Measurability in Modules

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Abstract

In this paper we prove that in modules, measurability (in the sense of Macpherson-Steinhorn) depends on being able to define a measure function on the p.p. definable subgroups. We give a classification of abelian groups in terms of measurability and U-rank. Finally we discuss the relation with $\mathbb{Q}(t)$-valued measures.

1 Introduction

A structure M is measurable (in the sense of Macpherson-Steinhorn) if one can definably assign an ($\mathbb{N}$-valued) dimension and an ($\mathbb{R}$-valued) measure to each definable set in M. This assignment must obey some basic axioms [Definition 2.2].

The motivating examples of measurable structures are pseudofinite fields (or ultraproducts of finite fields). In "One dimensional asymptotic classes of finite structures " [6] Macpherson and Steinhorn generalise from the specific case of finite fields, developing a notion of dimension and measure for definable subsets of finite structures. A 1-dimensional asymptotic class is a collection of finite L-structures, for some language L, to which one can assign (in a definable way) a dimension d and measure μ such that for any formula $\phi(\bar{x}, \bar{y}) \in L$, $||\phi(M^n, \bar{a})| - \mu|M|^d| \leq C|M|^{d-\frac{1}{2}}$, where M is any finite structure in the collection, and C a positive constant.

In [3] Elwes and Macpherson develop the more general notion of an N-dimensional asymptotic class. They prove any ultraproduct of an N-dimensional asymptotic class is measurable. As well as finite fields, finite cyclic groups are also an example of asymptotic classes, and therefore ultraproducts of finite cyclic groups are measurable (this is used in Section 4). However, it is not the case that all measurable structures are ultraproducts of asymptotic classes. We know, for example, that vector spaces are measurable in this sense.

Any measurable structure is supersimple of finite SU-rank, in fact the dimension behaves essentially like SU-rank. In this paper we consider measurability in the case of modules, which are stable, and therefore measurable modules are superstable. It is well known [7] that in modules every formula is equivalent to a Boolean combination of positive primitive (p.p.)-formulas [Definition 2.4]. We

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use this to show our main result [Theorem 3.1] that measurability of a complete theory of modules relies entirely on properties of the subgroups defined by p.p. formulas without parameters.

In section 4 we restrict our attention to abelian groups (or $\mathbb{Z}$-modules) and classify the measurable abelian groups. We also remark that the measurable abelian groups are precisely the pseudofinite abelian groups, where a pseudofinite structure is one which is infinite and elementarily equivalent to an ultraproduct of finite structures. Section 5 makes connections with the notion of $\mathbb{Q}(t)$-valued measures on Boolean combinations of cosets of $\mathbb{Z}^n$ defined in [2].

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2 Preliminaries

Throughout, unless otherwise specified, T is a complete theory in signature L, M a model of T.

Definition 2.1. *Def(M) is the set of definable (with parameters) sets in M.*

In [3] Elwes and Macpherson define measurability (for any language L) as follows:

Definition 2.2. *An infinite L-structure M is* MEASURABLE *if there is a function* $h : Def(M) \longmapsto \mathbb{N}\times\mathbb{R}^{>0}\cup\{(0,0)\}$ *(we write* $h(X) = (Dim(X), Meas(X))$ *such that the following holds:*

(i) For each L-formula $\varphi(\bar{x}, \bar{y})$ *there is a finite set* $D \subset \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0, 0)\}$ *so that for all* $\bar{a} \in M^m$ *we have* $h(\varphi(M^n, \bar{a})) \in D$.

(ii) If $\varphi(M^n, \bar{a})$ *is finite then* $h(\varphi(M^n, \bar{a})) = (0, |\varphi(M^n, \bar{a})|)$.

(iii) For every L-formula $\varphi(\bar{x}, \bar{y})$ *and all* $(d, \mu) \in D$*, the set* $\{\bar{a} \in M^m : h(\varphi(M^n, \bar{a})) = (d, \mu)\}$ *is* $\emptyset$*-definable.*

(iv) (Fubini) Let $X, Y \in Def(M)$ *and* $f : X \mapsto Y$ *be a definable surjection. Then there is an* $r \in \omega$ *and* $(d_1, \mu_1), ..., (d_r, \mu_r) \in \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0, 0)\}$ *so that if* $Y_i = \{\bar{y} \in Y : h(f^{-1}(\bar{y})) = (d_i, \mu_i)\}$ *then* $Y = Y_1 \cup ... \cup Y_r$ *is a partition of Y into non-empty disjoint definable sets. Let* $h(Y_i) = (e_i, \nu_i)$ *for* $i \in \{1, ..., r\}$.

Also let $c := Max\{d_1e_1 + ... + d_re_r\}$*, and suppose (without loss) that this maximum is attained by* $d_1+e_1, ..., d_s+e_s$*. Then* $h(X) = (c, \mu_1+\nu_1, ..., \mu_s+\nu_s)$.

If $X \in Def(M)$ and $h(X) = (d, \mu)$, we call d the dimension of X and μ the measure of X, and h the measuring function.

Remark 2.3. *1.If M is measurable, then for any* $N \models T$*, N will be measurable (by clause (iii)). So we call a complete theory T measurable if it has a measurable model.*
2. Any sets in definable bijection will clearly have the same dimension and measure (clause(iv)).

2.1 Model theory of modules

We now assume that M is a left R-module over a ring R and $L = L_R\{+, 0, r\}_{r\in R}$ is the language of left R-modules.

Definition 2.4. POSITIVE PRIMITIVE (P.P.) FORMULAS *(without parameters) are formulas of the form* $\exists \bar{y}(\psi_1(\bar{x}, \bar{y}) \wedge ... \wedge \psi_k(\bar{x}, \bar{y}))$ *where* $\psi_i(\bar{x}, \bar{y})$ *are atomic formulas. In the language of modules these are of the form* $\exists w_1....w_k \bigwedge_{j=1}^{m} (\Sigma v_i r_{ij} + \Sigma w_l s_{lj} = \bar{0})$ *where* $r_{ij}, s_{lj} \in R$*. Or equivalently they are of the form* $\exists \bar{w}(A\bar{v} + B\bar{w} = \bar{0})$ *where A and B are matrices of coefficients.*

Remark 2.5. *The set of p.p. formulas is clearly closed under conjunction.*

Remark 2.6. *1. Let* $\psi(\bar{x})$ *be a p.p. formula (without parameters) in the language of modules. Then* $\psi(\bar{x})$ *will define a subgroup of* M^n *(where n is the length of* $\bar{x}$*). We call these p.p. subgroups of M.*

2. Let $\psi(\bar{x}, \bar{y})$ *be a p.p. formula (without parameters) in the language of modules. If* $\bar{a} \in M^m$*, then* $\psi(\bar{x}, \bar{a})$ *will define a coset of* $\psi(\bar{x}, \bar{0})$

3. Let $f : X \longrightarrow Y$ *be a function such that f (i.e. its graph), X and Y are all p.p. definable without parameters. Let* $\bar{y} \in Y$ *then*

i) $f^{-1}(\bar{y})$ *is a coset of* $f^{-1}(\bar{0}) = ker(f)$ *(by above (2)).*

ii)Suppose M is measurable, Remark 2.3(2) gives that all cosets must have the same dimension and measure. That is to say, $h(f^{-1}(\bar{y})) = h(f^{-1}(\bar{0}))$ *and* $Y = \{\bar{y} \in Y : h(f^{-1}(\bar{y})) = h(ker(f))\}$*.*

Definition 2.7. *An* INVARIANT SENTENCE *is one which expresses a fact of the form* $|G : H \cap G| \leq m$*, where H and G are p.p. subgroups of M. These are fixed if we fix a complete theory of modules.*

Theorem 2.8. *(Baur, Monk, also see [7], [5]) In the language of left R-modules for every formula* $\phi(\bar{x})$ *(without parameters) of L there is a formula* $\psi(\bar{x})$ *which is a boolean combination of p.p formulas and invariant sentences such that* $\models \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$*.*

N.B. If T is a complete theory then the invariant sentences are fixed, so every formula is equivalent to Boolean combination of p.p. formulas. If we allow parameters in the formulas we get that every formula is equivalent to a boolean combination of cosets of p.p. definable subgroups.

