Small representations of SL_2 in the finite Morley rank category

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In this article we consider representations of SL_2 which are interpretable in finite Morley rank theories, meaning that inside a universe of finite Morley rank we shall study the following definable objects: a group G isomorphic to SL_2, an abelian group V, and an action of G on V; V is thus a definable G-module on which G acts definably. Our goal will be to identify V with a standard G-module, under an assumption on its Morley rank. (A word on this notion will be said shortly, after we have stated the results.)

It will be convenient to work with a faithful representation, possibly replacing SL_2 by the quotient PSL_2, and we shall write $G \simeq (\mathrm{P})\mathrm{SL}_2$ to cover both cases.

Theorem. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a faithful action of G on V for which V is G-minimal. Assume $\operatorname{rk} V \leq 3 \operatorname{rk} \mathbb{K}$. Then V bears a structure of $\mathbb{K}$-vector space such that:*

- *either $V \simeq \mathbb{K}^2$ is the natural module for $G \simeq \mathrm{SL}_2(\mathbb{K})$, or*
- *$V \simeq \mathbb{K}^3$ is the irreducible 3-dimensional representation of $G \simeq \mathrm{PSL}_2(\mathbb{K})$ with $\operatorname{char} \mathbb{K} \neq 2$.*

On the way we shall establish the following interesting results.

Lemma 1.6. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a simple algebraic group G over $\mathbb{K}$, a torsion-free abelian group V, and a faithful action of G on V for which V is G-minimal. Then $V \rtimes G$ is algebraic.*

Proposition 2.3. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a non-trivial action of G on V. Then for v generic in V, $C_G^\circ(v)$ is semi-simple or unipotent (possibly trivial).*

All statements above involve the Morley rank of a structure; the reader should bear in mind that this is an abstract analog of the Zariski dimension, which can be axiomatized by some natural properties [3]. The Morley rank is however not necessarily related to any geometry or topology, being a purely model-theoretic notion. Yet in general if a field $\mathbb{K}$ has Morley rank k and V is an algebraic variety of Zariski-dimension d over $\mathbb{K}$, then its Morley rank is dk. The rank hypothesis in the Theorem would thus amount, if the configuration were known to be algebraic, to assuming that $\dim V \leq \dim G$; but of course the possibility for a field to have a finite Morley rank $k > 1$ makes algebraic geometry less general than our context.

We work in a ranked universe as in [3]. Indeed, the semi-direct product $V \rtimes G$ is a ranked group in the sense of Borovik and Poizat [7, Corollaire 2.14 and Théorème 2.15]. We shall not go too deeply into purely model-theoretic arguments but will merely use the natural, intuitive properties of Morley rank as a notion of dimension.

Let us now say a word about the proof of the Theorem. As we have mentioned, there is no geometry a priori on $V \rtimes G$, and our efforts will be devoted to retrieving a suitable vector space structure on V which arises from the action of G. Model-theoretically speaking, the main tool is Zilber's so-called Field Theorem [7, Théorème 3.7], which enables one to find an (algebraically closed) field inside a solvable, non-nilpotent, infinite group of finite Morley rank. A major difficulty is that the action of an algebraic torus of G will not induce a vector space structure on all of V. And even if such a good structure exists, this does not mean that G itself is linear on V. The 2-dimensional case relies on a theorem by Timmesfeld (Fact 1.1 below); as for dimension 3, we extend the field action manually and some curious computations will, in the end, prove linearity of G.

Now that we have said what the present paper is, let us say what it is not: it does not relate directly to the classification project for simple groups of finite Morley rank, although some rudimentary aspects of representation theory have been used there, via the amalgam method.

1 Preparatory Remarks

We shall use throughout a characterization of the natural module which is due to Timmesfeld.

Fact 1.1 ([9, Chapter I, Theorem 3.4]). *Let $\mathbb{K}$ be a field and let $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$. Let V be a faithful G-module. Suppose the following:*

(i). $C_V(G) = 0$

(ii). $[U, U, V] = 1$, *where U is a maximal unipotent subgroup of G.*

Let $0 \neq v \in C_V(U)$ and $W = \langle v^G \rangle$. Then there exists a field action of $\mathbb{K}$ on W such that W is the natural G-module. In particular $G \simeq \mathrm{SL}_2(\mathbb{K})$.

We shall use the non-standard notation $(+)$ to denote quasi-direct sum, i.e. the sum of two submodules (of a fixed module) which have a finite, possibly non-trivial, intersection.

1.1 On Malcev's Theorem

Fact 1.2 ([7, Théorème 3.18]). *Let G be a connected, solvable group of finite Morley rank acting definably and faithfully on a definable, abelian group A. If a definable subgroup $B \leq A$ is G- or G'-minimal, then B is centralized by G'.*

Lemma 1.3. *In a universe of finite Morley rank, consider the following definable objects: a reductive algebraic group G, a nilpotent group V, and an action of G on V. Let U be a unipotent subgroup of G. Then $V \rtimes U$ is nilpotent.*

Proof. We may assume that U is a maximal unipotent subgroup. In this case, and by reductivity of G, U is the commutator subgroup of the Borel subgroup $B = N_G(U)$ [1, top of p. 65]. Now consider $H = V \rtimes B$ and write $F^\circ(H) = V \rtimes K$ with $K \leq B$. The quotient $H/F^\circ(H) \simeq B/K$ is abelian by [3, Theorem 9.21], so $U = B' \leq K$. □

Corollary 1.4. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a quasi-simple algebraic group G over $\mathbb{K}$, an abelian group V, and a non-trivial action of G for which V is G-minimal. Then V has the same characteristic as $\mathbb{K}$.*

Proof. Let p denote the characteristic of $\mathbb{K}$. Fix a maximal unipotent subgroup U of G. By Lemma 1.3, $V \rtimes U$ is nilpotent. If $p = 0$ and V is torsion or if $p \neq 0$ and $pV = V$, then Nesin's structure theorem for nilpotent groups [3, Theorem 6.8] yields $[V, U] = 0$. As conjugates of U generate G, the action is trivial, a contradiction. □

1.2 Algebraicity in characteristic 0

We specialize [6] to our context.

Fact 1.5 (special case of [6, Theorem 4]). *In a universe of finite Morley rank, consider the following definable objects: an abelian, torsion-free group A, an infinite group S, and a faithful action of S on A for which A is S-minimal. Then there is a subgroup $A_1 \leq A$ and a field $\mathbb{K}$ such that $A_1 \simeq \mathbb{K}_+$ definably, and S embeds into $\mathrm{GL}_n(\mathbb{K})$ for some n.*

Lemma 1.6. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a simple algebraic group G over $\mathbb{K}$, a torsion-free abelian group V, and a faithful action of G on V for which V is G-minimal. Then $V \rtimes G$ is algebraic. Moreover, any definable subgroup of V has rank a multiple of* $\operatorname{rk} \mathbb{K}$.

Proof. The assumptions imply that G is interpretable in $\mathbb{K}$ as a pure field. By Fact 1.5, there is a field structure $\mathbb{L}$ such that $V \simeq \mathbb{L}_+^n$ and $G \hookrightarrow \mathrm{GL}_n(\mathbb{L})$ definably. $\mathbb{L}$ has of course characteristic 0. By a result of Macpherson and Pillay (see [8, Theorem 3]), G is Zariski-closed in $\mathrm{GL}_n(\mathbb{L})$; so far G and $V \rtimes G$ are algebraic groups over $\mathbb{L}$. In particular G *as a pure group* interprets $\mathbb{L}$, so $\mathbb{K}$ *as a pure field* interprets $\mathbb{L}$. It follows that $\mathbb{K} \simeq \mathbb{L}$ definably by [7, Théorème 4.15].

Now consider a definable subgroup V_1 of V. Then the setwise stabilizer of V_1 in $\mathbb{K}$ is a definable, non-trivial subgroup of $\mathbb{K}$, whence equal to $\mathbb{K}$ by [7, Corollaire 3.3]. Hence V_1 is a vector space over $\mathbb{K}$, which proves that its rank is a multiple of $\operatorname{rk} \mathbb{K}$. □

As a consequence, one can drastically simplify certain identification results *in characteristic* 0. For example, the following simplification of part of [5] results.

