THE TORSION-FREE PART OF THE ZIEGLER SPECTRUM OF THE KLEIN FOUR GROUP.

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ABSTRACT. We will describe the torsion-free part of the Ziegler spectrum, both the points and the topology, over the integral group ring of the Klein group. For instance we will show that the Cantor–Bendixson rank of this space is equal to 3.

1. INTRODUCTION

The Ziegler spectrum of a ring R, Zg_R, is a topological space whose points are (isomorphism types) of indecomposable pure injective (= algebraically compact) R-modules. The set is endowed with a (compact) topology whose basic open sets may be defined in a number of equivalent ways. The original definition [21] uses positive primitive formulas — a notion from model theory but one whose algebraic meaning is easily understood (such a formula defines the projection, to one or more specified components, of the solution set of a specified system of linear equations). Precisely, if ψ and φ are positive primitive formulas then the basic open set (φ/ψ) is defined to be the set of those $M \in \mathrm{Zg}_R$ which contain an element m which satisfies in M φ but not ψ. Many specific examples can be seen in Sections 3 and 5 of this paper. Alternative definitions of the basic (= compact) open sets of the topology can be given in terms of homomorphisms between modules (see [3] or [14, 5.1.3]) or in terms of finitely presented functors (see [14, 13.1.3]).

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Ziegler's paper and subsequent work has shown the usefulness, for the model theory of modules, of describing this space but such description also has algebraic consequences. Indeed, the problem of describing the spectrum is an algebraic one, though concepts from model theory often are applied in solving it, as in this paper. Usually Zg_R is not so nice from the topological point of view: for instance it is rarely Hausdorff (see [11, 8.2.12] for some exotic examples of Ziegler spectra). However in many 'natural' settings this space is T_0, in particular this is the case when the Cantor–Bendixson rank of Zg_R, $\mathrm{CB}(\mathrm{Zg}_R)$, is defined.

For example, if $R = \mathbb{Z}$ is the ring of integers, then the CB-rank of Zg_R is equal to 2 with a standard division of points into finite, Prüfer, adic and generic (= divisible torsion-free): in more detail, $\mathbb{Z}/p^n\mathbb{Z}$ for p a prime and n a positive integer (these points have CB-rank 0), then, again for p a prime, $\mathbb{Z}_{p^\infty}$ (p-Prüfer group) and $\widehat{\mathbb{Z}}_p$ (p-adic integers), both of CB-rank 1, finally $\mathbb{Q}$ (the only point with CB-rank 2). It follows that $\mathrm{Zg}_{\mathbb{Z}}$ is T_0 but not Hausdorff. To have an example of a basic open set, take $\varphi \doteq (xp = 0)$ with p a prime, $\psi \doteq (x = 0)$, then (φ/ψ) consists of the modules in the spectrum with (nontrivial) p-torsion, that is, the $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{Z}_{p^\infty}$.

Due to results by Prest and Ringel (see [12], [18]) the picture is very similar for any tame hereditary finite dimensional algebra A over a field: the CB-rank of the Ziegler spectrum of A is 2 and the points of Zg_A are divided into finite length points, Prüfer and adic points (parameterized by the simple regular A-modules) and a unique (if A is connected) generic point. Note that for a finite dimensional algebra A the case of $\mathrm{CB}(\mathrm{Zg}_A) = 1$ is not possible (see Krause [9] and Herzog [6]), and $\mathrm{CB}(\mathrm{Zg}_A) = 0$ if and only if A is of finite representation type. Thus the simplest case of nontrivial behavior of Zg_A is when the CB-rank is equal to 2.

As already said, the general structure of the Ziegler spectrum of a ring R is often too involved, but it gets clearer when restricted to more manageable subcategories of R-modules. For instance, when $R = D$ is a commutative noetherian domain, or an order over a commutative Dedekind domain D, one could try to describe the closed subset $\mathrm{Zg}_{\mathrm{tf}}(R)$ of Zg_R consisting of D-torsion-free R-modules. There are just a few papers addressing this question. For instance Herzog and Puninskaya [7] proved that, if D is a complete local commutative noetherian domain of Krull dimension 1, then the category of finitely generated torsion-free D-modules is of finite representation type iff $\mathrm{Zg}_{\mathrm{tf}}(D)$ is a discrete (finite) space.

For our purposes more relevant is a paper by Marcja, Prest and Toffalori [10] that investigates the (D-) torsion-free part of the Ziegler spectrum over the group ring DG, where D is a commutative Dedekind domain of characteristic zero and G is a finite group. For instance they proved that every D-torsion-free indecomposable pure injective R-module is either divisible or reduced, therefore carries a natural structure of an (indecomposable pure injective) module over a completion $\widehat{D}_P G$ for some maximal ideal P of D. Furthermore they showed that the topology on $\mathrm{Zg}_{\mathrm{tf}}(DG)$ is obtained by gluing together 'P-patches' topological spaces associated to maximal ideals P of D. Despite [10] contains a lot of information on $\mathrm{Zg}_{\mathrm{tf}}(DG)$, the context seems to be too general to give a detailed description of this space.

In this note we will refine the analysis of [10] when D is the ring $\mathbb{Z}$ of integers and $G = C(2)^2$ is the Klein group, so for $R = \mathbb{Z}C(2)^2$, to give a very explicit description of $\mathrm{Zg}_{\mathrm{tf}}(R)$. For instance we will show that the CB-rank of this space is equal to 3, the only points of maximal rank being the modules $\mathbb{Q}\,Ge_i$ where the e_i are the indecomposable idempotents of $\mathbb{Q}\,G$; moreover those are the only closed points of $\mathrm{Zg}_{\mathrm{tf}}(\mathbb{Z}G)$.

Using the existence of almost split sequences (for lattices over orders over complete discrete valuation domains) we will also prove that the isolated points in $\mathrm{Zg}_{\mathrm{tf}}(\mathbb{Z}G)$ are exactly the indecomposable lattices over $\widehat{\mathbb{Z}}_p G$, but in contrast to the case of finite dimensional algebras none of them is closed (or finitely generated) as a $\mathbb{Z}G$-module.

The crucial point of our proofs is to show that a large locally closed subset of $\mathrm{Zg}_{\mathrm{tf}}(\mathbb{Z}G)$ is homeomorphic to a cofinite (clopen) subset of the Ziegler spectrum of the 4-subspace quiver $k\widetilde{D}_4$ (for k a field). This will be proved via a functor Δ providing a representation equivalence between certain categories of torsion-free $\widehat{\mathbb{Z}}_2 G$-modules and $k\widetilde{D}_4$-modules for $k = GF(2)$ the field with 2 elements, first exploited by Butler [2] to classify $\widehat{\mathbb{Z}}_2G$-lattices. We will show that Δ behaves well (in particular remains full) when restricted to a suitable category of pure injective torsion-free modules over $\widehat{\mathbb{Z}}_2G$. This allows us to recover pure injective modules in this category from corresponding pure injective $k\widetilde{D}_4$-modules. Thus $\mathrm{Zg}_{\mathrm{tf}}(\mathbb{Z}G)$ contains a large subspace homeomorphic to a clopen subset of $\mathrm{Zg}_{k\widetilde{D}_4}$. Therefore in addition to $\widehat{\mathbb{Z}}_2G$-lattices it contains Prüfer and adic points (parameterized by simple regular $k\widetilde{D}_4$-modules) each having CB-rank 1, and a unique point $\mathcal{G}'$ of CB-rank 2 (corresponding via Δ to the generic point over $k\widetilde{D}_4$). Note that $\mathcal{G}'$ itself is not of finite endolength as an R-module. The only generic (= closed) points of $\mathrm{Zg}_{\mathrm{tf}}(\mathbb{Z}G)$ are the modules $\mathbb{Q}Ge$ (for e an indecomposable idempotent of $\mathbb{Q}G$) and these have CB-rank 3.

Note that reductions (via functors) of categories of lattices over orders into categories of finite dimensional modules over finite dimensional algebras is an important tool in classifying lattices. As we will show, in our particular case, that is, $R = \mathbb{Z}C(2)^2$, the Ziegler spectrum is quite rigid with respect to such a functor. Recall that a similar (even more transparent) effect on the Ziegler spectrum has been observed by Puninski and Toffalori [16] for the so-called Klein rings (a special class of commutative artinian rings), where as the modeling example the Kronecker algebra $k\widetilde{A}_1$ was used. In this paper we will exploit a very general result of Prest [14, 18.2.5] on definable functors between definable categories.

It is well known that Butler-like functors exist in the wider context of $\mathbb{Z}$-orders; so one may expect a similar description of $\mathrm{Zg}_{\mathrm{tf}}(\ldots)$ in this broader setting. For instance our results may suggest that the least positive value of the CB-rank of $\mathrm{Zg}_{\mathrm{tf}}(\ldots)$ for various similar orders over Dedekind domains is 3. However the arguments we apply in this paper seem to be too 'ad hoc' to admit an easy generalization.

In the remainder of the paper, unless otherwise stated, G denotes the Klein group $C(2)^2$ and R is the integral group ring $\mathbb{Z}G$; moreover $\mathrm{Zg}_{\mathrm{tf}}$ abbreviates $\mathrm{Zg}_{\mathrm{tf}}(R)$. Modules are assumed to be right modules.

Here is the plan of the next sections. After recalling in Section 2 some basic facts about G and modules over $R = \mathbb{Z}G$, we begin in Section 3 our analysis of $\mathrm{Zg}_{\mathrm{tf}}$; in particular we deal with those points of the spectrum that are $\mathbb{Q}G$-modules or $\widehat{\mathbb{Z}}_pG$-modules for some odd prime p. This reduces our investigation to the key case $p = 2$. In Section 4 we summarize the picture of the Ziegler spectrum of the quiver algebra $k\widetilde{D}_4$ where k is any field. In Section 5 Butler's functor Δ is introduced in the wider setting linking certain indecomposable pure injective representations of $k\widetilde{D}_4$ (over $k = GF(2)$) and certain indecomposable pure injective $\widehat{\mathbb{Z}}_2G$-modules. This will ultimately provide the expected description of the 2-patch of $\mathrm{Zg}_{\mathrm{tf}}$. The topology of $\mathrm{Zg}_{\mathrm{tf}}$ will be treated in the final Section 6.

We assume some basic familiarity with the model theory of modules, just regarding positive primitive (pp for short) formulas and types, pure injective modules, pure injective hulls and so on. Classical sources for these matter include [8], [11] or [21], but [13] provides a shorter and more fitting introduction for algebraists.