Remark 2.9. *All definable sets in M are defined by formulas of the form:*

$$\bigvee_{i=1}^{n} \phi_{i0}(\bar{x}, \bar{a}_{i0}) \wedge (\neg\phi_{i1}(\bar{x}, \bar{a}_{i1}) \wedge ... \wedge \neg\phi_{in_i}(\bar{x}, \bar{a}_{in_i}))$$

where $\phi_{ij}(\bar{x}, \bar{y})$ *are p.p., and* $M \models \phi_{ij}(\bar{x}, \bar{a}_{ij}) \rightarrow \phi_{i0}(\bar{x}, \bar{a}_{i0})$ *for all j.*

3 Main Result

Let $Def_{p.p.}(M)$ be the set of subgroups defined by p.p. formulas(without parameters).

Theorem 3.1. *Let M be a module.*
Suppose we have a function $h_p : Def_{p.p.}(M) \longrightarrow \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$ such that the following hold:

i) If $|X|$ is finite then $h_p(X) = (0, |X|)$.

ii) For $X, Y \in Def_{p.p.}(M)$, $f : X \to Y$ surjective, $\emptyset$-p.p. definable (i.e. without parameters), if $h_p(Y) = (d, \mu)$ and $h_p(ker(f)) = (e, \nu)$ then $h_p(X) = (d + e, \mu\nu)$.

iii) Let X and Y be p.p. subgroups, with $h_p(Y) = (m, \nu)$. Then

1. *If $|X : Y| = n < \omega$ and then $h_p(X) = (m, n \times \nu)$.*
2. *$|X : Y|$ infinite then $h_p(X) = (m', \nu')$ with $m' > m$.*

Then h_p extends uniquely to a measuring function on the whole of Def (M).

Proof:
Let $h_p = (dim_p, meas_p)$

First we need to show how we would assign dimension and measure to a definable set. From Remark 2.3 we know that all definable sets in M are defined be formulas of the form:

$$\bigvee_{i=1}^{n} \phi_{i0}(\bar{x}, \bar{a}_{i0}) \wedge (\neg\phi_{i1}(\bar{x}, \bar{a}_{i1}) \wedge ... \wedge \neg\phi_{in_i}(\bar{x}, \bar{a}_{in_i}))$$

where $\phi_{ij}(\bar{x}, \bar{y})$ are p.p. definable.

STEP ONE: (cosets)

Suppose X is defined by $\phi(\bar{x}, \bar{a})$ ($\phi(\bar{x}, \bar{y})$ p.p.), then X is a coset of the p.p. definable subgroup Y defined by $\phi(\bar{x}, \bar{0})$. We define $h(X) = h_p(Y)$.

STEP TWO:(one disjunct)

Let $Z = \{\bar{x} \in M^n : M \models \phi_0(\bar{x}, \bar{a}_0) \wedge \neg\phi_1(\bar{x}, \bar{a}_1) \wedge ... \wedge \neg\phi_m(\bar{x}, \bar{a}_m)\}$
$X_i = \{\bar{m} \in M^n : M \models \phi_i(\bar{m}, \bar{a}_i)\}$
with $\phi_i(\bar{x}, \bar{y})$ p.p. formulas

So $Z = X_0 \backslash (X_1 \cup ... \cup X_m)$.

By Clause (iii) we may assume that each X_i, $(i > 0)$ has finite index in X_0.
DEFINE:
$dim(Z) = dim(X_0)$
$meas(Z) = \mu(X_0) + \Sigma_{\Delta \subseteq \{1...n\}} meas(\cap_{i \in \Delta} X_i)(-1)^{|\Delta|}$

Remark 3.2. *1. We know $meas(\cap_{i\in\Delta} X_i)$ because the intersection of two cosets Ha and Hb is either empty or a coset of $H \cap G$. As in this case H and G are both p.p definable $H \cap G$ will also be p.p. definable.(Remark 2.5)*
2. If $\chi = \{\bar{x} \in M : M \models \cap_{i\in\{1...m\}} \phi(\bar{x}, \bar{0})\}$, and $|Z : \chi| = k$ (will be finite by clause (iii)) we get that $meas(Z) = k \times meas(\chi)$

STEP THREE: (Finite number of disjuncts)

Let, $Z = \{\bar{x} \in M^n : M \models \bigvee_{i=0}^{m}(\phi_{i0}(\bar{x}, \bar{a}_{i0}) \wedge \bigwedge_{j=1}^{n_i} \neg\phi_{ij}(\bar{x}, \bar{a}_{ij}))\}$
$X_{ij} = \{\bar{x} \in M^n : M \models \phi_{ij}(\bar{x}, \bar{a}_{ij})\}$
$Z_k = \{\bar{x} \in M^n : M \models (\phi_{k0}(\bar{x}, \bar{a}_{k0}) \wedge \bigwedge_{i=1}^{n_k} \neg\phi_{ki}(\bar{x}, \bar{a}_{ki}))\}$

So we have that: $Z_i = X_{i0} \setminus \bigcup_{j=1}^{n_i} X_{ij}$
$Z = \cup_{k=0}^{m} Z_k$

Let $d = max\{dim(Z_0), ..., dim(Z_m)\}$, and $dim(Z_0) = ... = dim(Z_t) = d$ and $dim(Z_{t+1}), ..., dim(Z_m) < d$.

DEFINE:

$$\begin{aligned} h(Z) &= h(\textstyle\bigcup_{k=0}^{t} Z_k) && \text{(By Clause (iii))} \\ &= (d, \Sigma_{\Delta \subseteq \{1...t\}} meas(\cap_{k\in\Delta} Z_k)(-1)^{|\Delta|+1}) \end{aligned}$$

N.B. We know $meas(\cap_{k\in\Delta} Z_k)$ as $\cap_{k\in\Delta} Z_k$ is a formula equivalent to one of the form in step two.

Remark 3.3. *Let* $K_{ij} = \{x \in M : M \models \phi_{ij}(x, \bar{0})\}$ $\chi = \bigcap_{i,j} K_{ij}$ *then* $meas(Z) = meas(\chi) \times |Z : \chi|$.

