Commutators in groups definable in o-minimal structures

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Abstract

We prove the definability, and actually the finiteness of the commutator width, of many commutator subgroups in groups definable in o-minimal structures. It applies in particular to derived series and to lower central series of solvable groups. Along the way, we prove some generalities on groups with the descending chain condition on definable subgroups and/or with a definable and additive dimension.

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1   Introduction

The development of model theoretic stability since the 70's provided an effective bridge from classical geometric objects to general first-order definable sets. For example, the voluminous theory of groups of finite Morley rank can be seen as a large generalization of that of algebraic groups over algebraically closed fields. More recently, the theory of groups definable in o-minimal structures has had important developments as well, often recovering some aspects of stable groups theory. The present paper continues in this vein.

Groups definable in o-minimal structures and groups of finite Morley rank share many properties. They are both equipped with a finite dimension which is definable and additive, satisfy the descending chain condition on definable subgroups, and have definably connected components, i.e., smallest definable subgroups of finite index. These properties suffice for many developments of the

theory of groups of finite Morley rank. However, if one restricts our attention to the natural examples of groups definable over fields of characteristic 0, the category of groups definable in o-minimal structures appears to be strictly larger since it contains all semi-algebraic groups over real closed fields in addition to algebraic groups over algebraically closed fields of characteristic 0.

In fact, the main difference between groups of finite Morley rank and groups definable in an o-minimal structure is in the behaviour of the dimension. The dimension of a definable set A in the o-minimal case fails the main property of the Morley rank: $\dim(A) \geq n + 1$ if and only if A contains infinitely many pairwise disjoint definable subsets A_i with $\dim(A_i) \geq n$. This essential definition of the Morley rank is crucial in Zilber's stabilizer argument, and consequently Zilber's generation lemma on indecomposable sets in the finite Morley rank context [3, Section 5.4]. And this is what allows one to prove the definability of many commutators subgroups, and in particular of derived subgroups, in this context.

For groups definable in o-minimal structures, derived subgroups need not be definable. Using recent results from [11] on central extensions in the o-minimal case, Annalisa Conversano exhibits in [6, Example 3.1.7] a definably connected group G definable in an o-minimal expansion of the reals with G' not definable. Furthermore, G is a central extension, by an infinite center isomorphic to $[0, 1[$, of the simple group $\mathrm{PSL}_2(\mathbb{R})$. In the present paper we will essentially prove that central extensions of this type are the only obstructions to the definability of commutator subgroups. In order to state our main theorem we consequently need the following definition.

Definition 1.1 *We say that a definably connected group G definable in an o-minimal structure is a* strict *central extension of a definably simple group if $Z(G)$ is infinite and $G/Z(G)$ is infinite nonabelian and definably simple.*

Definability, in Definition 1.1 and throughout the paper, always refers to a fixed structure, typically an o-minimal structure.

We call *section* of a group G any quotient H/K where $K \triangleleft H \leq G$, and we speak of *definable* section when both K and H are definable. In view of the previous example of a group definable in an o-minimal structure and with a non definable derived subgroup, we will mostly work with the following assumption. It consists merely in excluding "bad" sections of this type.

Definition 1.2 *We say that a group G definable in an o-minimal structure satisfies the* assumption (*) *whenever the derived subgroup $(H/K)'$ is definable for every definable section H/K of G which is a strict central extension of a definably simple group.*

Our main theorem is the following.

Theorem 1.3 *Let G be a group definable in an o-minimal structure, and A and B two definable subgroups which normalize each other and such that $A^\circ B^\circ$ satisfies the assumption (*). Then the subgroup $[A, B]$ is definable and $[A, B]^\circ =$*

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$[A^\circ, B][A, B^\circ]$. *Furthermore, any element of $[A, B]^\circ$ can be expressed as the product of at most $\dim([A, B]^\circ)$ commutators from $[A^\circ, B]$ or $[A, B^\circ]$ whenever A° or B° is solvable.*

We will see in Corollary 7.6 below that if A° and B° are solvable in Theorem 1.3, then $A^\circ B^\circ$ satisfies the assumption $(*)$ and thus $[A, B]$ is always definable in this case. In particular, Theorem 1.3 implies that if G is a group definable in an o-minimal structure with G° solvable, then the lower central series and derived series of G consist of definable subgroups (see Corollary 8.8 below). In view of Conversano's example on the other hand, the assumption $(*)$ is necessary if one wants a statement for definability as general as in Theorem 1.3, though of course the conclusion of the theorem may remain valid without that assumption in more specific cases.

A commutator group $[A, B]$ as in Theorem 1.3 is the countable union of the definable sets $[A, B]_n$, consisting of the products of at most n commutators $[a, b]$, or their inverses $[a, b]^{-1}$, with a in A and b in B. Our proof of Theorem 1.3 will consist in showing that $[A, B]$ has *finite commutator width*, i.e., that $[A, B] = [A, B]_n$ for some n. This is equivalent to the definability of $[A, B]$ when the ground structure in which G is defined is ω-saturated, but since we make no saturation assumption we show directly the finiteness of the commutator width. Besides, we will always try to keep the best control on the commutator width, generally in terms of the dimension of the groups involved. We emphasize that even the existence of such a finite n did not seem to be known, even in the more classical case of semi-algebraic groups over real closed fields. Our result, when it applies, also says something apparently new about the commutator width in Lie groups.

Our argument for the proof of Theorem 1.3 will consist mainly in finding very rudimentary forms of Zilber's general stabilizer argument on generation by indecomposable sets in the finite Morley rank context. In fact, our bounds n as above we will be obtained ultimately with Corollary 5.4 below, concerning the generation by definable and definably connected subgroups in abelian groups.

We refer to [28] for a general introduction to o-minimal structures, and to [19] for groups definable in o-minimal structures. We will frequently point out analogy to the finite Morley rank case in [3]. All model theoretic notions used here are rather elementary and can be found in any introductory book on model theory. We simply recall that an *o-minimal* structure is a first-order structure $\mathcal{M}$ with a total, dense, and without end-points definable order and such that every definable subset of M is a boolean combination of intervals with end-points in $M \cup \{\pm\infty\}$.