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2. The Klein four group

Recall that $G = C(2)^2$ denotes the Klein four group, that is, the non-cyclic group of order 4 (hence the direct product of two copies of the group $C(2)$ with 2 elements). If a, b are generators of these copies of $C(2)$, then a and b generate G subject to relations $a^2 = b^2 = 1$ and $ab = ba$.

Since G is a finite abelian group, the integral group ring $R = \mathbb{Z}G$ is commutative and noetherian (but, say, $(1-a)(1+a) = 0$, hence R contains zero divisors). Furthermore the set of nonzero divisors of R contains $\mathbb{Z} \setminus \{0\}$, therefore inverting integers we obtain that $\mathbb{Q}G$ is a classical quotient ring of R. Since the characteristic of $\mathbb{Q}$ is zero this ring is semisimple artinian with the following list of indecomposable idempotents: $e_1 = 1/4 \cdot (1+a)(1+b)$, $e_2 = 1/4 \cdot (1+a)(1-b)$, $e_3 = 1/4 \cdot (1-a)(1+b)$ and $e_4 = 1/4 \cdot (1-a)(1-b)$. Observe that $4e_i \in R$ for every i, but R has no nontrivial idempotents.

For each prime p let $\mathbb{Z}_p$ be the localization of $\mathbb{Z}$ with respect to the prime ideal $p\mathbb{Z}$ and let $\widehat{\mathbb{Z}}_p$ be the completion of $\mathbb{Z}_p$ in the topology defined by powers of $p\mathbb{Z}$ (thus $\widehat{\mathbb{Z}}_p$ is a complete noetherian valuation domain). Let us put for simplicity $R_p = \widehat{\mathbb{Z}}_p G$.

Recall that an R-module M is said to be ($\mathbb{Z}$-) torsion-free if, whenever $mk = 0$ for some $m \in M$ and $0 \neq k \in \mathbb{Z}$, then $m = 0$. Most R-modules considered in this paper will be torsion-free. A finitely generated torsion-free R-module M is said to be an R-lattice. A similar terminology will be used for modules over $R_p = \widehat{\mathbb{Z}}_p G$ (clearly $\widehat{\mathbb{Z}}_p$-torsion-free is the same as $\mathbb{Z}$-torsion-free). However R_p-lattices are not finitely generated as R-modules.

If M is a torsion-free module over R, then M is a submodule of $M \otimes_{\mathbb{Z}} \mathbb{Q} = M\mathbb{Q}$, in particular one can form Me_i which is again an R-submodule of $M\mathbb{Q}$. Note that $e_i g = \pm e_i$ for any $g \in G$. For instance

$$e_1 a = 1/4 \cdot (1 + a + b + ab) \cdot a = 1/4 \cdot (a + 1 + ab + b) = e_1,$$

$$e_2 a = 1/4 \cdot (1 - a + b - ab) \cdot a = 1/4 \cdot (a - 1 + ab - b) = -e_2$$

and similarly $e_1 b = e_1$, $e_2 b = e_2$. In other words a, b act identically on Me_1, while on Me_2 the action of a is 'skewed' in the sense just explained. A skewed action of b or both a and b characterizes also Me_3, Me_4.

Thus the R-module structure on Me_i is always obtained from its abelian group structure by taking into account a trivial or skewed action of G. For instance the R-module Re_2 can be identified with $\mathbb{Z}$ on which a acts as multiplication by -1 and b acts trivially (hence ab acts as multiplication by -1). Thus, for n and n_i integers, $n \cdot (n_1 + n_2 a + n_3 b + n_4 ab) = n \cdot n_1 - n \cdot n_2 + n \cdot n_3 - n \cdot n_4$. Similarly Re_1 is isomorphic to $\mathbb{Z}$ where a and b act trivially. Therefore every homomorphism between the underlying abelian groups of these modules can be regarded in a natural way as an R-module homomorphism.

Finally observe that, if M is a torsion-free R-module then Me_i ($i = 1, \ldots, 4$) is isomorphic (as an R-module) to $M \cdot 4e_i \subseteq M$, and in this way is pp-interpretable in M.

3. Pure injectivity

Recall that a module M over an arbitrary ring L is said to be pure injective if every finitely satisfiable (in M) system of linear equations over L has a solution in M. For various reformulations of this notion see [11, Chapter 4]. For instance (see [14, Theorem 4.3.6]) M is pure injective iff any summation map $M^{(I)} \to M$ can be lifted along the natural inclusion into the direct product module $M^{(I)} \subseteq M^I$. Every pp-definable submodule of a pure injective module M is pure injective. For instance, if $L = R = \mathbb{Z}G$ and M is a pure injective $\mathbb{Z}$-torsion-free R-module, then for every $i = 1, \ldots, 4$ Me_i (as an R-module isomorphic to $4Me_i$) is pure injective.

By [14, Theorem 4.4.8] every module with the descending chain condition on pp-definable subgroups is pure injective (in particular this is the case for $\widehat{\mathbb{Z}}_p$ considered as a module over R_p or over R with G acting trivially). It follows that every module of finite endolength is pure injective. Therefore (being of endolength 1) all the modules $\mathbb{Q}Ge_i$ are pure injective and indecomposable.

By [14, Lemma 4.2.8] every module linearly compact over its endomorphism ring is pure injective. This can be applied as follows.

Fact 3.1. (see [10, Theorem 2.1]) Every R_p-lattice M is a pure injective module over R_p and R.

Proof. Since M is finitely generated and torsion-free, it is isomorphic to $\widehat{\mathbb{Z}}_p^{(k)}$ for some k as a module over $\mathbb{Z}$ and $\widehat{\mathbb{Z}}_p$. It is easily seen that $\widehat{\mathbb{Z}}_p$ is linearly compact over itself, therefore (since linear compactness is closed with respect to extensions) M is linearly compact over $\mathbb{Z}_p$. It follows that M is linearly compact over the larger ring $\operatorname{End}_{R_p} M$. Thus (by the above remark) M is pure injective over R_p, hence over R. $\square$

Thus indecomposable R_p-lattices are an important source of indecomposable pure injective torsion-free R-modules. As we have already seen, the modules $\mathbb{Q}Ge_i$, $i = 1, \ldots, 4$ are also pure injective and indecomposable. Later we will construct more (infinitely generated) pure injective R-modules as a part of the Ziegler spectrum of R.

In fact it is time to start our analysis of Zg_R, more precisely of its torsion-free part $\mathrm{Zg}_{\mathrm{tf}}$. First note that for every positive integer n $\big((xn = 0)/(x = 0)\big)$ defines a basic open set of Zg_R consisting of points with n-torsion. It follows that the torsion-free part $\mathrm{Zg}_{\mathrm{tf}}$ of Zg_R, as the intersection of complements of these open sets, is closed (but, by compactness of Zg_R, not open) in Zg_R. We will be interested in the topology of $\mathrm{Zg}_{\mathrm{tf}}$ induced from Zg_R. Indeed in this paper we will relativize everything to the closed subset $\mathrm{Zg}_{\mathrm{tf}}$ of the Ziegler spectrum of R. Being a closed subset of a compact space, $\mathrm{Zg}_{\mathrm{tf}}$ is itself compact.

Important information on $\mathrm{Zg}_{\mathrm{tf}}$ can be extracted from [10], where the more general case of $L = DH$, with D a commutative Dedekind domain of characteristic zero and H an arbitrary finite group, is considered. The authors of that paper call an L-module M reduced if $\cap_n MP^n = 0$ for every maximal ideal P of D. The following two facts correspond to [10, Theorems 2.1 and 2.2] as applied to our situation.

Fact 3.2. Every indecomposable pure injective torsion-free R-module M is either a simple $\mathbb{Q}G$-module (so is isomorphic to $\mathbb{Q}Ge_i$ for some $i = 1, \ldots, 4$) or is an indecomposable pure injective torsion-free reduced module over R_p for a unique prime p.

For every prime p let $\mathrm{Zg}_{\mathrm{tf}p}$ abbreviate $\mathrm{Zg}_{\mathrm{tf}}(R_p)$, the torsion-free part of the Ziegler spectrum of R_p.

Fact 3.3. Every indecomposable pure injective torsion-free reduced R_p-module is pure injective and indecomposable also when viewed as a module over R, via restriction of scalars. Furthermore the set of torsion-free reduced points of $\mathrm{Zg}_{\mathrm{tf}p}$ (as it is embedded in $\mathrm{Zg}_{\mathrm{tf}}$, with a divisible point excluded!) has the same topology whether it is considered as a subset of $\mathrm{Zg}_{\mathrm{tf}p}$ or $\mathrm{Zg}_{\mathrm{tf}}$.

We describe now the closed points of $\mathrm{Zg}_{\mathrm{tf}}$.

Proposition 3.4. *The modules $\mathbb{Q}Ge_i$, $i = 1, \ldots, 4$ are the only closed points of* Zg_{tf}.

Proof. One implication is easy. In fact, by [14, Theorem 5.1.12], every finite endolength module is closed in Zg_R, therefore in $\mathrm{Zg}_{\mathrm{tf}}$. For a lazy proof of the converse we could use [14, Corollaries 5.3.21, 5.3.23]: since R is countable, every closed point M in $\mathrm{Zg}_{\mathrm{tf}}$ is of finite endolength (relativized to the theory of torsion-free modules). But then for every prime p the descending chain of pp-subgroups Mp^n stabilizes, therefore (since M is torsion-free) $M = pM$. Thus M is divisible. It follows that

M has a natural structure of a $\mathbb{Q}G$-module and it remains to apply the semisimplicity of $\mathbb{Q}G$.

However we could avoid in this proof the countability hypothesis (and extract more information) as follows.

Let M be an indecomposable pure injective torsion-free R-module, we will prove that one of the points $\mathbb{Q}Ge_i$ is in the closure of M. As $M \neq 0$ and $1 = \sum_i e_i$, there is at least one $i = 1, \ldots, 4$ for which $Me_i \neq 0$. Also, by Fact 3.2 we may assume that M is a reduced R_p-module, whence (by Fact 3.3) it suffices to show the following (for $i = 1, \ldots, 4$):

$(*)$ if M is an R_p-module such that $Me_i \neq 0$, then $\mathbb{Q}Ge_i$ is in the closure of M (as an R_p-module).

