Fields interpretable in superrosy groups with NIP (the non-solvable case)

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Abstract

Let G be a group definable in a monster model $\mathfrak{C}$ of a rosy theory satisfying NIP. Assume that G has hereditarily finitely satisfiable generics and $1 < \mathrm{U}^{\text{þ}}(G) < \infty$. We prove that if G acts definably on a definable set of $\mathrm{U}^{\text{þ}}$-rank 1, then, under some general assumption about this action, there is an infinite field interpretable in $\mathfrak{C}$. We conclude that if G is not solvable-by-finite and it acts faithfully and definably on a definable set of $\mathrm{U}^{\text{þ}}$-rank 1, then there is an infinite field interpretable in $\mathfrak{C}$. As an immediate consequence, we get that if G has a definable subgroup H such that $\mathrm{U}^{\text{þ}}(G) = \mathrm{U}^{\text{þ}}(H) + 1$ and $G/\bigcap_{g \in G} H^g$ is not solvable-by-finite, then an infinite field interpretable in $\mathfrak{C}$ also exists.

0 Introduction

The paper is in some sense a continuation of [7]. But now we concentrate mainly on the existence of fields interpretable in groups. The general motivating question is

Question 1 *For a given infinite, pure group $\langle G, \cdot \rangle$ (say of finite dimension, whatever the dimension means), does there exist an infinite field interpretable in $\langle G, \cdot \rangle$?*

In this paper, the notion of dimension will be $\mathrm{U}^{\text{þ}}$-rank, and so G will be rosy. Recall that in the stable (finite Morley rank) context $\mathrm{U}^{\text{þ}}$-rank coincides with Lascar U-rank, and in the o-minimal context it coincides with the o-minimal dimension.

Notice that in order to have a positive answer to the above question, G cannot be abelian-by-finite. Indeed, otherwise $\langle G, \cdot \rangle$ would be 1-based of finite Lascar U-rank, so it would not interpret an infinite field (to see this, one should use the fact that abelian groups are 1-based together with some facts from [11, Chapters 2, 4], e.g. [11, Proposition 6.4]).

Assuming that G is not abelian-by-finite, one has three cases to consider:

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(1) G is not solvable-by-finite,

(2) G is solvable-by-finite but not nilpotent-by-finite,

(3) G is nilpotent-by-finite but not abelian-by-finite.

[10, Corollary 5.1] tells us that the answer to Question 1 is positive whenever G is a non-abelian-by-finite group definable in an o-minimal structure.

When G is of finite Morley rank, the answer to Question 1 in Case (2) is positive. In Case (1), it is open, but there are some partial results (assuming that G is not a bad group). Finally, in Case 3 the answer is negative by [3], but there are also some positive partial answers, e.g. in [14, 6]. For example, [6, Theorem 2.3] tells us that the answer is positive if the commutator subgroup of G is torsion-free.

A general goal is to answer Question 1 in a more general context than finite Morley rank or o-minimal structures, namely for G being dependent and of finite $U^{þ}$-rank. For some technical reason, in this paper, we will additionally assume that G has hereditarily fsg. This still covers the situations when G is superstable of finite U-rank and when G is a definably compact group definable in an o-minimal expansion of a real closed field. The definitions of rosiness, NIP, fsg, and other relevant notions are given in Section 1.

The main result of [7] says that if G is a dependent group of finite $U^{þ}$-rank, then Question 1 has a positive answer in Case (2).

Of course, Baudisch's theorem (see [3]) implies that the answer in Case (3) is negative for the class of dependent groups of finite $U^{þ}$-rank. However, a tiny modification of the proof of [6, Theorem 2.3] yields a positive answer under the additional assumption that the commutator subgroup of G is torsion-free. In fact, even more is true: every locally nilpotent, non-abelian group with icc on centralizers and with torsion-free commutator subgroup interprets an infinite field. Indeed, the proof of [6, Theorem 2.3] produces an integral domain F interpretable in $\langle G, \cdot\rangle$ whose additive group is a subgroup of G'. So, the field of fractions of F is also interpretable in $\langle G, \cdot\rangle$, and since G' is torsion-free, this field is of characteristic 0 and so infinite.

In this paper, we concentrate on the existence of a field both in Case (1) and also in the presence of a definable action of a definable group on a definable set of $U^{þ}$-rank 1.

For the next three results, we work in $\mathfrak{C}^{eq}$ where $\mathfrak{C}$ is a monster model of a (rosy) theory T satisfying NIP.

One of the main conclusions of this paper is the following theorem concerning Case (1) and generalizing the part of Cherlin's result (see [12, Corollary 3.28]) concerning the existence of a field (we do not generalize the part yielding the description of G).

Theorem 2 *Suppose T is rosy with NIP, and G is a definable group having hereditarily fsg and with $U^{þ}(G) = n + 1 < \infty$. Assume that G has a definable subgroup H of $U^{þ}$-rank n, and $G/\bigcap_{g\in G} H^g$ is not solvable-by-finite. Then there is an infinite field interpretable in $\mathfrak{C}$ [in $\langle G, \cdot\rangle$, if H is definable in $\langle G, \cdot\rangle$].*

As in [12], in order to prove the above theorem, we investigate definable actions of definable groups on sets of $\mathrm{U}^{\text{þ}}$-rank 1. However, in the finite Morley rank situation, it was sufficient to consider actions on strongly minimal sets, whereas in our case, we need to work with sets of $\mathrm{U}^{\text{þ}}$-rank 1. This and the fact that we do not have definable connected components makes our situation more complicated and calls for different arguments.

The main effort is put into proving the following theorem, generalizing the part of the Hrushovski's result (see [12, Theorem 3.27]) concerning the existence of a field (we do not generalize the part yielding the description of permutation groups). A similar result has also been proved for o-minimal structures in [8, Theorem 1.5]. In the whole paper, if G is a group acting on a set S and $R \subseteq S$, then G_R denotes the pointwise stabilizer of R; if $R = \{s\}$, then $G_s := G_R$.

Theorem 3 *Assume T is rosy with NIP, and let G be a definable group having hereditarily fsg and with $1 < U^{\text{þ}}(G) < \infty$. Assume that G acts definably on a definable set S of $U^{\text{þ}}$-rank 1 so that there is $s \in S$ for which no finite index, definable subgroup of G_s has normalizer of finite index in G. Then there is an infinite field interpretable in $\mathfrak{C}$; in fact, it is interpretable in the two-sorted structure with sorts G and S equipped with the group operation on G and the action of G on S.*

In particular, assuming the first sentence of the theorem, if G acts definably on a definable set S of $\mathrm{U}^{\text{þ}}$-rank 1 so that at least one orbit is infinite, and G does not have a definable subgroup H with $\mathrm{U}^{\text{þ}}(H) = \mathrm{U}^{\text{þ}}(G) - 1$ and whose normalizer is of finite index in G, then an infinite interpretable field exists.

From Theorem 3, we will get the following corollary, which implies Theorem 2.

Theorem 4 *Assume T has NIP. Let G be a definable group having hereditarily fsg and with $U^{\text{þ}}(G) < \infty$. Assume that G acts faithfully and definably on a definable set S of $U^{\text{þ}}$-rank 1. If G is not solvable-by-finite, then there is an infinite field interpretable in $\mathfrak{C}$; in fact, it is interpretable in the two-sorted structure with sorts G and S equipped with the group operation on G and the action of G on S. Moreover, if T is rosy, then $U^{\text{þ}}(G) < \omega$.*

It is worth mentioning that in the proofs of the above three theorems, the only places where the fsg assumption is used are the applications of [4, Theorem 2] saying that any dependent group of $\mathrm{U}^{\text{þ}}$-rank 2 and satisfying hereditarily fsg is solvable-by-finite. So, if one was able to remove the fsg assumption from [4, Theorem 2], then it could be automatically removed from all the results of this paper.

