \cup (\bar{e}_9, \bar{e}_{10})$ are both partial isomorphisms. We may suppose that $\bar{w}^* = (\bar{a}_9, \bar{a}_{10}) \cup (\bar{b}_9, \bar{b}_{10})$, and $\bar{x}^* = (\bar{a}_9, \bar{a}_8) \cup (\bar{c}_9, \bar{c}_8)$, for some $\bar{b}_9, \bar{b}_{10}, \bar{c}_9, \bar{c}_8$. Observe that $\bar{e}_{10}$ and $\bar{b}_{10}$ have no common entries, and $\bar{e}_8$ and $\bar{c}_8$ have no common entries, by the choice of $\bar{e}_{11}$ and $\bar{e}_7$ realising good extensions.

Claim 2. $\bar{b}_9 \downarrow_{\bar{a}_9} \bar{c}_9$.

Proof of Claim 2. By (4), if $d \in \bar{b}_9 \cap \bar{c}_9$ then $d \in \bar{a}_9$. Therefore, the claim follows from (3).

Given Claim 2, choose $\bar{e}_9'$ so that $x^* \cup (\bar{e}_9', \bar{e}_8)$ is a partial isomorphism; that is, $\bar{a}_9\bar{c}_9\bar{e}_9' \equiv \bar{a}_8\bar{c}_8\bar{e}_8$. Then $\bar{a}_9\bar{e}_9{}' \equiv \bar{a}_{10}\bar{e}_{10}$, as $\bar{a}_9\bar{e}_9' \equiv \bar{a}_8\bar{e}_8$ and $((g^{-1})^{z^*} \circ (g^{-1})^{y^*} \circ (x^*)^{-1}g^{-1})(\bar{a}_8\bar{e}_8) = \bar{a}_0\bar{e}_0 \equiv \bar{a}_{18}\bar{e}_{18} = (g^{u^*} \circ g^{v^*} \circ (w^*)^{-1}g)(\bar{a}_{10}\bar{e}_{10})$. Thus, there is $\bar{b}_9'$ such that $a_9\bar{e}_9'\bar{b}_9' \equiv \bar{a}_{10}\bar{e}_{10}\bar{b}_{10}$, and $\bar{b}_9' \downarrow_{\bar{a}_9\bar{e}_9'} \bar{c}_9$. By the remark before Claim 2, $\bar{e}_9'$ does not meet $\bar{c}_9$ or $\bar{b}_9'$. Thus, $\bar{b}_9' \downarrow_{\bar{a}_9} \bar{c}_9$. Hence, by Claim 2, as $\bar{b}_9'\bar{a}_9 \equiv \bar{b}_{10}\bar{a}_{10} \equiv \bar{b}_9\bar{a}_9$, we have $\bar{a}_9\bar{b}_9\bar{c}_9 \equiv \bar{a}_9\bar{b}_9'\bar{c}_9$ so there is $h \in G$ with $h(\bar{a}_9\bar{b}_9'\bar{c}_9) = \bar{a}_9\bar{b}_9\bar{c}_9$. Put $\bar{e}_9 = h(\bar{e}_9')$. Then $\bar{a}_9\bar{b}_9'\bar{c}_9\bar{e}_9' \equiv \bar{a}_9\bar{b}_9\bar{c}_9\bar{e}_9$. In particular, $\bar{a}_9\bar{c}_9\bar{e}_9 \equiv \bar{a}_9\bar{c}_9\bar{e}_9' \equiv \bar{a}_8\bar{c}_8\bar{e}_8$ and $\bar{a}_9\bar{b}_9\bar{e}_9 \equiv \bar{a}_9\bar{b}_9'\bar{e}_9' \equiv \bar{a}_{10}\bar{b}_{10}\bar{e}_{10}$. Thus, $\bar{e}_9$ has the required properties. $\square$

Remark 3.2 Let M be the universal countable homogeneous tournament. Since any two vertices must be related by an arc, M is not a free homogeneous L-structure. However, there is an *asymmetric* notion of free amalgamation: given finite $A, B_1, B_2 \subset M$ write $B_1 \downarrow_A B_2$ if $B_1 \cap B_2 \subseteq A$ and for all $b_1 \in B_1 \setminus A$ and $b_2 \in B_2 \setminus A$ we have $b_1 \to b_2$.

With this notion of free amalgamation, the proof of Proposition 3.1 can be shown to hold with very minor modifications. In the definition of a *good* *extension*, when extending u, v, w or their inverses, if finding an element d in an appropriate orbit over a tuple $\bar{c}$, we choose d so that $d \downarrow_{\bar{c}} A$, where A is the set of old points. However, when extending x, y, z or their inverses, we would choose d so that $A \downarrow_{\bar{c}} d$. This ensures that property (4) in Claim 1 holds, with the adapted definition of $\downarrow$.