Choose $m \in M$ such that $0 \neq me_i \in Me_i$. Let T be the theory of M (as an R_p-module) and let p be the type consisting of the following formulae: $x \neq 0$, $e_i \mid x$ and $n \mid x$, n a positive integer. We prove that $T' = T \cup p$ is consistent. By the compactness theorem it suffices to check that every finite subset of T' is consistent. Clearly we can limit this check to finite subsets of the above form but with just one n. But such a set is realized in M by $n' = me_i \cdot 4n = m \cdot 4e_i n$, in fact $0 \neq n' \in Mn \cap Me_i$. Thus p is realized in some model M' of T by an element $0 \neq m' \in M'e_i$, $m' \in M'n$ for all n. It easily follows that there is an R_p-module morphism $f : \mathbb{Q}Ge_i \to M'$ sending e_i to m'. Since $\mathbb{Q}Ge_i$ is a simple $\mathbb{Q}Ge_i$-module, f is an isomorphism onto its image. Furthermore, since $\mathbb{Q}Ge_i$ is an injective R_p-module (see [10, p.1128, Claim 2]), it follows that M' splits off a copy of $\mathbb{Q}Ge_i$. $\square$

Thus we have just 4 closed points in $\mathrm{Zg}_{\mathrm{tf}}$. In fact $(*)$ says more: each point $\mathbb{Q}Ge_i$ cannot be separated by an open set from any point M in $\mathrm{Zg}_{\mathrm{tf}}$ with $Me_i \neq 0$. Note that the last condition on M can be written as a basic open set $O_i = \big((e_i \mid x)/(x = 0)\big)$ or rather (to stay within the language of R-modules) as $\big((x(4e_i - 4) = 0)/(x = 0)\big)$. It is easily checked that for $j \neq i$ the module $\mathbb{Q}Ge_j$ does not belong to O_i. Thus $\mathrm{Zg}_{\mathrm{tf}} = \cup_{i=1}^{4} O_i$ and for every i $\mathbb{Q}Ge_i$ is the unique closed point in O_i.

Having described the closed points of $\mathrm{Zg}_{\mathrm{tf}}$ we next deal with the open ones. Clearly the complement of the 4 divisible points $\mathbb{Q}Ge_i$ is open. Let us denote it by $\mathrm{Zg}_{\mathrm{red}}$ ('red' for reduced): Fact 3.2 explains this terminology. Note that [10, Theorem 2.4] claims that every indecomposable R_p-lattice is isolated in $\mathrm{Zg}_{\mathrm{red}}$. By Proposition 3.4 none of these points is closed in the whole space $\mathrm{Zg}_{\mathrm{tf}}$.

The following theorem also refines another aspect of the aforementioned result of [10].

Theorem 3.5. *The only isolated points of* Zg_{tf} *are (indecomposable)* R_p*-lattices.*

Proof. Let M be an indecomposable R_p-lattice. By [19] the category of R_p-lattices admits almost split sequences. Arguing as for modules over finite dimensional algebras (see [11, Proposition 13.11]) we conclude that M is isolated in $\mathrm{Zg}_{\mathrm{tf}p}$. By Fact 3.3 M is isolated in $\mathrm{Zg}_{\mathrm{red}}$. Since $\mathrm{Zg}_{\mathrm{red}}$ is open in $\mathrm{Zg}_{\mathrm{tf}}$, it follows that M is isolated in $\mathrm{Zg}_{\mathrm{tf}}$.

For the converse let M be an isolated point of $\mathrm{Zg}_{\mathrm{tf}}$. By Fact 3.2 M is either a reduced isolated point of $\mathrm{Zg}_{\mathrm{tf}p}$ or is isomorphic to one of the modules $\mathbb{Q}Ge_i$, $i = 1, \dots, 4$. In the former case let (φ/ψ) be a pair of pp-formulae that isolates M. Representing M as a direct limit of R_p-lattices (as in [10, Theorem 2.3]) it is easily seen that (φ/ψ) already opens on one of these lattices N, therefore M is isomorphic to a direct summand of N.

It remains to notice that by Proposition 3.4 the point $\mathbb{Q}Ge_i$ cannot be separated by an open set from, say, R_pe_i, therefore is not isolated. □

At this point the topology of $\mathrm{Zg}_{\mathrm{tf}p}$ for p an odd prime is easy to describe. In fact in this case each e_i is in R_p, whence $R_p = \oplus_{i=1}^4 R_pe_i$, where $R_pe_i \cong \widehat{\mathbb{Z}}_2$ and the action of G is trivial or skewed, as described in Section 2. In particular every torsion-free R_p-module is a direct sum of copies of the R_pe_i and $\mathbb{Q}Ge_i$, $i = 1, \dots, 4$. It follows:

Lemma 3.6. *If p is an odd prime, then the only points in* Zg_{tfp} *are* R_pe_i *and* $\mathbb{Q}Ge_i$*,* $i = 1, 2, 3, 4$*. The former ones (as* R_p*-lattices) are exactly the isolated points of* Zg_{tfp} *and the latter points have* CB *rank* 1*.*

So the only case to be examined is $p = 2$. We will deal with that in Section 5. But before that let us recall in the next Section 4 some basic facts about the 4-subspace quiver $\widetilde{D}_4$. We only recall here that, as G is a 2-group, by [4, Corollary 5.25], the ring $\widehat{\mathbb{Z}}_2 G$ is local with Jacobson radical consisting of elements $n_1 + n_2 a + n_3 b + n_4 ab$ with $n_i \in \widehat{\mathbb{Z}}_2$ and $\sum_i n_i \in 2\widehat{\mathbb{Z}}_2$.

4. Four subspace quiver

The representation theory of the 4-subspace quiver $\widetilde{D}_4$ (over an arbitrary field k) will play a crucial role in what follows. In this section we will recall for the reader some facts from this theory. More details on finite dimensional (f.d.) representations of $\widetilde{D}_4$ can be found in Simson and Skowroński [20, 13.3] (when k is algebraically closed) and in Brenner [1] (in the general case). Pure injective representations of $\widetilde{D}_4$ are treated in Prest [12] and Ringel [18] in the wider framework of modules over tame hereditary finite dimensional algebras.

Let A be the following matrix algebra

$$A = \begin{pmatrix} k & 0 & 0 & 0 & 0 \\ k & k & 0 & 0 & 0 \\ k & 0 & k & 0 & 0 \\ k & 0 & 0 & k & 0 \\ k & 0 & 0 & 0 & k \end{pmatrix}.$$

Then the representations of A are the 'same' as representations of the quiver $\widetilde{D}_4$ with the subspace orientation:

$$\begin{tikzcd} 1 \arrow[drr] & 2 \arrow[dr] & & 3 \arrow[dl] & 4 \arrow[dll] \\ & & 0 & & \end{tikzcd}$$

Therefore every A-module corresponds to a quintet of k-vector spaces $V^* = (V, V_1, \ldots, V_4)$ such that each arrow $i \to 0$ corresponds to a k-vector space morphism $f_i : V_i \to V$.

For instance, A has four simple injective modules $I(1), \ldots, I(4)$, where $I(1)$ is given by the following diagram

$$\begin{tikzcd} k \arrow[drr] & 0 \arrow[dr] & & 0 \arrow[dl] & 0 \arrow[dll] \\ & & 0 & & \end{tikzcd}$$

(with obvious values of morphisms) and the other modules $I(2)$, $I(3)$ and $I(4)$ are described similarly.

It follows that every A-module is a direct sum of copies of these 4 simple injectives and a module built from a diagram where all morphisms are injective, that is, from a diagram where each V_i is a subspace of V. In particular let $I(0)$ denote the (non-simple) injective module corresponding to the following diagram

$$\begin{tikzcd}
k \arrow[drr] & k \arrow[dr] & & k \arrow[dl] & k \arrow[dll] \\
& & k & &
\end{tikzcd}$$

where all the maps are identities.

It follows from the general theory (see [20]) that most indecomposable f.d. A-modules are uniquely determined by their *dimension vector* $\overline{x} = \left(\begin{smallmatrix} x_1 & x_2 & & x_3 & x_4 \\ & & x_0 & & \end{smallmatrix}\right)$ where $x_0 = \dim V$ and, for $i = 1, \ldots, 4$, $x_i = \dim V_i$. For example the dimension vector of $I(0)$ is $\left(\begin{smallmatrix} 1 & 1 & & 1 & 1 \\ & & 1 & & \end{smallmatrix}\right)$ and that of $I(1)$ is $\left(\begin{smallmatrix} 1 & 0 & & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right)$.

Let us propose as further examples the simple projective module $P(0)$ corresponding to $\left(\begin{smallmatrix} 0 & 0 & & 0 & 0 \\ & & 1 & & \end{smallmatrix}\right)$ and the projective modules $P(i)$ ($i = 1, \ldots, 4$) where $P(1)$ corresponds to $\left(\begin{smallmatrix} 1 & 0 & & 0 & 0 \\ & & 1 & & \end{smallmatrix}\right)$ and $P(2)$, $P(3)$, $P(4)$ are defined similarly.

Furthermore every indecomposable f.d. A-module is *preinjective*, *preprojective* or *regular*. An easy way to determine the type of a f.d. module M is to calculate its defect (see [20, p. 200]):

$$\delta(M) = -2x_0 + x_1 + x_2 + x_3 + x_4.$$

Then

- M is preinjective iff $\delta(M) > 0$ (for instance $\delta(I(0)) = -2 + 1 + 1 + 1 + 1 = 2$).
- Similarly an indecomposable f.d. module M is preprojective iff $\delta(M) < 0$ (and for those the $P(i)$ ($i = 0, \ldots, 4$) are examples).
- Finally an indecomposable f.d. module M is regular iff $\delta(M) = 0$.

An example of a regular representation $R(\lambda)$, $\lambda \in k \setminus \{0, 1\}$ of A is given by the following diagram,

$$\begin{tikzcd}
k \arrow[drr, "f_1"'] & k \arrow[dr, "f_2"] & & k \arrow[dl, "f_3"'] & k \arrow[dll, "f_4"] \\
& & k^2 & &
\end{tikzcd}$$

(where for all $r \in k$, $f_1(r) = (r, 0)$, $f_2(r) = (0, r)$, $f_3(r) = (r, r)$ and $f_4(r) = (\lambda r, r)$), therefore of the dimension vector $\left(\begin{smallmatrix} 1\,1 & & 1\,1 \\ & 2 & \end{smallmatrix}\right)$.