On the other hand, keeping the fsg assumption in the above theorems and analyzing our proofs, one can check that the NIP assumption can be replaced by the condition that G has icc.

1 Definitions and basic observations

In this section, we work in $\mathfrak{C}^{eq}$ where $\mathfrak{C}$ is a monster model of a theory T in a language $\mathcal{L}$.

First we recall some things about rosy theories. For details on rosy theories, the reader is referred to [1, 5, 9], and on rosy groups to [4].

T is rosy if there is a ternary relation $\mathop{\smile\!\!\!\!\!\!|}\,^*$ on small subsets of $\mathfrak{C}^{eq}$ satisfying all the basic properties of forking independence in simple theories except for the Independence Theorem. Such a relation will be called an independence relation.

A formula $\delta(x,a)$ strongly divides over A if the formula is not almost over A and $\{\delta(x,a')\}_{a'\models \mathrm{tp}(a/A)}$ is k-inconsistent for some $k \in \mathbb{N}$.

We say that $\delta(x,a)$ þ-divides over A if we can find some tuple c such that $\delta(x,a)$ strongly divides over Ac.

A formula þ-forks over A if it implies a (finite) disjunction of formulas which þ-divide over A.

We say that the type $p(x)$ þ-divides over A if there is a formula in $p(x)$ which þ-divides over A; þ-forking is similarly defined. We say that a is þ-independent from b over A, denoted $a \mathop{\smile\!\!\!\!\!\!|}\,^{þ}_{A} b$, if $\mathrm{tp}\,(a/Ab)$ does not þ-fork over A.

In rosy theories, þ-independence is the weakest independence relation in the sense that $a \mathop{\smile\!\!\!\!\!\!|}\,^{*}_{C} b$ implies $a \mathop{\smile\!\!\!\!\!\!|}\,^{þ}_{C} b$ for any independence relation $\mathop{\smile\!\!\!\!\!\!|}\,^*$.

In [5], the following has been noticed.

Fact 1.1 *T is rosy iff $\mathop{\smile\!\!\!\!\!\!|}\,^{þ}$ has local character.*

We define $\mathrm{U}^{þ}$-rank by means of $\mathop{\smile\!\!\!\!\!\!|}\,^{þ}$ in the same way as Lascar U-rank is defined in terms of $\mathop{\smile\!\!\!\!\!\!|}$. In rosy theories, this rank shares many nice properties of U-rank in stable theories. In particular, it satisfies Lascar Inequalities.

If D is A-definable, we put $\mathrm{U}^{þ}(D) = \sup\{\mathrm{U}^{þ}(d/A) : d \in D\}$. If T is rosy, this definition, not depending on A, provides a good, general notion of dimension of a definable set. Without rosiness, it is not clear whether it depends on A (because it is not clear whether $\mathrm{U}^{þ}$-rank is preserved under taking non-þ-forking extensions). So, whenever $\mathrm{U}^{þ}(D)$ is considered without rosiness, we assume that the set A over which D is defined is distinguished. Sometimes A is clear from the context, e.g. computing $\mathrm{U}^{þ}(\mathfrak{C})$, we have $A = \emptyset$. If $\mathfrak{C}$ is many-sorted, then $\mathrm{U}^{þ}(\mathfrak{C})$ is the supremum of the $\mathrm{U}^{þ}$-ranks of all sorts. In particular, $\mathrm{U}^{þ}(\mathfrak{C}) < \infty$ means that the $\mathrm{U}^{þ}$-rank of every sort is less than ∞.

Some nice properties of $\mathrm{U}^{þ}$-rank in rosy theories are listed in Section 1 of [7]. But even without rosiness, we still have some good properties. For example, analyzing the proof of Lascar Inequalities [13, Theorem 5.1.6] and using [9, Lemma 2.1.6], we get that $\mathrm{U}^{þ}(ab/A) \leq \mathrm{U}^{þ}(a/bA) \oplus \mathrm{U}^{þ}(b/A)$ holds in any theory. Also, it follows easily that if $a \in acl(Ab)$, then $\mathrm{U}^{þ}(b/A) \geq \mathrm{U}^{þ}(a/A)$.

Below we prove two basic facts, which are not written down explicitly anywhere. They are a piece of folklore that has already been used in [4] and [7], and will be used in this paper in order to skip the assumption of rosiness of the theory T in some results. The first remark is pretty obvious. The proof of the second one, however, turns out to be a bit more complicated and it leads to some basic questions formulated right below it.

Remark 1.2 *Any theory with $U^{þ}(\mathfrak{C}) < \infty$ is rosy.*

Proof. By assumption, $\mathrm{U}^{\mathrm{þ}}(a) < \infty$ for every $a \in \mathfrak{C}$. Thus, we get $\mathrm{U}^{\mathrm{þ}}(a/A) < \infty$ for any $a \in \mathfrak{C}$ and $A \subseteq \mathfrak{C}^{eq}$. We know that $\mathrm{U}^{\mathrm{þ}}(ab/A) \leq \mathrm{U}^{\mathrm{þ}}(a/bA) \oplus \mathrm{U}^{\mathrm{þ}}(b/A)$ holds in any theory (without rosiness). Therefore, for any finite tuple a in $\mathfrak{C}$ and any $A \subseteq \mathfrak{C}^{eq}$, we have $\mathrm{U}^{\mathrm{þ}}(a/A) < \infty$.

Now if $a = [a_0]_E$ is imaginary (a_0 is a real tuple), then $a \in acl^{eq}(a_0)$. So, $\mathrm{U}^{\mathrm{þ}}(a/A) \leq \mathrm{U}^{\mathrm{þ}}(a_0/A)$ for any $A \subseteq \mathfrak{C}^{eq}$.

We have proved that every type in $\mathfrak{C}^{eq}$ has an ordinal $\mathrm{U}^{\mathrm{þ}}$-rank. This implies that there is no infinite sequence $p_0 \subsetneq p_1 \subsetneq \dots$ of þ-forking extensions, and so $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{þ}}$ has local character. We finish using Fact 1.1. ■

The theories of ordinal $\mathrm{U}^{\mathrm{þ}}$-rank (i.e. with $\mathrm{U}^{\mathrm{þ}}(\mathfrak{C}) < \infty$) are called superrosy. We say that T is of $\mathrm{U}^{\mathrm{þ}}$-rank α if $\mathrm{U}^{\mathrm{þ}}(\mathfrak{C}) = \alpha$.

It has been noticed in [2] that rosiness is preserved under taking reducts. Our second remark is related to this observation.

Remark 1.3 *Assume $\mathcal{M}$ with the universe M is interpretable in $\mathfrak{C}$. Let $\mathcal{L}'$ be the signature of $\mathcal{M}$. Working in the language $\mathcal{L}'$, we will use the subscript $\mathcal{L}'$. Otherwise we work in the language $\mathcal{L}$. $U^{þ}_{\mathcal{M}}$ stands for $U^{þ}$-rank computed in $\mathcal{M}$.*
(i) If T is (super)rosy, so is $Th_{\mathcal{L}'}(\mathcal{M})$.
(ii) Assume T is rosy. Consider any $m \in M^{eq}$ and $A \subseteq M^{eq}$. If $U^{þ}_{\mathcal{M}}(tp_{\mathcal{L}'}(m/A)) = n < \omega$, then there is a completion p of $tp_{\mathcal{L}'}(m/A)$ over A in the sense of $\mathfrak{C}$ such that $U^{þ}(p) \geq n$. In particular, if T' is a reduct of T and T is of $U^{þ}$-rank $n < \omega$, then T' is of $U^{þ}$-rank at most n.
(iii) The conclusion of (ii) holds with the assumption that T is rosy replaced by $U^{þ}(M) < \infty$.
(iv) If $U^{þ}(M) < \infty$, then $Th_{\mathcal{L}'}(\mathcal{M})$ is superrosy.

