MODEL THEORY OF A NON-DEGENERATE REPRESENTATION OF A UNITAL C^*-ALGEBRA

CAMILO ARGOTY

ABSTRACT. We study the theory of a Hilbert space H as a module for a unital C^*-algebra $\mathcal{A}$ from the point of view of continuous logic. We show this theory, in an appropiate lenguage, has quantifier elimination and it is superstable. We show that for every $v \in H$ the type $tp(v/\emptyset)$ is in correspondence with the positive linear functional over $\mathcal{A}$ defined by v. Finally, we characterize forking, orthogonality and domination of types and show the theory has weak elimination of imaginaries.

1. INTRODUCTION

Let $\mathcal{A}$ be a unital C^*-algebra and let $\pi : \mathcal{A} \to B(H)$ be a C^*-algebra nondegenerate isometric homomorphism, where $B(H)$ is the algebra of bounded operators over a Hilbert space H. The goal of this paper is to study H as a metric structure expanded by $\mathcal{A}$ from the point of view of continuous logic (see [6] and [5]). In order to describe the structure of H as a module for $\mathcal{A}$, we include a symbol $\dot{a}$ in the language of the Hilbert space structure whose interpretation in H will be $\pi(a)$ for every a in the unit ball of $\mathcal{A}$. Following [5], we study the theory of H as a metric structure of only one sort:

$$(Ball_1(H), 0, -, i, \tfrac{x+y}{2}, \|\cdot\|, (\pi(a))_{a\in Ball_1(\mathcal{A})})$$

where $Ball_1(H)$ and $Ball_1(\mathcal{A})$ are the corresponding unit balls in H and $\mathcal{A}$ respectively; 0 is the zero vector in H; $- : Ball_1(H) \to Ball_1(H)$ is the function that to any vector $v \in Ball_1(H)$ assigns the vector $-v$; $i : Ball_1(H) \to Ball_1(H)$ is

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the function that to any vector $v \in Ball_1(H)$ assigns the vector iv where $i^2 = -1$; $\frac{x+y}{2} : Ball_1(H) \times Ball_1(H) \to Ball_1(H)$ is the function that to a couple of vectors v, $w \in Ball_1(H)$ assigns the vector $\frac{v+w}{2}$; $\| \cdot \| : Ball_1(H) \to [0,1]$ is the norm function; $\mathcal{A}$ is an unital C^*-algebra; $\pi : \mathcal{A} \to B(H)$ is a C^*-algebra isometric homomorphism. The metric is given by $d(v,w) = \|\frac{v-w}{2}\|$. Briefly, the structure will be refered to as (H, π).

It is worthy noting that with this language, we can define the inner product taking into account that for every v, $w \in Ball_1(H)$,

$$\langle v \mid w \rangle = \|\tfrac{v+w}{2}\|^2 - \|\tfrac{v-w}{2}\|^2 + i(\|\tfrac{v+iw}{2}\|^2 - \|\tfrac{v-iw}{2}\|^2)$$

Because of this reason, we will make free use of the inner product as if it were included in the language. In most arguments, we will forget this formal point of view, and will treat H directly. To know more about the continuous logic point of view of Banach spaces please see [5], Section 2.

The main results of this paper are the following:

Theorem 1.1. *The theory of (H, π) admits quantifier elimination and is superstable.*

Theorem 1.2. *Let $v, w \in H$. Then $tp(v/\emptyset) = tp(w/\emptyset)$ if and only if $\phi_v = \phi_w$, where ϕ_v and ϕ_w are the positive linear functionals on $\mathcal{A}$ corresponding to the vectors v and w (see Lemma 2.38).*

Theorem 1.3. *Let $\bar{v} \in H^n$ and $E \subseteq H$. Then the type $tp(\bar{v}/E)$ has a canonical base and therefore, the theory of (H, π) has weak elimination of imaginaries.*

Theorem 1.4. *Let $E \subseteq H$, $p, q \in S_1(E)$ be stationary and v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \perp_E q$ if and only if $\phi_{P^{\perp}_{acl(E)}(v_e)} \perp \phi_{P^{\perp}_{acl(E)}(w_e)}$ (see Definition 2.35 and Definition 2.46).*

Theorem 1.5. *Assume (H, π) is saturated. Let E, F and G be small subsets of H such that $E \subseteq G$ and $F \subseteq G$. Let $p \in S_1(E)$ and $q \in S_1(F)$ be two stationary types. Then $p \rhd_G q$ if and only if there exist v, $w \in H$ such that $tp(v/G)$ is a non-forking extension of p, $tp(w/G)$ is a non-forking extension of q and $\phi_{P^{\perp}_{acl(F)}(w_e)} \leq \phi_{P^{\perp}_{acl(E)}(v_e)}$ (see Definition 2.35 and Definition 2.46).*

The author and Berenstein ([3]) studied the theory of the structure $(H, +, 0, \langle | \rangle, U)$ where U is a unitary operator in the case where the spectrum is countable and characterized prime models and orthogonality of types. The author and Ben Yaacov ([4]) studied the more general case of a Hilbert space expanded by a normal operator N. Most results in this paper are generalizations of results present in [3, 4]. Previous to that, Henson and Iovino in [19], observed that the theory of a Hilbert space expanded with a family of bounded operators is stable. A geometric characterization of forking in such structures was first done by Berenstein and Buechler [9]. In [8] Ben Yaacov, Usvyatsov and Zadka characterized the unitary operators corresponding to generic automorphisms of a Hilbert space as those unitary transformations whose spectrum is S^1 and gave the key ideas used in this paper to characterize domination and orthogonality of types.

A work related to this one is the one of Farah, Hart and Sherman who recently have showed that the theory of a C^*-algebra is not stable (See [15] and [16]). These papers and Farah's work point out one phenomenon: C^*-algebras have complicated model theoretical structure but their representations are very well behaved. This is similar to the case of the integers $\mathbb{Z}$: The theory $Th(\mathbb{Z})$ is quite difficult from the model theoretic point of view, but some of its representations like torsion free abelian groups are very well behaved.

This paper is divided as follows: In Section 2 we give a summary of the tools of C^*-algebras that we will use in this paper. In Section 3, we give an explicit axiomatization of $Th(H, \pi)$ and build the monster model for the theory. In Section 4, we characterize the types over the emtpy set as positive linear functionals on $\mathcal{A}$ and prove quantifier elimination. In section 5, we characterize definable and algebraic closures. In Section 6, we give a geometric interpretation of forking and show weak elimination of imaginaries. Finally, in Section 7, we characterize orthogonality and domination of types.

2. REPRESENTATIONS OF C^*-ALGEBRAS AND BOUNDED POSITIVE LINEAR FUNCTIONALS

This section deals with the representations of a C^*-algebra and with bounded positive linear functionals. The main theorems here are Theorem 2.43 which gives a canonical way to build representations of a C^*-algebra called the *Gelfand-Naimark-Segal* construction; Theorem 2.49 that generalizes Radon Nikodim Theorem; and Theorem 2.32, which states that a representations of an algebra of compact operators can be seen as a direct sum of representations on finite dimensional Hilbert spaces.

Gelfand-Naimark-Segal construction will be very helpful in defining definable closures and forking between types. Theorem 2.32 we will be used in Section 3 to characterize the theory of (H, π). The Gelfand-Naimark-Segal construction and Theorem 2.49 will be used in Section 4 to show that positive linear functionals will correspond to types of vectors in H, and in Section 7 to prove that the relations of almost domination and orthogonality between positive linear functionals over $\mathcal{A}$ characterize domination and orthogonality between types.

Definition 2.1. Let $\mathcal{A}$ be a complex Banach algebra. $\mathcal{A}$ is called a C^*-algebra if there exists a map $^* : \mathcal{A} \to \mathcal{A}$, called *involution* such that for all a, $b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$:

(1) $(a+b)^* = a^* + b^*$
(2) $(ab)^* = b^*a^*$
(3) $(\alpha a)^* = \bar{\alpha} a^*$
(4) $(a^*)^* = a$
(5) $|a^*a| = |a|^2$

Fact 2.2. $\mathbb{C}$ is a C^*-algebra under complex conjugation.

Definition 2.3.

- Let S be a linear operator from H into H. The operator S is called *bounded* if the set $\{\|Su\| : u \in H, \|u\| = 1\}$ is bounded in $\mathbb{C}$. If S is bounded we define the *norm* of S by:

$$\|S\| = \sup_{u \in H, \|u\|=1} \|Su\|$$

  Let H be a Hilbert space. We denote by $B(H)$ the algebra of all bounded linear operators from H to H.

- Given a linear operator $S : H \to H$, its *adjoint operator*, denoted S^* is the unique linear operator $S^* : H \to H$ such that for every $u, v \in H$, $\langle Su|v\rangle = \langle u|S^*v\rangle$.

Remark 2.4. The unicity of the adjoint comes from a duality relation between H and H'. See [21], Volume 1, Chapter VI, Section 2.

Fact 2.5. $B(H)$ is a C^*-algebra under the adjoint operation.

Remark 2.6. There are three important topologies on $B(H)$: The norm topology, the *strong* and the *weak*. Strong topology is the topology of pointwise convergence. In weak topology $T_k \to T$ if for all v and $w \in H$, $\langle T_k v \,|\, w\rangle \to \langle T v \,|\, w\rangle$

Definition 2.7. Given a subset $\mathcal{M} \subseteq B(H)$, we define de *commutant* $\mathcal{M}'$ of $\mathcal{M}$ the set,

$$\mathcal{M}' = \{S \in B(H) \mid \forall T \in M, ST = TS\}$$

Theorem 2.8 (Von Neumann Bicommutant Theorem. Theorem 2.2.2 in [20])**.** *Let $\mathcal{M}$ be a sub C^*-algebra of $B(H)$ containing the identity. Then the following are equivalent:*

(1) $\mathcal{M} = \mathcal{M}''$.
(2) $\mathcal{M}$ is weakly closed.
(3) $\mathcal{M}$ is strongly closed.

Definition 2.9. A C^*-subalgebra of $B(H)$ satisfying any of this equivalent condition is called a *Von Neumann algebra*.

Theorem 2.10 (Kaplanski densitiy theorem. Theorem 2.3.3. in [20])**.** *Let $\mathcal{A}$ be a C^*-subalgebra of $B(H)$ with strong closure $\mathcal{M}$. Then the unit ball $Ball_1(\mathcal{A})$ of $\mathcal{A}$ is strongly dense in the unit ball $Ball_1(\mathcal{M})$ of $\mathcal{M}$. Furthermore, the set of selfadjoint elements in $Ball_1(\mathcal{A})$ is strongly dense in the set of selfadjoint elements of $Ball_1(\mathcal{M})$.*

Definition 2.11. Let $\mathcal{A}$ be a C^*-algebra. A *representation* is an algebra homomorphism $\pi : \mathcal{A} \to B(H)$ such that for all $a \in \mathcal{A}$, $\pi(a^*) = (\pi(a))^*$. In this case H is called an $\mathcal{A}$-*module*. A Hilbert subspace $H' \subseteq H$ is called an $\mathcal{A}$-*submodule* or a *reducing* $\mathcal{A}$-*subspace* of H if H' is closed under π. H is called $\mathcal{A}$-*irreducible* or

$\mathcal{A}$-*minimal* if H has no proper non trivial $\mathcal{A}$-submodules. The set of representations of an algebra $\mathcal{A}$ on $B(H)$ is denoted $rep(\mathcal{A}, B(H))$.

Definition 2.12. Let (H, π) be a representation of a C^*-algebra $\mathcal{A}$. (H, π) is called *non-degenerate* if for every nonzero vector $v \in H$, there exists $a \in \mathcal{A}$ such that $\pi(a)v \neq 0$.