In a standard way indecomposable f.d. A-modules are organized in the Auslander–Reiten (AR for short) quiver of A (that is, a locally finite graph whose vertices are indecomposable f.d. A-modules and morphisms correspond to a basis of irreducible maps). In this global picture the preinjective modules form a connected component of the following shape:

$$\begin{tikzcd}
& & \tau I(1) \arrow[ddr] & & I(1) \\
& & \tau I(2) \arrow[dr] & & I(2) \\
\cdots & \tau I(0) \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & I(0) \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \\
& & \tau I(3) \arrow[ur] & & I(3) \\
& & \tau I(4) \arrow[uur] & & I(4)
\end{tikzcd}$$

where τ stands for the AR-translate. In a similar way preprojective A-modules form a connected component in the AR-quiver which starts with the projective module $P(0)$.

$$\begin{tikzcd}
& P(1) \arrow[ddr] & & \tau^{-1}P(1) & \\
& P(2) \arrow[dr] & & \tau^{-1}P(2) & \\
P(0) \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau^{-1}P(0) \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \cdots \\
& P(3) \arrow[ur] & & \tau^{-1}P(3) & \\
& P(4) \arrow[uur] & & \tau^{-1}P(4) &
\end{tikzcd}$$

The (indecomposable f.d.) regular A-modules are organized into infinitely many tubes (parameterized by irreducible polynomials over k), that is, they can be drawn on cylinders. Most of the tubes are homogeneous, that is they have just one simple regular module on the mouth and look like a line:

$$\begin{tikzcd}
\circ \arrow[r, bend left] & \circ \arrow[l, bend left] \arrow[r, bend left] & \circ \arrow[l, bend left] \arrow[r, bend left] & \circ \arrow[l, bend left] & \cdots
\end{tikzcd}$$

However A has 3 exceptional tubes of period 2. A typical representative on the mouth of such a tube is the regular A-module $R(\infty)$ given by the following diagram

$$R(\infty) = \begin{tikzcd}[column sep=small]
0 \arrow[drr] & k \arrow[dr] & & 0 \arrow[dl] & k \arrow[dll] \\
& & k & &
\end{tikzcd}$$

where the second module on this mouth is

$$\tau R(\infty) = \begin{tikzcd}[column sep=small]
k \arrow[drr] & 0 \arrow[dr] & & k \arrow[dl] & 0 \arrow[dll] \\
& & k & &
\end{tikzcd}$$

In drawing the AR-quiver of A the preprojective modules are usually put on the left of regular modules, and preinjective modules are put on the right of regulars. Then all morphisms will go from the left to the right, for instance there are no nonzero morphisms from regular to preprojective modules.

To calculate the dimensions of Hom spaces, the bilinear form associated to A is very useful. If M and N are f.d. A-modules with the dimension vectors $\overline{x}$ and $\overline{y}$, then let us define a bilinear form $q(\overline{x}, \overline{y})$ as $\dim \operatorname{Hom}(M, N) - \operatorname{Ext}^1(M, N)$ (where Hom, Ext^1 abbreviate here for simplicity Hom_A, Ext^1_A respectively). An easy calculation shows that $q(\overline{x}, \overline{y}) = \sum_{i=0}^4 x_i y_i - y_0 \cdot \sum_{i=1}^4 x_i$.

As an example, if N is preinjective, then $\operatorname{Ext}^1(M, N) = 0$, therefore $q(\overline{x}, \overline{y}) = \dim \operatorname{Hom}(M, N)$. For instance, when $N = I(0) = \left(\begin{smallmatrix}1&1&&1&1\\&&1&&\end{smallmatrix}\right)$ and, say, $M = R(\infty) = \left(\begin{smallmatrix}0&1&&0&1\\&&1&&\end{smallmatrix}\right)$, then

$$\dim \operatorname{Hom}(R(\infty), I(0)) = 3 - 1 \cdot 2 = 1.$$

Similarly for $N = I(1) = \left(\begin{smallmatrix}1&0&&0&0\\&&0&&\end{smallmatrix}\right)$ we obtain

$$\dim \operatorname{Hom}(R(\infty), I(1)) = 0 - 0 = 0$$

while, for $N = I(2)$, $\dim \mathrm{Hom}(R(\infty), I(2)) = 1$. The remaining Hom's from $R(\infty)$ to preinjective modules could be calculated using the τ-periodicity of $R(\infty)$ (see [18, after Proposition 6]).

Thus we know the structure of the category of the f.d. A-modules. Being of finite endolength every f.d. A-module M is pure-injective, hence (if indecomposable) it determines a closed point of Zg_A. Since the category of f.d. A-modules admits almost split sequences, by [11, Corollary 13.4], the f.d. points of Zg_A are just the isolated points . The following fact (which is again true for any f.d. algebra – see [11, Proposition 13.2 and Corollary 13.3]) will be useful later.

Fact 4.1. Every pure injective A-module is a direct summand of a direct product of f.d. A-modules, whence f.d. points are dense in Zg_A.

The description of the remaining (infinitely generated) points of Zg_A and the topology of this space we borrow from [18] (see also [12]). For each regular A-module S which is simple as an object in the category of regular modules there is a ray of irreducible monomorphisms $S = S(1) \to S(2) \to S(3) \to \ldots$ winding around the tube. The direct limit along this ray is a pure injective indecomposable A-module $S(\infty)$ called the (S-) Prüfer module. This module is countably (infinitely) generated and has CB-rank 1. Note that $S(\infty)$ is in the closure of the regular modules $S(i)$. Another way to obtain $S(\infty)$ is the following. By [18, Proposition 1] if V_j^*, $j \in J$ is an infinite set of (nonisomorphic) preinjective A-modules with $\mathrm{Hom}(S, V_j^*) \neq 0$ for all j, then $S(\infty)$ is a direct summand of $\prod_{j \in J} V_j^*$ (thus $S(\infty)$ is in the closure of the $V_j^\star$). For instance, if S is from a homogeneous tube, then $\mathrm{Hom}(S, V_j^*) \neq 0$ for every preinjective module V_j^*, therefore $S(\infty)$ is in the closure of any infinite set of preinjective points.

Thus for every homogeneous tube we have exactly one Prüfer module, and each nonhomogeneous tube produces two Prüfer modules.

Dually every simple regular A-module S is included in a coray of irreducible epimorphisms $S = S(1) \leftarrow S(2) \leftarrow S(3) \leftarrow \ldots$. The inverse limit along this coray, $\widehat{S}$, is said to be the S-adic module. This module is pure injective, indecomposable and has CB-rank 1. Furthermore, $\widehat{S}$ is in the closure of the $S(i)$, and in the closure of any infinite set of preprojective modules V_j^*, $j \in J$ with $\mathrm{Hom}(V_j^*, S) \neq 0$ (see [18, Proposition 2].

Finally there is a unique point of Zg_A of CB-rank 2 — the generic module $\mathcal{G}$ with the following diagram

$$\begin{tikzcd}
k(x) \arrow[drr, "f_1"'] & k(x) \arrow[dr, "f_2"] & & k(x) \arrow[dl, "f_3"'] & k(x) \arrow[dll, "f_4"] \\
& & k(x)^2 & &
\end{tikzcd}$$

where $k(x)$ is the field of rational functions in x over k and for every $q = q(x) \in k(x)$, $f_1(q) = (q, 0)$, $f_2(q) = (0, q)$, $f_3(q) = (q, q)$ and $f_4(q) = (q, qx)$. For instance (see [18, Proposition 3]) $\mathcal{G}$ is a direct summand of any infinite power of an arbitrary Prüfer module $S(\infty)$. Being of finite endolength, $\mathcal{G}$ is a closed point of Zg_A.

The description of the topology on Zg_A is given by the following result.

Fact 4.2. (see [18, Theorem]) A subset X of Zg_A is closed iff the following conditions are satisfied.

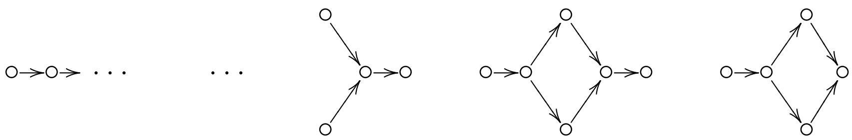
1) If S is a simple regular A-module and if there are infinitely many f.d. A-modules $V^* \in X$ with $\mathrm{Hom}(S, V^*) \neq 0$, then $S(\infty) \in X$.

2) If S is a simple regular A-module and if there are infinitely many f.d. A-modules $V^* \in X$ with $\mathrm{Hom}(V^*, S) \neq 0$, then $\widehat{S} \in X$.

3) If there are infinitely many f.d. A-modules in X or if there exists at least one infinite dimensional module in X, then $\mathcal{G} \in X$.

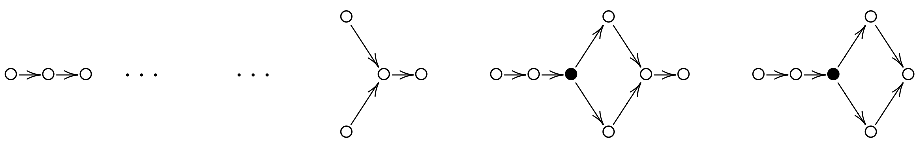
For instance, the basis of open sets for $\mathcal{G}$ is given by $\mathrm{Zg}_A \setminus F$, F a finite set of f.d. points in Zg_A.

As has been noticed in [18], the functors $\mathrm{Hom}(S, -)$ with S simple regular (from the category of f.d. A-modules to the category of k-vector spaces) play a crucial role in the classification of indecomposable pure injective A-modules. These functors can be described by the so-called patterns (see [18] or rather [17, 3] for a definition and numerous examples). For instance (see [17, p. 254]) if $S = R(\infty)$ (or any simple regular A-module from a nonhomogeneous tube) then its pattern has the following shape



(its vertices correspond to nonzero morphisms f from S to indecomposable f.d. A-modules V^* and the arrows correspond to irreducible maps $g : V^* \to W^*$ such that $gf \neq 0$). Notice that in this case all nonzero spaces $\mathrm{Hom}(S, V^*)$ are 1-dimensional. We will give a model-theoretic interpretation of the above diagram.