From [6] we have the following:

Theorem 3.4. *If we have* $h : Def(M) \longmapsto \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$ *with* $h(X) = (Dim(X), Meas(X))$ *such that the following hold:*

(i) For each L-formula $\varphi(x, \bar{y})$ *there is a finite set* $D \subset \mathbb{N} \times \mathbb{R}^{>0}$*, so that for all* $\bar{a} \in M^m$ *if* $\varphi(x, \bar{a}) \neq \emptyset$ *we have* $h(\varphi(M, \bar{a})) \in D$*. For each* $(n_i, \mu_i) \in D$ *the set* $\{\bar{y} : h(\varphi(M, \bar{y})) = (n_i, \mu_i)\}$ *is* $\emptyset$*-def in M.*

(ii) For each $n \in \omega$ *and* $\bar{a} \in M^n$ *we have* $h(\{\bar{a}\}) = (0, 1)$

(iii) For all $n \in \omega$*, and all disjoint def sets* $X_1, X_2 \subseteq M^n$ *we have that:*
if $dim(X_1) = dim(X_2)$ *then* $meas(X_1 \cup X_2) = meas(X_1) + meas(X_2)$
if $dim(X_1) < dim(X_2)$ *then* $meas(X_1 \cup X_2) = meas(X_2)$

(iv) For each $n \in \omega$ *and* $i \in \{1...n\}$ *the following hold.*
Let $X \subset M^n$ *be definable,* $\pi : M^n \to M$ *projection onto the* i^{th} *co-ordinate. And suppose there is a* (d, μ) *such that* $\forall a \in \pi(X)$ *we have* $h(\pi^{-1}(a) \cap X) = (d, \mu)$*. Then* $dim(X) = dim(\pi(X)) + d$ *and* $meas(X) = meas(\pi(X)) \times \mu$

Then M is measurable.

It is clear that (ii) and (iii) hold (by construction) for our assignment of dimension and measure.

For (i) we need to show that for every formula $\psi(x, \bar{y}) = \bigvee_{i=0}^{n} (\phi_{i0}(x, \bar{y}) \wedge \neg\phi_{i1}(x, \bar{y}) \wedge ... \wedge \neg\phi_{in_i}(x, \bar{y}))$ we have finite $D_\psi \subseteq \mathbb{N} \times \mathbb{R}^{>0}$ such that $\forall \bar{a} \in M^m$, $h(\psi(x, \bar{a})) \in D_\psi$.

We may assume that all the $\phi_{ij}(x, \bar{0})$ have the same dimension, that is to say (Clause (iii)) $| \cup_i \phi_{i0}(x, \bar{0}) : \phi_{ij}(x, \bar{0})| < \infty$. Now let $\chi = \bigcap_{i,j} \phi_{ij}(x, \bar{0})$, we know that for each $|\phi_{i0}(\bar{x}, \bar{0}) : \chi| = k_i$ (k_i finite). The possible values for $h(\psi(x, \bar{a}))$ are:

1) $(0, 0)$ if $M \models \neg\exists x \psi(x, \bar{a})$

2)$(dim(\phi_{00}(\bar{x}, \bar{0})), m \times meas(\chi))$, where $0 \leq m < \Sigma_i k_i$

This is because $\cup_i \phi_{i0}(x, \bar{a})$ is the union of up to $\Sigma_i k_i$ cosets of χ (sometimes there will be overlapping). In $\psi(x, \bar{a})$ some of these cosets could be empty, but clearly won't be the union of more than $\Sigma_i k_i$ cosets of χ. That is to say, $|\psi(x,\bar{a}) : \chi| = m \leq \Sigma_i k_i$, so by remark 3.3 $meas(\psi(x,\bar{a})) = m \times meas(\chi)$.

Clearly this gives a finite number of possibilities for D_ψ.

For (i) we have shown that we have a finite D_ψ, definability of this dimension and measure remains to be shown.

DEFINABILITY:

We want to show that any formula $\psi(x,\bar{y})$ and any $(d,\mu) \in D_\psi$ the set $\{\bar{y} : h(\psi(M,\bar{y})) = (d,\mu)\}$ is $\emptyset$-def in M. By Theorem 3.4(i) we can disregard the case where $\psi(M,\bar{y})$ is empty.

Case 1: $\psi(x,\bar{y})$ is p.p.

If $\psi(x,\bar{a})$ is consistent, then $\psi(x,\bar{a})$ is a coset of $\psi(x,\bar{0})$, $h(\psi(x,\bar{a})) = h(\psi(x,\bar{0}))$. $\{\bar{y} : h(\psi(M,\bar{y})) = h(\psi(x,\bar{0}))\}$ is defined by $\exists x(\psi(x,\bar{y}))$

Case 2: $\psi(x,\bar{y}) = \phi_0(x,\bar{y}) \wedge \neg\phi_1(x,\bar{y}) \wedge ... \wedge \neg\phi_m(x,\bar{y})$

Let $K_i = \{x \in M : M \models \phi_i(x,\bar{0})\}$ and $\chi = \bigcap_{i=1}^{n} K_i$ (this is a p.p. formula).We may assume $|K_0 : K_i|$ is finite. So $|K_0 : \chi|$ is finite. Suppose $|K_0 : \chi| = k$ and $h(\chi) = (d,\mu)$. Also let $X^i_{\bar{a}} = \{x \in M : M \models \phi_i(x,\bar{a})\}$, and $X_{\bar{a}} = \bigcup_{i\geq 1} X^i_{\bar{a}}$.

For any $\bar{a} \in M^m$, $h(\psi(x,\bar{a}))$ will depend on $|X_{\bar{a}} : \chi|$. If $|X_{\bar{a}} : \chi| = k_a$, then $h(\phi(x,\bar{a})) = (d,(k-k_a)\mu)$. We need to find $\emptyset$-def conditions for $|\bigcup_{i\geq 1} \phi_i(M,\bar{y}) : \chi| = k_y$.

We know from Ziegler [9] that as $X^i_{\bar{a}} \subseteq X_{\bar{a}}$, and $X_{\bar{a}}$ is the union of the $X^i_{\bar{a}}$ the following holds;

$$\Sigma_{\Delta\subseteq\{1,\dots,m\}}(-1)^{|\Delta|+1}|\cap_{i\in\Delta} X^i_{\bar{a}} : \chi| = |X_{\bar{a}} : \chi|$$

So it is enough to find $\emptyset$-def conditions for determining $|\cap_{i\in\Delta} X^i_{\bar{a}} : \chi|$.

Claim: Let $\bar{b}$ be another suitable length parameter set, $X^i_{\bar{b}}$ and $X_{\bar{b}}$ as above. For each $\Delta \subseteq \{1,...,m\}$,the following are equivalent:

1)$|\cap_{i\in\Delta} X^i_{\bar{a}} : \chi| = |\cap_{i\in\Delta} X^i_{\bar{b}} : \chi|$

2)$\bigcap_{i\in\Delta} X^i_{\bar{a}} = \emptyset$ if and only if $\bigcap_{i\in\Delta} X^i_{\bar{b}} = \emptyset$.