The paper is organized as follows. In Section 2 we recall or develop the background used on commutators in arbitrary groups. In Sections 3 and 4 we take the opportunity to recast, when this is possible, various facts typical of groups of finite Morley rank to the more general context of groups with various descending chain conditions on definable subgroups. In Section 5 we continue with an axiomatic treatment of groups with a definable and additive dimension. In Section 6 we insert groups definable in o-minimal structures in

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the preceding abstract machinery. Section 7 is devoted to a preliminary analysis of the structure of solvable groups. In Section 8 we prove our main results about commutators and finally in Section 9 we conclude with further results and questions.

2 Prelude to commutators

We fix the following notation as far as commutators are concerned in groups. If a and b are two elements of a group G, then we let $a^b = b^{-1}ab$ and $[a, b] = a^{-1}a^b$. If A and B are two arbitrary subsets of G, then $[A, B]_n$ denotes, for $n \geq 1$, the set of products of at most n elements of the form $[a, b]$ or $[a, b]^{-1}$, with (a, b) in $A \times B$. We denote by $[A, B]_\infty$, or simply by $[A, B]$ when there is no risk of confusion, the group generated by all commutators $[a, b]$, that is the union of all sets $[A, B]_n$. The *commutator width*, also frequently called the *commutator length*, of $[A, B]$ is the smallest n such that $[A, B] = [A, B]_n$ if such an n exists, and infinite otherwise.

Since our proofs by induction will use passages to quotients we will frequently "lift" commutators. This simply means that if N is a normal subgroup of G and $(a, b) \in G^2$, then $[aN, bN] = [a, b]N$ in G/N. In order to keep track of the commutator width when passing to quotients, the following lemma will be helpful.

Lemma 2.1 *Let G be a group, N a normal subgroup of G, and A and B two subsets of G such that, modulo N, $[A, B] = [A, B]_k$ for some k. Suppose also that $[A, B] \cap N = [A, B]_s \cap N$ for some s. Then $[A, B] = [A, B]_{k+s}$.*

Proof. Clearly, $[A, B]N/N = [AN/N, BN/N]$ by lifting of commutators. Our assumption on the commutator width of $[A, B]$ modulo N reads as

$$[AN/N, BN/N] = [AN/N, BN/N]_k.$$

Take any element g in $[A, B]$. Since g is also in $[A, B]N$, $g = \alpha u$ for some α in $[A, B]_k$ and some u in N. But now $u = \alpha^{-1}g$ is in $[A, B] \cap N$, and by assumption u is in $[A, B]_s$. It follows that $g = \alpha u$ is in $[A, B]_{k+s}$. $\qquad \square$

The following consequence of a theorem of Baer will be used for dealing with non-connected "finite bits" in our proofs.

Lemma 2.2 *Let A and B be two subgroups of a group, with A normalizing B. Suppose $C_A(B)$ of finite index in A and $C_B(A)$ of finite index in B. Then $[A, B]$ is a finite subgroup of B.*

Proof. Let n be the index of $C_A(B)$ in A and m the index of $C_B(A)$ in B. Fix a_i, $1 \le i \le n$, a set of representatives in A of $A/C_A(B)$, and b_j, $1 \le j \le m$, a set of representatives in B of $B/C_B(A)$.

We claim that the set of commutators $\{[a,b] \mid (a,b) \in A \times B\}$ is finite. In fact, any element a in A has the form $a_i\alpha$ for some i and some α in $C_A(B)$ and any element b in B has the form $b_j\beta$ for some j and some β in $C_B(A)$. Now $[a,b] = [a_i\alpha, b_j\beta] = \alpha^{-1}a_i^{-1}\beta^{-1}b_j^{-1}a_i\alpha b_j\beta$. Since α centralizes B and β centralizes A, we get $[a,b] = \beta^{-1}\alpha^{-1}[a_i,b_j]\alpha\beta = [a_i,b_j]^{\alpha\beta}$, and since $[A,B] \le B$ we also get

$$[a,b] = [a_i,b_j]^\beta = [a_i, b_j^\beta].$$

Hence $[a,b] \in a_i^{-1}a_i{}^B$. But $C_B(A) \le C_B(a_i)$ and it follows that a_i has a centralizer of finite index in B, and in other words that $a_i{}^B$ is finite. Hence $[a,b]$ varies in a finite set, and this proves our claim.

Now it suffices to apply a result of Baer saying that if A and B are two subgroups of a group, with A normalizing B and such that the set of commutators $\{[a,b] \mid (a,b) \in A \times B\}$ is finite, then the group $[A,B]$ is finite [26]. □

We recall two classical formulas which are valid for any elements a, b, and c in a group G:

$$[a,bc] = [a,c][a,b]^c \quad \text{and} \quad [ab,c] = [a,c]^b[b,c].$$

With these two formulas one sees that if A and B are two arbitrary subgroups of a group G, then the subgroup $[A,B]$ is always normalized by A and B. The proof of our main theorems will reduce in a crucial way to the second of these formulas via the following lemma.

Fact 2.3 *Let G be a group, x an element of G, and H a subgroup such that $\{[h,x] \mid h \in H\} \subseteq C_G(H)$. Then the map*

$$\mathrm{ad}\,_x: \begin{array}{ccc} H & \longrightarrow & G \\ h & \longmapsto & [h,x]. \end{array}$$

is a group homomorphism, with $C_H(x)$ as kernel and $\{[h,x] \mid h \in H\}$ as image.