Fact 2.13 (Remark 2.2.4 in [20])**.** A representation (H, π) of an unital C^*-algebra $\mathcal{A}$ is non-degenerate if and only if $\pi(e) = I$, where e is the identity of $\mathcal{A}$ and I is the identity of $B(H)$.

Assumption 2.14. From now on, every C^*-algebra $\mathcal{A}$ will be assumed to have identity e and every representation will be asumed to be non-degenerate.

Fact 2.15 (Corollary 2.2.5 in [20])**.** Let $\mathcal{A}$ be a C^*algebra and $\pi : \mathcal{A} \to B(H)$ a nondegenerate representation of $\mathcal{A}$. Let $\mathcal{M}$ be the strong closure of $\pi(\mathcal{A})$. Then $\mathcal{M}$ is weakly closed and $\mathcal{M} = \mathcal{A}''$.

Let (H, π) be a fixed representaton for a C^*-algebra $\mathcal{A}$.

Definition 2.16. Two subrepresentations (H_1, π_1), (H_2, π_2) of (H, π) are said to be *disjoint* if no subrepresentation of (H_1, π_1) is unitarily equivalent to any subrepresentation of (H_2, π_2).

Fact 2.17 (Proposition 3 in [11], Chapter 5, Section 2)**.** Two subrepresentations (H_1, π_1), (H_2, π_2) of (H, π) are disjoint if and only if there is a projection P in $\pi(\mathcal{A})' \cap \pi(\mathcal{A})''$ such that if P_1 and P_2 are the projections on H_1 and H_2 respectively, we have that $PP_1 = P_1$ and $(I - P)P_2 = P_2$.

Definition 2.18. Given $E \subseteq H$ and $v \in H$, we denote by:

(1) H_E, the Hilbert subspace of H generated by the elements $\pi(a)v$, where $v \in E$ and $a \in \mathcal{A}$.
(2) $\pi_E := \{\pi(a) \restriction H_E \mid a \in \mathcal{A}\}$.
(3) (H_E, π_E), the subrepresentation of (H, π) generated by E.
(4) H_v, the space H_E when $E = \{v\}$ for some vector $v \in H$
(5) $\pi_v := \pi_E$ when $E = \{v\}$.
(6) (H_v, π_v), the subrepresentation of (H, π) generated by v.

(7) $H_E^\perp$, the orthogonal complement of H_E

(8) P_E, the projection over H_E.

(9) $P_{E^\perp}$, the projection over $H_E^\perp$.

Remark 2.19. For a tuple $\bar{v} = (v_1, \dots, v_n)$, by $P_E \bar{v}$ we denote the tuple $(P_E v_1, \dots, P_E v_n)$.

Definition 2.20. (H, π) is called *cyclic* if there exists a vector v_π such that $\pi(\mathcal{A})v_\pi$ is dense in H. Such a vector is called a *cyclic vector* for the representation (H, π).

Remark 2.21. For $v \in H$, it is clear that v is a cyclic vector for $\mathcal{A}$ on H_v.

Notation 2.22. We say that (H, π, v_π) is a *cyclic representation* if v_π is a cyclic vector for (H, π).

Theorem 2.23 (Remark 3.3.1. in [20])**.** *Every representation can be seen as a direct sum of cyclic representations.*

Definition 2.24. Two representations (H_1, π_1) and (H_2, π_2) are said to be *unitarily equivalent* if there exists an isometry U from H_1 to H_2 such that for every $a \in \mathcal{A}$, $U\pi_1(a)U^* = \pi_2(a)$.

Definition 2.25. Two cyclic representations (H_1, π_1, v_1) and (H_2, π_2, v_2) are said to be *isometrically isomorphic* if there is an isometry U from H_1 to H_2 such that for every $a \in \mathcal{A}$, $U\pi_1(a)U^* = \pi_2(a)$ and $Uv_1 = v_2$.

Theorem 2.26 (Proposition 3.3.7 in [20])**.** *Two cyclic representations* (H_1, π_1, v_1) *and* (H_2, π_2, v_2) *are isometrically isomorphic if and only if for all* $a \in \mathcal{A}$, $\langle \pi_1(a)v_1 | v_1 \rangle = \langle \pi_2(a)v_2 | v_2 \rangle$.

Definition 2.27. Two representations (H_1, π_1) and (H_2, π_2) are said to be *approximately unitarily equivalent* if there exists a sequence of unitary operators $(U_n)_{n<\omega}$ from H_1 to H_2 such that for every $a \in \mathcal{A}$ $\pi_2(a) = \lim_{n\to\infty} U_n \pi_1(a) U_n^*$ in the norm topology.

Theorem 2.28 (Theorem II.5.8 in [12])**.** *Two nondegenerate representations* (H_1, π_1) *and* (H_2, π_2) *of a separable* C^**-algebra on separable Hilbert spaces are approximately unitarily equivalent if and only if, for all* $a \in \mathcal{A}$, *rank*$(\pi_1(a))$ = *rank*$(\pi_2(a))$

Definition 2.29. A representation (H, π) of $\mathcal{A}$ is called *compact* if $\pi(\mathcal{A}) \subseteq \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the algebra of compact operators on H.

Lemma 2.30 (Lemma I.10.7 in [12])**.** *Let $\mathcal{A}$ be an algebra of compact operators on a Hilbert space H. Every non-degenerate representation of $\mathcal{A}$ is a direct sum of irreducible representations which are unitarily equivalent to subrepresentations of the identity representation.*

Notation 2.31. For a Hilbert space H and a positive integer n, $H^{(n)}$ denotes the direct sum of n copies of H. If $S \in B(H)$, $S^{(n)}$ denotes the operator on $H^{(n)}$ given by $S^n(v_1, \cdots, v_n) = (Sv_1, \cdots, Sv_n)$. If $\mathcal{B} \subseteq B(H)$, $\mathcal{B}^{(n)}$ is the set $\{S^{(n)} \mid S \in \mathcal{B}\}$.

Theorem 2.32 (Theorem I.10.8 in [12])**.** *Let (H, π) be a compact representation of $\mathcal{A}$. Then for every $i \in \mathbb{Z}^+$, there are a Hilbert spaces H_i and positive integers n_i and k_i such that $dim(H_i) = n_i$ and*

$$H \simeq ker(\pi(\mathcal{A})) \oplus \bigoplus_{i \in \mathbb{Z}^+} H_i^{(k_i)}$$

and

$$\pi(\mathcal{A}) \simeq 0 \oplus \bigoplus_{i \in \mathbb{Z}^+} \mathcal{K}(H_i)^{(k_i)}$$

Remark 2.33. In case that $ker(\mathcal{A}) = 0$, ($\mathcal{A}$ no necessarilly unital) we have that this representation is non-degenerate.

Remark 2.34. Recall that if $R \in \bigoplus_{i \in \mathbb{Z}^+} \mathcal{K}(H_i)^{(k_i)}$, then there is a sequence $(R_i)_{i \in \mathbb{Z}^+}$ such that $R_i \in \mathcal{K}(H_i)^{(k_i)}$ and $R = \sum_{i \in \mathbb{Z}^+} R_i$ in the norm topology. This means, in particular, that $\lim_{i \to \infty} \|R_i\| = 0$.

Definition 2.35. Let (H, π) be a representation of $\mathcal{A}$. We define:

The *essential part* of π: It is the C^*-algebra homomorphism,

$$\pi_e := \rho \circ \pi : \mathcal{A} \to B(H)/\mathcal{K}(H)$$

of $\pi(\mathcal{A})$, where ρ is the canonical proyection of $B(H)$ onto the Calkin Algebra $B(H)/\mathcal{K}(H)$.

The *discrete part* of π: It is the restriction,

$$\pi_d : ker(\pi_e) \to \mathcal{K}(H)$$

$$a \to \pi(a)$$

The *discrete part* of $\pi(\mathcal{A})$: It is defined in the following way:

$$\pi(\mathcal{A})_d := \pi(\mathcal{A}) \cap \mathcal{K}(H).$$

The *essential part* of $\pi(\mathcal{A})$: It is the image $\pi(\mathcal{A})_e$ of $\pi(\mathcal{A})$ in the Calkin Algebra.

The *essential part* of H: It is defined in the following way:

$$H_e := \ker(\pi(\mathcal{A})_d)$$

The *discrete part* of H: It is defined in the following way:

$$H_d := \ker(\pi(\mathcal{A})_d)^{\perp}$$

The *essential part* of a vector $v \in H$: It is the projection v_e of v over H_e.

The *discrete part* of a vector $v \in H$: It is the projection v_d of v over H_d.

The *essential part* of a set $E \subseteq H$: It is the set

$$E_e := \{v_e \mid v \in E\}$$

The *discrete part* of a set $G \subseteq H$: It is the set

$$E_d := \{v_d \mid v \in G\}$$

Remark 2.36. Let (H, π) be a non-degenerate representation of $\mathcal{A}$. By Theorem 2.32, for every $i \in \mathbb{Z}^+$, there are a Hilbert spaces H_i and positive integers n_i and k_i such that $\dim(H_i) = n_i$ and

$$H_d \simeq \bigoplus_{i \in \mathbb{Z}^+} H_i^{(k_i)}$$

Definition 2.37. Let $\mathcal{A}'$ be the dual space of $\mathcal{A}$. An element $\phi \in \mathcal{A}'$ is called *positive* if $\phi(a) \geq 0$ whenever $a \in \mathcal{A}$ is positive, i.e. there is $b \in \mathcal{A}$ such that $a = b^*b$. The set of positive functionals is denoted by $\mathcal{A}'_+$.

Lemma 2.38. *Let $\mathcal{A}$ be a C^*-algebra of operators on a Hilbert space H, and let $v \in H$. Then the function ϕ_v on $\mathcal{A}$ such that for every $S \in \mathcal{A}$, $\phi_v(S) = \langle Sv \mid v \rangle$ is a positive linear functional.*

Proof. Linearity is clear. Let S be a positive selfadjoint operator in $\mathcal{A}$, let Q be its square root, that is, an operator such that $S = QQ^*$. Let $v \in H$; then $\langle Sv \mid v \rangle = \langle Q^*Qv \mid v \rangle = \langle Qv \mid Qv \rangle \geq 0$ □

Definition 2.39. Let ϕ be a positive linear functional on $\mathcal{A}$. Let

$$\Lambda^2(\mathcal{A},\phi) = \{a \in \mathcal{A} \mid \phi(a^*a) < \infty\}/ \sim_\phi,$$

where $a_1 \sim_\phi a_2$ if $\phi(a_1^* a_2) = 0$. For $(a)_{\sim_\phi}$, $(b)_{\sim_\phi} \in \Lambda^2(\mathcal{A},\phi)$, let

$$\langle (a)_{\sim_\phi} \mid (b)_{\sim_\phi} \rangle_\phi = \phi(a^*b).$$

Remark 2.40. The product $\langle \cdot \mid \cdot \rangle_\phi$ is a natural inner product on the space $\Lambda^2(\mathcal{A},\phi)$(see [11] page 472).

Definition 2.41. We define the space $L^2(\mathcal{A},\phi)$ to be the completion of $\Lambda^2(\mathcal{A},\phi)$ under the norm defined by $\langle \cdot \mid \cdot \rangle_\phi$.

Definition 2.42. Let ϕ be a positive linear functional on $\mathcal{A}$. We define the representation $M_\phi : \mathcal{A} \to B(L^2(\mathcal{A},\phi))$ in the following way: For every $a \in \mathcal{A}$ and $(b)_{\sim_\phi} \in L^2(\mathcal{A},\phi)$, let $M_\phi(a)((b)_{\sim_\phi}) = (ab)_{\sim_\phi}$.