Proof of claim:

1) $\Longrightarrow$ 2): Suppose $|\cap_{i\in\Delta} X^i_{\bar{a}} : \chi| = |\cap_{i\in\Delta} X^i_{\bar{b}} : \chi|$. Then $\cap_{i\in\Delta}X^i_{\bar{a}} = \emptyset$ iff $0 = |\cap_{i\in\Delta} X^i_{\bar{a}} : \chi| = |\cap_{i\in\Delta} X^i_{\bar{b}} : \chi|$ iff $\cap_{i\in\Delta}X^i_{\bar{b}} = \emptyset$.

2) $\Longrightarrow$ 1): Suppose $\cap_{i\in\Delta}X_{a_i} = \emptyset$ iff $\cap_{i\in\Delta}X_{b_i} = \emptyset$. If $|\cap_{i\in\Delta} X_{a_i} : \chi| = 0$ then (1) is obvious. Suppose $|\cap_{i\in\Delta} X_{a_i} : \chi| = n \neq 0$, so $\cap_{i\in\Delta}X^i_{\bar{a}} \neq \emptyset$ and $\cap_{i\in\Delta}X^i_{\bar{b}} \neq \emptyset$ and therefore both are cosets of $\cap_{i\in\Delta}K_i$ (See Remark 3.2.1 above). So,

$$|\cap_{i\in\Delta} X_{a_i} : \chi| = |\cap_{i\in\Delta} K_i : \chi| = |\cap_{i\in\Delta} X_{b_i} : \chi|$$

.

Clearly $\bigcap_{i\in\Delta} X^i_{\bar{y}} = \emptyset$ is a $\emptyset$-definable condition.

Case 3: Disjunctions.

$$\begin{aligned} \textit{Let} \quad & \psi(x,\bar{y}) = \bigvee_{i=0}^{n}(\phi_{i0}(x,\bar{y}) \wedge \bigwedge_{j=1}^{n_i} \neg\phi_{ij}(x,\bar{y})). \\ & K_{ij} = \{x \in M : M \models \phi_{ij}(x,\bar{0})\}, \\ & X_{ij} = \{x \in M : M \models \phi_{ij}(x,\bar{a})\} \\ & Z = \bigcup_i^n (X_{i0} \backslash X_{i1} \ldots . X_{in_i}) \\ & \chi = \cap_{i,j} K_{ij}. \end{aligned}$$

We may assume that all the K_i have the same dimension, d say, and $h(\chi) = (d, \mu)$. Similarly to case 2, $h(Z) = (d, |X : \chi|\mu)$. As above this is determined by which intersections of the X_{ij} are empty, which is a $\emptyset$-def property. How the general case works can be seen through examples:

Examples: 3.5. *a) Suppose that* $\psi(x,\bar{y}) = \phi_0(x,\bar{y}) \vee \phi_1(x,\bar{y})$. *Define* $K_i = \{x \in M : M \models \phi_i(x,\bar{0})\}$. *Suppose that* $h(K_i) = (d_i, \mu_i)$ *and that* $h(K_0 \cap K_1) = (d_2, \mu_2)$. *Let* $X_i = \{x \in M : M \models \phi_i(x,\bar{a})\}$.

Then $X_0 \cup X_1$ *will have one of the following forms:*

	$\mathbf{h(X_0 \cup X_1)}$		$\emptyset-$***def formula***
1)$X_1, X_2 = \phi$	$(0,0)$		$\neg\exists x\phi_0(x,\bar{y}) \wedge \neg\exists x\phi_1(x,\bar{y})$
2)$X_0 = \phi X_1 \neq \phi$	(d_1, μ_1)		$\neg\exists x\phi_0(x,\bar{y}) \wedge \exists x\phi_1(x,\bar{y})$
3)$X_0, X_1 \neq \phi, X_1 \cap X_0 = \phi$	*a)*$d_0 > d_1$	(d_0, μ_0)	$\exists x\phi_0(x,\bar{y})$
	b)$d_1 = d_2$	$(d_1, \mu_0 + \mu_1)$	$\exists x\phi_0(x,\bar{y}) \wedge \exists x\phi_1(x,\bar{y})$ $\wedge\neg\exists x(\phi_0(x,\bar{y}) \wedge \phi_1(x,\bar{y}))$
4)$X_1 \cap X_2 \neq \phi$	*a)*$d_0 > d_1$	(d_0, μ_0)	*See* 3)*a)*
	b)$d_1 = d_2$	$(d_1, \mu_0 + \mu_1 - \mu 3)$	$\exists x(\phi_0(x,\bar{y}), \phi_1(x,\bar{y}))$

b) Take the formula $\psi(x,\bar{y}) = (\phi_{00}(x,\bar{y}) \wedge \neg\phi_{01}) \vee \phi_{10}(x,\bar{y})$. *Let* $|K_{ij} : \chi| = l_{ij}$, $|K_{00} \cap K_{10} : \chi| = t$, *and let* $meas(\chi) = \mu$. *Then* $(X_{00} \backslash X_{01}) \cup X_{10}$ *will have one of the following forms:*

		$\mathbf{h((X_{00}) \backslash X_{01}) \cup X_{10})}$	$\phi-$***def formula***
1)	$X_{00}, X_{10} = \phi$	$(0,0)$	$\neg\exists x\phi_{00}(x,\bar{y}) \wedge \neg\exists x\phi_{10}(x,\bar{y})$
2)	$X_{00} \neq \phi$ $X_{10}, X_{01} = \phi$	$(d, l_{00}\mu)$	$\exists x\phi_{00}(x,\bar{y}) \wedge \neg\exists x\phi_{10}(x,\bar{y})$ $\wedge\neg\exists x\phi_{01}(x,\bar{y})$
3)	$X_{00}, X_{01} \neq \phi$ $X_{10} = \phi$	$(d, (l_{00} - l_{01})\mu)$	$\exists x\phi_{00}(x,\bar{y}) \wedge \neg\exists x\phi_{10}(x,\bar{y})$ $\wedge\exists x\phi_{01}(x,\bar{y})$

4)	$X_{00}, X_{10} \neq \phi, X_{01} = \phi$		*See a)*	
5)	$X_{00}, X_{01}, X_{10} \neq \phi$ $X_{10} \cap X_{01}, X_{10} \cap X_{00} = \phi$	$(d,$	$(l_{00} - l_{01}$ $+l_{10})\mu)$	$\exists x\phi_{00}(x, \bar{y}) \wedge \exists x\phi_{10}(x, \bar{y})$ $\wedge\exists x\phi_{01}(x, \bar{y})$ $\wedge\neg\exists x(\phi_{10}(x, \bar{y}) \wedge \phi_{01}(x, \bar{y}))$ $\wedge\neg\exists x(\phi_{00}(x, \bar{y}) \wedge \phi_{10}(x, \bar{y}))$
6)	$X_{00}, X_{01}, X_{10} \neq \phi$ $X_{10} \cap X_{01}, X_{10} \cap X_{00} \neq \phi$	$(d,$	$(l_{00} - l_{01}$ $+l_{10} - t + 1)\mu)$	$\exists x\phi_{00}(x, \bar{y}) \wedge \exists x\phi_{10}(x, \bar{y})$ $\wedge\exists x\phi_{01}(x, \bar{y})$ $\wedge\exists x(\phi_{10}(x, \bar{y}) \wedge \phi_{01}(x, \bar{y}))$ $\wedge\exists x(\phi_{00}(x, \bar{y}) \wedge \phi_{10}(x, \bar{y}))$
7)	$X_{00}, X_{01}, X_{10}, X_{10} \cap X_{00} \neq \phi$ $X_{10} \cap X_{01}, = \emptyset$	$(d,$	$(l_{00} - l_{01}$ $+l_{10} - t + 1)\mu)$	$\exists x\phi_{00}(x, \bar{y}) \wedge \exists x\phi_{10}(x, \bar{y})$ $\wedge\exists x\phi_{01}(x, \bar{y})$ $\wedge\neg\exists x(\phi_{10}(x, \bar{y}) \wedge \phi_{01}(x, \bar{y}))$ $\wedge\exists x(\phi_{00}(x, \bar{y}) \wedge \phi_{10}(x, \bar{y}))$

We have now showed (i), (ii), and (iii) of Theorem 3.3. for this h.