Choose a set of generators $\overline{x}$ for S and let the pp-formula φ generate the pp-type of $\overline{x}$ in S. Thus φ is equivalent to the annihilator condition $\overline{x}B = 0$ describing the relations on $\overline{x}$. Since $\mathrm{Hom}(S, V^*)$ is (at most) 1-dimensional for every indecomposable f.d. module V^*, it follows from general theory (see [15, Chapter 11]) that the interval $[0, \varphi]$ is a distributive lattice which is obtained from the above pattern by making 'free sums' of points (but taking into account all order relations on the original poset). Thus this interval of pp-formulae has the following shape (where the order relation goes from the left to the right)



(the new 'free sums' formulae are marked by bullets).

Let p be the pp-type of the tuple $\overline{x} \in S(\infty)$ (via the natural inclusion $S \subseteq S(\infty)$ along the ray of irreducible monomorphisms). Clearly p defines a cut between two dots in the above diagram. Furthermore p is the unique non-finitely generated pp-type containing φ.

The pattern of a simple regular A-module S from a homogeneous tube is more involved. For instance, (see [17, p. 149]) some spaces $\mathrm{Hom}(S, V^*)$ are 2-dimensional (e.g., $\dim \mathrm{Hom}(R(\lambda), I(0)) = 2$), hence the interval below φ is not a distributive lattice, but the conclusion stays the same:

Lemma 4.3. *For every simple regular A-module S there is a unique non-finitely generated indecomposable pp-type p containing φ.* (Recall that φ is isomorphic to $\mathrm{Hom}(S,-)$).

Proof. If p is the pp-type of the generating set $\overline{x}$ of $S = S(1)$ in $S(\infty)$, then p is not finitely generated and contains φ. Furthermore, if φ_n generates the pp-type of $\overline{x}$ in $S(n)$ (via the standard embedding $S(1) \subseteq S(n)$), then p is generated by the φ_n.

Suppose that q is another indecomposable non-finitely generated pp-type over A containing φ. Since q is not finitely generated, using almost split sequences (see [16, Proof of Proposition 7.3] for similar arguments) we conclude that $\varphi_n \in q$ for every n, therefore $p \subseteq q$. Let $(V^*, \overline{y})$ be a realization of q in an indecomposable pure injective module V^*. From $\varphi \in q$ it follows that $\mathrm{Hom}(S, V^*) \neq 0$. By the classification of indecomposable pure injective A-modules we obtain that $V^* \cong S(\infty)$, therefore there is a morphism $f : S(\infty) \to S(\infty)$ sending $\overline{x}$ to $\overline{y}$. If $p \subset q$ (strict inclusion) then $f \in \mathrm{Jac}(S(\infty))$, therefore f annihilates the simple regular socle $S(1)$ of $S(\infty)$, in particular $\overline{y} = f(\overline{x}) = 0$, a contradiction. □

5. The Butler functor

In this section we will recall Butler's functor and some of its properties (see [2]) and extend this functor to a certain category of pure injective modules over $R_2 = \widehat{\mathbb{Z}}_2 C(2)^2$ (indeed Butler's original theory applies to the category of R_2-lattices, as $\widehat{\mathbb{Z}}_2$ is a complete noetherian commutative valuation domain).

We say that a ($\mathbb{Z}$)-torsion-free R_2-module is b-reduced (reduced in the terminology of [2] in case of lattices) if $M \cap Me_i = 2Me_i$ for every $i = 1, \ldots, 4$.

Warning: The reader should take care of not confusing this notion of b-reduced with the notion of reduced which was defined in Section 3 before Fact 3.2.

Let $\mathcal{C}$ denote the category of b-reduced torsion-free R_2-modules. Observe that $\mathcal{C}$ is definable (in the sense of [14, 3.4.1]). Indeed the b-reduction condition defines a closed set of Zg_R, that is, the intersection (for $i = 1, \ldots, 4$) of $\big((e_i \mid x)/(2e_i \mid x)\big)^c$ –or rather $\big((4e_i \mid 4x \wedge 4 \mid x)/(8e_i \mid 4x)\big)^c$– where '$c$' stands for the complement. Let us denote by $\mathrm{Zg}_{\mathcal{C}}$ this closed subset.

Note also that $R_2 \cap R_2e_i = 4R_2e_i \neq 2R_2e_i$ and $R_2e_i \cap (R_2e_i)e_i = R_2e_i \neq 2R_2e_i$, therefore R_2 and R_2e_i, $i = 1, \ldots, 4$ are not b-reduced.

On the other hand the four modules $\mathbb{Q}Ge_i$ ($i = 1, \ldots, 4$) are in $\mathcal{C}$. Thus the set of reduced b-reduced indecomposable pure injective torsion-free R_2-modules is the intersection of $\mathrm{Zg}_{\mathcal{C}}$ with the open set $\big((x = x)/(2 \mid x)\big)$. In particular it is locally closed, but neither open nor closed in $\mathrm{Zg}_{\mathrm{tf}}$. In the following we will be mainly concerned with this locally closed set, rather than with the whole $\mathrm{Zg}_{\mathcal{C}}$.

The following proposition extends [2, Proposition 1.5] from lattices to torsion-free pure injective modules and has a similar proof.

Proposition 5.1. *Let M be a $\mathbb{Z}$-torsion-free R_2-module.*

1) If $2Me_i \not\subseteq M$ for some $i = 1, \ldots, 4$, then M contains a copy of R_2 as a direct summand.

2) If $i = 1, \ldots, 4$ and $2Me_i \subset M \cap Me_i$ (a strict inclusion) then M contains a copy of the rank 1 lattice R_2e_i as a direct summand.

3) Otherwise M is b-reduced.

Proof. 1) Choose $m \in M$ such that $2me_i \notin M$, therefore (since M is torsion-free) $4me_i \notin 2M$. From $4e_1 - 4e_i \in 2R_2$ it follows that $n = 4me_1$ is in M but not in $2M$ (otherwise $4me_i = 4me_1 + m(4e_i - 4e_1) \in 2M + M \cdot 2R_2 \subseteq 2M$). We claim that $N = n\widehat{\mathbb{Z}}_2 \cong \widehat{\mathbb{Z}}_2$ is a direct summand of M (as a $\widehat{\mathbb{Z}}_2$-module). Since $\widehat{\mathbb{Z}}_2$ is pure injective (over itself), it suffices to prove that the pp-type p (over $\widehat{\mathbb{Z}}_2$) of n in N is equal to $q = pp_M(n)$. Clearly $p \subseteq q$. Let $\varphi \in q$ be a pp-formula. By an elimination procedure for commutative valuation domains (see [11, Theorem 2.$\mathbb{Z}$.1]) and the fact that M is torsion-free we may assume that $\varphi \doteq 2^l \mid x$ for some $l \in \mathbb{N}$. Since $n \in M \setminus 2M$ it follows that $l = 0$, therefore φ becomes trivial. Thus $N = n\widehat{\mathbb{Z}}_2 \cong \widehat{\mathbb{Z}}_2$ is a direct summand of M as a $\widehat{\mathbb{Z}}_2$-module.

Therefore there exists a $\widehat{\mathbb{Z}}_2$-morphism $\pi : M \to \widehat{\mathbb{Z}}_2$ such that $\pi(n) = 1$. Let us define a map $\tau : M \to R_2$ by putting, for every $s \in M$, $\tau(s) = \sum_{g \in G} \pi(sg^{-1})g$. It is easily checked that τ is an R_2-morphism.

Recall that $n = 4me_1$. Since $e_1 g = e_1$ for every $g \in G$ we conclude that $ng = n$, therefore

$\tau(n) = \sum_{g\in G} \pi(ng^{-1})g = \sum_{g\in G} \pi(n)g = \sum_{g\in G} g = 4e_1$.

It follows that $\tau(m) \cdot 4e_1 = \tau(m \cdot 4e_1) = \tau(n) = 4e_1$. Since R_2 is a local ring we conclude that $\tau(m)$ is invertible in R_2, therefore τ is onto. Since R_2 is projective, τ splits, therefore M contains a direct summand isomorphic to R_2.

2) Suppose that $2Me_i \subset M \cap Me_i$ (a strict inclusion) and choose $m \in (M \cap Me_i) \setminus 2Me_i$. Since $m \in Me_i \setminus 2Me_i$ then (as above) we conclude that $N = m\widehat{\mathbb{Z}}_2 = mR_2 \cong R_2e_i$ is a pure submodule of Me_i as a module over $\widehat{\mathbb{Z}}_2$ or R_2. Because R_2e_i is pure injective, it follows that N is a direct summand of Me_i, therefore of $M^* = \oplus_{i=1}^4 Me_i$. Because $N \subseteq M \subseteq M^*$ it follows that $N \cong R_2e_i$ is a direct summand of M. $\square$

Note that in the above proof we used only the fact that M is a $\mathbb{Z}$-torsion-free R_2-module (although the result obviously applies to $\mathbb{Z}$-torsion-free pure-injective modules). Another elegant way to formulate the previous proposition is the following (see [14, 18.2.4]): any torsion-free R_2-module not in $\mathcal{C}$ contains one of the lattices R_2, R_2e_i, $i = 1, \ldots, 4$ as a pure submodule (and indeed as a direct summand).

In fact (again using Prest's remark [14, 18.2.4, p. 691]) we can single out minimal pairs associated with these modules.

Lemma 5.2. *The following are minimal pairs in the theory* T_{tf} *of torsion-free modules over* R_2 *(or* R*):*

1) $\big((8e_1 \mid 4x)/(4 \mid x \wedge 4e_1 \mid x)\big)$ *isolates* R_2*;*

2) $\big((4 \mid x \wedge 4e_i \mid x)/(8e_i \mid 4x)\big)$ *isolates* R_2e_i*,* $i = 1, \ldots, 4$.

Proof. 1) The set of indecomposable torsion-free R_p-lattices (with p ranging over primes) is dense in $\mathrm{Zg}_{\mathrm{tf}}$. Evaluating this pair (φ/ψ) on these lattices we see that it takes a nontrivial value just on R_2, and this value $(\varphi/\psi)(R_2)$ is a one-dimensional vector space over $k = GF(2)$. The conclusion follows easily (were $\varphi > \theta > \psi$ for some pp-formula θ, then we would obtain $(\varphi/\theta)(R_2) \neq 0$ and $(\theta/\psi)(R_2) \neq 0$, a contradiction).

Similar arguments prove 2). $\square$

Now, following Butler [2], we consider b-reduced R_2-modules M in the definable category $\mathcal{C}$ and we extend Butler's functor Δ in their setting. Let us preliminarily recall the notation $M^* = \oplus_{i=1}^4 Me_i$. Note that M b-reduced implies that $2M^* \subset M \subset M^*$ (strict inclusions).