Theorem 2.43 (Theorem 3.3.3. and Remark 3.4.1. in [20])**.** *Let ϕ be a positive functional on $\mathcal{A}$. Then there exists a cyclic representation $(H_\phi, \pi_\phi, v_\phi)$ such that for all $a \in \mathcal{A}$, $\phi(a) = \langle \pi_\phi(a) v_\phi | v_\phi \rangle$. This representation is called the Gelfand-Naimark-Segal construction.*

Proof. Take $(L^2(\mathcal{A},\phi_v), M_{\phi_v}, (e)_{\sim_{\phi_v}})$. Note that

$$\langle M_{\phi_v}(a)(e)_{\sim_{\phi_v}} \mid (e)_{\sim_{\phi_v}} \rangle = \langle (a)_{\sim_{\phi_v}} \mid (e)_{\sim_{\phi_v}} \rangle = \phi_v(a \cdot e) = \phi_v(a).$$

□

Theorem 2.44. *Let $v \in H$. Then $(H_v, \pi_v, v) \simeq (L^2(\mathcal{A},\phi_v), M_{\phi_v}, (e)_{\sim_{\phi_v}})$.*

Proof. By Gelfand-Naimark-Segal Theorem 2.43 and Theorem 2.26. □

Definition 2.45. We define the following (see [20]):

(1) A positive linear functional ϕ on $\mathcal{A}$ is called a *quasistate* if $\|\phi\| \leq 1$.
(2) The set of the of quasistates on $\mathcal{A}$ is denoted by $Q_\mathcal{A}$.
(3) In the case where $\|\phi\| = 1$, the positive linear functional ϕ is called a *state*.
(4) The set of states is denoted by $S_\mathcal{A}$.
(5) A state is called *pure* if it is not a convex combination of other states.
(6) The set of pure states is denoted by $PS_\mathcal{A}$.

Definition 2.46. Let ϕ and ψ be positive linear functionals on $\mathcal{A}$.

(1) They are called *orthogonal* ($\phi \perp \psi$) if $\|\phi - \psi\| = \|\phi\| + \|\psi\|$.
(2) Also, ϕ is called *dominated* by ψ ($\phi \leq \psi$) if there exist $\gamma > 0$ such that the functional $\gamma\psi - \phi$ is positive.

Fact 2.47 (Lemma 3.2.3 in [20])**.** Let ϕ and ψ be two positive linear functionals on $\mathcal{A}$. Then, $\phi \perp \psi$ if and only if for all $\epsilon > 0$ there exists a positive element $a \in \mathcal{A}$ with norm less than or equal to 1, such that $\phi(e - a) < \epsilon$ and $\psi(a) < \epsilon$.

Lemma 2.48. *Let ϕ_1, ϕ_2, ψ_1 and ψ_2 be positive linear functionals on $\mathcal{A}$ such that $\phi_1 \leq \phi_2$ and $\psi_1 \leq \psi_2$. If $\phi_2 \perp \psi_2$, then $\phi_1 \perp \psi_1$.*

Proof. Let $\gamma_1 > 0$ and $\gamma_2 > 0$ be such that $\gamma_1\phi_2 - \phi_1$ and $\gamma_2\psi_2 - \psi_1$ are positive. By Fact 2.47, for $\epsilon > 0$ there exists a positive $a \in \mathcal{A}$ with norm less than or equal to 1 such that $\phi_2(e - a) < \frac{\epsilon}{\gamma_1+\gamma_2}$ and $\psi_2(a) < \frac{\epsilon}{\gamma_1+\gamma_2}$. Then $\phi_1(e - a) \leq \gamma_1\phi_2(e - a) < \frac{\gamma_1\epsilon}{\gamma_1+\gamma_2} < \epsilon$ and $\psi_1(a) \leq \gamma_2\psi_2(a) < \frac{\gamma_2\epsilon}{\gamma_1+\gamma_2} < \epsilon$. □

Theorem 2.49 (Generalized Radon-Nikodim Theorem in [17])**.** *Let $\pi : \mathcal{A} \to B(H)$ be a representation and let v, $w \in H$. Then $\phi_v \leq \phi_w$ if and only if there exists a bounded positive operator $P : H_w \to H_v$ that commutes with $\pi(\mathcal{A})$ and $P(w) = v$.*

Definition 2.50. Let (H_i, π_i) for $i \in I$ be a family of representations of $\mathcal{A}$. We define a representation $\oplus\pi_i$ on $\oplus H_i$ in the following way: Let $v = \sum_i v_i$ and $a \in \mathcal{A}$, $\oplus\pi_i(a)v = \sum_i \pi_i(a)v_i$.

Definition 2.51. We define the following:

- A subset $F \subseteq S_{\mathcal{A}}$ is called *separating* if for every $a \in \mathcal{A}$, $\phi(a) = 0$ for every $\phi \in F$ implies that $a = 0$.
- Let $\phi \in S_{\mathcal{A}}$. ϕ is said to be *faithful* if for every $a \in \mathcal{A}_+$, $\phi(a) = 0$ implies that $a = 0$. A *faithful representation* is a representation (H, π) such that if $\pi(a) = 0$ then $a = 0$ for $a \in \mathcal{A}_+$.

Notation 2.52. For each $\phi \in S_{\mathcal{A}}$, let (H_ϕ, π_ϕ) be the Gelfand-Naimark-Segal construction of ϕ. For $F \subseteq S_{\mathcal{A}}$ let $(H_F, \pi_F) = (\oplus_{\phi\in F} H_\phi, \oplus_{\phi\in F}\pi_\phi)$.

Theorem 2.53 (Proposition 3.7.4 in [20])**.** *If $F \subseteq S_{\mathcal{A}}$ is separating, then (H_F, π_F) is a faithful representation.*

Definition 2.54. The representation $(H_{S_{\mathcal{A}}}, \pi_{S_{\mathcal{A}}})$ is called the *universal representation.*

3. THE THEORY OF (H, π)

In this section we use some results from section 2 to provide an explicit axiomatization of $Th(H, \pi)$. The main tool here is Theorem 2.28 which is mainly a consecuence of Voiculescu's theorem (see [12]). This Theorem states that two separable representations (H_1, π_1) and (H_2, π_2) of a separable C^*-algebra $\mathcal{A}$ are approximately unitarily equivalent if and only if for every $a \in \mathcal{A}$, rank$(\pi_1(a))$ =rank$(\pi_2(a))$. This last statement can be expressed in continuous first order logic and is the first step to build the axiomatization we mentioned above. Lemma 3.7, Theorem 3.8 and Corollary 3.14 are remarks and unpublished results from C. Ward Henson.

Lemma 3.1. *If $S : H \to H$ is a bounded operator, S is non-compact if and only if for some $\lambda_S > 0$, $S(Ball_1(H))$ contains an isometric copy of the ball of radius λ_S of ℓ^2 i.e., there exists an orthonormal sequence $(w_i)_{i\in\mathbb{N}} \subseteq S(Ball_1(H))$ and a vector sequence $(u_i)_{i\in\mathbb{N}} \subseteq Ball_1(H)$ such that for every $i \in \mathbb{N}$, $Su_i = \lambda_S w_i$.*

Proof. Suppose S is non-compact. Then there is a sequence $(u'_i)_{i\in\mathbb{N}} \subseteq Ball_1(H)$ such that no subsequence of $(Su'_i)_{i\in\mathbb{N}}$ is convergent. By Grahm-Schmidt process we can assume that $(Su'_i)_{i\in\mathbb{N}}$ is an orthogonal sequence. Since no subsequence of $(Su'_i)_{i\in\mathbb{N}}$ converges, we have that $\liminf\{\|Su'_i\| \mid i \in \mathbb{N}\} > 0$ (otherwise there wuould be a subsequence of Su'_i converging to 0). Let $\lambda_S := \frac{\liminf\{\|Su'_i\| \mid i\in\mathbb{N}\}}{2} > 0$. For $i \in \mathbb{N}$, let $u_i := \frac{\lambda_S u'_i}{\|Su_i\|}$ and $w_i := \frac{Su'_i}{\|Su'_i\|}$. Without loss of generality, we can asume that for all $i \in \mathbb{N}$, $\|Su'_i\| > \lambda_S$ and therefore $\|u_i\| \leq 1$. Then, $Su_i = S(\frac{\lambda_S u_i}{\|Su_i\|}) = \lambda_S \frac{Su'_i}{\|Su'_i\|} = \lambda_S w_i$.

On the other hand, suppose there are $\lambda_S > 0$, an orthonormal sequence $(w_i)_{i\in\mathbb{N}} \subseteq S(Ball_1(H))$ and a vector sequence $(u_i)_{i\in\mathbb{N}} \subseteq Ball_1(H)$ such that for every $i \in \mathbb{N}$, $Su_i = \lambda_S w_i$. Then no subsequence of $(Su_i)_{i\in\mathbb{N}}$ converges and S is non-compact. $\square$

Remark 3.2. If in Lemma 3.1 $\|S\| \leq 1$, it is clear that $\lambda_S \leq 1$.

Lemma 3.3. *Let $a \in Ball_1(\mathcal{A})$ be such that $\pi(a)$ is a non-compact operator on H. Let $\lambda_{\pi(a)}$, $(u_i)_{i\in\mathbb{N}}$ and $(w_i)_{i\in\mathbb{N}}$ as described in Lemma 3.1. Then, for every $n \in \mathbb{N}$*

$$(1) \quad (H,\pi) \models \inf_{u_1,u_2\cdots u_n} \inf_{w_1,w_2\cdots w_n} \max_{i,j=1,\cdots,n} \big(|\langle w_i \mid w_j\rangle - \delta_{ij}|, |au_i - \lambda_{\pi(a)} w_i|\big) = 0$$

Proof. This condition is a continuous logic condition for:

$$(2) \quad \exists u_1 u_2 \cdots u_n \exists w_1 w_2 \cdots w_n \wedge \big(\bigwedge_{i,j=1,\cdots,n} \langle w_i \mid w_j\rangle = \delta_{ij}\big) \wedge \wedge \big(\bigwedge_{i=1,\cdots,n} \dot{a} u_i = \lambda_{\pi(a)} w_i\big)$$

where δ_{ij} is Kronecker's delta. By Lemma 3.1, this set of conditions says that $\pi(a)(Ball_1(H))$ contains an isometric copy the ball of radius $\lambda_{\pi(a)}$ of ℓ^2. □

Remark 3.4. It is an easy consecuence of Riesz representation theorem that if $S : H \to H$ is an operator with rank n, then there exist two orthonormal families $E_1 := \{u_1, \cdots, v_n\}$, $E_2 := \{w_1, \cdots, w_n\}$ and a family $\{\alpha_i, \dots, \alpha_n\}$ of non-zero complex numbers such that for every $v \in H$, $Sv = \sum_{i=1}^n \alpha_i \langle v \mid u_i\rangle w_i$. Furthermore, if R is a compact operator, there is a complex sequence $(\alpha_i)_{i\in\mathbb{N}^+}$ such that for every $v \in H$, $Sv = \sum_{i=1}^n \alpha_i \langle v \mid u_i\rangle w_i$. If $\|S\| \leq 1$, then for every i, $|\alpha_i| \leq 1$.