Proof. Assume for simplicity that $\mathcal{M}$ is interpretable in $\mathfrak{C}$ over $\emptyset$. Of course, $\mathcal{M}^{eq}$ is also interpretable in $\mathfrak{C}$ over $\emptyset$.

Claim Let a, A, B be from M^{eq} and suppose that $tp_{\mathcal{L}'}(a/AB)$ þ-divides over A in the sense of $\mathcal{M}$. Then there are a' and B' in M^{eq} such that $tp_{\mathcal{L}'}(a'B'/A) = tp_{\mathcal{L}'}(aB/A)$, and for any realization a'' of $tp_{\mathcal{L}'}(a'/AB')$ one has that $tp_{\mathcal{L}}(a''/AB')$ þ-divides over A in the sense of $\mathfrak{C}$.

Proof of Claim. By the assumption, there is a formula $\varphi(x, y) \in \mathcal{L}'$ and $b \in M^{eq}$ such that $\varphi(x, b) \in tp_{\mathcal{L}'}(a/AB)$ and $\varphi(x, b)$ þ-divides over A in the sense of $\mathcal{M}$. Thus, there is $c \in M^{eq}$ such that $tp_{\mathcal{L}'}(b/Ac)$ is non-algebraic and $\{\varphi(x, b') : b' \models tp_{\mathcal{L}'}(b/Ac)\}$ is k-inconsistent for some k. Hence, there is $b' \models tp_{\mathcal{L}'}(b/Ac)$ such that $tp_{\mathcal{L}}(b'/Ac)$ is non-algebraic. We conclude that $\varphi(x, b')$ þ-divides over A in the sense of $\mathfrak{C}$.

To finish, choose any automorphism f of $\mathcal{M}$ fixing A pointwisely and mapping b to b', and put $a' = f(a)$, $B' = f[B]$. □

We argue that under the assumption of (ii) or (iii), $Th_{\mathcal{L}'}(\mathcal{M})$ is rosy. The case when T is rosy follows easily using local þ-ranks (see [2, Corollary 5.3]). Now consider the case when T is not necessarily rosy, but $U^{þ}(M) < \infty$. Then the $U^{þ}$-rank of M equipped with the induced structure (i.e. with predicates for the traces of all $\emptyset$-definable subsets of the appropriate sorts of $\mathfrak{C}$) is also less than ∞, and hence by Remark 1.2, M with the induced structure is rosy. Thus, by [2, Corollary 5.3], $Th_{\mathcal{L}'}(\mathcal{M})$ is rosy.

Now we give an inductive, with respect to n, proof of (ii) and (iii). The case $n = 0$ is obvious. For the induction step, assume $U^{þ}_{\mathcal{M}}(tp_{\mathcal{L}'}(a/A)) \geq n + 1$ for some a, A in M^{eq} and $n \in \omega$. Then there is $B \supseteq A$ such that a is þ-dependent on B over A in the sense of $\mathcal{M}$ and $U^{þ}_{\mathcal{M}}(tp_{\mathcal{L}'}(a/B)) \geq n$. Extending B and using rosiness of $Th_{\mathcal{L}'}(\mathcal{M})$ (in order to have that taking non-þ-forking extensions preserves $U^{þ}$-rank), we can assume that $tp_{\mathcal{L}'}(a/B)$ þ-divides over A in the sense of $\mathcal{M}$. By the Claim, we get that there are $a'B' \models tp_{\mathcal{L}'}(aB/A)$ such that for any $a'' \models tp_{\mathcal{L}'}(a'/B')$, $tp_{\mathcal{L}}(a''/B')$ þ-divides, and hence þ-forks over A in the sense of $\mathfrak{C}$. So, by the inductive hypothesis, $U^{þ}(tp_{\mathcal{L}}(a''/A)) \geq n + 1$ for some $a'' \models tp_{\mathcal{L}'}(a'/B')$.

The proof of (i) is similar. As in [2], using local þ-ranks, we get that $Th_{\mathcal{L}'}(\mathcal{M})$ is rosy. Now, suppose for a contradiction that T is superrosy, but $Th_{\mathcal{L}'}(\mathcal{M})$ is not superrosy. Using rosiness of $Th_{\mathcal{L}'}(\mathcal{M})$, we get an infinite sequence $\{\emptyset\} = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq M^{eq}$ and $a \in M$ such that $tp_{\mathcal{L}'}(a/B_{i+1})$ þ-divides over B_i in the sense of $\mathcal{M}$. Using recursively the Claim, we find a_i, $i \in \omega$, and an increasing sequence of sets B'_i, $i \in \omega$, in M^{eq} so that:

(a) $a_0 = a$,

(b) $a_i B'_i \models tp_{\mathcal{L}'}(aB_i)$,

(c) $tp_{\mathcal{L}'}(a_i/B'_i) \subseteq tp_{\mathcal{L}'}(a_{i+1}/B'_{i+1})$,

(d) for every $a'_{i+1} \models tp_{\mathcal{L}'}(a_{i+1}/B'_{i+1})$, $tp_{\mathcal{L}}(a'_{i+1}/B'_{i+1})$ þ-divides, and hence þ-forks over B'_i in the sense of $\mathfrak{C}$.

By compactness and (c), there is $a' \in M$ realizing all types $tp_{\mathcal{L}'}(a_i/B'_i)$, $i \in \omega$. Thus, by (d), $U^{þ}(a') = \infty$, a contradiction with superrosiness of $\mathfrak{C}$.
The proof of (iv) is almost the same as the proof of (i) ■

It is not clear, however, whether the above remark holds when n is an infinite ordinal (in particular, whether taking reducts always decreases $U^{þ}$-rank, even for infinite ordinal ranks). A similar remark can also be proved for SU-rank in simple theories, and one can also ask whether taking reducts necessarily decreases SU-rank (even for infinite ordinal ranks).

In Section 2, T will be assumed to have the non independence property (NIP).

Definition 1.4 *We say that T has the NIP if there is no formula $\varphi(x, y)$ and sequence $\langle a_i \rangle_{i<\omega}$ such that for every $w \subseteq \omega$, there is b_w such that $\models \varphi(a_i, b_w)$ iff $i \in w$.*

In the paper, whenever H and G are groups, then $H < G$ means that H is a (not necessarily proper) subgroup of G.

Recall that we say that a definable group G has icc (the uniform chain condition on intersections of uniformly definable subgroups) if for every formula φ, there is $n_\varphi \in \omega$ such that any chain of intersections of φ-definable subgroups of G has length at most n_φ.

Recall that the centralizer connected component of a definable group G is the intersection of all centralizers $C(g)$, $g \in G$, of finite index in G. The group G is said to be centralizer connected if it is equal to its centralizer connected component. If we have icc (or only icc on centralizers), then the centralizer connected component is a definable, finite index subgroup of G. In the proof of Theorem 2.1, we will need an easy fact that if G is centralizer connected and $Z(G)$ is finite, then $G/Z(G)$ is centerless.