To show 3.3.(iv), let $\psi(\bar{x}, \bar{a})$, K_{ij}, X_{ij}, χ, X defined as above (case 3). Let $\pi : M^n \to M$ be the projection onto the i^{th} co-ordinate (note this is a p.p. definable function defined on M^n). We may assume (Theorem 3.3.) that all the fibres of π restricted to X have the same dimension and measure. Let $\pi(X) = Y$ so we have that $h(\pi^{-1}(y) \cap X) = (d, \mu)$, $\forall y \in Y$.

Consider the projection π restricted to χ call this π' and let $\pi'(\chi) = \Upsilon$ as χ is p.p. definable we have that

$$h_p(\chi) = (dim_p(\Upsilon) + dim_p(ker(\pi')), meas_p(\Upsilon)meas_p(ker(\pi')))$$

Because χ has finite index in X and fibres of π on X are constant, each fibre $\pi^{-1}(a) \cap X, (a \in Y)$ must be covered by a constant (finite) number, m say, of translations of $ker(\pi')$. We therefore have that $dim(\pi^{-1}(a) \cap X) = dim(ker(\pi'))$ and $meas(\pi^{-1}(a) \cap X) = m \times meas(ker(\pi'))$. Similarly, $|Y : \Upsilon|$ must be finite (in fact $|Y : \Upsilon| \leq |X : \chi|$), so $dim_p(\Upsilon) = dim(Y)$.

$$\begin{aligned} dim(X) &= dim_p(\chi) \\ &= dim_p(\Upsilon) + dim_p(ker(\pi') \\ &= dim(Y) + dim(\pi^{-1}(a) \cap X) \end{aligned}$$

So we have what we want for the dimension part.
Now suppose $|Y : \Upsilon| = n$. So $meas(Y) = meas(\Upsilon) \times n$.

Claim: X is covered by mn cosets if χ.

Proof: Each of the m translates of $ker(\pi')$ that cover $\pi^{-1}(a) \cap X$ will map to the same coset of Υ in Y (i.e. the one that contains a, call this Υ_a). Also if x is in a translate of χ whose intersection with $\pi^{-1}(a) \cap X$ is not empty(but x

not necessarily in $\pi^{-1}(a) \cap X$), then x will also be mapped to Υ_a [a projection of a coset of χ is either empty or a coset of the projection of χ, i.e. Υ]. There are n such cosets of Υ. So $m \times n$ translates cover X. So:

$$\begin{aligned} meas(X) &= mn \times meas_p(\chi) \\ &= mn \times meas_p(\Upsilon) \times meas_p(ker(\pi')) \\ &= n \times meas_p(\Upsilon) \times m \times meas_p(ker(\pi')) \\ &= meas(Y) \times meas(\pi^{-1}(a) \cap X) \end{aligned}$$

So by Theorem 3.3. M is measurable.

Remark 3.6. *This extension is clearly unique.*

Remark 3.7. *We know from Lemma 3.3. in [3] that if Y is a definable subgroup of a measurable module M with measuring function h,then:*

$$h(\bigcup_{i=1}^{n} Ya_i) = (dim(Y), n \times meas(Y))$$

where Ya_i are disjoint cosets of Y. We therefore get that clauses (i)-(iii) in Theorem 3.1 are fulfilled by any measure function on a module, the measuring function on a measurable module is therefore unique (up to multiplication by constants).

Corollary 3.8. *If G and H are both measurable modules then so is $G \oplus H$.*

Proof:
Using Theorem 3.1. it is easy to show that if groups G and H are measurable then so is $G \oplus H$. By 3.1. it is enough to look at the p.p. definable subgroups and functions of $G \oplus H$.

We have measuring functions:
$h_G : Def(G) \longmapsto \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$
$h_H : Def(H) \longmapsto \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$

If $\varphi(\bar{x})$ defines a p.p subset of $(G \oplus H)^m$ then:

$$\begin{array}{cccc} \varphi((G \oplus H)^m) \equiv & \varphi(G^m) & \oplus & \varphi(H^m) \\ & \| & & \| \\ & G_\varphi & & H_\varphi \end{array}$$

So p.p. definable subsets of $G \oplus H$ will be of the form $G_\varphi \oplus H_\varphi$ where G_φ, H_φ are p.p. def subsets of G,H, respectively.

Now we need to define $h_p : Def_{p.p.}(G \oplus H) \mapsto \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$.

Suppose $h_G(G_\varphi) = (d_{G_\varphi}, m_{G_\varphi})$ and $h_H(H_\varphi) = (d_{H_\varphi}, m_{H_\varphi})$.

DEFINE $h_p(G_\varphi \oplus H_\varphi) = (d_{G_\varphi} + d_{H_\varphi}, m_{G_\varphi} \cdot m_{H_\varphi})$.

It is obvious that this will will fulfill clause (i) and (ii) of Theorem 3.1. Suppose that X, Y are p.p. subsets of $G \oplus H$ defined by ψ and ϕ respectively (i.e. $X = G_\psi \oplus H_\psi$ and $Y = G_\phi \oplus H_\phi$. Firstly suppose $|X : Y| = n$, then $|G_\psi : G_\phi| = n_1$ and $|H_\psi : H_\phi| = n_2$ with $n = n_1 n_2$. So,

$$\begin{aligned} dim_p(X) &= dim_G(G_\psi) + dim_H(H_\psi) \\ &= dim_G(G_\phi) + dim_H(H_\phi) \\ &= dim_p(Y) \end{aligned}$$

Similarly $meas(X) = n \times meas(Y)$.

Secondly, suppose $|X : Y|$ is infinite, then either $|G_\psi : G_\phi|$ or $|H_\psi : H_\phi|$ is infinite, so $dim_p(X) > dim_p(Y)$.

So by Theorem 3.1 $G \oplus H$ will be measurable.

4 Classification of Abelian Groups

In the following A will always be an abelian group. Also:

$\mathbb{Z}(n)$ will be the cyclic groups with n elements
$\mathbb{Z}(p^\infty)$ will be the p-prufer group.
$\mathbb{Z}_{(p)}$ will be the p-adics integers, or the integers localised at p.
$\mathbb{Q}$ will be the rationals.