Lemma 3.5. *Let $n \in \mathbb{N}$ and $a \in Ball_1(\mathcal{A})$ be such that rank$(\pi(a)) = n$. Let $\{\alpha_i, \dots, \alpha_n\}$ complex numbers as described in 3.4. Then*

$$(3) \quad (H,\pi) \models \inf_{u_1 u_2\cdots u_n} \inf_{w_1 w_2\cdots w_n} \sup_{v} \max_{i,j=1\cdots n} \big(|\langle u_i \mid u_j\rangle - \delta_{ij}|, |\langle w_i \mid w_j\rangle - \delta_{ij}|, , \|\dot{a}v - \sum_{k=1}^n \alpha_i \langle v \mid u_i\rangle w_i)\|\big) = 0$$

Proof. This condition is a continuous logic condition for:

$$(4) \quad \exists u_1 u_2 \cdots u_n \exists w_1 w_2 \cdots w_n \big(\bigwedge_{i,j=1,\cdots,n} \langle u_i \mid u_j\rangle = \delta_{ij} \wedge \langle w_i \mid w_j\rangle = \delta_{ij}\big) \wedge \wedge \forall v (\dot{a} v = \sum_{k=1}^n \alpha_i \langle v \mid u_i\rangle w_i)$$

where δ_{ij} is Kronecker's delta. □

Remark 3.6. If Condition 3 is valid for some $a \in \mathcal{A}$, by Remark 3.4, it is clear that $\pi(a)$ has rank n.

Recall that all C^*-algebras under consideration are unital and all representations are nondegenerate. However, in the next lemma we do not use the hypothesis that $\mathcal{A}$ is unital.

Lemma 3.7. *Let $\mathcal{A}$ be a separable C^*-algebra of operators on the separable Hilbert space H, and π_1 and π_2 two non-degenerate representations of $\mathcal{A}$ on H. Then the structures (H, π_1) and (H, π_2) are elementarily equivalent if and only if π_1 and π_2 are approximately unitarily equivalent.*

Proof. $\Rightarrow$ Suppose $(H, \pi_1) \equiv (H, \pi_2)$. Let $a \in Ball_1(\mathcal{A})$ and assume that $\text{rank}(\pi_1(a)) = n < \infty$ then Condition (3) will hold for $\pi = \pi_1$. By elementary equivalence, Condition (3) will hold for $\pi = \pi_2$ and therefore $\text{rank}(\pi_1(a)) = n$. In the same way, if $\text{rank}(\pi_1(a)) = \infty$, Condition (1) will hold for every n with respect to π_1. By elementary equivalence, Condition (1) will hold for every n with respect to π_2 and $\text{rank}(\pi_2(a)) = \infty$. This implies that the hypotesis of Theorem 2.28 hold, and therefore π_1 and π_2 are approximately unitarily equivalent.

$\Leftarrow$ Suppose π_1 and π_2 are approximately unitarily equivalent. Then, there exists a sequence of unitary operators $(U_n)_{n<\omega}$ such that for every $a \in \mathcal{A}$, $\pi_2(a) = \lim_{n\to\infty} U_n \pi_1(a) U_n^*$. Let $\mathcal{F}$ be a non-principal ultrafilter over $\mathbb{N}$. Let $(\tilde{H}_1, \tilde{\pi}_1) = \Pi_{\mathcal{U}}(H, U_n \pi_1(A) U_n^*)$ and let $(\tilde{H}_2, \tilde{\pi}_2) = \Pi_{\mathcal{U}}(H, \pi_2)$. It follows that $(\tilde{H}_1, \tilde{\pi}_1) \simeq (\tilde{H}_2, \tilde{\pi}_2)$ and $(H, \pi_1) \equiv (H, \pi_2)$.

□

Theorem 3.8. *Let $\mathcal{A}$ be a C^*-algebra, H_1 and H_2 be Hilbert spaces, and π_1 and π_2 be two representations of $\mathcal{A}$ on H_1 and H_2 respectively. Then the structures (H_1, π_1) and (H_2, π_2) are elementarily equivalent if and only if for all $a \in \mathcal{A}$, $\text{rank}(\pi_1(a)) = \text{rank}(\pi_2(a))$.*

Proof. **$\Rightarrow$:** Suppose (H_1, π_1) and (H_2, π_2) are elementarily equivalent and let $a \in \mathcal{A}$. By Theorem 3.3 and Theorem 3.5, $\text{rank}(\pi(a)) = n$ or ∞ is a set of conditions in $L(\mathcal{A})$. By elementary equivalence, $\text{rank}(\pi_1(a)) = \text{rank}(\pi_2(a))$.

$\Leftarrow$: Let (H_1, π_1) and (H_2, π_2) be such that $\text{rank}(\pi_1(a)) = \text{rank}(\pi_2(a))$, and let $\phi(a_1, \cdots, a_n) = 0$ be a condition in $L(\mathcal{A})$. Let $\hat{\mathcal{A}} \subseteq \mathcal{A}$ be the unital sub C^*-algebra of $\mathcal{A}$ generated by $\bar{a} = (a_1, \cdots, a_n)$, and $\hat{\pi}_1$ and $\hat{\pi}_2$ be the restrictions of π_1 and π_2 to $\hat{\mathcal{A}}$ (note that $\hat{\pi}_1(e) = I = \hat{\pi}_1(e)$). Then $\hat{\mathcal{A}}$ is separable and by Löwenheim-Skolem Theorem and Fact 2.13, there are two separable non-degenerate representations $(\tilde{H}_1, \tilde{\pi}_1)$ and $(\tilde{H}_2, \tilde{\pi}_2)$ which are elementary substructures of $(H_1, \hat{\pi}_1)$ and $(H_2, \hat{\pi}_2)$ respectively. By Theorem 2.28 $(\tilde{H}_1, \tilde{\pi}_1)$ is approximately unitarily equivalent to $(\tilde{H}_2, \tilde{\pi}_2)$. By the previous lemma, $(\tilde{H}_1, \tilde{\pi}_1)$ and $(\tilde{H}_2, \tilde{\pi}_2)$ are elementary equivalent.

Then, $(\hat{H}_1, \hat{\pi}_1) \models \phi(a_1, \cdots, a_n) = 0$ if and only if $(\hat{H}_2, \hat{\pi}_2) \models \phi(a_1, \cdots, a_n) = 0$. But $(H, \pi_1) \models \phi(a_1, \cdots, a_n) = 0$ if and only if $(\hat{H}_1, \hat{\pi}_1) \models \phi(a_1, \cdots, a_n) = 0$ and $(H, \pi_2) \models \phi(a_1, \cdots, a_n) = 0$ if and only if $(\hat{H}_2, \hat{\pi}_2) \models \phi(a_1, \cdots, a_n) = 0$. Then $(H, \pi_1) \models \phi(a_1, \cdots, a_n) = 0$ if and only if $(H, \pi_2) \models \phi(a_1, \cdots, a_n) = 0$.

□

Definition 3.9. Let T_π be the theory of Hilbert spaces together with the following conditions:

(1) For $v \in Ball_1(H)$ and a, $b \in Ball_1(\mathcal{A})$:

$$(\dot{ab})v = (\dot{a}\dot{b})v = \dot{a}(\dot{b}v)$$

(2) For $v \in Ball_1(H)$ and a, $b \in Ball_1(\mathcal{A})$:

$$(\dot{\tfrac{a+b}{2}})(v) = \tfrac{\dot{a}+\dot{b}}{2}(v) = \tfrac{\dot{a}v+\dot{b}v}{2}$$

(3) For v, $w \in Ball_1(H)$, and $a \in Ball_1(\mathcal{A})$:

$$\dot{a}\left(\tfrac{v+w}{2}\right) = \tfrac{\dot{a}v+\dot{a}w}{2}$$

(4) For $v \in Ball_1(H)$ and $a \in Ball_1(\mathcal{A})$:

$$\langle \dot{a}v \mid w\rangle = \langle v \mid \dot{a^*}w\rangle$$

(5) For $v \in Ball_1(H)$ and $a \in Ball_1(\mathcal{A})$:

$$\sup_v (\|\dot{a}v\| \dot{-} \|a\|\|v\|) = 0$$

$$\inf_v \max(|\|v\| - 1|, |\|\dot{a}v\| - \|a\| \,|) = 0$$

(6) For $v \in Ball_1(H)$ and e the identity element in $\mathcal{A}$:

$$(\dot{ie})v = iv$$

$$\dot{e}v = v$$

(7) For $a \in Ball_1(\mathcal{A})$ such that $\pi(a)$ is a non-compact operator on H. Let $\lambda_{\pi(a)}$, $(u_i)_{i\in\mathbb{N}}$ and $(w_i)_{i\in\mathbb{N}}$ as described in Lemma 3.1. For $n \in \mathbb{N}$

$$\inf_{u_1,u_2\cdots u_n} \inf_{w_1,w_2\cdots w_n} \max_{i,j=1,\cdots,n} \big(|\langle w_i \mid w_j\rangle - \delta_{ij}|, |au_i - \lambda_{\pi(a)} w_i|\big) = 0$$

(8) For $a \in Ball_1(\mathcal{A})$, such that $\operatorname{rank}(\pi(a)) = n \in \mathbb{N}$. Let $\alpha_1, \cdots, \alpha_n$ be complex number as described in Remark 3.4.

$$\exists u_1 u_2 \cdots u_n \exists w_1 w_2 \cdots w_n \big(\bigwedge_{i,j=1,\cdots,n} \langle w_i \mid w_j\rangle = \delta_{ij}\big) \wedge$$

$$\wedge \forall v (\dot{a} v = \sum_{k=1}^{n} \alpha_i \langle v \mid u_i\rangle w_i)$$

Remark 3.10. We gave in Lemmas 3.3 and 3.5 the complete continuous logic formalism only for the last two conditions. Conditions in Item (5), are natural continuous logic conditions that say that $\|\pi(a)\| = \|a\|$. The translations for the other conditions to the continuous logic formalism are straightforward and are left to the reader.

Remark 3.11. Second condition in Item (6) implies that the representation is non-degenerate.

Remark 3.12. Since the rationals of the form $\frac{k}{2^n}$ are dense in $\mathbb{R}$, Item (3) and Item (6) are enough to show that for all $v \in Ball_1(H)$ and all $a \in Ball_1(\mathcal{A})$, we have that $(\dot{\lambda a})v = \lambda(\dot{a}v)$.

Remark 3.13. We omit an explicit condition describing compact infinite rank operators in $\pi(\mathcal{A})$ because they completely determined by the finite rank operators in $\pi(\mathcal{A})$.

Corollary 3.14. *T_π axiomatizes the theory $Th(H, \pi)$.*

Proof. By Theorem 3.8. □

Lemma 3.15. *Let (H_1, π_1) and (H_2, π_2) be two non-degenerate representations of $\mathcal{A}$. If $(H_1, \pi_1) \equiv (H_2, \pi_2)$ then $\big((H_1)_d, (\pi_1)_d\big) \simeq \big((H_2)_d, (\pi_2)_d\big)$.*

Proof. For a given representation π, let $\pi(\mathcal{A})_f$ be the (not necessarilly closed) algebra of finite rank operators in $\pi(\mathcal{A})$. If $(H_1, \pi_1) \equiv (H_2, \pi_2)$, by Lemma 3.5,

$\pi_1(\mathcal{A})_f \simeq \pi_2(\mathcal{A})_f$ and by density of $\pi(\mathcal{A})_f$ in $\pi(\mathcal{A})_d$, we have that $\pi_1(\mathcal{A})_d \simeq \pi_2(\mathcal{A})_d$. Let $\mathcal{B} := \pi_1(\mathcal{A})_d \simeq \pi_2(\mathcal{A})_d$. Since $(H_1)_d$ and $(H_2)_d$ are the orthogonal complements of $ker(\mathcal{B})$ in H_1 and H_2 respectively, we get that $\big((H_1)_d, (\pi_1)_d\big)$ and $\big((H_2)_d, (\pi_2)_d\big)$ are non-degenerate representations of $\mathcal{B}$. Then by Lemma 2.30 and Theorem 2.32, $\big((H_1)_d, (\pi_1)_d\big) \simeq \big((H_2)_d, (\pi_2)_d\big)$. □