For this paper, an important consequence of NIP and rosiness is [4, Proposition 1.7]:

Fact 1.5 *Suppose T is rosy and has NIP. Then any definable group G has icc.*

We will also use [7, Theorem 3]:

Fact 1.6 *Assume T has NIP and G is a definable group of finite $U^þ$-rank. Assume that G is solvable-by-finite but not nilpotent-by-finite. Then there is an infinite field interpretable in $\langle G, \cdot \rangle$.*

In [7, Theorem 3], rosiness of T was additionally assumed, but we see, using Remark 1.3, that it is not necessary.

One more property that we are going to assume is fsg (finitely satisfiable generics). As was mentioned at the end of the introduction, we do not use this property anywhere in our proofs except for the applications of [4, Theorem 2]. That is why, in this paper, we restrict ourselves only to giving the definition of fsg and recalling [4, Theorem 2].

Definition 1.7 *A definable group G defined by a formula $G(x)$ has finitely satisfiable generics (or fsg) if there is a global type p containing $G(x)$ and a model $M \prec \mathfrak{C}$ of cardinality less than the degree of saturation of $\mathfrak{C}$, such that for all g, gp is finitely satisfiable in M (i.e. each formula in gp defines a set which intersects M). We say that G has hereditarily fsg if every definable subgroup of G also has fsg.*

It is easy to check that if G has fsg and N is a definable, normal subgroup of G, then G/N also has fsg.

Fact 1.8 *Assume T has NIP and G is a definable group satisfying hereditarily fsg. If $U^þ(G) = 2$, then G is solvable-by-finite.*

2 Getting fields

In this section, ***T* is a theory with NIP**, and we work in $\mathfrak{C}^{eq}$ where $\mathfrak{C}$ is a monster model of T. Whenever we have a definable group G acting definably on a definable set S, $\mathcal{M}$ will denote the two-sorted structure with sorts G and S equipped with the group operation on G and the action of G on S.

First we will show a variant of Theorem 3, which will allow us to prove all three theorems from the introduction.

Theorem 2.1 *Assume T is rosy, and let G be a definable group having hereditarily fsg and with $1 < U^þ(G) < \infty$. Assume that G acts definably on a definable set S of $U^þ$-rank 1 so that there is $s \in S$ for which no infinite, definable subgroup H of G_s of finite index in the intersection of stabilizers of some points in S and with $U^þ(G/H) \leq 2$ has normalizer of finite index in G. Then there is an infinite field interpretable in $\mathfrak{C}$.*

The following strengthening of Theorem 2.1 can be proved in almost the same way as Theorem 2.1 is proved below.

Theorem 2.2 *Assume T is rosy, and let G be a definable group having hereditarily fsg and with $1 < U^þ(G) < \infty$. Assume that G acts definably on a definable set S of $U^þ$-rank 1 so that there is $s \in S$ for which no infinite, definable in $\mathcal{M}^{eq}$ subgroup H of G_s of finite index in the intersection of stabilizers of some points in S and with $U^þ(G/H) \leq 2$ has normalizer of finite index in G. Then there is an infinite field interpretable in $\mathcal{M}$.*

Proof of Theorem 2.1. The proof is by induction on $\mathrm{U}^þ(G)$. First consider the case $\mathrm{U}^þ(G) = 2$. By Fact 1.8, G is solvable-by-finite (this is the only place in the proof where the fsg assumption is used). So, using Fact 1.6, either we get a field interpretable in the pure group G and we are done, or else G is nilpotent-by-finite. Since by rosiness and NIP we have icc on centralizers, using [7, Corollary 3.3(ii)] and replacing G by a definable subgroup of finite index, we can assume that G is nilpotent and centralizer connected. Then $Z(G)$ is infinite (otherwise $G/Z(G)$ would be centerless, a contradiction). Since $\mathrm{U}^þ(G) = 2 > \mathrm{U}^þ(S)$, G_s is infinite. So, by assumption, $[G : G_s]$ must be also infinite, and so Gs is infinite. Replacing S by Gs, the action becomes transitive. Since $G_S \lhd G$, by assumption, G_S is finite. Thus, replacing G by G/G_S, we can assume that the action is additionally faithful.

Notice that there is no non-trivial $z \in Z(G)$ and $x \in S$ with $zx = x$. Otherwise $z(gx) = (zg)x = (gz)x = g(zx) = gx$ for every $g \in G$, and so $z \in G_S$ by transitivity of the action. This is a contradiction with faithfulness.

We have proved that $G_s \cap Z(G) = \{e\}$. But $N(G_s) \supseteq \langle G_s, Z(G)\rangle$ and both G_s and $Z(G)$ are infinite, which implies that $\mathrm{U}^þ(N(G_s)) = 2$. Thus, $[G : N(G_s)]$ is finite, a contradiction.

Now assume that $\mathrm{U}^þ(G) \geq 3$ and the theorem holds for groups of smaller $\mathrm{U}^þ$-rank.

As in the base step, we see that Gs is infinite. So, replacing S by Gs, we can assume that the action is transitive. Since $G_S \lhd G$, by assumption, we get $\mathrm{U}^{\text{þ}}(G/G_S) \geq 3$. So, replacing G by G/G_S, we can assume that the action of G on S is also faithful. This gives us easily that for any infinite definable subgroup H of G, there is $t \in S$ with Ht infinite. Indeed, otherwise by icc, $\{e\} = H_S = \bigcap_{t \in S} H_t$ is an intersection of finitely many subgroups H_t, all of finite index in H. So, H is finite, a contradiction.

Claim 1 $\mathrm{U}^{\text{þ}}(G) < \omega$.

Proof of Claim 1. Suppose for a contradiction that $\mathrm{U}^{\text{þ}}(G) \geq \omega$. We have that $[G : G_s]$ is infinite, so $\mathrm{U}^{\text{þ}}(Gs) = 1$, and hence $\mathrm{U}^{\text{þ}}(G/G_s) = \mathrm{U}^{\text{þ}}(Gs) = 1$. Thus, $\mathrm{U}^{\text{þ}}(G_s) + 1 = \mathrm{U}^{\text{þ}}(G_s) + \mathrm{U}^{\text{þ}}(G/G_s) \leq \mathrm{U}^{\text{þ}}(G) \leq \mathrm{U}^{\text{þ}}(G_s) \oplus \mathrm{U}^{\text{þ}}(G/G_s) = \mathrm{U}^{\text{þ}}(G_s) + 1$. So, we get $\mathrm{U}^{\text{þ}}(G) = \mathrm{U}^{\text{þ}}(G_s) + 1$. Therefore,

$$\mathrm{U}^{\text{þ}}(G) > \mathrm{U}^{\text{þ}}(G_s) \geq \omega.$$

Since G_s is infinite, there is $s_1 \in S$ with $G_s s_1$ infinite and so of $\mathrm{U}^{\text{þ}}$-rank 1. As above, we get

$$\mathrm{U}^{\text{þ}}(G_s) > \mathrm{U}^{\text{þ}}(G_{ss_1}) \geq \omega.$$

We continue this procedure and obtain $s_i \in S$, $i \geq 1$, so that

$$\mathrm{U}^{\text{þ}}(G) > \mathrm{U}^{\text{þ}}(G_s) > \mathrm{U}^{\text{þ}}(G_{ss_1}) > \mathrm{U}^{\text{þ}}(G_{ss_1s_2}) > \dots,$$

a contradiction. □

Let $\mathcal{H}$ be the collection of all definable, finite index subgroups H of G_s with the property that for any $g \in G$, if $[H : H \cap H^g] < \omega$, then $H = H^g$.

Claim 2 $\mathcal{H}$ is nonempty.