We refer to the field $k = GF(2)$ and we associate to M a module over $k\widetilde{D}_4$ ($\widetilde{D}_4$-module for short) $\Delta(M) = V^*$ where $V = M^*/M$ and, for every $i = 1, \dots, 4$, $V_i = (Me_i + M)/M \cong Me_i/M \cap Me_i = Me_i/2Me_i$.

Note also that, since $m = \sum_{1 \leq i \leq 4} me_i$ for every $m \in M$, it easily follows that for every i, $V = \sum_{j \neq i} V_j$. This leads us to consider the definable category $\mathcal{D}$ consisting of the $\widetilde{D}_4$-modules V^* with $V = \sum_{j \neq i} V_j$ for every $i = 1, \dots, 4$. Observe that this condition determines a closed subset $\mathrm{Zg}_{\mathcal{D}}$ of $\mathrm{Zg}_{k\widetilde{D}_4}$.

The map Δ clearly extends also to morphisms, therefore gives rise to a functor from $\mathcal{C}$ into the category of $\widetilde{D}_4$-modules, and indeed into $\mathcal{D}$. Butler [2] noticed that Δ is full when restricted to the category of R_2-lattices and gives a representation equivalence from this category to the category of f.d. representations of $\widetilde{D}_4$ in $\mathcal{D}$. It is easily seen that a f.d. $\widetilde{D}_4$-module is in the image of Δ iff it contains no modules $I(1), \dots, I(4)$ and $P(0), P(1), \dots, P(4)$ as direct summands.

It follows from general theory that the image of $\mathcal{C}$ with respect to Δ is a definable category of $\widetilde{D}_4$-modules (and defines a clopen cofinite subset if we think in terms of the Ziegler spectrum). Since Δ commutes with direct products and direct limits, it provides an example of a definable functor between definable categories (see [14, 18.2] or just notice that $\Delta(M)$ is clearly definable in M, hence Δ is an interpretation).

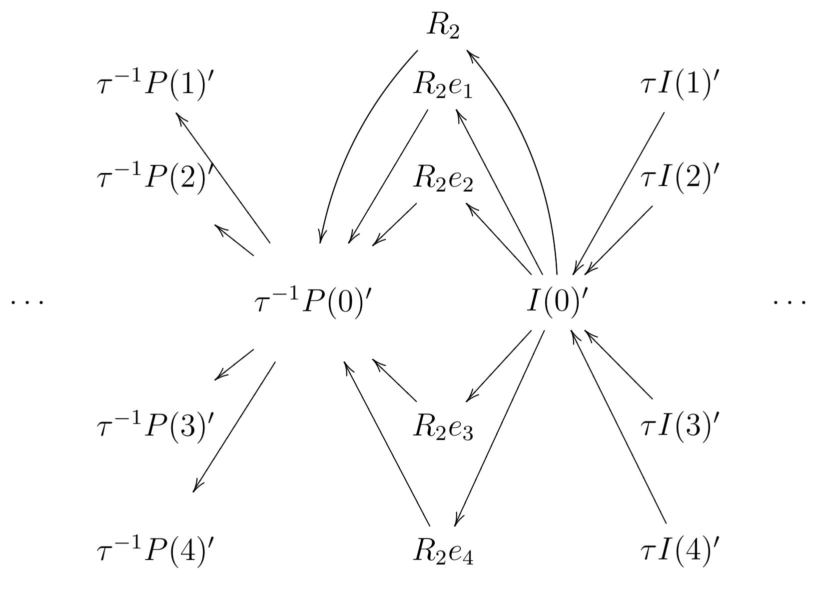
We will prove later that the image of Δ is just $\mathcal{D}$ and we will also describe in more detail the preimage of any f.d. point in $\mathcal{D}$. But to illustrate here Butler's ideas let us construct an R_2-lattice whose image will be a simple regular module $R(1)$ with the following diagram

$$\begin{tikzcd}
k \arrow[drr] & k \arrow[dr] & & 0 \arrow[dl] & 0 \arrow[dll] \\
& & k & &
\end{tikzcd}$$

(living on a non-homogeneous tube). What we have to do is to construct a module $R_2e_1 \oplus R_2e_2$ (so $\widehat{\mathbb{Z}}_2 \oplus \widehat{\mathbb{Z}}_2$ where G acts on the two copies of $\widehat{\mathbb{Z}}_2$ as suggested by e_1, e_2 respectively) and to choose the elements in this direct sum which are identified via embeddings $\widehat{\mathbb{Z}}_2/2\widehat{\mathbb{Z}}_2 \cong k \to k^2 \leftarrow k \cong \widehat{\mathbb{Z}}_2/2\widehat{\mathbb{Z}}_2$ as $\widetilde{D}_4$-modules. Hence we are led to define $M = \{(n,m) \in \widehat{\mathbb{Z}}_2 \oplus \widehat{\mathbb{Z}}_2 \mid n - m \in 2\widehat{\mathbb{Z}}_2\}$. In particular, $Me_1 = \widehat{\mathbb{Z}}_2 \oplus 0$, $Me_2 = 0 \oplus \widehat{\mathbb{Z}}_2$, therefore $\Delta(M) \cong R(1)$.

Note also that the kernel of the induced map from $\operatorname{Hom}(M,N)$ to $\operatorname{Hom}(\Delta(M),\Delta(N))$ consists of morphisms f such that $f(Me_i) \subseteq N$ for every i, and that it equals $\sum_i 2\operatorname{Hom}(Me_i, Ne_i)$.

Dieterich [5] showed that Butler's functor Δ (in its original framework of lattices) induces an isomorphism from the AR-quiver of $\mathcal{C}$ to the AR-quiver of its image. By adjoining the points out of the domain of Δ one gets the complete AR-quiver of R_2-lattices (see [5, p. 54]). Similar to $\widetilde{D}_4$ this quiver contains infinitely many regular tubes (and 3 of them are nonhomogeneous), but the end of the preinjective component is sewn with the beginning of the preprojective component as follows



(where to the right and to the left we denote by N' the R_2-lattice corresponding to a given $\widetilde{D}_4$-module N). Thus the category of R_2-lattices is essentially richer than the category of $\widetilde{D}_4$-modules, in particular there are some nonzero morphisms from “preinjectives” to “preprojective” lattices.

Furthermore to get an irreducible morphism, say from $I(0)'$ to R_2, we represent $I(0)'$ as the submodule of $R_2^* = \oplus_{i=1}^4 R_2 e_i$ consisting of tuples $(m_1, \ldots, m_4)$ such that $2 \mid m_i - m_j$ for all $i \neq j$, and then multiply it by 2 to get inside R_2.

Now we are going to prove the main result of this section: that Δ is full when restricted to the category of b-reduced (and reduced) pure injective torsion-free R_2-modules in $\mathcal{C}$. The following remark will be helpful.

Remark 5.3. Every indecomposable pure injective torsion-free module over $\widehat{\mathbb{Z}}_2$ is isomorphic to $\widehat{\mathbb{Z}}_2$ or $\mathbb{Q}$. Thus every reduced pure injective torsion-free R_2-module M, if viewed as a $\widehat{\mathbb{Z}}_2$-module, is the pure injective envelope of a module $K = \widehat{\mathbb{Z}}_2^{(\alpha)}$ for some α, in particular M is pp-essential over K (as a $\widehat{\mathbb{Z}}_2$-module).

Theorem 5.4. *The functor Δ is full when restricted to the category of reduced b-reduced pure injective torsion-free R_2-modules.*

Proof. Let M, N be reduced (and b-reduced) pure injective torsion-free R_2-modules, we wish to prove that the induced map from $\mathrm{Hom}(M, N)$ to $\mathrm{Hom}(\Delta(M), \Delta(N))$ is onto. Put $\Delta(M) = V^*$, $\Delta(N) = W^*$. From the construction of Δ we have that $V = M^*/M$ and $V_i = Me_i/M \cap Me_i = Me_i/2Me_i \subseteq V$ $(i = 1, \ldots, 4)$ are k-vector spaces, similarly $W = N^*/N$ and $W_i = Ne_i/2Ne_i$. Thus a morphism $f : \Delta(M) \to \Delta(N)$ is given by a morphism of vector spaces $V \to W$ such that $f(V_i) \subseteq W_i$ for every i (thus $f_i : V_i \to W_i$ will denote the induced map). Let π_M denote the projection $Me_i \to Me_i/2Me_i = V_i$ and similarly for $\pi_N : Ne_i \to W_i$.

By Remark 5.3 each Me_i (if viewed as a $\widehat{\mathbb{Z}}_2$-module) is isomorphic to the pure injective hull of a submodule $K = \widehat{\mathbb{Z}}_2^{(\alpha)}$ (as a $\widehat{\mathbb{Z}}_2$-module) for some α. Since $\widehat{\mathbb{Z}}_2^{(\alpha)}$ is a projective $\widehat{\mathbb{Z}}_2$-module, f_i can be lifted to a morphism $g_i' : \widehat{\mathbb{Z}}_2^{(\alpha)} \to Ne_i$ of $\widehat{\mathbb{Z}}_2$-modules such that $f_i \pi_M(m) = \pi_N g_i'(m)$ for every $m \in K = \widehat{\mathbb{Z}}_2^{(\alpha)}$ (see the following diagram).

$$\begin{tikzcd}
\widehat{\mathbb{Z}}_2^{(\alpha)} \subseteq \arrow[d, "g'_i"'] & Me_i \arrow[r, "\pi_M"] \arrow[dl, dashed, "g_i"] & V_i \arrow[d, "f_i"] \\
Ne_i \arrow[rr, "\pi_N"] & & W_i
\end{tikzcd}$$

Since K is a pure submodule of Me_i, and Ne_i is pure injective, there is a $\widehat{\mathbb{Z}}_2$- module morphism $g_i : Me_i \to Ne_i$ extending g'_i, that is, $g_i \mid_K = g'_i$. This morphism g_i can be regarded in a natural way as an R_2-module morphism with respect to the action of a and b corresponding to e_i. We claim that g_i makes the above diagram commutative, that is, $\pi_N g_i(m) = f_i \pi_M(m)$ for every $m \in Me_i$.