Theorem 1.7 ([5, Theorem A in char. 0])**.** *Let G be a connected, non-solvable group of finite Morley rank acting definably and faithfully on a* torsion-free *connected abelian group V of Morley rank 2. Then there is an algebraically closed field $\mathbb{K}$ of Morley rank 1 and characteristic 0 such that $V \simeq \mathbb{K}_+^2$, and G is isomorphic to $\mathrm{GL}_2(\mathbb{K})$ or $\mathrm{SL}_2(\mathbb{K})$ in its natural action.*

Proof. V is clearly G-minimal. By Fact 1.5, there is an interpretable field structure $\mathbb{K}$ such that $G \hookrightarrow \mathrm{GL}_n(\mathbb{K})$ with $V \simeq \mathbb{K}^n$. Clearly the dimension must be 2, making the rank of the field 1. So there is a field $\mathbb{K}$ of rank 1 such that $V \simeq \mathbb{K}_+^2$ and $G \hookrightarrow \mathrm{GL}_2(\mathbb{K})$. But definable subgroups of $\mathrm{GL}_2(\mathbb{K})$, especially over a field of rank 1, are known: [8, Theorem 5] together with connectedness and non-solvability of G this forces either $G \simeq \mathrm{GL}(V)$ or $G \simeq \mathrm{SL}(V)$. □

1.3 Around tori

Fact 1.8 ([10, Corollary 9])**.** *Let $\mathbb{K}$ be a field of finite Morley rank of characteristic $p > 0$. Then $\mathbb{K}^\times$ has no torsion-free definable section.*

A *good torus* is a definable, abelian, divisible group with no torsion-free definable section; the latter condition being equivalent to: every definable subgroup is the definable hull of its torsion subgroup. If one relaxes the requirement to: a definable, abelian, divisible group with no torsion-free definable *quotient*, one gets the definition of a *decent torus*; equivalently: the whole group is the definable hull of its torsion subgroup.

Wagner's Theorem 1.8 states that in finite Morley rank, the multiplicative group of a field of characteristic p is a good torus.

Lemma 1.9. *In a universe of finite Morley rank, consider the following definable objects: two infinite, abelian groups K and H, and a faithful action of K on H for which H is K-minimal. Suppose that H has exponent p and that K contains a non-trivial q-torus for each $q \neq p$. Then $\operatorname{rk} H = \operatorname{rk} K$.*

Proof. By Zilber's Field Theorem, there is a field structure $\mathbb{L}$ such that $K \hookrightarrow \mathbb{L}^\times$ and $H \simeq \mathbb{L}_+$. In particular, $\operatorname{char} \mathbb{L} = p$. Now $\mathbb{L}^\times/K$ is torsion-free, so by Wagner's Theorem, K cannot be proper in $\mathbb{L}^\times$. Hence $\operatorname{rk} K = \operatorname{rk} \mathbb{L} = \operatorname{rk} H$. □

Lemma 1.10. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$ of characteristic p, a definable subgroup $\Theta \leq \mathbb{K}^\times$, an abelian group V, and an action of Θ on V. Then there is $\theta \in \operatorname{Tor} \Theta$ such that $C_V(\Theta) = C_V(\theta)$ and $[V, \Theta] = [V, \theta]$.*

Proof. By Wagner's Theorem (Fact 1.8), $\Theta = d(\operatorname{Tor}\Theta)$. By the descending chain condition on centralizers, $C_V(\Theta) = C_V(\operatorname{Tor}\Theta) = C_V(\theta_1, \ldots, \theta_n)$ for torsion elements, and we take a generator θ_0 of the finite cyclic group $\langle\theta_1, \ldots, \theta_n\rangle$: one has $C_V(\Theta) = C_V(\theta_0)$, and this holds true of any root of θ_0.

Now the group $[V, \operatorname{Tor}\Theta]$ is definable, so $\Sigma = \{t \in \Theta : [V, t] \leq [V, \operatorname{Tor}\Theta]\}$ is a definable subgroup of Θ containing $\operatorname{Tor}\Theta$. Again, as $\Theta = d(\operatorname{Tor}\Theta)$, it follows that $\Sigma = \Theta$, that is $[V, \Theta] = [V, \operatorname{Tor}\Theta]$. We turn to the lattice of connected groups $\{[V, t] : t \in \operatorname{Tor}\Theta\}$: if t_1 is a root of t_2, then $[V, t_1] \geq [V, t_2]$. So by the ascending chain condition, there is $\theta \in \operatorname{Tor}\Theta$ such that $[V, \theta] = [V, \operatorname{Tor}\Theta] = [V, \Theta]$. We may assume that θ is a root of θ_0, and we are done. $\square$

1.4 Cohomological computations

Fact 1.11. *Let A be a connected, abelian group of finite Morley rank of bounded exponent and α a definable automorphism of finite order coprime to the exponent of A. Then $A = C_A(\alpha) \oplus [A, \alpha]$. Moreover, if $A_0 < A$ is a definable, connected, α-invariant subgroup, then $[A, \alpha] \cap A_0 = [A_0, \alpha]$.*

Proof. Let ad_α and Tr_α be the adjoint and trace maps, that is

$$\operatorname{ad}_\alpha(x) = x^\alpha - x \qquad \text{and} \qquad \operatorname{Tr}_\alpha(x) = x + \cdots + x^{\alpha^{n-1}}$$

where n is the order of α. It is easily seen, as A has no n-torsion, that $\ker \operatorname{ad}_\alpha \cap \ker \operatorname{Tr}_\alpha = 0$. In particular, $\operatorname{rk} A \geq \operatorname{rk}(\ker \operatorname{ad}_\alpha) + \operatorname{rk}(\ker \operatorname{Tr}_\alpha)$. Moreover, $\operatorname{im} \operatorname{ad}_\alpha \leq \ker \operatorname{Tr}_\alpha$ and $\operatorname{im} \operatorname{Tr}_\alpha \leq \ker \operatorname{ad}_\alpha$. It follows therefore that $\operatorname{rk} A \geq \operatorname{rk}(\ker \operatorname{ad}_\alpha) + \operatorname{rk}(\ker \operatorname{Tr}_\alpha) \geq \operatorname{rk}(\ker \operatorname{ad}_\alpha) + \operatorname{rk}(\operatorname{im} \operatorname{ad}_\alpha) = \operatorname{rk} A$, so $\operatorname{im} \operatorname{ad}_\alpha = \ker \operatorname{Tr}_\alpha$. Hence $A = \ker \operatorname{ad}_\alpha \oplus \ker \operatorname{Tr}_\alpha = \ker \operatorname{ad}_\alpha \oplus \operatorname{im} \operatorname{ad}_\alpha = C_A(\alpha) \oplus [A, \alpha]$.

Let $a_0 \in A_0$; then $a_0 \in \operatorname{ad}_\alpha(A_0)$ iff $\operatorname{Tr}_\alpha(a_0) = 0$ iff $a_0 \in \operatorname{ad}_\alpha(A)$. $\square$

Corollary 1.12. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$ of characteristic p, a connected subgroup T of $\mathbb{K}^\times$, an abelian p-group A, and an action of T on A. Then $A = C_A(T) \oplus [A, T]$. Let $A_0 < A$ be a definable, connected, T-invariant subgroup. Then $C_A(T)$ covers $C_{A/A_0}(T)$. Moreover, if T is a good torus, $C_T(A) = C_T(A_0, A/A_0)$.*

Proof. Since T is a decent torus we may apply Lemma 1.10 and find a torsion element $t_0 \in T$ such that $C_A(T) = C_A(t_0)$ and $[A, T] = [A, t_0]$. We use Fact 1.11 and deduce that $A = C_T(A) \oplus [A, T]$.

If $x \in A$ maps to an element in $C_{A/A_0}(t_0)$, then denoting the canonical projection by π one has $\pi \operatorname{ad}_{t_0}(x) = \operatorname{ad}_{t_0} \pi(x) = 0$. Hence $\operatorname{ad}_{t_0}(x) \in A_0$ and by Fact 1.11 there is $x_0 \in A_0$ such that $\operatorname{ad}_{t_0}(x) = \operatorname{ad}_{t_0}(x_0)$, whence $x \in x_0 + \ker \operatorname{ad}_{t_0}$, and $\ker \operatorname{ad}_{t_0} = C_A(t_0)$.