Proof of Claim 2. By icc, the intersection of all G_s^g such that $[G_s : G_s \cap G_s^g] < \omega$ is a definable, finite index subgroup of G_s. Denote this subgroup by H. We see that if $[H : H \cap H^g] < \omega$, then $H = H \cap H^g$, and so $H < H^g$. This implies $H^{g^{-1}} < H$. On the other hand, $\mathrm{U}^{\text{þ}}(H^{g^{-1}}) = \mathrm{U}^{\text{þ}}(H)$, so by Lascar Inequalities for groups, $[H : H^{g^{-1}}] < \omega$. Thus, $H^{g^{-1}} = H$, so $H = H^g$. We have seen that $H \in \mathcal{H}$. □

Consider any $H \in \mathcal{H}$. It is infinite, so there is $t \in S$ with Ht infinite. Since $\mathrm{U}^{\text{þ}}(S) = 1$, there are only finitely many infinite orbits on S under H; denote them by $o_0, \dots, o_n$. For $i = 0, \dots, n$, put

$$H_i = H_{o_i} \lhd H$$

and

$$\delta(H) = \min_{0 \leq i \leq n} \mathrm{U}^{\text{þ}}(H_i).$$

From now on, we choose $H \in \mathcal{H}$ with maximal possible $\delta(H)$ (we can do it by Claims 1 and 2).

Claim 3 Assume $\delta(H) \leq \mathrm{U}^{\text{þ}}(H) - 2$. Then there is i such that H/H_i acting on o_i satisfies the assumptions of the theorem. Thus, since $\mathrm{U}^{\text{þ}}(H/H_i) \leq \mathrm{U}^{\text{þ}}(H) < \mathrm{U}^{\text{þ}}(G)$, by the inductive hypothesis, there is an infinite, interpretable field.

Proof of Claim 3. Let $H_{i_1}, \dots, H_{i_k}$ be all H_i's for which $\mathrm{U}^{\text{þ}}(H_i) = \delta(H)$. Suppose that for every $j = 1, \dots, k$, H/H_{i_j} acting on o_{i_j} does not satisfy the assumption of the theorem (it is enough to show that this leads to a contradiction). Then for every $j = 1, \dots, k$, there is a nonempty $S_j \subseteq o_{i_j}$ and an infinite, definable, finite index subgroup K_j/H_{i_j} of $(H/H_{i_j})_{S_j} = H_{S_j}/H_{i_j}$ such that $[H/H_{i_j} : N_{H/H_{i_j}}(K_j/H_{i_j})]$ is finite. This implies $[H : N_H(K_j)] < \omega$. So, by an argument similar to the proof of Claim 2, we can find $L \in \mathcal{H}$ such that $L < \bigcap_{j=1}^{k} N_H(K_j)$ (namely, define L as the intersection of all $(\bigcap_{j=1}^{k} N_H(K_j))^g$ such that $[\bigcap_{j=1}^{k} N_H(K_j) : \bigcap_{j=1}^{k} N_H(K_j) \cap (\bigcap_{j=1}^{k} N_H(K_j))^g] < \omega$.). By the definition of $\mathcal{H}$, $L \lhd H$.

We will be done if we prove that $\delta(L) > \delta(H)$ because this will be a contradiction with maximality of $\delta(H)$.

Since $[H : L] < \omega$, each o_i is a union of finitely many infinite orbits $o_0^i \cup \dots \cup o_{n_i}^i$ under L. For $i = 0, \dots, n$ and $j = 0, \dots, n_i$, define

$$L_j^i = L_{o_j^i}.$$

We need to prove that for any $i \in \{0, \dots, n\}$ and $j \in \{0, \dots, n_i\}$, $\mathrm{U}^{\text{þ}}(L_j^i) > \delta(H)$. So, consider any i, j as above.

Case 1 $i \notin \{i_1, \dots, i_k\}$.
Put $H_j^i = H_{o_j^i}$. Then $H > H_j^i > H_i$. Since $[H : L] < \omega$, we also have that $L \cap H_j^i = L_j^i$ has finite index in H_j^i. Thus, $\mathrm{U}^{\text{þ}}(L_j^i) \geq \mathrm{U}^{\text{þ}}(H_i) > \delta(H)$ (the last inequality is true because of the assumption of Case 1).

Case 2 $i = i_l$ for some $l \in \{1, \dots, k\}$.
Then $\delta(H) = \mathrm{U}^{\text{þ}}(H_i)$, so it is enough to show that $\mathrm{U}^{\text{þ}}(L_j^i) > \mathrm{U}^{\text{þ}}(H_i)$.

Take any $x \in S_l$. Then $x \in o_m^i$ for some $m \in \{0, \dots, n_i\}$.

Subclaim $\mathrm{U}^{\text{þ}}(L_m^i) > \mathrm{U}^{\text{þ}}(H_i)$.

Proof of Subclaim. By the choice of K_j's, $[K_l : H_i] \geq \omega$. On the other hand, since $[H : L] < \omega$ and $K_l < H$, we have $[K_l : L \cap K_l] < \omega$. From these two observations, we obtain

$$(*) \qquad \mathrm{U}^{\text{þ}}(L \cap K_l) > \mathrm{U}^{\text{þ}}(H_i).$$

As L normalizes K_l, $L \cap K_l \lhd L$. We also know that $L \cap K_l$ stabilizes x. Therefore, for any $h_1 \in L$ and $h_2 \in L \cap K_l$, we have $h_2 h_1 x = h_1 h_2' x = h_1 x$ for some $h_2' \in L \cap K_l$. Moreover, $Lx = o_m^i$. Hence, $L \cap K_l < L_m^i$. In virtue of $(*)$, the proof of the Subclaim is completed. $\square$

There is $h \in H$ such that $ho^i_m \cap o^i_j \neq \emptyset$. Since $L \lhd H$, we easily get $ho^i_m = o^i_j$. So, for any $g \in L$, we have the following equivalences:

$$g \in L^i_m \iff (\forall y \in o^i_m)(hgy = hy) \iff (\forall z \in o^i_j)(g^h z = z) \iff g^h \in L^i_j.$$

This means that $(L^i_m)^h = L^i_j$. Hence, by the Subclaim, $\mathrm{U}^\flat(L^i_j) = \mathrm{U}^\flat(L^i_m) > \mathrm{U}^\flat(H_i)$, which completes the proof of Claim 3. □

By Claim 3, in order to finish the proof of the theorem, it remains to show that the condition $\delta(H) = \mathrm{U}^\flat(H) - 1$ leads to a contradiction. So, assume $\delta(H) = \mathrm{U}^\flat(H) - 1$. Then $\mathrm{U}^\flat(H_i) = \mathrm{U}^\flat(H) - 1 = \mathrm{U}^\flat(G) - 2$ for every $i = 0, \dots, n$.

Claim 4 There are infinite, definable normal subgroups L_i of H for $i = 0, \dots, n$ such that:
(i) L_i is a finite index subgroup of H_i,
(ii) for any $g \in G$, L_i^g acts trivially on go_i,
(iii) for any $i, j \in \{0, \dots, n\}$ and $g \in G$, if $[L_i : L_i \cap L_j^g] < \omega$, then $L_i = L_j^g$.

Proof of Claim 4. We define L_i as the intersection of all groups H_j^g for which $[H_i : H_i \cap H_j^g]$ is finite. By icc, each L_i is definable and of finite index in H_i. Since $H_i \lhd H$, we see that $L_i \lhd H$. Thus, (i) and so (ii) holds.