Using Theorem 3.1 and Corollary 3.8 it is now easy to classify the Abelian groups into measurable and non measurable groups. We know that a theory being measurable implies that it is superstable of finite U-rank, so any abelian groups not of that form are discounted. We may also discount any group whose p.p. functions don't obey Fubini. We also know [6] that any ultraproduct of finite cyclic groups is measurable. From Szmielew [8] we know that the complete theories of abelian groups are of the form:

$$Th(\bigoplus_{\text{p prime}} [\oplus_{n>0}\mathbb{Z}(p^n)^{K_{(p,n)}} \oplus \mathbb{Z}(p^\infty)^{\lambda_p} \oplus \mathbb{Z}_{(p)}^{\mu_p}] \oplus \mathbb{Q}^\nu)$$

where $K_{(p,n)}, \lambda_p, \mu_p, \nu$ are cardinals $\leq \omega$.

So we are able to get a complete list of complete theories of abelian groups of finite U-rank, and decide which are measurable.

$GROUP$		$U-rank$	$Measurable$
$\mathbb{Z}(p)^\omega$		1	Y
$\mathbb{Z}(p^\infty)$		1	N
$\mathbb{Q}$		1	Y
$\mathbb{Z}_{(p)}$		1	N
$\oplus_{p \text{ prime}}\mathbb{Z}(p)$		1	Y
$\oplus_{i\in I}\mathbb{Z}(p_i^{n_{p_i}})^{k_{p_i}}$	$k_{p_i} < \omega$ $\lvert I\rvert$ infinite	1	Y
$\oplus_{i\in I}\mathbb{Z}(p_i{}^\infty)^{k_i}$	$k_i < \omega$	1	N
$\oplus_{i\in I}\mathbb{Z}_{(p_i)}^{k_i}$	$k_i < \omega$	1	N
$\mathbb{Z}_{(p)} \oplus \mathbb{Z}(p^\infty)$		1	Y
$\oplus_{i\in I}(\mathbb{Z}_{(p_i)} \oplus \mathbb{Z}(p_i^\infty))^{n_i}$	$n_i < \omega$	1	Y
$\oplus_{i\in I}\mathbb{Z}_{(p_i)}^{n_i} \oplus_{j\in J} \mathbb{Z}(p^\infty)^{m_j}$	with $n_i, m_j < \omega$ and either $I \neq J$ or $n_i \neq m_i$ for some i	1	N
$\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p)^\omega$		2	N
$\mathbb{Z}_{(p)} \oplus \mathbb{Z}(p)^\omega$		2	N
$\mathbb{Z}(p^\infty) \oplus \mathbb{Z}_{(p)} \oplus \mathbb{Z}(p)^\omega$		2	Y
$\oplus_{i=1}^m\mathbb{Z}(p_i^{n_i})^\omega$		$\Sigma_{i=1}^m n_i$	Y
$\oplus_{i\in I}\mathbb{Z}(p^\infty)^{n_i} \oplus_{j\in J} \mathbb{Z}(p_j^{m_j})^\omega$	$\lvert I\rvert, \lvert J\rvert$ finite	$\Sigma_{j\in J}m_j + 1$	N
$\oplus_{i\in I}\mathbb{Z}_{(p_i)}^{n_i} \oplus_{j\in J} \mathbb{Z}(p_j^{m_j})^\omega$	$\lvert I\rvert, \lvert J\rvert$ finite	$\Sigma_{j\in J}m_j + 1$	N
$\oplus_{i\in I}(\mathbb{Z}_{(p_i)} \oplus \mathbb{Z}(p_i^\infty))^{n_i} \oplus_{j\in J} \mathbb{Z}(p_j^{m_j})^\omega$	$n_i, m_j, \lvert J\rvert < \omega$	$\Sigma_{j\in J}m_j + 1$	Y

Remark 4.1. *By looking at this list we can see that the measurable abelian groups are precisely the pseudofinite abelian groups.*

5 $\mathbb{Q}(t)$-valued measure

In "Invariant measures on groups satisfying chain conditions"[2] van den Dries and Cifu Lopes introduced a $\mathbb{Q}(t)$-valued measure on boolean combinations of cosets of $\mathbb{Z}^n$. The set up in this paper is rather different from ours. The idea is to start with a group Ω and consider the following sets:

Set-up in [2]	*Equivalent in our set up*
$\mathcal{C}$: a set of subgroups of Ω closed under $\cap$	p.p. definable subgroups of M^n
$\mathcal{G} = \{aA : A \in \mathcal{C}, a \in \Omega\} \cup \{\emptyset\}$	p.p definable cosets of M^n
$\mathcal{A}$: the collection of finite unions of sets $X \backslash (Y_1 \cup ... \cup Y_m),\ X, Y_i \in \mathcal{G}$	Definable subsets of M^n

Proposition 5.1. *Let $\mathcal{A}_n$ be the algebra on $\mathbb{Z}^n$ generated by cosets of subgroups of $\mathbb{Z}^n$. In [2] [Proposition 1.1] it is proved that you can put a $\mathbb{Q}(t)$-valued measure, μ_n on each $\mathcal{A}_n$ with the following properties:*

$$(i)\mu_0(\mathbb{Z}^0) = 1$$
$$(ii)\mu_1(\mathbb{Z}^1) = t$$
$$(iii)\mu_n(X) = \mu_n(a + X), X \in \mathcal{A}_n$$
$$(iv)\mu_n(X) = \mu_{n+1}(X \times \{0\}), X \in \mathcal{A}_n$$
$$(v) \textit{if } X \in \mathcal{A}_n,\ X \neq \emptyset \textit{ then } \mu_n(X) \neq 0$$
$$(vi)\mu_{n+m}(X \times Y) = \mu_n(X) \times \mu_m(Y)$$

Definition 5.2. *Here a* MEASURE ON *$\mathcal{A}$ is a function $\mu : \mathcal{A} \to U$ (U an additive abelian group) such that for all disjoint $X, Y \in \mathcal{A}$ we have $\mu(X \cup Y) = \mu(X) + \mu(Y)$.*

This also being a measure on boolean combinations of cosets, it would make sense for there to be some sort of correspondence between our measuring function in modules and a $\mathbb{Q}(t)$-valued measure on definable subsets of modules. However, the Fubini property does not follow from being able to assign a measure as in 5.1 (for example $\mathbb{Z}$ doesn't obey Fubini) so we will have to add some extra conditions.

Theorem 5.3. *Let $\mathcal{A}_n$ be the definable sets in n variables, where M is a module as in section 3. M is measurable if and only if for every n, we have a measure $\mu_n : \mathcal{A}_n \to \mathbb{Q}(t)$ with the following properties:*

Let $X \in \mathcal{A}_n, Y \in \mathcal{A}_m$
$$(i)\mu_1(\{a\}) = 1$$
$$(ii)\mu_n(X) = \mu_n(aX)$$
$$(iii)\mu_n(X) = \mu_{n+1}(X \times \{0\})$$
$$(iv) \textit{If } X \neq \emptyset \textit{ then } \mu_n(X) \neq 0$$
$$(v)\mu_{n+m}(X \times Y) = \mu_n(X)\mu_m(Y)$$

(vi)Fubini: Let $f : X \to Y$ be definable and surjective, there are a finite number of $m_i \in \mathbb{Q}(t)$ such that $Y_i = \{\bar{y} \in Y : \mu_n(f^{-1}(\bar{y})) = m_i\}$ are disjoint and partition Y. We get $\mu_n(X) = \Sigma_i m_i \mu_m(Y_i)$.