Now suppose that T is a good torus and let $\Theta = C_T(A_0, A/A_0)$; by assumption, Θ is a decent torus. Then $C_A(\Theta)$ covers $C_{A/A_0}(\Theta) = A/A_0$; it follows that $A = C_A(\Theta) + A_0 \leq C_A(\Theta)$. $\square$

1.5 Automorphisms of semi-direct products

Lemma 1.13. *In a universe of finite Morley rank, let A, T be definable, abelian, infinite groups such that A is T-minimal and the action is faithful. Let K be a definable group normalizing A and T. Then K centralizes T.*

Proof. We let K act on $\operatorname{End} A$ by:

$$s^{\varphi}(a) := (s(a^{\varphi^{-1}}))^{\varphi}$$

By assumption, K normalizes the image of T in $\operatorname{End} A$, which additively generates a definable algebraically closed field (this is the proof of Zilber's field theorem). In particular, as there are no definable groups of automorphisms of a field of finite Morley rank [3, Theorem 8.3], K acts trivially on T. $\square$

1.6 A three fields configuration

The following lemma will appear at a crucial moment in the proof of our main theorem, when dealing with the Cartan subalgebra of the adjoint representation of $(\mathrm{P})\mathrm{SL}_2$ (Proposition 3.13 below).

Lemma 1.14. *In a universe of finite Morley rank, consider the following definable objects: three infinite fields $\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3$, a connected group T acting on the underlying additive groups, and a map $B : \mathbb{K}_1 \times \mathbb{K}_2 \to \mathbb{K}_3$.*

Suppose that for each $i = 1, 2, 3$, $T/C_T(\mathbb{K}_i)$ acts on $(\mathbb{K}_i, +)$ as an infinite subgroup of $\mathbb{K}_i^{\times}$. Suppose further that $C_T^{\circ}(\mathbb{K}_1)$ is non-trivial in its action on $(\mathbb{K}_2, +)$. If B is bi-additive and globally T-covariant (in the sense that $B(k_1^t, k_2^t) = B(k_1, k_2)^t$), then either B is identically 0 or gives rise to a definable isomorphism $\mathbb{K}_1 \simeq \mathbb{K}_3$.

Proof. For the sake of clarity we shall write $k_1 \otimes k_2$ for $B(k_1, k_2)$. Moreover, we shall drop field multiplication operations. Last but not least, the action of t on k_i will be denoted by $t \cdot k_i$; as $T/C_T(\mathbb{K}_i)$ acts as a subgroup of $\mathbb{K}_i^{\times}$, one has $t \cdot (k_i k_i') = (t \cdot k_i)k_i'$, which allows simply writing $t \cdot k_i k_i'$.

Let $T_1 = C_T^{\circ}(K_1)$ and Θ be its image in $\mathbb{K}_2^{\times}$; by assumption, $\Theta \neq 1$. It follows that Θ additively generates $\mathbb{K}_2$.

First suppose that there exist $(k_1, k_2) \in K_1 \times K_2$ both non-zero such that $k_1 \otimes k_2 = 0$. By T_1-covariance and right additivity, it follows that $k_1 \otimes \mathbb{K}_2 = 0$. Now by T-covariance and left additivity, $\mathbb{K}_1 \otimes \mathbb{K}_2 = 0$: B is identically zero.

We may therefore suppose that for any $(k_1, k_2) \in \mathbb{K}_1 \times \mathbb{K}_2$ both non-zero, $k_1 \otimes k_2 \neq 0$. So any $k_2 \in \mathbb{K}_2 \setminus \{0\}$ induces a function $f_{k_2} : \mathbb{K}_1 \to \mathbb{K}_3$ given by

$$f_{k_2}(k_1) = (k_1 \otimes k_2)/(1 \otimes k_2)$$

We claim that this function actually does not depend on the choice of $k_2 \neq 0$. Let $k_2' \in \mathbb{K}_2$ be non-zero. As Θ additively generates $\mathbb{K}_2$, there are finitely many $t_i \in T_1$ such that $k_2' = \sum_i t_i \cdot k_2$. Let $k_1 \in \mathbb{K}_1$. Then by T_1-covariance,

$$\begin{aligned} k_1 \otimes k_2' &= \textstyle\sum_i [t_i \cdot (k_1 \otimes k_2)] = \sum_i [t_i \cdot (1 \otimes k_2) f_{k_2}(k_1)] \\ &= f_{k_2}(k_1) \textstyle\sum_i [t_i \cdot (1 \otimes k_2)] = f_{k_2}(k_1)(1 \otimes k_2') \end{aligned}$$

Since $k_2' \neq 0$, $1 \otimes k_2' \neq 0$, and dividing one finds $f_{k_2'}(k_1) = f_{k_2}(k_1)$, as desired.

So let $f : \mathbb{K}_1 \to \mathbb{K}_3$ be this function. Clearly $f(k_1) = f_1(k_1) = (k_1 \otimes 1)/(1 \otimes 1)$ is additive; we now show that it is multiplicative.

As the image of T in $\mathbb{K}_1^\times$ was asssumed to be non-trivial, it additively generates $\mathbb{K}_1$. It therefore suffices to show that f is multiplicative on (the image of) T. We shall denote by $\bar{t}$ the elements induced by t in $\mathbb{K}_1^\times$ and in $\mathbb{K}_2^\times$; in context, there is no risk of confusion. Let $s, t \in T$. Then

$$\begin{aligned}
f(\bar{s}\bar{t}) &= (\bar{s}\bar{t} \otimes 1)/(1 \otimes 1) = t \cdot (\bar{s} \otimes \bar{t}^{-1})/(1 \otimes 1) \\
&= t \cdot \left[(\bar{s} \otimes \bar{t}^{-1})/(1 \otimes \bar{t}^{-1})\right] \left[(1 \otimes \bar{t}^{-1})/(1 \otimes 1)\right] \\
&= t \cdot \left[f_{\bar{t}^{-1}}(\bar{s})(1 \otimes \bar{t}^{-1})/(1 \otimes 1)\right] \\
&= f(\bar{s}) \left[t \cdot (1 \otimes \bar{t}^{-1})/(1 \otimes 1)\right] \\
&= f(\bar{s}) \left[(\bar{t} \otimes 1)/(1 \otimes 1)\right] \\
&= f(\bar{s}) f(\bar{t})
\end{aligned}$$

So the function $f : \mathbb{K}_1 \to \mathbb{K}_3$ is a non-zero definable ring homomorphism between two infinite definable fields of finite Morley rank. It follows that it is a definable isomorphism. □

2 Actions of (P)SL$_2$

The present section is devoted to general actions of (P)SL$_2$ in the finite Morley rank category, with no assumption on the rank itself. Proposition 2.3 is our main result. The following notations will be adopted in §§2 and 3.

Notation 2.1. Let $G \simeq$ (P)SL$_2$. Fix a Borel subgroup B of G and let $U = B'$ be its unipotent radical. Let T be an algebraic torus such that $B = U \rtimes T$. Let i be the involution in T, and $\zeta \in N(T)$ a 2-element inverting T (the order of ζ depends on the isomorphism type of G).

Let us start with a classical observation.

Lemma 2.2. *A definable, connected subgroup of* (P)SL$_2$ *is semi-simple, unipotent, or contains a maximal unipotent subgroup of* (P)SL$_2$.