Proof. For any h_1 and h_2 in H we have

$$[h_1h_2, x] = [h_1,x]^{h_2}[h_2,x] = [h_1,x][h_2,x]$$

by assumption. This shows that $\mathrm{ad}\,_x$ is a group homomorphism and the rest is clear. □

For any group G we let $G^1 = G^{(1)} = G' = [G,G]$, the *derived* subgroup of G, and we define by induction on $n \ge 1$ the *lower central series* and *derived series* as follows: $G^{n+1} = [G, G^n]$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. We say that G is *n-nilpotent* (resp. *n-solvable*) whenever $G^n = 1$ (resp. $G^{(n)} = 1$), and we say

that G is *nilpotent* (resp. *solvable*) whenever it is n-nilpotent (resp. n-solvable) for some $n \geq 1$. The nilpotency (resp. solvability) *class* of G is the smallest n such that G is n-nilpotent (resp. n-solvable). We also define the *upper central series* as follows: $Z_1(G) = Z(G)$ is the center of G, and by induction $Z_{n+1}(G)$ is the preimage in G of $Z(G/Z_n(G))$. As far as nilpotent groups are concerned, our inductive proofs will naturally use the following elementary fact.

Fact 2.4 *Let $n \geq 1$.*

(a) *If G is a nilpotent group of class n, then $G^i \leq Z_{n-i}(G)$, for $0 \leq i \leq n$.*

(b) *A group G is nilpotent of class n if and only if $Z_n(G) = G$ and $Z_{n-1}(G) < G$. In this case $G/Z(G)$ is nilpotent of class $n - 1$.*

Proof. The inclusions of first claim are easily proved by induction on n; see for instance [29, Lemma 0.1.7]. See [29, Corollary 0.1.8] for the first point of the second claim; the second point then follows. $\qquad\square$

3 Groups with *dcc*

In this section we work with a group definable in a structure $\mathcal{M}$ under the mere assumption that it satisfies the *descending chain condition*, *dcc* for short, on definable subgroups: any strictly descending chain of $\mathcal{M}$-definable subgroups is finite. Groups definable in an o-minimal structure $\mathcal{M}$ satisfy the *dcc* on definable subgroups [23, Remark 2.13 (ii)], and consequently all the results of the present section will apply to such groups.

In what follows definability always refers to definability from the original universe $\mathcal{M}$. Actually everything in the present section can be done for a group interpretable in $\mathcal{M}$, and hence definability here may refer to definability in $\mathcal{M}^{\text{eq}}$. Any definable group with no proper definable subgroup of finite index is said to be *definably connected*. Notice that a definable group can be definably connected even without satisfying the *dcc*. We start with two general facts about definably connected groups (independent of the *dcc*).

Fact 3.1 *Let G be a definably connected group.*

(a) *Any definable action of G on a finite set is trivial.*

(b) *If G' is finite, then G is abelian.*

(c) *If $Z(G)$ is finite, then $G/Z(G)$ is centerless.*

Proof. (a). The fixator of any element is a definable subgroup of finite index in G, and equals G by definably connectedness of the latter.

(b). Let g be any element in G. Notice that the G-conjugacy class g^G lies in the coset gG', since G/G' is abelian. Since G' is finite, the coset gG' is finite and it follows that the G-conjugacy class g^G is finite as well. As G acts definably on

g^G by conjugation, claim (a) shows that g^G is central in G, and in particular g is central in G.

(c). One may argue exactly as in [3, Lemma 6.1]. In fact it suffices to apply Fact 2.3 to elements $x \in Z_2(G)$, in a fashion very similar to what is done in Lemma 4.1 below. $\qquad\square$

Fact 3.2 *Let A and B be two definably connected definable subgroups of a definable group G, with A normalizing B. Then AB is a definably connected definable subgroup of G.*

Proof. Clearly AB is a definable subgroup since A normalizes B, and thus we only have to check its definable connectedness.

Let H be a definable subgroup of AB of finite index in AB. Since $H \cap A$ is a definable subgroup of finite index of A, it must be A by definable connectedness of A. Hence $A \leq H$. Similarly, $B \leq H$. We have shown that any definable subgroup H of finite index in AB satisfies $AB \leq H$. Hence AB has no proper definable subgroup of finite index, and thus it is definably connected. $\qquad\square$

From now on we consider only groups with the *dcc*. If H is a definable subgroup of a group with the *dcc* on definable subgroups, then H has a smallest definable subgroup of finite index, the intersection of all of them. It is denoted by H° and called the *definably connected component* of H. Clearly H° is definably characteristic in H, and in particular normal in H.

Any subset X of a group G with the *dcc* on definable subgroups is contained in a smallest definable subgroup, the intersection of all of them. It is called the *definable hull* of X and denoted by $H(X)$.

Remark 3.3 *Let G be a definable group with the dcc on definable subgroups, X and K two arbitrary subsets of G with X K-invariant. Then $H(X)$ is K-invariant.*

Proof. We have $X^k = X$ for any element k of the arbitrary subset K. It follows that $X \subseteq H(X) \cap H(X)^k$ for every such element k. Since $H(X) \cap H(X)^k$ is a definable subgroup containing X, the definition of $H(X)$ implies that $H(X) = H(X) \cap H(X)^k$, and thus $H(X) \leq H(X)^k$. If $H(X) < H(X)^k$, then conjugating by k^{-1} we find

$$X \subseteq H(X)^{k^{-1}} < H(X)$$

(since $X = X^{k^{-1}}$), a contradiction to the definition of $H(X)$. Hence $H(X) = H(X)^k$. $\qquad\square$

We now restate a typical fact about groups of finite Morley rank which indeed uses only the *dcc* on definable subgroups.

Fact 3.4 *Let G a group with the dcc on definable subgroups. If X and Y are two arbitrary subgroups of G, then $[H(X), H(Y)] \leq H([X, Y])$. In particular if X is an n-nilpotent (resp. n-solvable) subgroup, then $H(X)$ is n-nilpotent (resp. n-solvable) as well.*

Proof. Since X and Y normalize $[X, Y]$, they also normalize $H([X, Y])$ by Remark 3.3. As $N(H([X, Y]))$ is definable, we get that $H(X)$ and $H(Y)$ are both in $N(H([X, Y]))$. Now working modulo $H([X, Y])$, the proof works formally exactly as in [3, Lemma 5.37] (using the notation "H" instead of the notation "d" used there for the definable hull). □

We mention that Fact 3.4 was also observed in [7, Lemma 6.8] when the ambient group G is definable in an o-minimal structure, referring to the same proof.