To see (iii), consider any $i, j \in \{0, \dots, n\}$ and $g \in G$ such that $[L_i : L_i \cap L_j^g] < \omega$. Then the property $[H_i : L_i] < \omega$ implies $[H_i : L_i \cap L_j^g] < \omega$. We also know that $L_i \cap L_j^g$ is an intersection of groups of the form H_k^a. Therefore, $L_i = L_i \cap L_j^g$, i.e. $L_i < L_j^g$. Thus, $L_i^{g^{-1}} < L_j$. Since $\mathrm{U}^\flat(L_i^{g^{-1}}) = \mathrm{U}^\flat(L_i) = \mathrm{U}^\flat(H_i) = \mathrm{U}^\flat(H) - 1 = \mathrm{U}^\flat(L_j)$, we get $[L_j : L_i^{g^{-1}}] < \omega$. On the other hand, $L_i^{g^{-1}}$ is an intersection of groups of the form H_k^a. Hence, $L_j = L_i^{g^{-1}}$, i.e. $L_i = L_j^g$. □

By transitivity of the action of G on S, we can choose $g \in G$ so that $gs \in o_0$. Since o_0 is an infinite orbit under H and $Hs = \{s\}$, we see that $g \notin N(H)$. But $H \in \mathcal{H}$, so we conclude that $[H : H \cap H^g]$ is infinite.

Now we will show that for every $i \in \{0, \dots, n\}$,

$$(!) \qquad [L_0 : L_0 \cap L_i^g] \geq \omega \text{ and } [L_i^g : L_0 \cap L_i^g] \geq \omega.$$

If any of the above conditions is false, then $[L_0 : L_0 \cap L_i^g] < \omega$ or $[L_i : L_i \cap L_0^{g^{-1}}] < \omega$. Thus, by Claim 4, $L_0 = L_i^g$. It follows that $\langle H, H^g \rangle < N(L_0)$. Since $[H : H \cap H^g] \geq \omega$, we get $[N(L_0) : H^g] \geq \omega$. Hence, $\mathrm{U}^\flat(N(L_0)) \geq \mathrm{U}^\flat(H^g) + 1 = \mathrm{U}^\flat(G)$. So,

$$[G : N(L_0)] < \omega.$$

On the other hand, the fact that $H_0 = G_{\{s\} \cup o_0} \cap H$ has finite index in $G_{\{s\} \cup o_0}$ and $[H_0 : L_0] < \omega$ implies that

$$[G_{\{s\} \cup o_0} : L_0] < \omega.$$

Moreover, $\mathrm{U}^\flat(L_0) = \mathrm{U}^\flat(G) - 2 \geq 1$, so

$$\mathrm{U}^\flat(G/L_0) = 2 \text{ and } L_0 \text{ is infinite.}$$

All these three observations together give us a contradiction with the assumption of the theorem. So, (!) has been proved.

Notice that there are only finitely many orbits on S under H. We know that there are finitely many infinite orbits, so we need to check that there are finitely many finite orbits. For any $a \in G$, we have $G_{as} = G_s^a$. Thus, we see that Has being finite is equivalent to any of the following:

$$[H : H \cap G_s^a] < \omega \iff [H : H \cap H^a] < \omega \iff H = H^a \iff a \in N(H),$$

and the last condition implies $as \in N(H)s$. By the assumption of the theorem, $[G : N(H)] \geq \omega$. But also $\mathrm{U}^{\flat}(H) = \mathrm{U}^{\flat}(G) - 1$. Thus, $[N(H) : H] < \omega$. So, from the previous computation, there are only finitely elements in S with finite orbits under H.

By the above observation, we can choose $i \in \{0, \ldots, n\}$ so that $o_0 \cap go_i \neq \emptyset$. Choose an element $g_1 s \in o_0 \cap go_i$.

Recall that two subgroups G_1 and G_2 of a given group are said to be commensurable (symbolically $G_1 \sim G_2$) if $[G_1 : G_1 \cap G_2]$ and $[G_2 : G_1 \cap G_2]$ are finite. $\sim$ is always an equivalence relation.

Since $gs \in o_0$, $L_0 < G_s^g$. Also, H^g is a finite index subgroup of G_s^g. Hence,

$$(!!) \qquad L_0 \sim L_0 \cap H^g.$$

Since $g_1 s \in o_0$, $L_0 < G_s^{g_1}$. So, as above

$$(!!!) \qquad L_0 \sim L_0 \cap H^{g_1}.$$

Since $g_1 s \in go_i$ and L_i^g acts trivially on go_i, we get $L_i^g < G_s^{g_1}$. Therefore,

$$(!!!!) \qquad L_i^g \sim L_i^g \cap H^{g_1}.$$

We know that $L_0^{g_1} \lhd H^{g_1}$. So, $\langle L_0 \cap H^{g_1}, L_i^g \cap H^{g_1} \rangle < N(L_0^{g_1})$. In particular, by (!!!!), $L_i^g \sim L_i^g \cap H^{g_1} < N(L_0^{g_1}) \cap H^g$. Similarly, by (!!) and (!!!), $L_0 \sim L_0 \cap H^{g_1} \cap H^g < N(L_0^{g_1}) \cap H^g$. Using the last two observations together with (!), we get $[N(L_0^{g_1}) \cap H^g : L_0 \cap H^{g_1} \cap H^g] \geq \omega$. On the other hand, $\mathrm{U}^{\flat}(L_0 \cap H^{g_1} \cap H^g) = \mathrm{U}^{\flat}(L_0) = \mathrm{U}^{\flat}(H) - 1 = \mathrm{U}^{\flat}(H^g) - 1$. So, we conclude that

$$(\diamond) \qquad [H^g : N(L_0^{g_1}) \cap H^g] < \omega.$$

We claim that

$$(\diamond\diamond) \qquad [H^g : H^g \cap H^{g_1}] \geq \omega.$$

Suppose it is not true. Then $[H : H \cap H^{g^{-1}g_1}] < \omega$. So, $H = H^{g^{-1}g_1}$, which implies $H^g = H^{g_1}$. Hence, $H^g(g_1 s) = g_1 H s = \{g_1 s\}$. But $g_1 s \in go_i$, so there is $t \in o_i$ such that $g_1 s = gt$, and so $H^g(g_1 s) = gHt = go_i$ is infinite. This is a contradiction.

Of course, we have $H^{g_1} \cup (N(L_0^{g_1}) \cap H^g)) \subseteq N(L_0^{g_1})$. So, by $(\diamond)$ and $(\diamond\diamond)$, we get $[N(L_0^{g_1}) : H^{g_1}] \geq \omega$, and so $[N(L_0) : H] \geq \omega$. Since $\mathrm{U}^{\flat}(H) = \mathrm{U}^{\flat}(G) - 1$, we see that $[G : N(L_0)] < \omega$. It was shown in the proof of (!) that this gives us a contradiction with the assumption of the theorem. So, the proof of the theorem is completed. ■

Proof of Theorem 2.2. The proof is almost the same as the proof of Theorem 2.1. The only difference is that we have to work with objects definable in $\mathcal{M}^{eq}$ (whereas $\mathrm{U}^{\text{þ}}$-ranks should still be computed in $\mathfrak{C}$). ■

Before we turn to further results, let us make a few comments concerning Theorem 2.1.

Notice that, using only icc instead of the full NIP assumption, the above inductive proof always reduces the situation to a smaller $\mathrm{U}^{\text{þ}}$-rank, and finally leads to the case $\mathrm{U}^{\text{þ}}(G) = 2$. Then we get a field via Fact 1.6 or even [4, Theorem 4.5]. Since in [4, Theorem 4.5], it is enough to assume icc instead of NIP, we see that in Theorems 2.1 and 2.2, NIP can be also replaced by icc. Looking at the proofs that follows, this will easily imply that in Theorems 2, 3, 4, one can also replace NIP by icc (keeping the fsg assumption).