In the proof the subscripts are occasionally omitted to make the proof clearer.

($\Longleftarrow$) Suppose we have $\mu : \mathcal{A} \to \mathbb{Q}(t)$. Consider a p.p. definable set X. Define $dim_p(X) = deg(\mu(X))$ and $meas_p(X) =$ leading coefficient(lc) of $\mu(X)$. By conditions (i)-(vi) it is clear that this function on p.p. definable subsets, will fulfill clause (i) and (ii) of Theorem 3.1.

Suppose $X \supseteq Y$, let Ya_i be disjoint cosets of Y in X. If $|X : Y| = n$ then $X = \bigcup_{i=1}^{n} Ya_i$, we have $\mu(Ya_i) = \mu(Y)$ so by definition 5.2, $\mu(X) = \Sigma_{i=1}^{n}\mu(Y) = n\mu(Y)$. So $dim_p(X) = deg(\mu(X)) = deg(n\mu(Y)) = dim_p(Y)$ and $meas_p(X) = lc(\mu(X)) = n \times lc(\mu(Y)) = n \times meas_p(Y)$. If $|X : Y|$ is infinte, suppose for contradiction $deg(\mu(X)) = deg(\mu(Y)) = d$ then for all $t \in \omega$ $X \supseteq \cup_{i=1}^{t} Ya_i$, so $lc(\mu(X)) > t \times lc(\mu(Y))$, clearly a contradiction.

So by Theorem 3.1 M will be measurable.

($\Longrightarrow$) Recall that in this set-up $\mathcal{C}$ is the set of p.p definable subgroups of M^n. We make use of the following lemma from [2].

Lemma 5.4. *Let $\mu : \mathcal{C} \to \mathbb{Q}(t)$ be a function such that $\mu(A) = d\mu(B)$ whenever $A, B \in \mathcal{C}$ and B is a subgroup of A of index d (finite). Then there is a unique left invariant $\mathbb{Q}(t)$-valued measure on $\mathcal{A}$ extending μ.*

For X a p.p definable subgroup of M define $\mu_n(X) = meas(X)t^{dim(X)}$. By Remark 3.7 we have that the condition in lemma 5.4 is satisfied (from Remark 3.7 we can also show that $meas(X) \in \mathbb{Q}$ (up to multiplication by constants)).

So by Lemma 5.4 we have a suitable left invariant μ. (i) is clear from the fact that $(dim, meas)(0) = (0, 1)$ and left invariance, (ii) is left invariance, (iii) follows from (v) and (i), (iv) are obvious.

For (v) we need $\mu(X \times Y) = \mu(X)\mu(Y)$. If X and Y are both p.p. definable (or cosets of p.p. def subgroups) then this is clear from our assignment. For simplicity let us consider the case where $X = X_0 \backslash X_1$, $Y = (Y_0 \setminus Y_1) \cup (Y_2 \setminus Y_3)$ where $X_0, X_1, Y_0, Y_1, Y_2, Y_3$ are all cosets of p.p. subgroups, and $X_1 \subseteq X_0$, $Y_1 \subseteq Y_0$, $Y_3 \subseteq Y_2$. The general case is a clear extension of this case. Using remark 2.5. and the fact that we know this is a measure we get the following:

$$\begin{aligned}
\mu(X) &= \mu(X_0) - \mu(X_1) \\
\mu(Y) &= \mu(Y_0) + \mu(Y_2) - \mu(Y_0 \cap Y_2) - \mu(Y_1 \setminus (Y_1 \cap Y_2)) \\
&\quad -\mu(Y_3 \setminus (Y_3 \cap Y_0)) - \mu(Y_1 \cap Y_3) \\
&= \mu(Y_0) + \mu(Y_2) - \mu(Y_0 \cap Y_2) - \mu(Y_1 \setminus Y_1) \\
&\quad + \mu(Y_1 \cap Y_2) - \mu(Y_3) + \mu(Y_3 \cap Y_0) - \mu(Y_1 \cap Y_3) \\
X \times Y &= X_0 \times Y_0 \cup X_0 \times Y_2 \setminus (X_1 \times Y_0) \cup (X_1 \times Y_2) \\
&\quad \cup (X_0 \times (Y_1 \cap Y_3)) \cup (X_0 \times (Y_1 \setminus (Y_1 \cap Y_2))) \\
&\quad \cup (X_0 \times (Y_3 \setminus (Y_3 \cap Y_0)))
\end{aligned}$$

Now we can calculate:

$$
\begin{aligned}
\mu(X \times Y) = \quad & \mu(X_0 \times Y_0) + \mu(X_0 \times Y_2) - \mu(X_0 \times (Y_1 \cap Y_2)) \\
& - \mu((X_1 \times Y_0) \cup (X_1 \times Y_2) \cup (X_0 \times (Y_1 \cap Y_3)) \\
& \cup (X_0 \times (Y_3 \setminus (Y_3 \cap Y_0))) \cup (X_0 \times (Y_1 \setminus (Y_1 \cap Y_2)))) \\
= \quad & \mu(X_0)\mu(Y_0) + \mu(X_0)\mu(Y_2) - \mu(X_0)\mu(Y_1 \cap Y_2) \\
& - \mu(X_1)\mu(Y_0) - \mu(X_1)\mu(Y_2) - \mu(X_0)\mu(Y_1 \cap Y_2) - \mu(X_0)\mu(Y_1) \\
& + \mu(X_0)\mu(Y_1 \cap Y_2) - \mu(X_0)\mu(Y_3) + \mu(X_0)\mu(Y_3 \cap Y_0) \\
& + \mu(X_1)\mu(Y_0 \cap Y_2) + \mu(X_1)\mu(Y_1 \cap Y_3) + \mu(X_1)\mu(Y_0 \cap Y_3) \\
& - \mu(X_1)\mu(Y_0 \cap Y_3) + \mu(X_1)\mu(Y_1) - \mu(X_1)\mu(Y_1 \cap Y_2) \\
& + \mu(X_1)\mu(Y_1 \cap Y_3) + \mu(X_1)\mu(Y_3) + \mu(X_1)\mu(Y_0 \cap Y_3) \\
& + \mu(X_1)\mu(Y_1 \cap Y_2) - \mu(X_1)\mu(Y_1 \cap Y_2) - \mu(X_1)\mu(Y_1 \cap Y_3) \\
= \quad & \mu(X_0)\mu(Y_0) + \mu(X_0)\mu(Y_2) - \mu(X_0)\mu(Y_1 \cap Y_2) \\
& - \mu(X_1)\mu(Y_0) - \mu(X_1)\mu(Y_2) - \mu(X_0)\mu(Y_1 \cap Y_2) - \mu(X_0)\mu(Y_1) \\
& + \mu(X_0)\mu(Y_1 \cap Y_2) - \mu(X_0)\mu(Y_3) + \mu(X_1)\mu(Y_0 \cap Y_2) \\
& + \mu(X_1)\mu(Y_1 \cap Y_3) + \mu(X_1)\mu(Y_0 \cap Y_3) + \mu(X_1)\mu(Y_1) \\
& - \mu(X_1)\mu(Y_1 \cap Y_2) + \mu(X_1)\mu(Y_3) - \mu(X_1)\mu(Y_0 \cap Y_3) \\
= \quad & \mu(X)\mu(Y)
\end{aligned}
$$

We now need to prove the fubini condition.