If G is any group, then $F(G)$ denotes the subgroup generated by all normal nilpotent subgroups of G and is called the *Fitting* subgroup of G. It is clearly a normal subgroup of G. For the final fact of this section we use a theorem valid for all $\mathcal{M}_c$-*groups*, that is groups with the mere *dcc* on centralizers, as opposed to the *dcc* on all definable subgroups.

Fact 3.5 *Let G be a group with the dcc on centralizers. Then $F(G)$ is nilpotent. In particular if G is a group with the dcc on definable subgroups, then $F(G)$ is nilpotent and definable.*

Proof. The nilpotency of $F(G)$ for all $\mathcal{M}_c$-groups is shown in [29, Theorem 1.2.11]. When G has the *dcc* on all definable subgroups, the definable hull $H(F(G))$ exists. Since $F(G) \trianglelefteq G$, Remark 3.3 implies that $H(F(G))$ is normal in G as well. But by Fact 3.4 $H(F(G))$ is also nilpotent. Hence in this case $H(F(G)) \leq F(G)$, and both groups are equal and definable. □

4　Nilpotent groups with dcc

We continue in the same spirit as in the previous section, analyzing the structure of definable groups with the *dcc*, now specializing to the case of nilpotent and abelian groups. As in the previous section, definability may refer here to definability from $\mathcal{M}^{\mathrm{eq}}$ for some fixed structure $\mathcal{M}$. We start with a lemma inspired by [3, Ex. 5 p. 98] in the finite Morley rank case.

Lemma 4.1 *Let G be a nilpotent group with the dcc on definable subgroups, and H an infinite normal subgroup of G. Then $H \cap Z(G)$ is infinite.*

Proof. We proceed by induction on the nilpotency class of G. Of course it starts with G abelian, and thus we may assume that our counterexample of minimal nilpotency class is nonabelian.

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We suppose $H \cap Z(G)$ finite. Applying the induction hypothesis in $G/Z(G)$, which has smaller nilpotency class by Fact 2.4 and clearly also satisfies the *dcc*, we see that the image of H in $G/Z(G)$ has an infinite intersection with $Z(G/Z(G))$. In other words,

$$HZ(G)/Z(G) \cap Z(G/Z(G))$$

is infinite. Replacing H by $H \cap Z_2(G)$, we may thus assume $H \leq Z_2(G)$ without loss of generality.

Let $x \in G$. Then the application $\mathrm{ad}_x : h \longmapsto [h, x]$, defined for $h \in H$, takes its values in $H \cap Z(G)$ since $H \trianglelefteq G$ and $H \leq Z_2(G)$. By Fact 2.3 this is a group homomorphism, with $C_H(x)$ as kernel. As $H \cap Z(G)$ is finite, its image is finite, and thus the kernel $C_H(x)$ has finite index in H.

Besides, $Z(G) = C_G(x_1, \cdots, x_n)$ for finitely many elements $x_1, ..., x_n \in G$ by *dcc* on centralizers. In particular

$$H \cap Z(G) = C_H(x_1) \cap \cdots \cap C_H(x_n)$$

has finite index in H. But since H is infinite and $H \cap Z(G)$ is finite this is a contradiction. $\qquad\square$

It is a question whether Lemma 4.1 can be strengthened to groups with the mere *dcc* on centralizers. We note here that there exist groups G with the *dcc* on centralizers but such that $G/Z(G)$ does not have this property [4].

We now prove an "infinite" version of the classical normalizer condition for finite nilpotent groups. We will not use it in the present paper but we expect it to be as useful as in [12] for a potential theory of Carter subgroups in groups definable in o-minimal structures.

Lemma 4.2 *Let G be a nilpotent group with the dcc on definable subgroups, and H a definable subgroup of infinite index in G. Then $N_G(H)/H$ is infinite.*

Proof. Our proof can be compared to that of [3, Lemma 6.3] in the finite Morley rank case, but here we argue by induction on the nilpotency class rather than on the dimension, and we deal with "non-connected finite bits" throughout. We note also that we do not need the full *dcc* on definable subgroups, but merely the existence of definably connected components of definable subgroups.

Our counterexample G of minimal nilpotency class is of course not abelian. Take such a G and the corresponding $H \leq G$. Let $Z = Z(G)$. Since G is a counterexample to our statement and since $Z^\circ \leq N(H)$, we have $Z^\circ \leq H^\circ \leq H$. If HZ had finite index in G, then we would get $H^\circ = (HZ)^\circ = G^\circ$, and then H could not be of infinite index in G. This shows that HZ has infinite index in G, and it follows that HZ/Z has infinite index in G/Z as well. By Fact 2.4, G/Z has smaller nilpotency class, and the induction hypothesis applied in this quotient now implies that HZ has infinite index in $N(HZ)$ (since $N(HZ)$ is

exactly the preimage in G of $N_{G/Z}(HZ/Z)$). It follows that $(HZ)^\circ$ has infinite index in $N^\circ(HZ)$ and since $H^\circ = (HZ)^\circ$ we get

$$[N^\circ(HZ) : H^\circ] = \infty.$$

We are done if we show that $N^\circ(HZ)$ normalizes H. Of course, $N^\circ(HZ)$ normalizes $(HZ)^\circ = H^\circ$. Take now any element h in H. Then the coset hH° is also a coset of the form $h(HZ)^\circ$, with h in HZ. By Fact 3.1 (a), the definably connected group $N^\circ(HZ)$ induces by conjugation a trivial action on the finite quotient $(HZ)/(HZ)^\circ$. In other words, $N^\circ(HZ)$ setwise stabilizes by conjugation the coset hH°. We have shown that $N^\circ(HZ)$ normalizes H, and this finishes our proof. □

We now pass to a finer analysis of abelian groups with the *dcc*, essentially as in [16] in the finite Morley rank case; see also [3, Theorem 6.7].