Theorem 3.16. *Let (H_1, π_1) and (H_2, π_2) be two representations of $\mathcal{A}$. Then $(H_1, \pi_1) \equiv (H_2, \pi_2)$ if and only if*

$$\big((H_1)_d, (\pi_1)_d\big) \simeq \big((H_2)_d, (\pi_2)_d\big)$$

and

$$((H_1)_e, (\pi_1)_e) \equiv ((H_2)_e, (\pi_2)_e)$$

Proof. By Theorem 3.8, $(H_1, \pi_1) \equiv (H_2, \pi_2)$ if and only if $((H_1)_d, (\pi_1)_d) \equiv ((H_2)_d, (\pi_2)_d)$ and $((H_1)_e, (\pi_1)_e) \equiv ((H_2)_e, (\pi_2)_e)$. By Lemma 3.15, this is equivalent to $\big((H_1)_d, (\pi_1)_d\big) \simeq \big((H_2)_d, (\pi_2)_d\big)$ and $((H_1)_e, (\pi_1)_e) \equiv ((H_2)_e, (\pi_2)_e)$. □

Remark 3.17. For $E \subseteq H$, $(H_E)_e = H_{E_e}$ and $(H_E)_d = H_{E_d}$

Lemma 3.18. *Let $v \in H_d$. Then v is algebraic over $\emptyset$.*

Proof. If $v \in H_d$ by Theorem 2.32, there exist a sequence v_i of vectors in H_d such that $v_i \in H_i^{k_i}$, and $v = \sum_{i\geq 1} v_k$. Given that $\|v_k\| \to 0$ when $k \to \infty$, the orbit of v under any automorphism U of (H, π) is a Hilbert cube which is compact, which implies that v is algebraic. □

Lemma 3.19 (Proposition 2.7 in [18])**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures, $A \subseteq M$ and $B \subseteq N$. If $f : A \to B$ is an elementary map, then there is an elementary map $g : acl_{\mathcal{M}}(A) \to acl_{\mathcal{N}}(B)$ extending f. Moreover, if f is onto, then so is g.*

Remark 3.20. Any elementary map is 1-1 so the previous Lemma implies that a bijective elementary map can be extended by a bijective elementary map to the algebraic closures.

Recall from Definition 2.45 that $S_{\mathcal{A}}$ denotes the collection of states of $\mathcal{A}$.

Definition 3.21. Let $H_{S_{\mathcal{A}}}$ be the space,

$$H_{S_{\mathcal{A}}} = \oplus_{\phi \in S_{\mathcal{A}}} L^2(\mathcal{A}, \phi)$$

and let $\pi_{S_{\mathcal{A}}}$ be,

$$\pi_{S_{\mathcal{A}}} = \oplus_{\phi \in S_{\mathcal{A}}} M_\phi,$$

Theorem 3.22. *Let $\kappa \geq |S_{\mathcal{A}}|$ be such that $cf(\kappa) = \kappa$. Then the structure*

$$(\tilde{H}_\kappa, \tilde{\pi}_\kappa) = (H_d, \pi_d) \oplus \bigoplus_{\kappa} (H_{S_{\pi(\mathcal{A})_e}}, \pi_{S_{\pi(\mathcal{A})_e}})$$

is κ universal, κ homogeneous and is a monster model for $Th(H, \pi)$.

Proof. Let us denote $(\tilde{H}_\kappa, \tilde{\pi}_\kappa)$ just by $(\tilde{H}, \tilde{\pi})$.

$(\tilde{H}, \tilde{\pi}) \models Th(H, \pi)$: For every $a \in Ball_1(\mathcal{A})$, if $rank(a) = \infty$ in H_e, then $rank(a) = \infty$ in $\tilde{H}_e$ and if $rank(a) = 0$ in H_e, then $rank(a) = 0$ in $\tilde{H}_e$. By Theorem 3.16 $(H_e, \pi_e) \equiv (\tilde{H}_e, \tilde{\pi}_e)$. By Theorem 3.16, $(H, \pi) \equiv (\tilde{H}, \tilde{\pi})$.

κ-Universality: Let $(H', \pi') \models Th(H, \pi)$ be a model with density less than κ. Theorem 3.16, $(H', \pi'_d) \simeq (\tilde{H}_d, \tilde{\pi}_d) \simeq (H_d, \pi_d)$. Then without loss of generality we can asume that $\pi(\mathcal{A}) = \pi(\mathcal{A})_e$. By Theorem 2.23, there exists a set I and a family $(H_i, \pi_i, v_i)_{i \in I}$ of cyclic representations such that $(H', \pi') = \bigoplus_{i \in I} (H_i, \pi_i)$. By Theorem 2.44, $(H_{v_i}, \pi_{v_i}, v_i) \simeq (L^2(\mathcal{A}, \phi_{v_i}), M_{\phi_{v_i}}, (e)_{\sim_{\phi_{v_i}}})$. Since the density of (H', π') is less than κ, the size of I is less than κ and clearly (H', π') is isomorphic to a subrepresentation of $(\tilde{H}, \tilde{\pi})$.

κ-Homogeneity: Let U be a partial elementary map between E, $F \subseteq \tilde{H}$ with $|E| = |F| < \kappa$.

(1) We can extend U to an unitary equivalence between H_E and H_F: Let a_1, $a_2 \in \mathcal{A}$ and $e_1, e_2 \in E$. Then we define $U(\pi(a_1)(e_1)+\pi(a_2)(e_2)) := \pi(a_1)(U(e_1)) + \pi(a_2)(U(e_2))$. After this, we extend this constuction continuously to H_E.

(2) We can extend U to an unitary equivalence between $(H_d \oplus H_{E_e})$ and $(H_d \oplus H_{F_e})$: By Lemma 3.18, $(H_d \oplus H_{E_e}) \subseteq acl_{\tilde{H}}(E)$ and $(H_d \oplus H_{F_e}) \subseteq acl_{\tilde{H}}(F)$. By Lemma 3.19, we can extend U in the desired way.

(3) We can find an unitary equivalence between $(H_d \oplus H_{E_e})^\perp$ and $(H_d \oplus H_{F_e})^\perp$: Given that $|E| = |F| < \kappa$, there are two subsets C_1 and C_2 of κ such that $(H_d \oplus H_{E_e})^\perp = \bigoplus_{C_1} H_{PS_{\pi(\mathcal{A})_e}}$ and $(H_d \oplus H_{E_e})^\perp = \bigoplus_{C_2} H_{PS_{\pi(\mathcal{A})_e}}$. We have that $|C_1| = |C_2| = \kappa$ and therefore,

$$\bigoplus_{C_1} (H_{S_{\pi(\mathcal{A})_e}}, \pi_{S_{\pi(\mathcal{A})_e}}) \simeq \bigoplus_{C_2} (H_{S_{\pi(\mathcal{A})_e}}, \pi_{S_{\pi(\mathcal{A})_e}}).$$

Let U' an isomorphism between

$$\bigoplus_{C_1}(H_{S_{\pi(\mathcal{A})_e}}, \pi_{S_{\pi(\mathcal{A})_e}}) \text{ and } \bigoplus_{C_2}(H_{S_{\mathcal{A}_e}}, \pi_{S_{\pi(\mathcal{A})_e}}).$$

(4) Let $v \in \tilde{H}_\kappa$. Then $v = v_d + v_{E_e} + v_{E_e^\perp}$, where $v_{E_e} := P_{E_e}v$ and $v_{E_e^\perp} := P_{E_e^\perp}v$. Let $w := Uv_d + Uv_{E_e} + U'v_{E_e^\perp}$, and $U'' := U \oplus U'$. Then w and U'' are such that U'' is an automorphism of $\tilde{H}_\kappa$ extending U such that $U''v = w$.

□

Remark 3.23. By Remark 2.36, for every $i \in \mathbb{Z}^+$, there are a Hilbert spaces H_i and positive integers n_i and k_i such that $\dim(H_i) = n_i$ and

$$\tilde{H}_\kappa \simeq \bigoplus_{i\in\mathbb{Z}^+} H_i^{(k_i)} \oplus \bigoplus_{\kappa}(\oplus_{\phi\in S_\mathcal{A}} L^2(\mathcal{A}, \phi))$$

and

$$\pi(\mathcal{A}) \simeq \bigoplus_{i\in\mathbb{Z}^+} \mathcal{K}(H_i^{(k_i)}) \oplus \bigoplus_{\kappa}(\oplus_{\phi\in S_\mathcal{A}} M_\phi).$$

4. TYPES AND QUANTIFIER ELIMINATION

In this section we provide a characterization of types in (H, π). The main results here are Theorem 4.3 and Theorem 4.6 that characterize types in terms of subrepresentations of (H, π) and, its consecuence, Corollary 4.7 that states that T_π has quantifier elimination. As in the previous section, we denote by $(\tilde{H}, \tilde{\pi})$ the monster model for the theory T_π as constructed in Theorem 3.22.

Remark 4.1. An automorphism U of (H, π) is a unitary operator U on H such that $U\pi(a) = \pi(a)U$ for every $a \in Ball_1(\mathcal{A})$.

Proof. Asume U is an automorphism of (H, π). It is clear that U must be a linear operator. Also, for every $v, w \in H$ and $\pi(a) \in \mathcal{A}$, we must have that $U(\pi(a)v) = \pi(a)(Uv)$ and $\langle Uv \,|\, Uw\rangle = \langle v \,|\, w\rangle$ by definition of automorphism. Therefore U must be unitary and commutes with the elements of $\pi(\mathcal{A})$. Conversely, if U is an unitary operator commuting with the elements of $\pi(\mathcal{A})$, then U is clearly an automorphism of (H, π). □

Lemma 4.2. *Let*

$$H_d = \bigoplus_{i\in\mathbb{Z}^+} H_i^{(k_i)}$$

be as in Theorem 2.32. Let $v \in H_i^{(k_i)}$ *for some* $i \in \mathbb{Z}^+$ *and let* $U \in Aut(H, \pi)$*. Then* $Uv \in H_i^{(k_i)}$*.*

Proof. By Theorem 2.32, $\pi(\mathcal{A})_d = \pi(\mathcal{A}) \cap \mathcal{K}(H)$ can be seen as:

$$\pi(\mathcal{A})_d = \bigoplus_{i\in\mathbb{Z}^+} \mathcal{K}(H_i^{(k_i)}).$$

By Theorem 4.1, any automorphism $U \in \mathrm{Aut}((H, \pi))$ commutes with every element of $\pi(\mathcal{A})$, in particular with any element K of $\mathcal{K}(H_i^{(k_i)})$. Thus, if $v \in H_i^{(k_i)}$ and $K \in \mathcal{K}(H_i^{(k_i)})$, $KUv = UKv$. This implies that $Uv \in H_i^{(k_i)}$. □

Theorem 4.3. *Let* v, $w \in \tilde{H}$*. Then* $tp(v/\emptyset) = tp(w/\emptyset)$ *if and only if* (H_v, π_v, v) *is isometrically isomorphic to* (H_w, π_w, w)*.*

Proof. Let us suppose that $tp(v/\emptyset) = tp(w/\emptyset)$. Then there is an automorphism U of $(\tilde{H}, \tilde{\pi})$ such that $Uv = w$. Therefore the representations (H_v, π_v, v) and (H_w, π_w, w) are unitarily equivalent and therefore (H_v, π_v, v) is isometrically isomorphic to (H_w, π_w, w).