Proof. Let K be a definable, connected subgroup. We may assume that K is proper; as K is then solvable (see for instance [8, Théorème 4]), up to conjugacy $K \leq B$. Let U_1 be the unipotent radical of K; if K is not semi-simple, then $U_1 \neq 1$. If K is not unipotent either, that is if $K > U_1$, then we may split $K = U_1 \rtimes T_1$ for some non-trivial, semi-simple subgroup; so fixing $u \in U_1^\#$ one has $K \geq \langle u^K \rangle \geq \langle u^{T_1} \rangle = U_1$, as desired. □

2.1 Actions of (P)SL$_2$ and centralizers

Proposition 2.3. *In a universe of finite Morley rank, consider the following definable objects: a field* $\mathbb{K}$, *a group* $G \simeq$ (P)SL$_2(\mathbb{K})$, *an abelian group* V, *and a non-trivial action of* G *on* V. *Then for* v *generic in* V, $C_G^\circ(v)$ *is semi-simple or unipotent (possibly trivial).*

Proof. We first show that we may assume $C_V(G) = 0$. Assume the result holds when $C_V(G) = 0$ and let V be as in the statement. Let $V_0 = C_V(G) < V$. Since G is perfect, one has $C_{V/V_0}(G) = 0$, and the action of G on V/V_0 is non-trivial. By assumption, the result holds for V/V_0. Now let $v \in V$ be generic. Then $\bar{v} \in V/V_0$ is generic too, and in particular $C_G^\circ(\bar{v})$ is either semi-simple or unipotent. As $C_G^\circ(v) \leq C_G^\circ(\bar{v})$, we are done.

So from now on we suppose $C_V(G) = 0$. In Notation 2.1 we have fixed a maximal unipotent subgroup $U \leq G$, $B = N(U)$ its normalizer, T an algebraic torus such that $B = U \rtimes T$, and a 2-element ζ inverting T.

Let $v \in V$ be generic. $C_G^\circ(v)$ is proper in $(\mathrm{P})\mathrm{SL}_2$, hence solvable [8, Théorème 4]; up to conjugacy, $C_G^\circ(v) \leq B$. Assume that $C_G^\circ(v)$ is neither unipotent nor semi-simple. Then by Lemma 2.2, $C_G^\circ(v)$ contains U.

So $C_G^\circ(v) = U \rtimes T_v$ for some non-trivial $T_v \leq T$. The family $\{T_v : v \in V, U \leq C_G^\circ(v) \leq B\}$ of subgroups of T is uniformly definable; as $T \simeq \mathbb{K}^\times$ is a good torus, the family is finite [4, Rigidity II]. It follows that there is a *common* $T_0 \leq T$ such that generically, $C_G^\circ(v)$ is conjugate to $U \rtimes T_0$.

Now let $V_1 = C_V(U)$. Clearly V_1 is infinite, taking a B-minimal subgroup of V and applying Malcev's Theorem (Fact 1.2). As any two distinct conjugates of U generate G and $C_V(G) = 0$, V_1 must be disjoint from V_1^g for $g \notin B$. It follows that $N_G(V_1) = B$ and that V_1 is disjoint from its distinct conjugates. One therefore has

$$\operatorname{rk} V_1^G = \operatorname{rk} V_1 + \operatorname{rk} G - \operatorname{rk} B = \operatorname{rk} V_1 + \operatorname{rk} \mathbb{K}.$$

By assumption, the generic element of V is centralized by a conjugate of $U \rtimes T_0$. Thus V_1^G is generic in V. But furthermore, for v generic in V_1, $C_G^\circ(v)$ is a conjugate of $U \rtimes T_0$ containing U; conjugacy is therefore obtained by an element of $N(U) = B$. As $B' = U$, $U \rtimes T_0$ is normal in B; hence $C_G^\circ(v) = U \rtimes T_0$. This means that T_0 centralizes a generic subset X of V_1; as $X + X = V_1$ it follows that $V_1 = C_V(U \rtimes T_0)$.

Let $W = V_1 \oplus V_1^\zeta$ and $\check{W} = W \setminus (V_1 \cup V_1^\zeta)$. The generic element of W is in $\check{W}$. Let $v \in \check{W}$. Clearly $T_0 \leq C_G^\circ(v)$. Moreover, if $C_G^\circ(v)$ is not semi-simple, then it must meet a unipotent subgroup which can only be either U or U^ζ as $1 \neq T_0 \leq C_G^\circ(v)$. In that case, $C_G^\circ(v)$ contains either U or U^ζ by Lemma 2.2, against the definition of $\check{W}$. This means that for $v \in \check{W}$, one has $T_0 \leq C_G^\circ(v) \leq T$. In particular, $\check{W}^G$ is *not* generic in V.

It follows that $W < V$. As V_1^G is generic in V, W cannot be G-invariant. Therefore $T \cdot \langle\zeta\rangle \leq N_G(W) < G$, and equality follows from maximality of $T \cdot \langle\zeta\rangle$. As $T \cdot \langle\zeta\rangle$ also normalizes $V_1 \cup V_1^\zeta$, one sees that $N_G(\check{W}) = T \cdot \langle\zeta\rangle$.

Let $w \in \check{W}$. Assume that $w \in \check{W}^g$ for some $g \in G$. Then $C_G^\circ(v)$ is a non-trivial connected subgroup of T, so $C_G(C_G^\circ(v)) = T = T^g$, and $g \in N(T) = T \cdot \langle\zeta\rangle = N(\check{W})$. This implies that

$$\operatorname{rk} \check{W}^G = \operatorname{rk} \check{W} + \operatorname{rk} G - \operatorname{rk} T = 2 \operatorname{rk} V_1 + 2 \operatorname{rk} \mathbb{K} = 2 \operatorname{rk} V_1^G.$$

But V_1^G is already generic in V which is infinite: this is a contradiction. $\square$

Corollary 2.4. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a non-trivial action of G on V. Then* $\operatorname{rk} V \geq 2 \operatorname{rk} \mathbb{K}$.

2.2 Four-groups of PSL_2

We finish this section with an easy but useful relation on ranks when the characteristic is not 2. Given a definable, involutive automorphism j of an abelian group W of finite Morley rank with no involutions, one has $W = W^{+_j} \oplus W^{-_j}$ with obvious notations [3, Exercise 14 p.73].

Lemma 2.5. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq \mathrm{PSL}_2(\mathbb{K})$, an abelian group V* of characteristic not 2, *and a non-trivial action of G on V. Then ζ is an involution and*

$$\operatorname{rk} V = \operatorname{rk}\left(V^{+_i+_\zeta}\right) + \frac{3}{2}\operatorname{rk}\left(V^{-_i}\right)$$

Proof. Write $V = V^{+_i} \oplus V^{-_i}$, then $V^{+_i} = V^{+_i+_\zeta} \oplus V^{+_i-_\zeta}$ and $V^{-_i} = V^{-_i+_\zeta} \oplus V^{-_i-_\zeta}$. Let $a = \operatorname{rk} V^{+_i+_\zeta}$ and $b = \operatorname{rk} V^{+_i-_\zeta}$. Clearly, $b = \operatorname{rk} V^{-_i+_\zeta} = \operatorname{rk} V^{-_i-_\zeta}$. There follows $\operatorname{rk} V^{-_i} = 2b$ and $\operatorname{rk} V = a + 3b$. $\square$

3 Proof of the Theorem

We now attack our main result.

Theorem. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a faithful action of G on V for which V is G-minimal. Assume* $\operatorname{rk} V \leq 3 \operatorname{rk} \mathbb{K}$. *Then V bears a structure of $\mathbb{K}$-vector space such that:*

- *either $V \simeq \mathbb{K}^2$ is the natural module for $G \simeq \mathrm{SL}_2(\mathbb{K})$, or*
- *$V \simeq \mathbb{K}^3$ is the irreducible 3-dimensional representation of $G \simeq \mathrm{PSL}_2(\mathbb{K})$ with* $\operatorname{char} \mathbb{K} \neq 2$.

Notation 3.1. In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a non-trivial action of G on V for which V is G-minimal. Assume $\operatorname{rk} V \leq 3 \operatorname{rk} \mathbb{K}$.

One should also bear in mind Notation 2.1 which introduces the usual elements and subgroups of $(\mathrm{P})\mathrm{SL}_2$.

Notation 3.2. Let $k = \operatorname{rk} \mathbb{K}$ and write $\operatorname{rk} V = 2k + \nu$.

Notice that $0 \leq \nu \leq k$ by Corollary 2.4 and our assumption that $\operatorname{rk} V \leq 3 \operatorname{rk} \mathbb{K}$. Moreover, if $\nu = 0$, then by [5, Theorem B] (which is a consequence, in finite Morley rank, of Timmesfeld's identification result, Fact 1.1), we are done.