First we must show that given a formula, $\psi(\bar{x}, \bar{y})$, the set $\{\mu(\psi(\bar{x}, \bar{a})) : \bar{a} \in M^n\}$ is finite. Suppose we have the same set up as in chapter three, i.e.

$$
\begin{aligned}
let \quad & \psi(\bar{x}, \bar{y}) = \bigvee_{i=0}^{n} (\phi_{i0}(\bar{x}, \bar{y}) \wedge \bigwedge_{j=1}^{n_i} \neg\phi_{ij}(\bar{x}, \bar{y})). \\
& K_{ij} = \{x \in M : M \models \phi_{ij}(\bar{x}, \bar{0})\}, \\
& X_{ij} = \{x \in M : M \models \phi_{ij}(\bar{x}, \bar{a})\} \\
& X = \bigcup_i^n (X_{i0} \backslash X_{i1} \cup .... \cup X_{in_i}) \\
& Z_i = X_{i0} \backslash (X_{i1} \cup .... \cup X_{in_i}) \\
& \chi = \cap_{i,j} K_{ij}.
\end{aligned}
$$

We know from [2] that

$$\mu(X) = \Sigma_{\Delta \subseteq \{i \ldots n\}} \Sigma (-1)^{|\Delta|+1} \mu(\cap_{i \in \Delta} Z_i)$$

So it is enough to prove that there are a finite number of possible values for the measure of each $\cap_{i \in \Delta} Z_i$ as $\bar{a}$ varies. We can therefore consider just one disjunct:

$$
\begin{aligned}
& \zeta(\bar{x}, \bar{y}) = \phi_0(\bar{x}, \bar{y}) \wedge \bigwedge_{j=1}^{n} \neg\phi_j(\bar{x}, \bar{y}) \\
& \mu(\zeta(M^n, \bar{a})) = \mu(\phi_0(M^n, \bar{a})) + \Sigma_{\Delta \subset \{1 \ldots n\}} (-1)^{|\Delta|} \mu(\cap_{j \in \Delta} \phi_j(M^n, \bar{a}))
\end{aligned}
$$

Now as each of the $\cap_{j \in \Delta} \phi_j(M^n, \bar{a})$ are cosets of p.p. formulas (or empty), the possibilities for values of $\mu(\cap_{j \in \Delta} \phi_j(M^n, \bar{a}))$ are 0 (if empty) or $\mu(\cap_{j \in \Delta} \phi_j(M^n, \bar{0}))$. This gives us a finite number of possibilities for the values of $\mu(\psi(M^n, \bar{a}))$ as $\bar{a}$ varies.

Let $f : X \to Y$, there are a finite number of $m_i \in \mathbb{Q}(t)$ such that $Y_i = \{\bar{y} \in Y : \mu(f^{-1}(\bar{y})) = m_i\}$ are disjoint and partition Y. Also the $f^{-1}(Y_i)$ partition X, so $\mu(X) = \Sigma_{i=1}^{r} \mu(f^{-1}(Y_i))$. It is therefore enough to show $\mu(f^{-1}(Y_i)) = \mu(Y_i) m_i$. So it is enough to consider maps were all the fibres have the same measure, i.e. where $Y = \{\bar{y} \in Y : \mu(f^{-1}(\bar{y})) = m\}$.

By similar calculations we can assume that X is of the form $X_0 \setminus X_1 \cup ... \cup X_n$ where each X_i is a p.p. coset. For the case where $deg(\mu(X_i)) = deg(\mu(X_0))$ for all $i \in \{1, ..., n\}$, we know we have $\mu(X) = m\mu(Y)$, directly from measurability.

Suppose $f : X \to Y$, $X \subseteq M^p$, $Y \subseteq M^q$, consider $R = \{(\bar{x}, f(\bar{x}) : \bar{x} \in X\}$ and the projection π onto the last q coordinates. We have $\mu(R) = \mu(X)$ and $\mu(\pi^{-1}(y)) = \mu(f^{-1}(y))$ for $y \in Y$. We can therefore consider f to be a projection.

We are looking at $\pi : X(= X_0 \setminus X_1 \cup ... \cup X_n) \to Y$ a surjective projection with $|X_0 : X_i| = \infty$ for $i = 1, ..., t$ and $|X_0 : X_i| < \infty$ for $i = t + 1, ..., n$. When the X_i have the same degree as X_0 we get the required result. We can therefore incorporate these into X_0 and reduce to the case where $X = X_0 \setminus X_1 \cup ... \cup X_t$ with $|X_0 : X_i| = \infty$ for $i = 1, ..., t$. As π is a projection we may assume that $\pi = \pi_0|_{X_0}$ (i.e. π is defined on the whole of X_0). Let $\pi_0(X_0) = Y_0$, as all fibres of π have the same size $X_1 \cup ... \cup X_t$ must cover whole fibres of π_0, or the same size (proper) coset in each fibre of f_0.

Let F_0 be a fibre of π_0, $\mu(F_0)) = K_0$ (these are all the same size as X_0 is p.p), and F_1 be a fibre of π, $\mu(F_1) = K_1$. Suppose $X_1 \cup ... \cup X_l$ covers fibres of f_0 exactly, let $Y' = f_0(X_1 \cup ... \cup X_l)$. Each fibre has the same degree, so we know (by measurability) that $\mu(X_1 \cup ... \cup X_l) = \mu(Y')K$.

Suppose $X_{l+1} \cup ... \cup X_t$ covers the same sized (proper) coset in each fibre of $f_0|_{X_0 \setminus X_1 \cup ... \cup X_l}$. We may assume that $X_1 \cup ... \cup X_l$ and $X_{l+1} \cup ... \cup X_t$ are disjoint (as the fibres are definable and disjoint). Note $\mu(Y) = \mu(Y_0) - \mu(Y')$ as $X_{l+1} \cup ... \cup X_t$ doesn't cover any whole fibres so the image of the fibre containing these won't be empty. Also, $\mu(X_{l+1} \cup ... \cup X_t) = \mu(Y)\mu(F_0 \setminus F_1)$

$$
\begin{aligned}
\mu(X) &= \mu(X_0) - \mu(X_1 .... X_l) - \mu(X_{l+1} \cup ... \cup X_t) \\
&= \mu(Y_0)K_0 - \mu(Y')K_0 - \mu(X_{l+1} \cup ... \cup X_t) \\
&= (\mu(Y_0) - \mu(Y'))K_0 - \mu(X_{l+1} \cup ... \cup X_t) \\
&= \mu(Y)K_0 - \mu(X_{l+1} \cup ... \cup X_t) \\
&= \mu(Y)K_0 - \mu(Y)\mu(F_0 \setminus F_1) \\
&= \mu(Y)K_0 - \mu(Y)(K_0 - K_1) \\
&= \mu(Y)K_1
\end{aligned}
$$

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