Notice that if we strengthen the assumption of Theorem 2.1 by dropping the condition $\mathrm{U}^{\text{þ}}(G/H) \leq 2$, then the existence of an infinite field interpretable in $\mathcal{M}$ follows easily from this theorem and Remark 1.3. To see this, it is enough to show that G acting on S considered in $\mathcal{M}$ satisfies the assumptions of the theorem. By Remark 1.3, it is clear that $\mathcal{M}$ is rosy with NIP, and working in $\mathcal{M}$, we have that $\mathrm{U}^{\text{þ}}_{\mathcal{M}}(G) < \infty$, $\mathrm{U}^{\text{þ}}_{\mathcal{M}}(S) = 1$, G has hereditarily fsg, the action of G on S is definable, and no infinite, definable subgroup H of G_s of finite index in the intersection of stabilizers of some points in S has normalizer of finite index in G. The only remaining assumption is that $\mathrm{U}^{\text{þ}}_{\mathcal{M}}(G) > 1$. However, since by the assumption of the theorem G_s is infinite (otherwise $1 = \mathrm{U}^{\text{þ}}(S) \geq \mathrm{U}^{\text{þ}}(Gs) = \mathrm{U}^{\text{þ}}(G/G_s) \geq 2$, a contradiction) and $[G : G_s] \geq \omega$, we get $\mathrm{U}^{\text{þ}}_{\mathcal{M}}(G) \geq \mathrm{U}^{\text{þ}}_{\mathcal{M}}(G_s) + \mathrm{U}^{\text{þ}}_{\mathcal{M}}(G/G_s) \geq 2$.

Now we will apply Theorem 2.2 to get Theorem 3. In fact, we will use the weaker version of Theorem 2.2 in which the condition $\mathrm{U}^{\text{þ}}(G/H) \leq 2$ is dropped, and we will prove the following strengthening of Theorem 3.

Theorem 2.3 *Assume T is rosy, and let G be a definable group having hereditarily fsg and with $1 < \mathrm{U}^{\text{þ}}(G) < \infty$. Assume that G acts definably on a definable set S of $\mathrm{U}^{\text{þ}}$-rank 1 so that there is $s \in S$ for which no finite index, definable in $\mathcal{M}^{eq}$ subgroup of G_s has normalizer of finite index in G. Then there is an infinite field interpretable in $\mathcal{M}$.*

Proof. By the assumption of the theorem, $[G : G_s] \geq \omega$. So, Gs is infinite. Hence, replacing S by Gs, we can assume that the action of G on S is transitive. Also by the assumption, $\mathrm{U}^{\text{þ}}(G/G_S) > 1$. Indeed, if $\mathrm{U}^{\text{þ}}(G/G_S) = 1$, then $[G_s : G_S] < \omega$, which together with $G_S \lhd G$ contradicts the assumption. So, replacing G by G/G_S, we can assume that G acts faithfully on S. As in the proof of Theorem 2.1, this implies that $\mathrm{U}^{\text{þ}}(G) < \infty$.

Let $\mathcal{G}$ be the collection of all finite index subgroups of G which are definable in $\mathcal{M}^{eq}$.

Consider any $H \in \mathcal{G}$. Since $[G : H] < \omega$, Hs is infinite. By the assumption that $\mathrm{U}^{\text{þ}}(S) = 1$, there are only finitely many infinite orbits under H; denote them by $o_0, \dots, o_n$. For $i = 0, \dots, n$, put

$$H_i = H_{o_i} \lhd H$$

and

$$\Delta(H) = \max_{0 \leq i \leq n} \mathrm{U}^{\text{þ}}(H_i).$$

From now on, we choose $H \in \mathcal{G}$ with maximal possible $\Delta(H)$. Take i with $\Delta(H) = \mathrm{U}^{\text{þ}}(H_i)$.

By the assumption of the theorem, $\Delta(H) < \mathrm{U}^{\text{þ}}(H) - 1$. Indeed, since G acts transitively on S, there is $g \in G$ such that $s \in go_i$. As $G_s > H_i^g \lhd H^g$ and $[G : H^g] < \omega$, by the assumption of the theorem, we get $[G_s : H_i^g] \geq \omega$, and so $\Delta(H) = \mathrm{U}^{\text{þ}}(H_i) = \mathrm{U}^{\text{þ}}(H_i^g) < \mathrm{U}^{\text{þ}}(G) - 1 = \mathrm{U}^{\text{þ}}(H) - 1$.

Since H/H_i and the action of H/H_i on S are definable in $\mathcal{M}^{eq}$, the following claim will finish the proof.

Claim H/H_i acting on o_i satisfies the assumption of Theorem 2.2.

Proof of Claim. Of course, $\mathrm{U}^{\text{þ}}(H/H_i) \leq \mathrm{U}^{\text{þ}}(H) = \mathrm{U}^{\text{þ}}(G) < \omega$ and $\mathrm{U}^{\text{þ}}(H/H_i) = \mathrm{U}^{\text{þ}}(H) - \mathrm{U}^{\text{þ}}(H_i) = \mathrm{U}^{\text{þ}}(H) - \Delta(H) > 1$. Moreover, $\mathrm{U}^{\text{þ}}(o_i) = 1$. Thus, if the assumption of Theorem 2.2 fails, there is a nonempty $S_i \subseteq o_i$ and an infinite, definable in $\mathcal{M}^{eq}$, finite index subgroup K_i/H_i of $(H/H_i)_{S_i} = H_{S_i}/H_i$ such that $[H/H_i : N_{H/H_i}(K_i/H_i)]$ is finite. This implies $[H : N_H(K_i)] < \omega$. Put $N = N_H(K_i)$. We see that $N \in \mathcal{G}$.

We will be done if we prove that $\Delta(N) > \Delta(H)$ because this will be a contradiction with maximality of $\Delta(H)$.

Since $[H : N] < \omega$, we have that o_i is the union of finitely many infinite orbits $o_0^i \cup \dots \cup o_{n_i}^i$ under N. Take any $s_i \in S_i$. Then $s_i \in o_j^i$ for some j. Put $N_j^i = N_{o_j^i}$.

We have that $K_i \lhd N$ and K_i stabilizes s_i. Hence, K_i stabilizes o_j^i, and so $K_i < N_j^i$. On the other hand, $[K_i : H_i] \geq \omega$. So, $\mathrm{U}^{\text{þ}}(N_j^i) > \mathrm{U}^{\text{þ}}(H_i)$. Thus, we conclude that $\Delta(N) > \Delta(H)$. ■

Observe that in the above proof, one can use Theorem 2.1 (even with the stronger assumption obtained by dropping the condition $\mathrm{U}^{\text{þ}}(G/H) \leq 2$) instead of Theorem 2.2. Indeed, it is clear because applying Remark 1.3 at the beginning of the above proof, we can assume that $\mathfrak{C} = \mathcal{M}$.