Fact 4.3 *Let A be an abelian group with the dcc on definable subgroups. Then*

  (a) *$A = D \oplus B$ for some definable divisible subgroup D and some (interpretable but not necessarily definable) complement B of bounded exponent.*

  (b) *$A = DC$ for some definable and characteristic subgroups D and C, with D divisible as above and C of bounded exponent. Furthermore $D \cap C$ is finite whenever A contains no infinite elementary abelian p-subgroup.*

Proof. (a). Let $D = \bigcap_{n \in \mathbb{N}} A(n!)$, where $A(k) = \{a^k \mid a \in A\}$. By *dcc*, $D = A(n!)$ for some n and D is definable. It is clearly divisible. Now, since A is abelian, a well known theorem of Baer implies that any subgroup disjoint from the divisible subgroup D can be extended to a complement. In particular $A = D \oplus B$ for some complement B. Now, clearly, B must be of bounded exponent.

(b). Starting from the decomposition $A = D \oplus B$ as in claim (a), we can now take $C = \{a \in A \mid a^{\exp(B)} = 1\}$. Then all our statements are clear, knowing the decomposition of any abelian divisible group D as a direct product of quasicyclic Prüfer p-groups and of copies of $\mathbb{Q}$. □

Corollary 4.4 *Let A be a definably connected abelian group with the dcc on definable subgroups, and with no infinite elementary abelian p-subgroup. Then A is divisible.*

Proof. Let $A = DC$ be a decomposition of A as in Fact 4.3 (b), with D divisible and C of bounded exponent. For p any prime and any $n \geq 0$, let $C_{p^n} = \{c \in C \mid g^{p^n} = 1\}$. Since C has bounded exponent, the p-primary component of C has the form C_{p^N} for some N, and it is in particular definable. By assumption, C_p is finite. Considering the group homomorphism $x \longmapsto x^p$, we see by induction

on n that each subgroup C_{p^n} is finite. In particular, the p-primary component C_{p^N} of C is finite.

Suppose now towards a contradiction $D < A$. Since $A = DC$ and A is definably connected, we then get that C is infinite. Hence C is the direct product, for primes p varying in an infinite set I, of nontrivial finite p-groups, each of exponent p^N for some N depending on p. Now taking successive p^Nth powers of C, as p varies in I, we build an infinite descending chain of definable subgroups of C, as contradiction. Hence $A = D$ is divisible. $\square$

As a consequence of Fact 4.3 we also get a result of lifting of torsion. It can be compared to [3, Ex. 11 p. 93], which have had endless applications in the finite Morley rank case.

Fact 4.5 *Let G be a group with the dcc on definable subgroups, N a normal definable subgroup of G, and x an element of G such that x has finite order n modulo N. Then the coset xN contains an element of finite order, involving the same prime divisors as in n.*

Proof. The definable hull $H(x)$ is abelian by Fact 3.4. Replacing G by $H(x)$ and N by $N \cap H(x)$, we may assume G abelian. Now from Fact 4.3 we get a decomposition $N = D \oplus B$ where D is n-divisible and B is a direct sum of p-groups, for p prime dividing n (we may "transfer" from the original B to the original D given by Fact 4.3 (a) the Sylow q-subgroups for q not dividing n). Since x^n is in N, $x^n = d^n b$ for some d in D and some b in B. Now xd^{-1} is in the same N-coset as x, and since $(xd^{-1})^n = b$ the element xd^{-1} satisfies our requirement. $\square$

By [19, Theorem 5.1] (see also [27, Proposition 6.1]), any abelian group of bounded exponent *definable* in an o-minimal structure must be finite. Hence any definably connected abelian group definable in an o-minimal structure is divisible, by Fact 4.3 (b) or Corollary 4.4 (contrarily to the finite Morley rank case where infinite bounded exponent subgroups are a real possibility).

In the finite Morley rank case, Fact 4.3 (b) generalizes identically to nilpotent groups by a theorem of Nesin. For nilpotent groups G definable in o-minimal structures, we always have that G° is divisible by [7, Theorem 6.10]. In any case, all assumptions made in the next lemma, and in the reminder of this section, are met for groups definable in o-minimal structures.

Lemma 4.6 *Let G be a nilpotent group with the dcc on definable subgroups and with G° divisible. Let $\mathrm{Tor}\,(G)$ denote the set of torsion elements of G. Then*

(a) $\mathrm{Tor}\,(G)$ *is a subgroup of G, and it is the direct product of its Sylow p-subgroups. Furthermore it commutes with G°.*

(a′) $\mathrm{Tor}\,(G^\circ)$ *is in $Z^\circ(G^\circ)$, and in particular it is a divisible abelian subgroup of G°. Besides, any definable section of $G^\circ/Z^\circ(G^\circ)$ is torsion-free.*

(b) $G = G^\circ * B$ (central product) for some finite subgroup B of $\mathrm{Tor}\,(G)$.

(c) If G° has no infinite elementary abelian p-subgroup, then the subgroup B as in claim (b) may be chosen to be a finite characteristic subgroup.

Proof. (a) The fact that $\mathrm{Tor}\,(G)$ is a subgroup and the direct product of its Sylow p-subgroups is true in any nilpotent group ([25, 5.2.7]).

Since G° is p-divisible for any prime p, it commutes with any Sylow p-subgroup by [30, 4.2; 4.7; 6.13]. In particular $\mathrm{Tor}\,(G)$ commutes with G°.

(a') In particular, the set of torsion elements of G° forms a central subset of G°, and hence it is a central subgroup of G°. By divisibility of G°, $\mathrm{Tor}\,(G^\circ)$ is also a divisible (abelian) subgroup of G°. Hence any finite quotient of $\mathrm{Tor}\,(G^\circ)$ must be trivial, and thus $\mathrm{Tor}\,(G^\circ)Z^\circ(G^\circ)/Z^\circ(G^\circ)$ is trivial. Hence $\mathrm{Tor}\,(G^\circ) \leq Z^\circ(G^\circ)$. Our last claim follows then from the lifting of torsion given by Fact 4.5.