Indeed, since K is pp-essential in Me_i as a $\widehat{\mathbb{Z}}_2$-module, there exist $n \in K$ and a pp-formula $r \mid x + ys$ (to be read "r divides $x + ys$") over $\widehat{\mathbb{Z}}_2$ (in particular $r, s \in \widehat{\mathbb{Z}}_2$) that connects n and m in Me_i, meaning that $n + ms \in Me_i r$ but $n, mb \notin Me_i r$ (use [11, Theorem 4.10.(d)] and observe that, due to the torsion-free assumption, the only interesting pp-formulae over $\widehat{\mathbb{Z}}_2$ regard divisibility conditions). Choose $m' \in Me_i$ such that $m'r = n + ms$ and apply π_M. Clearly $2 \mid r$ (otherwise r is invertible), whence $m'r \in 2Me_i$ and we can conclude that $\pi_M(n) = \pi_M(ms)$. Therefore

$(*)\qquad f_i \pi_M(ms) = f_i \pi_M(n) = \pi_N g'_i(n).$

From $m'r \in 2Me_i$ it also follows that $g_i(m'r) = g_i(m')r \in 2Ne_i$. Thus applying π_N to the equality $g_i(m'r) = g'_i(n) + g_i(ms)$ we obtain $\pi_N g'_i(n) = \pi_N g_i(ms)$. Taking into account $(*)$ we conclude $f_i \pi_M(ms) = \pi_N g_i(ms)$, therefore (since M is torsion-free) $f_i \pi_M(m) = \pi_N g_i(m)$, as desired.

Let g be the R_2-module morphism from M^* to N^* whose restriction on Me_i, $i = 1, \ldots, 4$ is g_i. Then $f\pi_M = \pi_N g$. Finally for every $m \in M$ we have $\pi_M(m) = 0$, hence $\pi_N g(m) = 0$ and therefore $g(m) \in N$. Thus the restriction of g to M defines a morphism from M to N that lifts f. $\square$

Note that in the proof of the above theorem we only used that M is reduced (b-reduced pure injective torsion-free), but N could be an arbitrary (not necessarily reduced) b-reduced pure injective torsion-free R_2-module.

A small adjustment of the above theorem gives its full strength.

Corollary 5.5. *The functor Δ is full on b-reduced pure injective torsion-free R_2-modules.*

Proof. Note that every pure injective torsion-free R_2-module M is a direct sum of copies of the $\mathbb{Q}Ge_i$ $(i = 1, \dots, 4)$ and a reduced module. Indeed if M is not reduced, take a nonzero element m in its divisible part. Then for some i $n_i = 4me_i$ is a non-zero element in $Me_i \cap M$, moreover n_i is still divisible. Then standard arguments show that n_i is included into a direct summand of M isomorphic to $\mathbb{Q}Ge_i$. Now let N be a maximal direct sum of copies of the $\mathbb{Q}Ge_i$. Since each of these modules is of finite endolength, N is pure injective, therefore $M = N \oplus M/N$ and M/N is reduced.

It remains to notice that the divisible part of M is annihilated by Δ, therefore can be ignored when extending morphisms. $\square$

Since Δ preserves direct products and direct limits it follows that Δ preserves pure injectivity, that is, if M is a pure injective R_2-module, then $\Delta(M)$ is a pure injective $\widetilde{D}_4$-module. In fact (as we will show later) Δ preserves indecomposability, therefore induces a map $\mathrm{Zg}_{\mathcal{C}} \to \mathrm{Zg}_{\mathcal{D}}$. Again we will show later that this map is one-to-one (after removing the $\mathbb{Q}Ge_i$). But one important theorem can be already formulated.

Theorem 5.6. *Δ induces a homeomorphism from $\mathrm{Zg}_{\mathcal{C}} \setminus \{\mathbb{Q}Ge_i\}_{i=1}^4$ onto its image in $\mathrm{Zg}_{\mathcal{D}}$.*

Proof. Apply Corollary 5.5 and Prest [14, 18.2.27] $\square$

Indeed we are going to see that the image of Δ is just the whole space $\mathrm{Zg}_{\mathcal{D}}$. But for this we need one more property of Δ.

Proposition 5.7. *Let M be a pure injective b-reduced reduced torsion-free R_2-module. Then the kernel of the induced map from $\mathrm{End}(M)$ to $\mathrm{End}(\Delta(M))$ is contained in $\mathrm{Jac}(\mathrm{End}(M))$.*

Proof. Let $f \in \ker(\Delta_M)$. Extend f to $f^* : M^* \to M^*$ in the obvious way. Since $f \in \ker(\Delta_M)$, therefore $f^*(e_iM) \subseteq 2e_iM$, and then

$f^{*k}(e_iM) \subseteq 2^k e_iM$ for every positive integer k and every $i = 1, \dots, 4$. Clearly $M^* = \oplus_{i=1}^4 Me_i$ is pure injective reduced and torsion-free. It follows (see [10, p. 1128] for similar arguments) that a formal inverse $(1 - f^*)^{-1} = 1 + f^* + f^{*2} + \dots$ can be given the unique R_2-module action on M^* that inverts $1 - f^*$. Since $f^*(M) \subseteq M$ it follows that the above map sends M to M, therefore $1 - f$ is invertible in $\mathrm{End}(M)$. $\square$

The following corollary is standard.

Corollary 5.8. *Let M be a b-reduced pure injective reduced torsion-free R_2-module. Then there is a natural 1-1 correspondence between direct sum decompositions of M and $\Delta(M)$. In particular Δ preserves and reflects indecomposability (within this class of modules).*

Proof. Since (see [8, Corollary 7.5]) the endomorphism ring of every pure injective module is F-semiperfect (in particular, idempotents lift modulo the Jacobson radical) the statement of the corollary is a consequence of general theory (see [8, p. 212] for this kind of argument).

For instance, if M is indecomposable, then $\mathrm{End}(M)$ is a local ring. Since $\ker(\Delta_M) \subseteq \mathrm{Jac}(\mathrm{End}(M))$, it follows that $\mathrm{End}(\Delta(M))$ is local, therefore $\Delta(M)$ is indecomposable. For the converse, if $\Delta(M)$ is indecomposable, then $\mathrm{End}(M)$ cannot have non-trivial idempotents, otherwise they will survive when factoring out $\ker(\Delta_M)$. $\square$

Theorem 5.9. *Δ is a bijection between $\mathrm{Zg}_{\mathcal{C}} \setminus \{\mathbb{Q}Ge_i\}_{i=1}^4$ and $\mathrm{Zg}_{\mathcal{D}}$.*

Proof. At the level of lattices this was shown by Butler. Let us briefly recall his proof. We are given a 5-uple of vector spaces $V^* = (V, (V_i)_i)$ in $\mathcal{D}$ with V finite dimensional. For every $i = 1, \dots, 4$ let $V_i \cong GF(2)^{d_i}$ for some non-negative integer d_i. Lift V_i to $\widehat{\mathbb{Z}}_2^{d_i}$ viewed as an R_2-module with respect to the action of a and b determined by e_i. Let σ_i denote the canonical $\widehat{\mathbb{Z}}_2$- (indeed R_2-) module morphism of $\widehat{\mathbb{Z}}_2^{d_i}$ onto V_i. Thus $\sigma = \sum_i \sigma_i$ is an R_2-module morphism of $\oplus_i \widehat{\mathbb{Z}}_2^{d_i}$ to V. Then the kernel M of σ is a b-reduced R_2-lattice and $\Delta(M) \cong V^*$ (see [2, (1.3)] for more details).

In particular for each simple regular $\widetilde{D}_4$-module S there exists an indecomposable R_2-lattice S' such that $\Delta(S') \cong S$. We will also call this lattice simple regular, because it is on the mouth of a regular tube of R_2-lattices.

Now let us deal with infinite dimensional indecomposable pure injective $\widetilde{D}_4$-modules V^* in $\mathcal{D}$.

Use Fact 4.1 and represent V^* as a direct summand of a direct product $\prod_{j\in J} V_j^*$ of f.d. indecomposable pure injective $\widetilde{D}_4$-modules V_j^* (or of indecomposable pure injective $\widetilde{D}_4$-modules V_j^* already in the image of Δ) such that V_j^* is in $\mathcal{D}$ for every $j \in J$. Lifting the corresponding idempotent we obtain an indecomposable pure-injective torsion-free b-reduced reduced R_2-module M as a direct summand of $\prod_{j\in J} M_j$, where for every j M_j is an indecomposable pure injective R_2-module such that $\Delta(M_j) \cong V_j^*$. Since Δ reflects isomorphisms this module is even unique (up to isomorphism).

This is the general strategy. But let us treat separately the cases when V^* is S-Prüfer or S-adic for some simple regular S, or $\mathcal{G}$, to obtain more information on the topology $\mathrm{Zg}_{\mathcal{C}}$.

First take $V^* = S(\infty)$, the S-Prüfer module. Choose an infinite set of preinjective $\widetilde{D}_4$-modules V_j^*, $j \in J$ such that $\mathrm{Hom}(S, V_j^*) \neq 0$. Then (see [18, Proposition 1]) $S(\infty)$ is a direct summand of the (pure injective) direct product $\prod_{j\in J} V_j^*$. This provides a preimage M of $S(\infty)$ as illustrated above. Let us denote it $S'(\infty)$ and call it the S'-Prüfer module.

Now the generic module $\mathcal{G}$ is a direct summand of any infinite product of copies of $S(\infty)$. Lifting this decomposition we obtain an indecomposable pure injective torsion-free reduced module $\mathcal{G}'$ which we call *pseudo-generic* (since $\mathcal{G}'$ is reduced, the strictly descending chain $\mathcal{G}' \supset 2\mathcal{G}' \supset 4\mathcal{G}' \supset \ldots$ shows that it is not of finite endolength). Note that $\mathcal{G}'$ can be written in terms of generators and relations using the diagram for $\mathcal{G}$ (see above).

Finally, in the case of the S-adic $\widetilde{D}_4$-module $\widehat{S}$ we simply recall that it is (as any pure injective $\widetilde{D}_4$-module) a direct summand of a direct product of indecomposable f.d. $\widetilde{D}_4$-modules. Also, we can assume that these $\widetilde{D}_4$-modules are in $\mathcal{D}$, in other words exclude $P(0), \ldots, P(4)$ and $I(1), \ldots, I(4)$. Lifting this decomposition we obtain its R_2 analogue, the S'-adic module $\widehat{S'}$.