So we suppose $\nu > 0$ throughout. Our goal is to show that the characteristic is not 2, $\nu = k$, and $G \simeq \mathrm{PSL}_2$ acts on $V \simeq \mathbb{K}^3$ in the usual irreducible way.

If V has characteristic 0, then by Lemma 1.6, $V \rtimes G$ or $V \rtimes G/Z(G)$ is algebraic; $\dim_{\mathbb{K}} V$ is 2 or 3, and as irreducible algebraic representations of $(\mathrm{P})\mathrm{SL}_2$ are well-known, the analysis already ends. From now on, we suppose $\operatorname{char} \mathbb{K}$ to be a prime number p. The proof will involve studying various submodules of V, defining a field action piecewise, and eventually proving its linearity. On our way we shall prove $p \neq 2$.

Lemma 3.3. *We may suppose that $C_V(G) = 0$.*

Proof. Suppose our Theorem holds for modules with a trivial right-kernel. Notice that by G-minimality, $W = C_V(G)$ is finite. It follows that there is no right kernel for G on the G-minimal module $\bar{V} = V/W$; so the result holds for the action of G on $\bar{V}$. In particular, as we have assumed $\operatorname{rk} V > 2k$, we find that $\operatorname{char} \mathbb{K} \neq 2$ and $G \simeq \mathrm{PSL}_2(\mathbb{K})$, so that $\langle i, \zeta \rangle$ is a four-group. We also know that ζ inverts a set of rank $2k$ in $\bar{V}$.

It follows that ζ inverts a set of rank $\geq 2k$ in V. Hence $\operatorname{rk} V^{-\zeta} \geq 2k$, and Lemma 2.5 implies that $V^{+i+\zeta}$ is finite. As $\operatorname{char} \mathbb{K} \neq 2$, the group $V^{+i+\zeta}$ is clearly connected, and we deduce that $W \leq V^{+i+\zeta} = 0$. $\square$

We next need to introduce an ad hoc analog of the Cartan algebra (Proposition 3.6), and then a weight module decomposition (Proposition 3.16). Characteristic 2 will be eliminated shortly before structural identification (Proposition 3.17).

3.1 T-invariant sections

Our finer study of subspaces starts here. A word on terminology: if K is a group acting on a definable, connected, abelian group V, we shall call V a K-module. In particular, K-submodules are by definition definable and connected. The connectedness requirement reflects however less a necessary assumption than a general methodological line.

In this subsection only, we work with abstract T-modules of finite Morley rank which need not relate to our current representation V, but T is still the multiplicative group of a field of finite Morley rank and characteristic p.

Definition 3.4. Call a T-module X *degenerate* if $C_T^\circ(X) \neq 1$.

We now consider $\operatorname{cork} C_T^\circ(X) = \operatorname{rk}(T/C_T^\circ(X))$.

Lemma 3.5.

(i). *Let X be a T-module. Then* $\operatorname{cork} C_T^\circ(X) \leq \operatorname{rk} X$.

If X is degenerate, then $\operatorname{cork} C_T^\circ(X) < \operatorname{rk} X$.

(ii). *Let X be a $T \cdot \langle \zeta \rangle$-module. Then* $\operatorname{cork} C_T^\circ(X) \leq \frac{\operatorname{rk} X}{2}$.

If X is degenerate, then $\operatorname{cork} C_T^\circ(X) < \frac{\operatorname{rk} X}{2}$.

Proof. Let $\Theta = C_T^\circ(X)$.

(i). Let $0 = X_0 < X_1 < \cdots < X_n = X$ be a maximal series of T-modules, and $\Theta_i = C_T^\circ(X_i/X_{i-1})$. As $\Theta = (\cap_i \Theta_i)^\circ$ by Corollary 1.12, one has $\operatorname{cork}\Theta \leq \sum_i \operatorname{cork}\Theta_i$. So we may assume that X itself is T-minimal.

By Zilber's Field Theorem, there is a field structure $\mathbb{L}$ such that T/Θ embeds into $\mathbb{L}^\times$ and $X \simeq \mathbb{L}_+$; the first claim follows. If in addition X is degenerate, that is if $\Theta \neq 1$, then by Wagner's Theorem Θ must contain non-trivial torsion; as Θ is connected it follows that $T/\Theta \not\simeq \mathbb{L}^\times$, and the embedding is proper, whence the second claim.

(ii). Considering a maximal series of $T \cdot \langle \zeta \rangle$-modules, we may now assume that X is $T \cdot \langle \zeta \rangle$-minimal.

Let $Y \leq X$ be a T-minimal T-submodule. If $Y < X$, then $Y \cap Y^\zeta$ is finite, and $X = Y(+)Y^\zeta$. Moreover $C_T^\circ(Y) = \Theta$. Applying (i) we find $\operatorname{cork}\Theta \leq \operatorname{rk} Y = \frac{\operatorname{rk} X}{2}$, the inequality being strict if X is degenerate.

We now suppose that $Y = X$, that is X is T-minimal. But now Lemma 1.13 forces the action of ζ to be trivial on T/Θ, whence $\Theta = T$, and the claim is obvious. □

3.2 The largest degenerate sumodule

We specialize these ideas to our current setting (Notation 3.1).

Proposition 3.6. *The largest degenerate T-submodule X of V exists; it has rank ν, and its conjugates are generic in V. Moreover submodules of V/X have rank divisible by k.*

Proof.

Step 1. There is a non-trivial degenerate T-submodule of V.

Proof: Suppose not. Let $V_1 \leq V_2 \leq V$ be B-submodules, with V_1 and V_2/V_1 B-minimal. Notice that by Malcev's Theorem (Fact 1.2), both V_1 and V_2/V_1 are even T-minimal. Notice further that $V_2 < V$, as otherwise the action is quadratic, and Fact 1.1 yields a contradiction.

If $\operatorname{rk} V_1 \neq k$ then by Lemma 1.9 $T_1 = C_T^\circ(V_1)$ must be infinite; taking $C_V^\circ(T_1) \geq V_1$ we are done. So we may assume $\operatorname{rk} V_1 = k$.

Suppose $\operatorname{rk} V_2/V_1 \neq k$. As V_2/V_1 is T-minimal, the group $T_2 = C_T^\circ(V_2/V_1)$ is non-trivial by Lemma 1.9. Since T_2 is a decent torus, $C_{V_2}(T_2)$ covers V_2/V_1 by Corollary 1.12, so $C_{V_2}^\circ(T_2)$ is non-trivial; in particular $C_V^\circ(T_2) \neq 1$: we are done.

So suppose $\operatorname{rk} V_2/V_1 = k$, that is $\operatorname{rk} V_2 = 2k$, and let $W_2 = (V_2 \cap V_2^\zeta)^\circ$. Clearly $\operatorname{rk} W_2 \geq 2k - \nu > 0$. If $(V_1 \cap W_2)^\circ \neq 0$, then by T-minimality of V_1, one has $V_1 \leq W_2$. By T-minimality of V_2/V_1, one finds that W_2 is either V_1 or V_2, a contradiction as neither is ζ-invariant since they are B-invariant and proper.

Therefore $(V_1 \cap W_2)^\circ = 0$, and in particular $V_2 = V_1(+)W_2$; whence W_2 is T-minimal, and ζ-invariant. As ζ inverts T, Lemma 1.13 then forces T to centralize W_2: we are done. $\Diamond$

Step 2. Any degenerate T-submodule of V has rank $\leq \nu$; equality holds iff its G-conjugates are generic.

Proof: Let X be degenerate and $\Theta = C_T^\circ(X) \neq 1$. We first claim that for x generic in X, $C_G^\circ(x)$ is semi-simple. Otherwise, as $C_G^\circ(x)$ contains $\Theta \leq T$, it contains either U or U^ζ; we may assume that for x generic in X, U centralizes x. Thus U centralizes X. As the latter is ζ-invariant, it follows that $G = \langle U, U^\zeta \rangle$ centralizes X, a contradiction.