(b) Any extension of a locally finite group by a locally finite group is locally finite [14, Lemma 1.A.2 p. 2], and in particular any torsion solvable group is locally finite. It follows that $\mathrm{Tor}\,(G)$ is locally finite. But by lifting of torsion, Fact 4.5, we get that $G = G^\circ * \mathrm{Tor}\,(G)$, since G/G° is finite. Now we can also pick up finitely many representatives of finite orders of all cosets of G° in G; they generate a finite subgroup B of $\mathrm{Tor}\,(G)$ by local finiteness of the latter, and we also have that $G = G^\circ * B$.

(c) Let n be the order of B, or the minimum common multiple of orders of elements of B. Consider the set $\tilde{B}$ of elements g in $\mathrm{Tor}\,(G)$ such that $g^n = 1$. Clearly $B \subseteq \tilde{B}$. Any element g in $\tilde{B}$ has the form bh for some $b \in B$ and some element h in G°. Since $[b,h] = 1$, we see that $1 = g^n = (bh)^n = b^n h^n = h^n$.

Now the torsion subgroup of G° consists of direct products of quasicyclic Prüfer p-groups. Our extra assumption that G° contains no infinite elementary abelian p-subgroup for any prime p now implies that each p-primary component of G° is the direct product of at most finitely many copies of the quasicyclic Prüfer p-group. In particular, for n as above, there are only finitely many elements h in G° satisfying $h^n = 1$. Since B is finite, this shows that $\tilde{B}$ is also finite.

Since $\tilde{B}$ is a finite subset of $\mathrm{Tor}\,(G)$, we conclude by local finiteness of the latter that $\langle \tilde{B} \rangle$ is finite. Clearly, by definition, $\langle \tilde{B} \rangle$ is characteristic in G. So we may replace B by $\langle \tilde{B} \rangle$. $\qquad\square$

Corollary 4.7 *Let G be a definably connected group with the dcc on definable subgroups. Let H be a definable normal nilpotent subgroup of G, with H° divisible and without infinite elementary abelian p-subgroups. Then $H = (Z(G) \cap H)H^\circ$, and $\mathrm{Tor}\,(H) \leq Z(G)$.*

Proof. The first point follows from Lemma 4.6 (c) and Fact 3.1 (a).

For the second point, we have seen in the proof of Lemma 4.6 (c) that the set of elements of H° of order dividing n is finite, for every n, and thus Fact 3.1 (a) also gives that the torsion of H° is central in G. Now the full torsion

subgroup $\mathrm{Tor}\,(H)$ of H is in $Z(G)$ by Lemma 4.6 (c) and Fact 3.1 (a) as well.
□

Of course, Corollary 4.7 may apply with H the Fitting subgroup of G.

Corollary 4.8 *Let G be a definably connected group with the dcc on definable subgroups. Suppose that $F^\circ(G)$, which is definable and nilpotent by Fact 3.5, is divisible and without infinite elementary abelian p-subgroups. Then $F(G) = Z(G)F^\circ(G)$, and $\mathrm{Tor}\,(F(G)) \leq Z(G)$.*

Proof. Since $Z(G) \leq F(G)$, Corollary 4.7 applies directly. □

We conclude the present section with a flexible characterization of central extensions in the general context under consideration. Again, for the reasons mentioned before Lemma 4.6, everything here applies to any group definable in an o-minimal structure.

Lemma 4.9 *Let G be a definably connected group with the dcc on definable subgroups, and let H be a definable normal subgroup such that H° contains no infinite elementary abelian p-subgroups. Then $H \leq Z(G)$ if and only if $H^\circ \leq Z(G)$.*

Proof. Suppose $H^\circ \leq Z(G)$. Since H is finite modulo H°, Fact 3.1 (a) implies that $H/H^\circ \leq Z(G/H^\circ)$. Hence $[G, H] \leq H^\circ$, and in particular $H' \leq H^\circ$. But by assumption $H^\circ \leq Z(G)$, so in particular we get $H' \leq Z(H)$. So the definable subgroup H is 2-nilpotent. Besides, the definably connected abelian subgroup H° is divisible by Corollary 4.4. Now by Corollary 4.7 we get $H \leq Z(G)$. □

5 Groups with a definable and additive dimension

We continue as in the two previous sections with rather general considerations. We will see in the next section that this applies to groups definable in o-minimal structures. Throughout the present section, G is a group interpretable in a structure $\mathcal{M}$, and definability refers to $\mathcal{M}^{\mathrm{eq}}$. We suppose that to each definable set in Cartesian products of G is attached a dimension in $\mathbb{N}$ and denoted by dim. We essentially require the dimension to be definable and additive, and also require a natural additional axiom on the dimension of finite sets. Axioms are the following.

(A1) **(Definability)** If f is a definable function between two definable sets A and B, then for every m in $\mathbb{N}$ the set $\{b \in B \mid \dim(f^{-1}(b)) = m\}$ is a definable subet of B.

(A2) **(Additivity)** If f is a definable function between two definable sets A and B, whose fibers have constant dimension m in $\mathbb{N}$, then $\dim(A) = \dim(B) + m$.

(A3) **(Finite sets)** A definable set A is finite if and only if $\dim(A) = 0$.

Axioms A2 and A3 guarantee that if f is a definable bijection between two definable sets A and B, then $\dim(A) = \dim(B)$. In other words, they guarantee that the dimension is preserved under definable bijections.

Using axiom A2, one sees then that if $K \leq H \leq G$ are definable subgroups of G, then
$$\dim(H) = \dim(K) + \dim(H/K),$$
a special case of the additivity we will use freely throughout.

In general axioms A1-2 suffice for computing the dimensions of unions of uniformly definable families of sets (see [13, Fact 4 and Corollary 5] in the finite Morley context). Here we will merely use a computation of the dimension for products of definable subgroups.