Conversely, let (H_v, π_v, v) be isometrically isomorphic to (H_w, π_w, w). By Theorem 3.22, (H_v, π_v) and (H_w, π_w) can be seen as subrepresentations of $(\tilde{H}, \tilde{\pi})$. Given that (H_v, π_v, v) and (H_w, π_w, w) are isometrically isomorphic, by Theorem 2.32, Theorem 3.22 and Remark 3.23, the decompositions of (H_v, π_v) and (H_w, π_w) into cyclic representations are isometrically isomorphic too, and therefore $\tilde{H}_v^{\perp}$ and $\tilde{H}_w^{\perp}$ are isometrically isomorphic. Then we get an automorphism of $(\tilde{H}, \tilde{\pi})$ that sends v to w, and v and w have the same type over the empty set. □

Theorem 4.4. *Let* $v, w \in H$*. Then* $tp(v/\emptyset) = tp(w/\emptyset)$ *if and only if* $\phi_v = \phi_w$*, where* ϕ_v *denotes the positive linear functional on* $\mathcal{A}$ *defined by* v *as in Lemma 2.38.*

Proof. Let v and $w \in H$ be such that $\mathrm{tp}(v/\emptyset) = \mathrm{tp}(w/\emptyset)$. Then $\mathrm{qftp}(v/\emptyset) = \mathrm{qftp}(w/\emptyset)$ and therefore, for every $a \in \mathcal{A}$, $\langle \pi(a)v|v\rangle = \langle \pi(a)w|w\rangle$. But this means that $\phi_v = \phi_w$.

Conversely, if $\phi_v = \phi_w$, by Theorem 2.26, (H_v, π_v, v) is isometrically isomorphic to (H_w, π_w, w) and by Theorem 4.3 $\mathrm{tp}(v/\emptyset) = \mathrm{tp}(w/\emptyset)$. □

Lemma 4.5. *Let $E \subseteq H$, $U \in Aut(H,\pi)$. Then $U \in Aut((H,\pi)/E)$ if and only if $U \restriction (H_E,\pi_E) = Id_{(H_E,\pi_E)}$.*

Proof. Suppose that $U \restriction (H_E,\pi_E) = Id_{(H_E,\pi_E)}$. Then, U fixes H_E pointwise, and, therefore, fixes E pointwise. Conversely, suppose $U \in \mathrm{Aut}((H,\pi)/E)$. By Remark 4.1, U is an unitary operator that commutes with every $S \in \pi(\mathcal{A})$. Then for every $S \in \pi(\mathcal{A})$ and $v \in E$, we have that $U(Sv) = S(Uv) = Sv$. So U acts on H_E like the identity and the conclusion follows. □

Theorem 4.6. *Let v and $w \in \tilde{H}$ and $E \subseteq \tilde{H}$. Then tp(v/E) =tp(w/E) if and only if $P_E(v) = P_E(w)$ and tp$(P_E^{\perp}(v)/\emptyset)$ =tp$(P_E^{\perp}(w)/\emptyset)$.*

Proof. **⇒:** Suppose tp(v/E) =tp(w/E). Given that tp(v/E) =tp(w/E), there exists $U \in \mathrm{Aut}((\tilde{H},\tilde{\pi})/E)$ such that $Uv = w$. By Lemma 4.5, $U \restriction (H_E,\pi_E) = Id_{(H_E,\pi_E)}$ and $P_E(v) = U(P_E(v)) = P_E(w)$. On the other hand, $U(P_E^{\perp}(v)) = P_E^{\perp}(w)$ and therefore tp$(P_E^{\perp}(v)/\emptyset)$ =tp$(P_E^{\perp}(w)/\emptyset)$.

⇐: Asume $P_E(v) = P_E(w)$ and tp$(P_E^{\perp}(v)/\emptyset)$ =tp$(P_E^{\perp}(w)/\emptyset)$. Then there exists an automoprhism U of $(\tilde{H},\tilde{\pi})$ such that $U(P_E^{\perp}(v)) = P_E^{\perp}(w)$. Let $\tilde{U} = Id_{H_E} \oplus (U \restriction \tilde{H}_E^{\perp})$. Then, by Lemma 4.5, $\tilde{U}$ is an automorphism of $(\tilde{H},\tilde{\pi})$ that fixes E pointwise and $Uv = w$. This implies that tp(v/E) =tp(w/E). □

Corollary 4.7. *The structure (H,π) has quantifier elimination.*

Proof. This follows from Theorem 4.6 that shows that types are determined by quantifier-free conditions contained in it. □

Recall that weak* topology in $\mathcal{A}'$ (the Banach dual Algebra of $\mathcal{A}$) is the coarsest topology in $\mathcal{A}'$ such that for every $a \in A$, the function $F_a : \mathcal{A}' \to \mathbb{C}$ is continuous, where $F_a(\phi) = \phi(a)$ for $a \in \mathcal{A}$ and $\phi \in \mathcal{A}'$.

Theorem 4.8. *The stone space $S_1(Th(H,\pi))$ (i.e. the set of types of vectors of norm less than or equal to 1) with the logic topology is homeomorphic to the quasi state space $Q_{\mathcal{A}}$ with the weak* topology.*

Proof. We consider types of vectors with norm less than or equal to 1. Similarly, we consider positive linear functionals with norm less than or equal to 1, that is, the quasi state space $Q_{\mathcal{A}}$. By Theorem 4.4, types of vectors in H are determined by the corresponding positive linear functionals, so there is a bijection between $S_1(Th(H,\pi))$ and $Q_{\mathcal{A}}$.

To prove bicontinuity, let $h : S_1(Th(H,\pi)) \to Q_{\mathcal{A}}$ be the previously defined bijection. Let X be a weak* basic open set in $Q_{\mathcal{A}}$; then there exists an open set $V \subseteq \mathbb{C}$ and an element $a \in \mathcal{A}$ such that for every $\phi \in Q_{\mathcal{A}}$, we have that $\phi \in X$ if and only if $\phi(a) \in V$. For $\phi \in X$ let v_ϕ be a cyclic vector such that $\phi = \phi_{v_\phi}$. Then for every $\phi \in X$, $\langle \pi(a)v_\phi \mid v_\phi \rangle \in V$ but this condition defines an open set in $S_1(Th(H,\pi))$.

Conversely, by quantifier elimination, every basic open sets X in the logic topology in $S_1(Th(H,\pi))$ can be expresed as finite intersection of sets with the form:

$$\{p \in S_1(Th(H,\pi)) \mid v_\phi \models p \Rightarrow \langle \pi(a)v_\phi \mid v_\phi \rangle \in V\}$$

where $V \subseteq \mathbb{C}$ open. Each of this sets is in correspondence by h with a set of the form

$$\{\phi \in Q_{\mathcal{A}} \mid \langle \pi(a)v_\phi \mid v_\phi \rangle \in V\}$$

which defines an open set in $Q_{\mathcal{A}}$. □

5. DEFINABLE AND ALGEBRAIC CLOSURES

In this section we give a characterization of definable and algebraic closures. The results here are consequences of Theorem 2.43 and Theorem 2.32. Gelfand-Naimark-Segal construction is a tool for understanding definable closures (see Theorem 2.44). Algebraic closures are studied with the help of Theorem 2.32.

Theorem 5.1. *Let $E \subseteq H$. Then $dcl(E) = H_E$*

Proof. From Lemma 4.5, it is clear that $H_E \subseteq dcl(E)$. On the other hand, if $v \in H_E$, let $\lambda \in \mathbb{C}$ such that $l \neq 1$ and $|\lambda| = 1$. Then, the operator $U := Id_{H_E} \oplus \lambda Id_{H_E^\perp}$ is an automorphism of (H,π) fixing E such that $Uv \neq v$. □

Lemma 5.2. *Let $v \in H_e$. Then v is not algebraic over $\emptyset$.*

Proof. We can asume that (H,π) is the monster model with density $\kappa > 2^{\aleph_0}$. Then there are κ vectors v_i for $i < \kappa$ such that every v_i has the same type over $\emptyset$ as v. This means that the orbit of v under the automorphisms of (H,π) is unbounded and therefore v is not algebraic over the emptyset. □

Lemma 5.3. *Let $v \in H$ such that $v_e \neq 0$. Then v is not algebraic over $\emptyset$.*

Proof. Clear from previous Lemma 5.2. □

Theorem 5.4. $acl(\emptyset) = H_d$

Proof. By Lemma 3.18, $H_d \subseteq acl(\emptyset)$ and, by Lemma 5.3, $acl(\emptyset) \subseteq H_d$. □

Theorem 5.5. *Let $E \subseteq H$. Then $acl(E)$ is the Hilbert subspace of H generated by $dcl(E)$ and $acl(\emptyset)$.*

Proof. Let G be the Hilbert subspace of H generated by $dcl(E)$ and $acl(\emptyset)$. It is clear that $G \subseteq acl(E)$. Let $v \in acl(E)$. By Lemma 3.18, $v_d \in acl(\emptyset)$, and by Theorem 5.1 and Lemma 5.2, $v_e \in dcl(E) \setminus acl(\emptyset)$. Then $v_e \in dcl(E)$ and $acl(E) \subseteq G$. □

6. FORKING AND STABILITY

In this section we give an explicit characterization of non-forking and prove that $Th(H, \pi)$ is stable. Henson and Iovino in [19], observed that a Hilbert space expanded with a family of bounded operators is stable. Here, we give an explicit independence relation that has the properties of non-forking in superstable theories.

Definition 6.1. Let E, F, $G \subseteq H$. We say that E is *independent* from G over F if for all $v \in E$ $P_{acl(F)}(v) = P_{acl(F \cup G)}(v)$ and denote it $E \mathop{\smile\!\!\!\!\!\!|}^{*}_{F} G$.

Remark 6.2. Let $\bar{v}$, $\bar{w} \in H^n$. Then, $\bar{v}$ is independent from $\bar{w}$ over $\emptyset$ if and only if for every $j, k = 1, \ldots, n$, $H_{(v_j)_e} \perp H_{(w_k)_e}$.

Remark 6.3. Let $\bar{v}$, $\bar{w} \in H^n$ and $E \subseteq H$. Then, $\bar{v}$ is independent from $\bar{w}$ over E if and only if for every $j, k = 1, \ldots, n$, $H_{P_E^\perp(v_j)_e} \perp H_{P_E^\perp(w_k)_e}$.

Remark 6.4. Let $\bar{v} \in H^n$ and E, $F \subseteq H$. Then $\bar{v} \mathop{\smile\!\!\!\!\!\!|}^{*}_{E} F$ if and only if for every $j = 1, \ldots, n$ $v_j \mathop{\smile\!\!\!\!\!\!|}^{*}_{E} F$ that is, for all $j = 1, \ldots, n$ $P_{acl(E)}(v_j) = P_{acl(E \cup F)}(v_j)$

Theorem 6.5. *Let $E \subseteq F \subseteq H$, $p \in S_n(E)$ $q \in S_n(F)$ and $\bar{v} = (v_1, \ldots, v_n)$, $\bar{w} = (v_1, \ldots, v_n) \in H^n$ be such that $p = tp(\bar{v}/E)$ and $q = tp(\bar{w}/F)$. Then q is an extension of p such that $\bar{w} \mathop{\smile\!\!\!\!\!\!|}^{*}_{E} F$ if and only if the following conditions hold:*

(1) *For every $j = 1, \ldots, n$, $P_{acl(E)}(v_j) = P_{acl(F)}(w_j)$*

(2) *For every* $j = 1, \ldots, n$, $(H_{P^{\perp}_{acl(E)} v_j}, \pi_{P^{\perp}_{acl(E)} v_j}, P^{\perp}_{acl(E)} v_j)$ *is isometrically isomorphic to* $(H_{P^{\perp}_{acl(F)} w_j}, \pi_{P^{\perp}_{acl(F)} w_j}, P^{\perp}_{acl(F)} w_j)$

Proof. Clear from Theorem 4.3 and Remark 6.2 □

Remark 6.6. Recall that for every $E \subseteq H$ and $v \in H$, $P^{\perp}_{acl(E)} v = (P^{\perp}_{E} v)_e$.