Hence, the centralizer in G of the generic element of X is semi-simple. Let $x \in X$ be generic, and suppose that $g \in G$ is such that $x \in X^g$. Then $\langle \Theta, \Theta^g \rangle \leq C_G^\circ(x)$ which is semi-simple, so $C_G^\circ(\langle \Theta, \Theta^g \rangle)$ is an algebraic torus, which can be only $C_G^\circ(\Theta) = T$, and only T^g for a similar reason. Hence $g \in N_G(T) = T \cdot \langle \zeta \rangle = N_G(X)$. So X is generically disjoint from its distinct conjugates; it follows that

$$\operatorname{rk} X^G = 2k + \operatorname{rk} X \leq \operatorname{rk} V = 2k + \nu$$

Hence $\operatorname{rk} X \leq \nu$ and equality holds iff X^G is generic in V. $\Diamond$

Step 3. The sum of two degenerate T-submodules is degenerate.

Proof: Let X_1, X_2 be degenerate T-submodules, and $\Theta_i = C_T^\circ(X_i) \neq 1$. Considering $\hat{X}_i = C_V^\circ(\Theta_i) \geq X_i$, we may assume that the X_i are $T \cdot \langle \zeta \rangle$-modules.

By Lemma 3.5 (ii) $\operatorname{cork}_T \Theta_i < \frac{\operatorname{rk} X_i}{2}$, so using Step 2 $\operatorname{rk} \Theta_i > k - \frac{\nu}{2} \geq \frac{k}{2}$. It follows that $\Theta_{12} = (\Theta_1 \cap \Theta_2)^\circ$ is non-trivial. Now $X_{12} = C_V^\circ(\Theta_{12})$ contains $X_1 + X_2$. $\Diamond$

We may then let X be the sum of all degenerate T-submodules; by Step 2, $\operatorname{rk} X \leq \nu$.

Step 4. $\operatorname{rk} X = \nu$; non-trivial proper submodules of V/X have rank k.

Proof: We now consider a series $X = X_0 < X_1 < \dots X_m = V$ of T-modules with T-minimal factors X_i/X_{i-1} for $i \geq 1$ (X itself need not be T-minimal). Let $\Theta_i = C_T^\circ(X_i/X_{i-1})$. If there is $i \geq 1$ with $\Theta_i \neq 1$, then Θ_i centralizes X_i/X_{i-1} so by Corollary 1.12, $X_i = X_{i-1} + C_{X_i}(\Theta_i) \leq X_{i-1} + X = X_{i-1}$: a contradiction. So for any $i \geq 1$, one has $\Theta_i = 1$, meaning that $T/\Theta_i \simeq T \simeq \mathbb{K}^\times$.

Lemma 1.9 then implies $\operatorname{rk}(X_i/X_{i-1}) = k$. In particular, X has rank ν, and is followed by two T-minimal factors of rank k. $\Diamond$

This concludes the proof of Proposition 3.6. $\square$

Corollary 3.7. *Let $u \in U^\#$. Then $C_V(u, u^\zeta) = 0$.*

Proof. Let $v \in C_V^\#(u)$ and $H = C_G^\circ(v) < G$. Since u normalizes H, one has $H \leq B$. If in addition $v \in C_V(u^\zeta)$ then $H \leq B \cap B^\zeta = T$. So T, u, u^ζ normalize H, forcing $H = 1$. But then v^G is generic in V, so by Proposition 3.6 $H \neq 1$, a contradiction. $\square$

3.3 Notation storm

Notation 3.8. Let X be the largest degenerate T-submodule of V, and let $\Theta = C_T^\circ(X)$. Let $\theta_0 \in \operatorname{Tor}\Theta$ be given by Lemma 1.10 (for the action of Θ on V), so that $X = C_V(\theta_0)$ and $[V, \Theta] = [V, \theta_0]$.

We shall eventually prove that $T = \Theta$ centralizes X; the proof is a bit technical (Proposition 3.13). Let us first introduce a useful object.

Notation 3.9. Let $M = [V, \Theta] = [V, \theta_0]$ (see Notation 3.8).

One has $V = M \oplus X$ by Corollary 1.12. Moreover, by Proposition 3.6, non-trivial proper T-submodules of M are T-minimal and have rank k.

We also need to study V as a U-module.

Notation 3.10. For $i \geq 0$, Z_i denotes the connected component of the i^{th} center of the action of U on V.

The series is strictly increasing up to V by nilpotence of $V \rtimes U$.

We actually wish to study the interplay of our series Z_i with degenerate submodules. Each Z_i is B-invariant, so each Z_i is acted on by Θ and splits as $Z_i = C_{Z_i}(\Theta) \oplus [Z_i, \Theta]$ by Corollary 1.12.

Notation 3.11. Let $X_i = C_{Z_i}(\Theta) = C_{Z_i}^\circ(\Theta) = (Z_i \cap X)^\circ$ and $M_i = [Z_i, \Theta] = (Z_i \cap M)^\circ$.

3.4 Finer study of X

We prove in this section that T centralizes X (Proposition 3.13). This will involve a three-fields argument relying on Lemma 1.14.

Lemma 3.12. *$X = X_1 + C_X(T)$.*

Proof. By Corollary 1.12 it suffices to show that T centralizes X/X_1; by the same it actually suffices to show that T centralizes every quotient X_i/X_{i-1} (bear in mind Notation 3.11). So fix $i \geq 2$; we may assume that $\bar{X}_i = X_i/X_{i-1}$ is non-trivial. Let us prove that $[\bar{X}_i, T] = 0$.

Supposing the contrary, there exists a T-minimal submodule $A \leq [\bar{X}_i, T]$. By Corollary 1.12, $C_A(T) = 0$, and by Zilber's Field Theorem there is a definable field structure $\mathbb{L}$ such that $A \simeq \mathbb{L}_+$ and T induces an infinite subgroup of $\mathbb{L}^\times$.

For the sake of notation, we shall write $\bar{Z}_{i-1}$ for Z_{i-1}/Z_{i-2}. Commutation yields a map $X_i \times U \to Z_{i-1}$ which in turns gives rise to a map $\bar{X}_i \times U \to \bar{Z}_{i-1}$; the latter is bi-additive since U acts trivially on $\bar{Z}_{i-1}$. We restrict it to a definable, bi-additive map $B : A \times U \to \bar{Z}_{i-1}$, which is obviously T-covariant.

We wish to apply Lemma 1.14. A and U are the additive groups of fields on which T acts as desired, but we need also take care of the image group $\bar{Z}_{i-1}$. We shall retrieve a field structure by going to a suitable quotient, which will be a section of M.

Suppose that $a \in A, u \in U$ are both non-zero, and that $B(a, u) \in \bar{X}_{i-1}$. As Θ (Notation 3.8) additively generates $\mathbb{K}_+$, one has $B(a, U) \leq \bar{X}_{i-1}$. Since

$\mathrm{rk}(\bar{X}_{i-1}) < \mathrm{rk}\, X \leq k = \mathrm{rk}\, U$, there is therefore $u \in U$ not zero such that $B(a,u) = 0$. Using the same argument, we find $B(a,U) = 0$; this means that $[x,U] \subseteq Z_{i-2}$ for any $x \in X$ such that $a = (x \mod X_{i-1})$. But $X_{i-1} = (X \cap Z_{i-1})^\circ$, that is X_{i-1} is the connected component of $\{x \in X_i : [x,U] \subseteq Z_{i-2}\}$; since $X_i > X_{i-1}$, there exists $a \in A$ such that $B(a,U) \notin \bar{X}_{i-1}$ (and this is even true for generic a).

Let π be the canonical projection $\bar{Z}_{i-1} \to \bar{Z}_{i-1}/\bar{X}_{i-1} \simeq \bar{M}_{i-1}$, and consider the map $B' = \pi \circ B : A \times U \to \bar{M}_{i-1}$. We have just shown that this bi-additive, T-covariant map is non-zero. Let N be a T-minimal quotient of the submodule of $\bar{M}_{i-1}$ generated by the image of B'; composing B' with the projection $\pi_N : \bar{M_{i-1}} \to N$, we find a non-zero definable, bi-additive, T-covariant map B'' with image a T-minimal module N.