Lemma 5.1 *Let G be a group equipped with a dimension satisfying axioms A1-3. If A and B are two definable subgroups of G, then $\dim(AB) = \dim(A) + \dim(B) - \dim(A \cap B)$.*

Proof. It suffices to consider the definable map $(a, b) \longmapsto ab$ from $A \times B$ onto AB. Its fibers are in definable bijection with $A \cap B$. $\qquad\square$

Clearly any group equipped with a dimension satisfying axioms A1-3, and which satisfies the *dcc* on definable subgroups of the same dimension, satisfies the *dcc* on all definable subgroups. Indeed, if $H_1 > H_2 > \cdots$ is a strictly descending chain of definable subgroups, then subchains of subgroups with the same dimension must be finite by assumption, and chains of subgroups with strictly decreasing dimensions must be finite as well.

In the general context of groups with a dimension satisfying axioms A1-3 we do not have a good version of Zilber's stabilizer argument, and consequently Zilber's generation lemma for indecomposable sets as in the finite Morley rank context (see [3, Theorem 5.26]). Our work here will consist mainly in reductions to the following lemma, which can be seen as a rudimentary version of Zilber's generation lemma.

Lemma 5.2 *Let G be a group equipped with a dimension satisfying axioms A1-3, and let A_i, $i \in I$, be an arbitrary family of definably connected definable subgroups of G which pairwise normalize each others. Then $\langle A_i \mid i \in I \rangle = A_{i_1} \cdots A_{i_k}$ for finitely many $i_1, \ldots i_k \in I$, and in particular it is definable. Furthermore we can always take k to be smaller than or equal to the dimension of that definable subgroup.*

Proof. First notice that the definable subgroups A_i of G can be definably connected even if G does not satisfy the *dcc*, as in the context of Fact 3.2.

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Any finite product $A_{i_1} \cdots A_{i_k}$ is a definably connected and definable subgroup by Fact 3.2. Now one sees with Lemma 5.1 that any such finite product $A_{i_1} \cdots A_{i_k}$ of maximal dimension is equal to $\langle A_i \mid i \in I \rangle$.

Once we know that $\langle A_i \mid i \in I \rangle = A_{i_1} \cdots A_{i_k}$ is definable (and definably connected), we see again with Lemma 5.1 that we may choose $k \leq \dim(\langle A_i \mid i \in I \rangle)$. $\qquad \square$

Question 5.3 *Can one relax the normalization hypothesis in Lemma 5.2, assuming the ambient group, say, nilpotent?*

The following corollary of Lemma 5.2 is going to be the basic tool for the proof of our main theorem.

Corollary 5.4 *Let G be a group equipped with a dimension satisfying axioms A1-3, H a definably connected definable subgroup, and X an arbitrary subset of G. Suppose that $[H, X]$ is abelian and centralized by H. Then $[H, X]$ is a definably connected definable (abelian) subgroup of $C(H)$, and actually we have $[H, X] = [H, X]_{\dim([H,X])}$*

Proof. Fix any element x in X. Then the map $\mathrm{ad}_x : H \longrightarrow G$ is a group homomorphism by Fact 2.3, and clearly it is definable. Hence its image $[H, x]$ is a definably connected definable subgroup of G.

Now all such subgroups $[H, x]$ pairwise normalize each other since they all live in $[H, X]$ which is abelian. But $[H, X]$ is generated by these subgroups $[H, x]$, and hence it suffices now to apply Lemma 5.2. It implies that $[H, X]$ is a definably connected definable (abelian) subgroup of G.

Our last remark about the commutator width of $[H, X]$ corresponds to the last claim in Lemma 5.2. $\qquad \square$

If G is any group, $R(G)$ denotes the subgroup generated by all normal solvable subgroups of G and is called the *solvable radical* of G. We start with a general remark about solvable radicals in arbitrary definably connected groups.

Remark 5.5 *Let G be any definably connected group. If $R(G)$ is finite, then $R(G) = Z(G)$ and $G/R(G)$ has a trivial solvable radical.*

Proof. The first point follows from Fact 3.1 (a) and the second is obvious once we know that $R(G)$ is solvable. $\qquad \square$

We now prove the definability and the solvability of the solvable radical in a rather abstract context, which incorporates both groups of finite Morley rank and groups definable in o-minimal structures.

Lemma 5.6 *Let G be a group equipped with a dimension satisfying axioms A1-3 and with the dcc on definable subgroups of same dimension. Then $R(G)$ is definable and solvable.*

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Proof. By assumption G satisfies the *dcc* on all definable subgroups.

By Remark 3.3 and Fact 3.4, $R(G)$ is generated by the set of *definable* normal solvable subgroups of G, so it suffices to show that the group generated by such definable normal solvable subgroups is solvable. By Lemma 5.2, the product of all *definably connected* definable normal solvable subgroups of G is the product of finitely many of them, and hence it is definable and solvable. Let K denote this maximal definably connected definable normal solvable subgroup of G (which, of course, is going to be $R^{\circ}(G)$).