Theorem 6.7. $\mathop{\smile\!\!\!\!\!\!|}^*$ *is a freeness relation.*

Proof. By Remark 6.4, to prove local character, finite character and transitivity it is enough to show them for the case of a 1-tuple.

Local character: Let $v \in H$ and $E \subseteq H$. Let $w = (P_{acl(E)}(v))_e$. Then there exist a sequence of $(l_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$, a sequence of finite tuples $(a^k_1, \ldots, a^k_{l_k})_{k \in \mathbb{N}} \subseteq \mathcal{A}$ and a sequence of finite tuples $(e^k_1, \ldots, e^k_{l_k})_{k \in \mathbb{N}} \subseteq E$ such that if $w_k := \sum_{j=1}^{l_k} \pi(a^k_j) e^k_j$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$. Let $E_0 = \{e^k_j \mid j = 1, \ldots, l_k \text{ and } k \in \mathbb{N}\}$. Then $v \mathop{\smile\!\!\!\!\!\!|}^*_{E_0} E$ and $|E_0| = \aleph_0$.

Finite character: We show that for $v \in H$, $E, F \subseteq H$, $v \mathop{\smile\!\!\!\!\!\!|}^*_E F$ if and only if $v \mathop{\smile\!\!\!\!\!\!|}^*_E F_0$ for every finite $F_0 \subseteq F$. The left to right direction is clear. For right to left, suppose that $v \not\mathop{\smile\!\!\!\!\!\!|}^*_E F$. Let $w = P_{acl(E \cup F)}(v) - P_{acl(E)}(v)$. Then $w \in \mathrm{acl}(E \cup F) \backslash \mathrm{acl}(E)$.

As in the proof of local character, there exist a sequence of pairs $(l_k, n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}^2$, a sequence of finite tuples $(a^k_1, \ldots, a^k_{l_k + n_k})_{k \in \mathbb{N}} \subseteq \mathcal{A}$ and a sequence of finite tuples $(e^k_1, \ldots, e^k_{l_k}, f^k_1, \ldots, f^k_{n_k})_{k \in \mathbb{N}}$ such that $(e^k_1, \ldots, e^k_{l_k}) \subseteq E$, $(f^k_1, \ldots, f^k_{n_k})_{k \in \mathbb{N}} \subseteq F$ and if $w_k := \sum_{j=1}^{l_k} \pi(a^k_j) e^k_i + \sum_{j=1}^{n_k} \pi(a^k_{l_k + j}) f^k_j$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$.

If $v \not\mathop{\smile\!\!\!\!\!\!|}^*_E F$, then $w = P_{acl(E \cup F)}(v) - P_{acl(E)}(v) \neq 0$. For $\epsilon = \|w\| > 0$ there is k_ϵ such that if $k \geq k_\epsilon$ then $\|w - w_k\| < \epsilon$. Let $F_0 := \{f^1_1, \ldots, f^{n_{k_\epsilon}}_{k_\epsilon}\}$ Then F_0 is a finite subset such that $v \not\mathop{\smile\!\!\!\!\!\!|}^*_E F_0$.

Transitivity of independence: Let $v \in H$ and $E \subseteq F \subseteq G \subseteq H$. If $v \mathop{\smile\!\!\!\!\!\!|}^*_E G$ then $P_{acl(E)}(v) = P_{acl(G)}(v)$. It is clear that $P_{acl(E)}(v) = P_{acl(F)}(v) = P_{acl(G)}(v)$ so $v \mathop{\smile\!\!\!\!\!\!|}^*_E F$ and $v \mathop{\smile\!\!\!\!\!\!|}^*_F G$. Conversely, if $v \mathop{\smile\!\!\!\!\!\!|}^*_E F$ and $v \mathop{\smile\!\!\!\!\!\!|}^*_F G$, we have that $P_{acl(E)}(v) = P_{acl(F)}(v)$ and $P_{acl(F)}(v) = P_{acl(G)}(v)$. Then $P_{acl(E)}(v) = P_{acl(G)}(v)$ and $v \mathop{\smile\!\!\!\!\!\!|}^*_E G$.

Symmetry: It is clear from Remark 6.3.

Invariance: Let U be an automorphism of (H,π). Let $\bar{v}=(v_1,\dots,v_n)$,$\bar{w}=(w_1,\dots,w_n)\in H^n$ and $E\subseteq H$ be such that $\bar{v}\downarrow^*_E\bar{w}$. By Remark 6.3, this means that for every j, $k=1,\dots,n$ $H_{P^{\perp}_{acl(E)}(v_j)}\perp H_{P^{\perp}_{acl(E)}(w_k)}$. It follows that for every j, $k=1,\dots,n$ $H_{P^{\perp}_{acl(UE)}(Uv_j)}\perp H_{P^{\perp}_{acl(UE)}(Uw_k)}$ and, again by Remark 6.3, $Uv\downarrow^*_{acl(UE)}Uw$.

Existence: Let $(\tilde{H},\tilde{\pi})$ be the monster model and let $E\subseteq F\subseteq\tilde{H}$ be small sets. We show, by induction on n, that for every $p\in S_n(E)$, there exists $q\in S_n(F)$ such that q is a non-forking extension of p.

Case $n=1$**:** Let $v\in\tilde{H}$ be such that $p=tp(v/E)$ and let $(H',\pi',u):=(L^2(\mathcal{A},\phi_{(P^{\perp}_{acl(E)}v)})_e,M_{(P^{\perp}_{acl(E)}v)_e},(e)_{(P^{\perp}_{acl(E)}v)_e})$. Then, the model $(\hat{H},\hat{\pi}):=(H,\pi)\oplus(H',\pi')$ is an elementary extension of (H,π). Let $v':=P_{acl(E)}v+P^{\perp}_{acl(E)}v_d+u\in\hat{H}$. Then, by Theorem 6.5, the type $tp(v'/F)$ is a $\downarrow^*$-independent extension of $tp(v/E)$.

Induction step: Now, let $\bar{v}=(v_1,\dots,v_n,v_{n+1})\in\tilde{H}^{n+1}$. By induction hypothesis, there are $v'_1,\dots,v'_n\in H$ such that $tp(v'_1,\dots,v'_n/F)$ is a $\downarrow^*$-independent extension of $tp(v_1,\dots,v_n/E)$. Let U be a monster model automorphism fixing E pointwise such that for every $j=1,\dots,n$, $U(v_j)=v'_j$. Let $v'_{n+1}\in\tilde{H}$ be such that $tp(v'_{n+1}/Fv'_1\cdots v'_n)$ is a $\downarrow^*$-independent extension of $tp(U(v_{n+1})/Ev'_1,\cdots v'_n)$. Then, by transitivity, $tp(v'_1,\dots,v'_n,v'_{n+1}/F)$ is a $\downarrow^*$-independent extension of $tp(v_1,\dots,v_n,v_{n+1}/E)$.

Stationarity: Let $(\tilde{H},\tilde{\pi})$ be the monster model and let $E\subseteq F\subseteq\tilde{H}$ be small sets. We show, by induction on n, that for every $p\in S_n(E)$, if $q\in S_n(F)$ is a $\downarrow^*$-independent extension of p to F then $q=p'$, where p' is the $\downarrow^*$-independent extension of p to F built in the proof of existence.

Case $n=1$**:** Let $v\in H$ be such that $p=tp(v/E)$, and let $q\in S(F)$ and $w\in H$ be such that $w\models q$. Let v' be as in previous item. Then, by Theorem 6.5 we have that:

(1) $P_{acl(E)}v=P_{acl(F)}v'=P_{acl(F)}w=$

(2) $(H_{P^{\perp}_{acl(E)}v},\pi_{P^{\perp}_{acl(E)}v},P^{\perp}_{acl(E)}v)$ is isometrically isomorphic to both

$$(H_{P^{\perp}_{acl(F)}w},\pi_{P^{\perp}_{acl(F)}w},P^{\perp}_{acl(F)}w)$$

and

$$(H_{P^{\perp}_{acl(F)}v'}, \pi_{P^{\perp}_{acl(F)}v'}, P^{\perp}_{acl(F)}v')$$

This means that $P_{acl(F)}v' = P_{acl(F)}w$ and $(H_{P^{\perp}_{acl(F)}w}, \pi_{P^{\perp}_{acl(F)}w}, P^{\perp}_{acl(F)}w)$ is isometrically isomorphic to $(H_{P^{\perp}_{acl(F)}v'}, \pi_{P^{\perp}_{acl(F)}v'}, P^{\perp}_{acl(F)}v')$ and, therefore $q = tp(v'/F) = p'$.

Induction step: Let $\bar{v} = (v_1, \dots, v_n, v_{n+1})$, $\bar{v}' = (v_1', \dots, v_n, v_{n+1}')$ and $\bar{w} = (w_1, \dots, w_n) \in \tilde{H}$ be such that $\bar{v} \models p$, $\bar{v}' \models p'$ and $\bar{w} \models q$. By transitivity, we have that $tp(v_1', \dots, v_n'/F)$ and $tp(w_1, \dots, w_n/F)$ are $\downarrow^*$-independent extensions of $tp(v_1, \dots, v_n/E)$. By induction hypothesis, $tp(v_1', \dots, v_n'/F) = tp(w_1, \dots, w_n/F)$. Let U be a monster model automorphism fixing E pointwise such that for every $j = 1, \dots, n$, $U(v_j) = v_j'$ and let U' a monster model automorphism fixing F pointwise such that for every $j = 1, \dots, n$, $U'(v_j') = w_j'$. Again by transitivity, $tp(U^{-1}(v_{n+1}')/Fv_1 \cdots v_n)$ and $tp((U' \circ U)^{-1}(w_{n+1})/Fv_1, \cdots v_n)$ are $\downarrow^*$-independent extensions of $tp(v_{n+1}/Ev_1, \cdots v_n)$. By the case $n = 1$ $tp(U^{-1}(v_{n+1}')/Fv_1 \cdots v_n) = tp((U' \circ U)^{-1}(w_{n+1})/Fv_1, \cdots v_n)$ and therefore $p' = tp(v_1', \dots, v_n'v_{n+1}'/F) = tp(w_1, \dots, w_n, w_{n+1}/F) = q$.

□

Lemma 6.8 (Theorem 14.14 in [6]). *A first order continuous logic theory T is stable if and only if there is an independence relation $\downarrow^*$ satisying local character, finite character of dependence, transitivity, symmetry, invariance, existence and stationarity. In that case the relation $\downarrow^*$ coincides with non-forking.*

Theorem 6.9. *The theory T_π is superstable and the relation $\downarrow^*$ agrees with non-forking.*

Proof. By Lemma 6.8, T_π is stable and the relation $\downarrow^*$ agrees with non-forking. To prove superstability, we have to show that for every $\bar{v} = (v_1, \dots, v_n) \in H$, every $F \subseteq H$ and every $\epsilon > 0$, there exist a finite $F_0 \subseteq F$ and $\bar{v}' = (v_1', \dots, v_n') \in H^n$ such that $\|v_j - v_j'\| < \epsilon$ and $v_j' \downarrow_{F_0} F$ for every $j \leq n$. As in the proof of local character, for $j = 1, \dots, n$ let $(\,{}^j a_1^k, \dots, {}^j a_{l_k}^k)_{k \in \mathbb{N}}$, $(\,{}^j e_1^k, \dots, {}^j e_{l_k}^k)_{k \in \mathbb{N}}$, $w_j := (P_{acl(E)}(v_j))_e$

and $(w_j^k)_{k\in\mathbb{N}}$ be such that $w_j^k := \sum_{s=1}^{l_k} \pi({}^j a_s^k)\ {}^j e_s^k$ for $k \in \mathbb{N}$, and $w_j^k \to w_j$. For $j = 1, \dots, n$, let $K_j \in \mathbb{N}$ be such that $\|w_j - w_j^{K_j}\| < \epsilon$, let $v_j' := (P_{acl(E)} v_j)_d + w_j^{K_j}$ and let $F_0^j = \{e_s^k \mid k \leq K_j \text{ and } s = 1, \dots, l_k\}$. If we define $F_0 := U_{i=1}^n F_0^j$, then for every $j = 1, \dots, n$ we have that $v_j' \mathop{\smile\!\!\!\!|}^*_{F_0} F$, $|F_0| < \aleph_0$ and $\|v_j - v_j'\| < \epsilon$. □

Remark 6.10. According Theorem 6.7, Theorem 6.5 stablishes a characterization of non-forking extensions.