Now N being a section of M satisfies $C_T^\circ(N) = 1$; as N is T-minimal it has rank k. It follows that T induces on N a field structure of rank k. As Θ acts trivially on A but non-trivially on U, we may apply Lemma 1.14. We find a definable field isomorphism, forcing $\mathrm{rk}\, A = \mathrm{rk}\, N = k$. In particular, $\mathrm{rk}\, X = k$, $X_{i-1} = 0$, and $A = X$ is T-minimal. But since ζ normalizes X, Lemma 1.13 implies that T centralizes X, hence also $\bar{X}_i$, a contradiction. □

Proposition 3.13. *T centralizes X.*

Proof. By Lemma 3.12, $[X,T] \leq X_1$. But $X_1 \cap X_1^\zeta \leq C_V(U, U^\zeta) = 0$ and $[X,T]$ is ζ-invariant. Hence $[X,T] = 0$. □

3.5 Decomposing the module

Our study of the "Cartan subalgebra" X is almost done. We now move to what will turn out to be the positive weight submodule.

Notation 3.14. Let $Y = [X,U]$.

Lemma 3.15. *$Y \leq M$.*

Proof. Suppose $Y \not\leq M$; let $i > 0$ be minimal such that $Y \leq M \oplus X_i$. Let π denote the projection $M \oplus X_i \to X_i/X_{i-1} = \bar{X}_i$; our choice of i means that $\pi(Y) \neq 0$. Let $x \in X$ be such that $\pi([x,U]) \neq 0$.

Since U acts trivially on Z_i/Z_{i-1}, the non-trivial, definable function $\varphi = \pi \circ \mathrm{ad}_x : U \to \bar{X}_i$ is actually a morphism. Let j be minimal such that $X = X_j$. Clearly $i \leq j-1$, so $X_i < X$. In particular $\mathrm{rk}\, \bar{X}_i < \mathrm{rk}\, X \leq k = \mathrm{rk}\, U$, and it follows that $\ker \varphi \neq 0$. Now the latter is T-invariant, so $\ker \varphi = U$. This means that $[x,U] \subseteq M \oplus X_{i-1}$, against the choice of x. □

Proposition 3.16. *$V = X \oplus Y \oplus Y^\zeta$; U centralizes $Y, (X+Y)/Y$, and $V/(X+Y)$. Moreover, $C_X(U) = 0$ and $C_V(U) = Y$.*

Proof. We know that $Y \leq M$. As U does not centralize X, $Y \neq 0$. If $Y = M$, then $M = M^\zeta$ is U-invariant: a contradiction. So $0 < Y < M$; by Proposition 3.6, Y is T-minimal and has rank k. In particular, Y is B-minimal, and by

Malcev's Theorem (Fact 1.2) U centralizes Y. It is then clear that $Y \cap Y^\zeta = 0$, and therefore $M = Y \oplus Y^\zeta$. By construction, U centralizes $(X+Y)/Y$. Now the B-module $V/(X+Y)$ is isomorphic as a T-module to Y^ζ, so it is T-minimal, and by Fact 1.2 again it is centralized by U.

It remains to show that $C_V(U) = Y$, equivalently $C_X(U) = 0$. Choose $u \in U$ such that $u\zeta$ has order 3. Let $x \in C_X(U)$, $y = [x^\zeta, u] \in Y$ and $z = [y^\zeta, u] \in X + Y$. Then:

$$\begin{aligned} x^{u\zeta} &= x^\zeta; \\ x^{u\zeta u\zeta} &= (x^\zeta + y)^\zeta = y^\zeta \pm x; \\ x &= (y^\zeta + z)^\zeta \pm x^\zeta = \pm y + z^\zeta \pm x^\zeta \end{aligned}$$

Projecting onto Y we find $y = 0$, that is $x^\zeta \in C_V(u)$. Then by Corollary 3.7, one has $x = 0$, as desired. □

3.6 Getting rid of SL_2

Proposition 3.17. *The characteristic is not* 2.

Proof. Suppose it is. For any $u \in U^\times$ consider the map $\varphi : V \to V$ given by commutation with u. Since the characteristic is 2 one finds $\operatorname{im}\varphi \leq \ker\varphi$; in particular $\operatorname{rk} V \leq 2 \operatorname{rk}\ker\varphi$. But on the other hand one has $C_V(u, u^\zeta) = 0$ by Corollary 3.7; in particular, $2\operatorname{rk}\ker\varphi \leq \operatorname{rk} V$. All together, this shows that $\operatorname{im}\varphi = [V, u] = \ker^\circ\varphi = C_V^\circ(u)$ has rank $\frac{1}{2}\operatorname{rk} V = k + \frac{\nu}{2}$.

On the other hand by Proposition 3.16, one has $C_V^\circ(u) = [V, u] \leq X \oplus Y$; since $Y = C_V^\circ(U) \leq C_V^\circ(u)$, one finds $C_V^\circ(u) = Y \oplus C_X^\circ(u)$. As T centralizes X, one has $C_X^\circ(u) = C_X^\circ(U) = 0$ by Proposition 3.16. Hence $C_V^\circ(u) = Y$ has rank k: a contradiction. □

Corollary 3.18. *$G \simeq \mathrm{PSL}_2$; ζ has order* 2 *and inverts X.*

Proof. The characteristic is not 2. As T centralizes X (Proposition 3.13), the involution $i \in T$ cannot invert X. It follows that $G \simeq \mathrm{PSL}_2$. In particular ζ has order 2.

Now since Y is T-minimal, i must either invert or centralize it. If i centralizes Y, then it centralizes $M = Y \oplus Y^\zeta$ and X: so i centralizes V, a contradiction. Hence i inverts Y, and also Y^ζ: it follows that i inverts M. So ζ which is conjugate to i must also invert a module of rank $2k$. Let us write $M = M^{+_\zeta} \oplus M^{-_\zeta}$ under the action of ζ. Then Y is disjoint from both, showing that both have rank k. It follows that ζ must invert X. □

3.7 Identification

Let us serve some refreshments.

- ζ has order 2 (Corollary 3.18)
- $Y = C_V^\circ(U) = [X, U]$ is B-minimal (Notation 3.14 and Proposition 3.16)

- $V = Y \oplus X \oplus Y^\zeta$ (Notation 3.14 and Proposition 3.16)
- $X = C_V^\circ(T)$ is inverted by ζ (Proposition 3.13 and Corollary 3.18)
- X and Y have rank k.

We now work towards understanding the scalar action on X.

Corollary 3.19. *Let $x \in X$, $t \in T$, $u \in U^\#$. Then there is a unique $x' \in X$ such that $[x', u] = [x, u]^t = [x, t \cdot u]$; x' depends on x and t, but not on u.*

Proof. Fix $u_1 \in U^\#$ and consider the definable morphism from X to Y which maps x to $[x, u_1]$. This is injective, as the kernel lies in $C_X(u_1) = C_X(T, u_1) \leq C_X(U) = 0$. By equality of ranks, the map is a bijection. Now suppose another $u_2 \in U^\#$ is given, and we have elements x_1', x_2' such that $[x_i', u_i] = [x, u_i]^t$. Then there is $\tau \in T$ such that $u_2 = u_1^\tau$, and it follows that:

$$\begin{array}{rclclclcl} [x_2', u_2] & = & [x, u_2]^t & = & [x, u_1^\tau]^t & = & [x, u_1]^{\tau t} & & \\ & = & [x, u_1]^{t\tau} & = & [x_1', u_1]^\tau & = & [x_1', u_1^\tau] & = & [x_1', u_2] \end{array}$$

whence $x_1' = x_2'$, as claimed. □

And we can finally define a $\mathbb{K}$-scalar action. This is done on each component:

Notation 3.20.

- On Y, $k \cdot y$ is given by the action of T.
- On Y^ζ, we let $k \cdot y^\zeta = (k \cdot y)^\zeta$.
- On X, we let $k \cdot x$ be the unique $x' \in X$ such that $[x', u] = k \cdot [x, u]$ (Corollary 3.19; this does not depend on the choice of u).

We shall check that G acts linearly. We do it piecewise; notice that when we claim that U acts linearly on X, we mean that the operation induced by elements of U from X *to* V is linear, without claiming anything about invariance under the action.