Thus every module in $\mathrm{Zg}_{\mathcal{D}}$ has gotten its preimage in $\mathrm{Zg}_{\mathcal{C}}$, therefore the induced map $\Delta : \mathrm{Zg}_{\mathcal{C}} \setminus \{\mathbb{Q}Ge_i\}_{i=1}^4 \to \mathrm{Zg}_{\mathcal{D}}$ is a homeomorphism. $\square$

Note that on the way we proved the following remark.

Remark 5.10. *Every pure injective reduced torsion-free R_2-module is a direct summand of a direct product of R_2-lattices.*

Although the topology on $\mathrm{Zg}_{\mathcal{D}}$ and consequently that on $\mathrm{Zg}_{\mathcal{C}}$ is known, it may be worth recalling its concise description à la Ringel [18]. Let S' be a simple regular R_2-lattice corresponding to a simple regular $\widetilde{D}_4$-module $S = \Delta(S')$. Let us define a functor $\mathrm{Hom}^*_{R_2}(S', -)$ (from the category of R_2-lattices to the category of k-vector spaces) as follows. Let $\overline{x}$ be a set of generators for S' and let $\overline{x}B = 0$ be the set of relations defining S' (this gives a subfunctor of the forgetful functor $\mathrm{Hom}(R^k, -)$, where k is the length of $\bar{x}$). Then $\mathrm{Hom}^*_{R_2}(S', -)$ is a quotient of this functor given by the pp-pair $\big((\overline{x}B = 0)/(8 \mid 4\bar{x}e_i)\big)$, therefore Δ provides a natural isomorphism $\mathrm{Hom}^*_{R_2}(S', -) \cong \mathrm{Hom}_{k\widetilde{D}_4}(S, -)$ in particular these functors have the same lattice of finitely generated subfunctors.

Taking into account Fact 4.2 and Theorem 5.6, we obtain (here, again $\mathrm{Zg}_{\mathcal{C}}$ is considered within the category of R_2-modules, that is, as a subset of $\mathrm{Zg}_{\mathrm{tf}2}$ regarded as embedded in $\mathrm{Zg}_{\mathrm{tf}}$).

Theorem 5.11. *A subset X of $\mathrm{Zg}_{\mathcal{C}}$ is closed iff the following holds true.*

1) If S' is a simple regular R_2-lattice and there are infinitely many lattices $M_j \in X$, $j \in J$ with $\mathrm{Hom}^*(S', M_j) \neq 0$, then $S'(\infty) \in X$.

2) If S' is a simple regular R_2-lattice and there are infinitely many lattices $M_j \in X$, $j \in J$ with $\mathrm{Hom}^*(M_j, S') \neq 0$, then $\widehat{S}' \in X$.

3) If there are infinitely many R_2-lattices in X or X contains at least one non-finitely generated reduced module, then the pseudo-generic module $\mathcal{G}' \in X$.

4) If $Me_i \neq 0$ for some $M \in X$ and $i = 1, \dots, 4$, then $\mathbb{Q}Ge_i \in X$.

6. Topology

In this section we will describe the topology on $\mathrm{Zg}_{\mathrm{tf}}$, the torsion-free part of the Ziegler spectrum of $R = \mathbb{Z}C(2)^2$. Let X be a subset of $\mathrm{Zg}_{\mathrm{tf}}$; we have to decide whether X is open or not.

Recall that in Section 3 we defined open sets O_i, $i = 1, \dots, 4$ such that $\mathrm{Zg}_{\mathrm{tf}} = \cup_{i=1}^4 O_i$ and $\mathbb{Q}Ge_i$ is the only closed point within O_i. Thus X is open iff $X \cap O_i$ is open for every i, therefore we may assume that $X \subseteq O_i$ for some i. By the remark after Proposition 3.4, $\mathbb{Q}Ge_i$ cannot be separated by an open set from any point of O_i, therefore, if $\mathbb{Q}Ge_i \in X$ then X is open iff $X = O_i$.

Otherwise $X \subseteq O_i \setminus \{\mathbb{Q}Ge_i\}$. Since $\mathbb{Q}Ge_i$ is a closed point, the last set is open in $\mathrm{Zg}_{\mathrm{tf}}$. Thus we may assume that X consists of reduced points.

For each prime p let $\mathcal{O}_p$ denote the open set $\big((x = x)/(p \mid x)\big)$. Note that $\mathcal{O}_p$ is the torsion-free non-divisible part of the Ziegler spectrum of R_p, $\mathrm{Zg}_{\mathrm{tfp}}$ regarded as embedded in $\mathrm{Zg}_{\mathrm{tf}}$ (the divisible part is $\{\mathbb{Q}Ge_i\}_{i=1}^4$).

If $p \neq 2$, then, since X contains no divisible points, $X \cap \mathcal{O}_p$ consists of (at most four) lattices $R_p e_i$, therefore is open. Thus X is open iff $X \cap \mathcal{O}_2$ is open, and therefore (by [10, Theorem 2.2]) iff $X \cap \mathcal{O}_2$ is open considered as a subset of $\mathrm{Zg}_{\mathrm{tf2}}$. But Theorem 5.11 provides a complete answer to this question.

For instance, if X contains a pseudo-generic point $\mathcal{G}'$ then to be open X must contain all Prüfer and adic points and almost all R_2-lattices (and there are no further restrictions but, by the assumption, $\mathbb{Q}Ge_i \notin X$).

Now it is not difficult to execute the Cantor–Bendixson analysis for $\mathrm{Zg}_{\mathrm{tf}}$.

Theorem 6.1. *Let $R = \mathbb{Z}C(2)^2$ and let Zg_{tf} denote the torsion-free part of the Ziegler spectrum of R.*

1) The only isolated points of Zg_{tf} are $R_p = \widehat{\mathbb{Z}}_p C(2)^2$-lattices (again, there are infinitely many of them if $p = 2$, and just finitely many otherwise).

2) The only points of CB*-rank* 1 *of* Zg_{tf} *are the (S'-) Prüfer and adic points for every simple regular R_2-lattice S'.*

3) The only point of CB*-rank* 2 *is the pseudo-generic point $\mathcal{G}'$.*

4) The only points of CB*-rank* 3 *are the divisible modules $\mathbb{Q}Ge_i$, $i = 1, \ldots, 4$.*

Proof. 1) follows from Theorem 3.5. Furthermore by Proposition 3.4 and the discussion afterwards, the $\mathbb{Q}Ge_i$ are points of maximal CB-rank and they are the only closed points of $\mathrm{Zg}_{\mathrm{tf}}$. In particular 4) will follow from 2) and 3).

2) By 1) the CB-rank of any Prüfer or adic point of $\mathrm{Zg}_{\mathrm{tf}}$ is at least 1. Let $M' = S'(\infty)$ be an S'-Prüfer point over R_2. We claim that M' is isolated in the first derivative (that is, the subspace of non-isolated points) of $\mathrm{Zg}_{\mathrm{tf}}$ by the intersection of $\big((x = x)/(2 \mid x)\big)$ with the pp-pair defined by the functor $\mathrm{Hom}^*(S', -)$. Indeed, if M is an indecomposable pure injective torsion-free R-module in this intersection, then M has a natural structure of an R_2-module and $\mathrm{Hom}^*(S', M) \neq 0$. If $S = \Delta(S')$ and $V^* = \Delta(M)$ then from the functorial isomorphism we obtain $\mathrm{Hom}_{k\widetilde{D}_4}(S, V^*) \neq 0$. Since V^* is not finitely generated, it follows from the description of $\mathrm{Zg}_{k\widetilde{D}_4}$ that $V^* \cong S(\infty)$, therefore $M \cong S'(\infty)$ by the definition of that.

Note that there exists a homeomorphism of $\mathrm{Zg}_{\widetilde{D}_4}$ (given by elementary duality composed with standard duality $\mathrm{Hom}(-, k)$) that interchanges Prüfer and adic points, therefore there exists such a homeomorphism on $\mathrm{Zg}_{\mathcal{C}}$. It follows that each adic point has CB-rank 1.

Now the pseudo-generic point $\mathcal{G}'$ is a direct summand of any infinite product of copies of any Prüfer point $S'(\infty)$. It follows that $\mathcal{G}'$ is in the closure of any such point, therefore cannot be separated from $S'(\infty)$ at level one. Thus $\mathrm{CB}(\mathcal{G}') \geq 2$, and it is exactly 2 because the open set $\big((x = x)/(2 \mid x)\big)$ separates $\mathcal{G}'$ from the remaining points $\mathbb{Q}Ge_i$.

At the final step we can use the open sets O_i to separate the points $\mathbb{Q}Ge_i$, $i = 1, \ldots, 4$ from each other. □

Thus we have completed a description of the points and topology of $\mathrm{Zg}_{\mathrm{tf}}$. However, because the construction of points has not been direct, their algebraic structure is still enigmatic. To give an example, let us consider a simple regular R_2-module S'. Then there is a sequence of irreducible maps $S' = S'(1) \to S'(2) \to \dots$ in the category of R_2-lattices that goes (applying Δ) to the sequence of irreducible monomorphisms $S = S(1) \to S(2) \to \dots$ in the category of f.d. $\widetilde{D_4}$-modules. Let $T = \varinjlim S'_i$. Since Δ commutes with direct limits, it follows that $\Delta(T) \cong S(\infty)$.

Question 6.2. *Is it true that T is an indecomposable pure injective R_2-module?*

Recall that there exists an indecomposable pure injective torsion-free reduced R_2-module $S'(\infty)$ such that $\Delta(S'(\infty)) \cong S(\infty) \cong \Delta(T)$. The problem is that, without knowing that T is pure injective, it is not clear how to lift this isomorphism. The best we can do is the following.

Lemma 6.3. *$S'(\infty)$ is a direct summand of the pure injective envelope of T.*

Proof. As we have already mentioned Δ provides a natural isomorphism of functors $\mathrm{Hom}^*_{R_2}(S', -)$ and $\mathrm{Hom}_{k\widetilde{D}_4}(S, -)$. Suppose that the former functor is given by a pp-pair (φ/ψ) over R_2. By Lemma 4.3 there is a unique non-finitely generated pp-type over $k\widetilde{D}_4$ containing $\mathrm{Hom}(S, -)$, therefore there exists a unique non-finitely generated pp-type p over R_2 in the sort (φ/ψ). It follows easily that both T and $S'(\infty)$ realize p, therefore $S'(\infty) = \mathrm{PE}(p)$ is a direct summand in $\mathrm{PE}(T)$. $\square$

Is it true at least that $\mathrm{PE}(T)$ has no divisible part?

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