Now, we will show that if one considers the assumption $\mathrm{U}^{\text{þ}}(G) > 1$ in a stronger sense, namely that for every (not only for some) set A of parameters over which G is defined, there is $g \in G$ with $\mathrm{U}^{\text{þ}}(g/A) > 1$, then the assumption that T is rosy can be eliminated from Theorem 2.3. Indeed, by Remark 1.3, $\mathcal{M}$ is rosy, and working in $\mathcal{M}$, all assumptions of Theorem 2.3 except for $\mathrm{U}^{\text{þ}}_{\mathcal{M}}(G) > 1$ are clearly satisfied. So, it remains to show that $\mathrm{U}^{\text{þ}}_{\mathcal{M}}(G) > 1$. By the assumption of the theorem, $[G : G_s]$ is infinite. Moreover, there is a definable bijection between G/G_s and $o(s)$. So, computing $\mathrm{U}^{\text{þ}}$-rank over a set of parameters over which everything is defined,

$\mathrm{U}^{\flat}(G/G_s) = 1$. Thus, G_s is infinite because otherwise, computing $\mathrm{U}^{\flat}$-rank over some set of parameters, $\mathrm{U}^{\flat}(G) = 1$, a contradiction. So, working in $\mathcal{M}$, G has an infinite, definable subgroup of infinite index (namely G_s). Therefore, $\mathrm{U}^{\flat}_{\mathcal{M}}(G) \geq 2$.

Now, using Theorem 3, we will prove Theorem 4.

Proof of Theorem 4. By Remark 1.3, we see that $\mathcal{M}$ is superrosy and $\mathrm{U}^{\flat}_{\mathcal{M}}(S) = 1$. So, it is enough to prove Theorem 4 under the assumption that T is rosy.

To see that $\mathrm{U}^{\flat}(G) < \omega$, we use icc and faithfullness of the action in the same way as in the proof of Theorem 2.1.

In order to get the existence of a field, we argue by induction on $\mathrm{U}^{\flat}(G)$. Suppose the theorem is true for groups of $\mathrm{U}^{\flat}$-rank smaller than $\mathrm{U}^{\flat}(G)$. By Remark 1.3, we can assume that $\mathfrak{C} = \mathcal{M}$ (but even without this reduction, the argument below works using Theorem 2.3 insead of Theorem 3). Consider two cases.

Case 1 There is a definable (in $\mathcal{M}^{eq}$) subgroup H of G such that $\mathrm{U}^{\flat}(G) = \mathrm{U}^{\flat}(H)+1$ and $[G : N(H)] < \omega$.

If H is solvable-by-finite, then using icc and [7, Remark 3.3(i)], H has a definable in $\langle H, \cdot \rangle$, normal subgroup H_0 of finite index which is solvable. By icc, $H_1 := \bigcap_{g \in N(H)} H_0^g$ is a finite index, definable (in $\mathcal{M}^{eq}$) subgroup of H_0 whose normalizer contains $N(H)$. So, replacing H by H_1, we can assume that H is solvable. But $\mathrm{U}^{\flat}(N(H)/H) = 1$, so $N(H)/H$ is solvable-by-finite in virtue of Fact 1.8 (in fact, it is even abelian-by-finite by [4, Theorem 1]). Thus, G is solvable-by-finite, a contradiction.

We have proved that H is not solvable-by-finite, and so H acting on S satisfies the assumptions of the theorem. On the other hand, $\mathrm{U}^{\flat}(H) < \mathrm{U}^{\flat}(G)$. So, by the inductive hypothesis, we get an infinite field interpretable in $\mathcal{M}$.

Case 2 Case 1 does not hold.

Since G is not solvable-by-finite, $\mathrm{U}^{\flat}(G) > 1$. By faithfullness and icc, there is $s \in S$ for which Gs is infinite, i.e. $[G : G_s] \geq \omega$. Then s witnesses that G acting on S satisfies the assumption of Theorem 3. So, we get an infinite field interpretable in $\mathcal{M}$. ■

In fact, even Theorem 2.1 is strong enough to get Theorem 4. To see this, one should repeat the above proof of Theorem 4 modifying Case 1 in the following way: There is an infinite, definable subgroup H of G such that $1 \leq \mathrm{U}^{\flat}(G) - \mathrm{U}^{\flat}(H) \leq 2$ and $[G : N(H)] < \omega$.

Proof of Theorem 2. Put $Z = \bigcap_{g \in G} H^g$. It is clear that G/Z acting on G/H by $(aZ) \cdot (gH) = agH$ satisfies the assumption of Theorem 4. ■

It was mentioned in the introduction that some of our theorems generalize the appropriate results about the finite Morley rank case. Now, we will explain why.

We claim that Theorem 3 generalizes the part of [12, Theorem 3.27] concerning the existence of a field. More precisely, we will show that the assumption of [12, Theorem 3.27] implies the assumption of Theorem 3.

Recall that in [12, Theorem 3.27], T is stable and G is a definable, transitive group of permutations of a strongly minimal set S. It is checked right before [12, Theorem 3.27] that the Morley rank of G is finite. Thus, $\mathrm{U}^{\text{þ}}(G) < \omega$. In order to get a field, it is also assumed in [12] that the Morley rank of G is at least 2. Choose any $s \in S$. Then G_s must be infinite. On the other hand, $[G : G_s]$ is also infinite. Hence, $\mathrm{U}^{\text{þ}}(G) \geq 2$. So, in order to show that in this situation, the assumption of Theorem 3 is satisfied, it is enough to prove that there is no definable subgroup H of G_s such that $[G : N(H)] < \omega$. Suppose for a contradiction that such an H exists. By transitivity of the action, strong minimality of S and the fact that $[G : G^0] < \omega$, we get that G^0 acts transitively on S. Thus, $N(H)$ also acts transitively on S. Take any non-trivial $h \in H$. Then for any $g \in N(H)$, we have $h(gs) = g(h^{g^{-1}}s) = gs$ as $h^{g^{-1}} \in G_s$. So, $h \in G_S = \{e\}$, a contradiction.

Another remark from the introduction says that Theorem 2 generalizes the part of [12, Corollary 3.28] concerning the existence of a field. Indeed, in [12, Corollary 3.28], one has a simple group G of finite Morley rank $n > 0$ with a definable subgroup H of rank $n-1$. Then G/H is strongly minimal. So, $\mathrm{U}^{\text{þ}}(G/H) = 1$, and hence $\mathrm{U}^{\text{þ}}(G) = \mathrm{U}^{\text{þ}}(H) + 1 < \omega$. Moreover, $Z := \bigcap_{g \in G} H^g$ is normal in G, so it is trivial as G is simple. Thus, $G/Z = G$. Simplicity of G also implies that G is not solvable-by-finite, and we see that the assumption of Theorem 2 is satisfied.

We finish the paper with some open questions concerning the structure of groups and their possible $\mathrm{U}^{\text{þ}}$-ranks.

[12, Theorem 3.27] in the finite Morley rank case and [8, Theorem 1.5] in the o-minimal case describe the structure of the permutation groups in terms of an interpretable field. Notice that by the argument from the base induction step of the proof of Theorem 2.1, if $\mathrm{U}^{\text{þ}}(G) = 2$ in Theorem 3, then G is not nilpotent-by-finite. So, by [4, Theorem 4.5], after passing to a definable subgroup of finite index and quotienting by its finite center, G is definably the semidirect product of the additive and multiplicative groups of an algebraically closed field interpretable in $\mathfrak{C}$, and moreover $G = G^{00}$. However, a description of G in Theorem 3 in the case $\mathrm{U}^{\text{þ}}(G) = 3$ remains an open problem.

The assumptions of Theorem 2 or 4 imply that $\mathrm{U}^{\text{þ}}(G) \geq 3$, and the structure of G in both these theorems is also unknown.

Moreover, as in the finite Morley rank or o-minimal case, one could try to prove that the assumption of Theorem 3 together with transitivity and faithfulness of the action imply that $\mathrm{U}^{\text{þ}}(G) = 2$ or $\mathrm{U}^{\text{þ}}(G) = 3$. Similarly, in Theorem 4, and in Theorem 2 with the stronger assumption that G is simple, one could try to prove that $\mathrm{U}^{\text{þ}}(G) = 3$.

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