Lemma 3.21. *$T \cdot \langle \zeta \rangle$ acts linearly on V. U acts linearly on $Y \oplus X$.*

Proof. By construction, T is linear on Y and Y^ζ. It is linear on X, as it acts trivially! By construction, ζ is linear on $Y \oplus Y^\zeta$. As it inverts X, it is also linear on X. So $T \cdot \langle \zeta \rangle$ is linear on V.

As U acts trivially on Y, it is linear on Y. It remains to see that U is linear on X. Let $u \in U$, $x \in X$, and $k \in \mathbb{K}$. By definition of the action on X, one has $[k \cdot x, u] = k \cdot [x, u]$, and therefore:

$$k \cdot x^u - k \cdot x = k \cdot [x, u] = [k \cdot x, u] = (k \cdot x)^u - k \cdot x$$

Linearity follows. □

It remains to prove that U is linear on Y^ζ. As T is, and since T acts transitively on $U^\#$, it suffices to exhibit one non-trivial element of U which is linear on Y^ζ.

Notation 3.22 (Bryant Park element)**.** Let $w = \zeta$ (it is an involution, after all). Let $u \in U$ be such that (wu) has order 3.

Such an element exists (this may be viewed as a special case of the Steinberg relations). We shall prove that this particular u is linear on Y^ζ.

FUGA

Lemma 3.23. *For any $y \in Y$, there is a unique $x \in X$ such that $y^{wu} = y + x + y^w$.*

Proof. A priori, one has

$$y^{wu} = y_1 + x + y_2^w$$

for elements $y_1, y_2 \in Y$ and $x \in X$. But U centralizes Y, $(X+Y)/Y$, and $V/(X+Y)$ by Proposition 3.16. So $y_2 = y$. We push further, using the fact that w inverts X (Corollary 3.18).

$$\begin{aligned} y^{(wu)^2} &= y_1^{wu} + x^{wu} + y^{wwu} \\ &= y_1^{wu} - x^u + y \end{aligned}$$

and

$$y = y^{(wu)^3} = y_1^{wuwu} - x^{uwu} + y^{wu}$$

whence applying u^{-1},

$$y = y_1^{wuw} - x^{uw} + y^w$$

Now U^w centralizes Y^w, $(X+Y^w)/Y^w$, and $V/(X+Y^w)$ (Proposition 3.16), so $[u^w, y_1] \in X + Y^w$. It follows that y_1 is the projection on Y of y_1^{wuw}. On the other hand, $x^u \in X + Y$, so $x^{uw} \in X + Y^w$. Taking projections on Y modulo $X + Y^w$, one has $y_1 = y$. $\square$

Lemma 3.24. *Let $y \in Y$ and $x \in X$ be as in Lemma 3.23. Then $[x, u] = 2y$.*

Proof. By definition,

$$y^{wu} = y + x + y^w$$

Let us iterate:

$$\begin{aligned} y^{(wu)^2} &= y^{wu} + x^{wu} + y^{wwu} \\ &= (y + x + y^w) - x^u + y \\ &= 2y + x - x^u + y^w \end{aligned}$$

and

$$\begin{aligned} y^{(wu)^3} &= 2y^{wu} + x^{wu} - x^{uwu} + y^{wwu} \\ &= 2(y + x + y^w) - x^u - x^{uwu} + y \\ &= 3y + 2x - x^u - x^{uwu} + 2y^w \end{aligned}$$

As wu has order three, one has:

$$2y + 2x - x^u - x^{uwu} + 2y^w = 0$$

Now u centralizes $(Y+X)/Y$, so there is $y_1 \in Y$ such that $x^u = x + y_1$. Let x_1 be associated to y_1 by Lemma 3.23: one has $y_1^{wu} = y_1 + x_1 + y_1^w$. Hence

$$\begin{array}{rcl} x^{uwu} & = & x^{wu} + y_1^{wu} \\ & = & -x^u + (y_1 + x_1 + y_1^w) \\ & = & -x - y_1 + y_1 + x_1 + y_1^w \\ & = & x_1 - x + y_1^w \end{array}$$

It follows that

$$2y + 2x - (x + y_1) - (x_1 - x + y_1^w) + 2y^w = 0,$$

and projecting onto Y modulo $X + Y^w$,

$$y_1 = 2y$$

so that $[x, u] = y_1 = 2y$. □

Notation 3.25. For $y \in Y$, let $x(y)$ be the element x given by Lemma 3.23.

Lemma 3.26. *The function $x(y)$ is $\mathbb{K}$-linear.*

Proof. Let $k \in \mathbb{K}$. Then

$$[x(k \cdot y), u] = 2(k \cdot y) = k \cdot (2y) = k \cdot [x(y), u] = [k \cdot x(y), u]$$

And we are done. □

Corollary 3.27. *u is linear on Y^w.*

Proof. Let $y \in Y$ and $k \in \mathbb{K}$; let $y_2 = k \cdot y$, and $x_2 = x(y_2)$. Then

$$(k \cdot y^w)^u = y_2^{wu} = y_2 + x_2 + y_2^w = k \cdot y + x_2 + k \cdot y^w$$

On the other hand,

$$k \cdot y^{wu} = k \cdot (y + x + y^w) = k \cdot y + k \cdot x + k \cdot y^w$$

As x is $\mathbb{K}$-linear, both expressions are equal: u is linear on Y^w. □

It follows that $G = \langle T, \zeta, u \rangle$ is linear on V. We may now finish the proof. First, any irreducible representation of SL_2 is a tensor product of twists of algbraic irreducible (linear) representations by [2, Théorème 10.3]. As the algebraic dimension is 3 here, there can be only one factor. Either we untwist it, thus changing the linear structure before we reach a conclusion, or we observe that T has been proved to act algebraically in our construction, so that no untwisting is actually needed with our particular linear structure.

An alternative would be to argue as follows. As G acts linearly on V, there is an isomorphic embedding i of G into $\mathrm{GL}_3(\mathbb{K})$. The image G' of i is generated by the conjugates of its maximal torus, which is Zariski-closed: hence G' itself is closed in $\mathrm{GL}_3(\mathbb{K})$, whence algebraic. Now the isomorphism $i : G \simeq G'$ is the composition of an algebraic map and a field automorphism. Since i is actually algebraic on T, the field automorphism involved is the identity; i is algebraic. Now the representation of $G' \leq \mathrm{GL}_3(\mathbb{K})$ on V is algebraic; and so is that of G on V.

On the other hand, in the course of proving linearity we were forced to work out the action of T, w, and u explicitly, so we could even complete the identification by hand with concrete computations.

This concludes the identification together with the proof of our Theorem.

A final word. Our reader wonders whether anything similar can be obtained in higher rank, say for $\operatorname{rk} V \leq 4 \operatorname{rk} \mathbb{K}$. First of all, it should be borne in mind that *two families* of such objects will exist: the standard representation of SL_2 on homogeneous polynomials of degree 3, and the infinitely many, pairwise non-isomorphic tensor products of two natural representations twisted by distinct field automorphisms (or equivalently, take one of the twists to be the identity). Were one able to successfully analyze submodules – which in our case relied on the smallness assumption: see the proof of Proposition 3.6 – and taking a good direct sum decomposition for granted, identification issues would remain, as the iterations of the Fugue would no longer assemble in a linear pattern of weights. On the other hand, the polynomial case can be dealt with by the methods used here, and characterized specifically by a hypothesis of the form $C_V(U) = C_V(u)$ for all $u \in U^\#$, which amounts in a sense to assuming Corollary 3.7. The second author announces the following:

Proposition. *Let $\mathbb{K}$ be a field of finite characteristic $p \neq 2, 3$ which is quadratically and cubically closed. Suppose that $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$ acts on an abelian group V faithfully, irreducibly, and with the two following conditions:*

- $[V, U, U, U, U] = 0$ *but* $[V, U, U, U] \neq 0$*;*
- *for any* $u \in U^\#$*,* $C_V(u) = C_V(U)$*.*

Then V bears a structure of $\mathbb{K}$-vector space which makes it isomorphic (as a G-module) to the module of homogeneous polynomials of degree 3.

There are however no model-theoretic assumptions.

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