Theorem 6.11. *Let $\bar v = (v_1, \dots, v_n) \in H^n$ and $E \subseteq H$. Then $Cb(tp(\bar v/E)) := \{(P_E v_1, \dots, P_E v_n)\}$ is a canonical base for the type $tp(\bar v/E)$*

Proof. First of all, we consider the case of a 1-tuple. By Theorem 6.5 $tp(v/E)$ does not fork over $Cb(tp(v/E))$. Let $(v_k)_{k<\omega}$ a Morley sequence for $tp(v/E)$. We have to show that $P_E v \in dcl((v_k)_{k<\omega})$. By Theorem 6.5, for every $k < \omega$ there is a vector w_k such that $v_k = P_E v + w_k$ and $w_k \perp acl(\{P_E v\} \cup \{w_j \mid j < k\})$. This means that for every $k < \omega$, $w_k \in H_e$ and for all j, $k < \omega$, $H_{w_j} \perp H_{w_k}$. For $k < \omega$, let $v_k' := \frac{v_1 + \cdots + v_k}{n} = P_E v + \frac{w_1 + \cdots + w_k}{n}$. Then for every $k < \omega$, $v_k' \in dcl((v_k)_{k<\omega})$. Since $v_k' \to P_e v$ when $k \to \infty$, we have that $P_E v \in dcl((v_k)_{k<\omega})$.

For the case of a general n-tuple, by Remark 6.4, it is enough to repeat previous argument in every component of $\bar v$. □

Corollary 6.12. *The theory T_π has weak elimination of imaginaries.*

Proof. Clear by previous theorem. □

7. ORTHOGONALITY AND DOMINATION

In this section, we characterize domination, orthogonality of types in terms of similar relationships between positive linear functionals on $\mathcal{A}$. These are the statements Theorem 7.5 and Theorem 7.8. For a complete description of the relation of domination see [10], Definition 5.6.4.

Theorem 7.1. *Let $v, w \in H$. Then (H_v, π_v, v) is isometrically isomorphic to a subrepresentation of (H_w, π_w, w) if and only if $\phi_v \leq \phi_w$.*

Proof. Suppose (H_v, π_v, v) is isometrically isomorphic to a subrepresentation of (H_w, π_w, w). Then there exists a vector $v' \in H_w$ such that $(H_v, \pi_v, v) \simeq (H_{v'}, \pi_{v'}, v')$.

By Theorem 2.49, there exists a bounded positive operator $P : (H_w, \pi_w, w) \to (H_{v'}, \pi_{v'}, v')$ such that $Pw = v'$ and P commutes with every element of $\pi_v(\mathcal{A})$. Let $\gamma = \|P\|^2$. Then, for every positive element $a \in \mathcal{A}$, $\phi_v(a) = \phi_{v'}(a) = \langle \pi(a)v' \mid v' \rangle = \langle \pi(a)Pw \mid Pw \rangle = \langle P^* \pi(a)Pw \mid w \rangle = \langle \pi(a)\|P\|^2 w \mid w \rangle \leq \gamma \langle \pi(a)w \mid w \rangle = \gamma \phi_w(a)$ which means that $\gamma\phi_w - \phi_v$ is positive and $\phi_v \leq \phi_w$.

The converse is Corollary 3.3.8 in [20]. □

Lemma 7.2. *Let $v, w \in H$. If $\phi_v \perp \phi_w$, then (H_v, π_v, v) is not isometrically isomorphic to any subrepresentation of (H_w, π_w, w).*

Proof. Suppose $\phi_v \perp \phi_w$, and (H_v, π_v, v) is isometrically isomorphic to subrepresentation of (H_w, π_w, w). By Theorem 7.1 $\phi_v \leq \phi_w$; let $\gamma > 0$ be a real number such that $\gamma\phi_w - \phi_v$ is a bounded positive functional and let $u \in H$ be such that $\phi_u = \gamma\phi_w - \phi_v$. Then $\phi_v = \gamma\phi_w - \phi_u$, and $\|\phi_w - \phi_v\| = \|\phi_w - \gamma\phi_w + \phi_u\| = \|(1 - \gamma)\phi_w + \phi_u\| = |1-\gamma|\|\phi_w\| + \|\phi_u\| \neq |1+\gamma|\|\phi_w\| + \|\phi_u\| = \|\phi_w + \phi_v\| = \|\phi_w\| + \|\phi_v\|$, but this contradicts $\phi_v \perp \phi_w$. □

Theorem 7.3. *Let $v, w \in H$. $\phi_v \perp \phi_w$ if and only if no subrepresentation of (H_v, π_v, v) is isometrically isomorphic to a subrepresentation of (H_w, π_w, w).*

Proof. Suppose $\phi_v \perp \phi_w$. By Lemma 2.48, if $(H_{v'}, \pi_{v'}, v')$ is a subrepresentation of (H_v, π_v, v) and $(H_{w'}, \pi_{w'}, w')$ is a subrepresentation of (H_w, π_w, w), then $\phi_{v'} \perp \phi_{w'}$, By Lemma 7.2, $(H_{v'}, \pi_{v'}, v')$ is not isometrically isomorphic to $(H_{w'}, \pi_{w'}, w')$, and the conclusion follows.

Conversely, suppose no subrepresentation of (H_v, π_v, v) is isometrically isomorphic to a subrepresentation of (H_w, π_w, w). Then the representations (H_v, π_v) and (H_w, π_w) are disjoint. By Fact 2.17, there is a projection $P \in \pi(\mathcal{A})' \cap \pi(\mathcal{A})''$ such that $PP_v = P_v$ and $(I - P)P_w = P_w$. Then, $\phi_v(I - P) = \langle (I - P)v \mid v \rangle = \langle (v - PP_v v) \mid v \rangle = \langle (v - v) \mid v \rangle = 0$. On the other hand, $\phi_w(P) = \langle Pw \mid w \rangle = \langle w - (w - Pw) \mid w \rangle = \langle w - (I - P)w \mid w \rangle = \langle w - (I - P)P_w w \mid w \rangle = \langle w - P_w w \mid w \rangle = \langle w - w \mid w \rangle = 0$. By Fact 2.15 and Theorem 2.10, the projection P is strongly approximable by positive elements in $\pi(\mathcal{A})$ and therefore $\epsilon > 0$ there exists a positive element $a \in \mathcal{A}$ with norm less than or equal to 1, such that $\phi_v(e - a) < \epsilon$ and $\phi_w(a) < \epsilon$. By 2.47, $\phi_v \perp \phi_w$. □

Lemma 7.4. *Let $p, q \in S_1(\emptyset)$, let v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \perp^a q$ if and only if $\phi_{v_e} \perp \phi_{w_e}$.*

Proof. Suppose $p \perp^a q$. By Remark 6.2, this implies that $H_{v_e} \perp H_{w_e}$ for all $v \models p$ and $w \models q$. Let $v \models p$ and $w \models q$. Then no subrepresentation of $(H_{v_e}, \pi_{v_e}, v_e)$ is isometrically isomorphic to any subrepresentation of $(H_{w_e}, \pi_{w_e}, w_e)$. By Lemma 7.3, this implies that $\phi_{v_e} \perp \phi_{w_e}$.

Conversely, if $p \not\perp^a q$ there are v, $w \in H$ such that $v \models p$, $w \models q$ and $H_{v_e} \not\perp H_{w_e}$. This implies that there exists an element $a \in \mathcal{A}$ such that $v_e \not\perp \pi(a)w_e$. Since $v_e = P_{w_e}v_e + P_{w_e}^{\perp}v_e$ and $P_{w_e}v_e \neq 0$, we can prove that $\phi_{P_{w_e}v_e} \leq \phi_{v_e}$ by using a procedure similar to the one used in the proof of Theorem 7.1 and, since $P_{w_e}v_e \in H_{w_e}$, we get $\phi_{P_{w_e}v_e} \leq \phi_{w_e}$. By Lemma 2.48, this implies that $\phi_{v_e} \not\perp \phi_{w_e}$. □

Theorem 7.5. *Let $E \subseteq H$. Let $p, q \in S_1(E)$, let v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \perp_E^a q$ if and only if $\phi_{P_E^{\perp}(v_e)} \perp \phi_{P_E^{\perp}(w_e)}$*

Proof. Clear by Lemma 7.4. □

Theorem 7.6. *Let $E \subseteq H$. Let $p, q \in S_1(E)$. Then, $p \perp^a q$ if and only if $p \perp q$.*

Proof. Assume $p \perp^a q$, $E \subseteq F \subseteq H$ are small subsets of the monster model and p', $q' \in S_1(F)$ are non-forking extensions of p and q respectively. Let v, $w \in H$ be such that $v \models p'$ and $w \models q'$, then $\phi_{P_F^{\perp}(v_e)} = \phi_{P_E^{\perp}v_e} \perp \phi_{P_E^{\perp}w_e} = \phi_{P_F^{\perp}(w_e)}$. By Lemma 7.4, this implies that $p' \perp^a q'$. Therefore $p \perp q$.

The converse is trivial. □

Lemma 7.7. *Let $p, q \in S_1(\emptyset)$ and let v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \rhd_{\emptyset} q$ if and only if $\phi_{w_e} \leq \phi_{v_e}$.*

Proof. Suppose $p \rhd_{\emptyset} q$. Suppose that v' and w' are such that $v' \models p$, $w' \models q$ and if $v' \mathop{\smile\!\!\!\!\!\!|}^*_{\emptyset} E$ then $w' \mathop{\smile\!\!\!\!\!\!|}^*_{\emptyset} E$ for every E. Then for every $E \subseteq H$

$$P_E v'_e = 0 \Rightarrow P_E w'_e = 0$$

This implies that $w'_e \in H_{v'_e}$, and $H_{w'_e} \subseteq H_{v'_e}$. By Theorem 7.1, $\phi_{w_e} = \phi_{w'_e} \leq \phi_{v'_e} = \phi_{v_e}$.

For the converse, suppose $\phi_{w_e} \leq \phi_{v_e}$. Then, by Theorem 7.1 H_{w_e} is isometrically isomorphic to a subrepresentation of H_{v_e}, which implies that there is $w' \in H_v$ such that $w' \models tp(w/\emptyset)$ and for every $E \subseteq H$

$$P_E v_e = 0 \Rightarrow P_E w'_e = 0$$

This means than $tp(w/\emptyset) \lhd_\emptyset tp(v/\emptyset)$. □

Theorem 7.8. *Let E, F and G be small subsets of $\tilde{H}$ and $p \in S_1(E)$ and $q \in S_1(F)$ be two stationary types. Then $p \rhd_G q$ if and only if there exist v, $w \in \tilde{H}$ such that $tp(v/G)$ is a non-forking extension of p, $tp(w/G)$ is a non-forking extension of q and $\phi_{P^\perp_{acl(G)} w_e} \leq \phi_{P^\perp_{acl(G)} v_e}$.*

Proof. Clear by Lemma 7.7. □

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Camilo Argoty, Departamento de Matemáticas, Universidad Sergio Arboleda, Bogotá, Colombia