The following lemma encapsulates the idea we have taken from Lascar [16] (where it is applied with G the automorphism group of a strongly minimal set, g an 'unbounded' strong automorphism, and $n = 2$). Of course, in the statement below, some occurrences of g could be replaced by g^{-1}.

Lemma 3.3 *Let G be a Polish group, let $g \in G \setminus \{1\}$, let n be a positive integer, and define $\beta : G^n \to G$ by $\beta(h_1, \ldots, h_n) = g^{h_1} \ldots g^{h_n}$. Suppose that for any non-empty open $U_1, \ldots, U_n \subseteq G$ there is non-empty $Y \subset G$ such that $\beta(U_1 \times \ldots \times U_n)$ is dense in Y. Then any normal subgroup K of G containing g is open.*

Proof. As β is continuous, $E = \mathrm{Im}(\beta)$ is an analytic subset of G, so has the Baire property. Furthermore, the group H generated by E has the Baire property. For example this holds since $H = \bigcup_{k \geq 1} X_k$, where X_k is the set of elements of H expressible by a word of length k in $E \cup E^{-1}$: each X_k is analytic so has the Baire property, and H is a countable union of such sets so has the Baire property by Proposition 3.5.1 of [20].

If F is a closed nowhere-dense subset of G then $F' := \beta^{-1}(F)$ is closed in G^n. Also, F' is nowhere dense: for suppose that F' is dense in $U_1 \times \ldots \times U_n$, a non-empty open subset of G^n. Let Y be as in the lemma. Then $\beta(F') = F$ is dense in Y, a contradiction.

It follows that $E := \mathrm{Im}(\alpha)$ is not meagre. For otherwise, $E \subseteq \bigcup_{k \in \omega} F_k$ where the F_k are closed nowhere-dense. Then $G^n = \beta^{-1}(E) \subseteq \bigcup_{k \in \omega} \beta^{-1}(F_k)$. Each $\beta^{-1}(F_k)$ is closed nowhere-dense by the last paragraph, and this contradicts the Baire Category Theorem.

Thus, H, the group generated by E, is not meagre. Hence, as H has the Baire property, by Proposition 1.3 it is open. Thus, since any normal subgroup of G containing g must contain H, the group K must also be open. $\square$

Finally, we restate and prove our main theorem.

Theorem 3.4 *The homogeneous structures of each of the following kinds have simple automorphism group.*

(i) Any transitive free homogeneous structure whose automorphism group is not the full symmetric group.

(ii) The universal homogeneous tournament.

(iii) For any integer $n \geq 3$, the homogeneous digraph which is universal subject to omitting an independent set of size n.

Parts (ii) and (iii) above also follow from [19]. The examples in (ii) arise in Cherlin's classification of homogeneous digraphs – see the family I_n in [4, p.74].

Proof. In each case, let M be the homogeneous structure under consideration, and put $G := \mathrm{Aut}(M)$.

(i) Let N be a non-trivial normal subgroup of G. By Lemmas 2.10 and 2.11, there is $g \in N$ which is fixed point free and has no 2-cycles. Thus, we may define the map $\alpha : G^6 \to G$ as in Proposition 3.1. By that proposition and Lemma 3.3, N is open. Hence, by Lemma 2.8, $N = G$.

(ii) Let $h \in G \setminus \{1\}$. We first claim that h does not fix pointwise any infinite definable set. Indeed, let $X \subset M$ be infinite and $\bar{a}$-definable. Easily, there are distinct $b, c \in M$ outside $\bar{a}$ such that $h(b) = c$. Then by the universal property characterising the tournament M, there is $d \in X$ with $b \to d$ and $d \to c$; then $h(d) \neq d$.

It follows from Lemma 2.11 that there is fixed-point-free $g \in \langle h \rangle^G$ with no 2-cycles (the latter is automatic for automorphisms of tournaments). Thus, by Remark 3.2, the conclusion of Proposition 3.1 holds with respect to g, so by Lemma 3.3, any normal subgroup of G containing h is open. The proof of Lemma 2.8, working with the notion $\downarrow$ from Remark 3.2, easily shows that G has no proper non-trivial open normal subgroups.

(iii) This is essentially as in (ii). Again for finite $A, B, C \subset M$ we put $B \downarrow_A C$ if for any $b \in B \setminus A$ and $c \in C \setminus A$ we have $b \to c$, and argue as in Proposition 3.1 and Remark 3.2. $\square$

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