Any definable normal solvable subgroup N of G has a finite image in G/K. So replacing G by G/K we may assume that all normal solvable subgroups are finite. If N is such a normal solvable subgroup, then as N is normalized by G° we get that $N \leq C_G(G^{\circ})$ by Fact 3.1 (a). But $C_G(G^{\circ})$ is a finite subgroup of G, as otherwise $C_G^{\circ}(G^{\circ})$ is a nontrivial definably connected definable subgroup of G, hence in G°, and then in the center of G° which is finite, a contradiction. Hence any normal solvable subgroup of G is in the finite (normal) subgroup $C_G(G^{\circ})$ of G, and in particular in $R(C_G(G^{\circ}))$, which is solvable. This completes the proof. $\qquad\square$

Remark 5.7 *Lemma 5.6 generalizes the definability and the solvability of $R(G)$, noticed by Belegradek in the finite Morley rank context. As we will see in the next section that groups definable in o-minimal structures satisfy all our current assumptions, it also gives the definability and the solvability of the full solvable radical $R(G)$ in this context. We note that in the literature on groups in o-minimal structures it is usually only $R^{\circ}(G)$ which is defined, essentially by the same argument as in the second paragraph of the proof of Lemma 5.6. See for example [5, Fact 2.3].*

We finish the present section with a mere definition. If G is a group as in Lemma 5.6, then $R(G)$ is a solvable definable subgroup, and $F(G)$ is a nilpotent definable subgroup by Fact 3.5. Of course, we have

$$F(G) \trianglelefteq R(G) \trianglelefteq G$$

and the same inclusions hold with definably connected components. If G is definably connected and $R(G)$ is finite (equivalently $R^{\circ}(G) = 1$), then Remark 5.5 shows that $Z(G) = F(G) = R(G)$ and $G/R(G)$ has no nontrivial normal abelian subgroup. In this case G can have only finite normal abelian subgroups (in $Z(G)$).

The definition of semisimplicity for groups may vary in the literature, depending on whether authors admit a finite center or not. Here we will admit such finite centers in our definition of semisimplicity.

Definition 5.8 *A definably connected group G as in Lemma 5.6 is said to be semisimple whenever $R^{\circ}(G) = 1$. By Remark 5.5, this is equivalent to requiring that $Z(G) = R(G)$ is finite and $G/R(G)$ has trivial abelian normal subgroups.*

6   Groups definable in o-minimal structures

From now on we specialize our analysis to groups *definable* in o-minimal structures. Here we insist on definability, as opposed to interpretability, because since [23] the whole theory of groups in o-minimal structures has been developed in this restricted context. Hence the framework from now on is the following: $\mathcal{M}$ is an o-minimal structure and G is a group definable in $\mathcal{M}$. In general the o-minimal structure $\mathcal{M}$ itself may not eliminate imaginaries, but in some sense G does. In fact, G has *strong definable choice* in the following sense.

Fact 6.1 [7, Theorem 7.2] *Let G be a group definable in an o-minimal structure $\mathcal{M}$, and let $\{T(x) \mid x \in X\}$ be a definable family of non-empty definable subsets of G. Then there is a definable function $t : X \longrightarrow G$ such that for all $x, y \in X$ we have $t(x) \in T(x)$ and if $T(x) = T(y)$ then $t(x) = t(y)$.*

In particular, a definable section H/K of a group G definable in an o-minimal structure $\mathcal{M}$ can also be regarded as definable in $\mathcal{M}$, even though it has a priori to be considered as definable in $\mathcal{M}^{\mathrm{eq}}$. Hence when making proofs by induction below we will freely pass to definable sections of groups definable in o-minimal structures, and still consider them as definable.

As seen already in Sections 3 and 4, groups definable in o-minimal structures satisfy the *dcc* on definable subgroups, and all the results of these two sections apply to them. Similarly, all results of Section 5 apply to groups definable in o-minimal structures: for the definition of the dimension of definable subsets, which uses the cell decomposition, we refer to [28]. The definability and the additivity (our axioms A1 and A2 respectively) can be found in [28, Chapter 4 (1.5)], and the fact that finite sets are exactly the 0-dimensional sets (our axiom A3) is noticed in [28, Chapter 4 (1.1)].

A group is *definably simple* if it is nonabelian and has no proper nontrivial normal definable subgroup (in the specific universe considered, typically our ground o-minimal structure here). As seen at the end of Section 5, if G is a group definable in an o-minimal structure then $R(G)$ is definable and solvable. When G is definably connected, then $G/R^\circ(G)$ is semisimple and $G/R(G)$ has a nice description, following a solution of a version of the Cherlin-Zilber conjecture for groups in o-minimal structures.

Fact 6.2 [20, Theorem 4.1] *Let G be an infinite definably connected group definable in an o-minimal structure, with $R(G) = 1$. Then G is the direct product of definable definably simple subgroups $G_1, ..., G_k$.*

For Fact 6.2, see also [20, Remark 4.2]. There is in fact much more information known about the (finitely many) definably simple factors G_i appearing in Fact 6.2, and we can see them in at least two ways. On the one hand, for any definably simple group G definable in an o-minimal structure $\mathcal{M}$, there is a definable real closed field $\mathcal{R}$ and a real algebraic group H defined over the subfield of real algebraic numbers $R_{alg} \subseteq R$, such that G is definably isomorphic

in $\mathcal{M}$ to $H(R)^\circ$, the definably connected component of $H(R)$ (See [20, 4.1] for the existence of H and the proof of [22, 5.1] for the fact that H can be defined over R_{alg}; see also [11, Fact 1.1]). On the other hand, we can see G as a group elementarily equivalent to a simple (centerless) Lie group by [22, Theorem 5.1].

We now prove that such definably simple groups have finite commutator width. Recall that a group G is *perfect* if $G' = G$.

Fact 6.3 *Let G be a definably simple group definable in an o-minimal structure. Then $G = [G,G]_k$ for some k, and in particular $G = G'$ is perfect.*

Proof. Clearly, it suffices to prove the first statement.

If G is finite, then of course the definable simplicity of G implies its abstract group theoretic simplicity. Since G' is a nontrivial normal subgroup, we get $G = G' = [G,G]_k$ for some k.

Suppose now G infinite. Our assumptions pass to elementary extensions, so taking a sufficiently saturated elementary extension of the ground o-minimal structure, we can work in a sufficiently saturated elementary extension G^* of G. If we show there that $G^{*\prime} = [G^*, G^*]_k$ for some k, then $G^{*\prime}$ is definable, and $G^* = G^{*\prime} = [G^*, G^*]_k$ by definable simplicity of G^*, showing then by elementary equivalence that $G = [G,G]_k$ as well. So we may assume that G is already sufficiently saturated (actually ω-saturated suffices in what follows).

Consider first the case G not definably compact. Then by [22, Corollary 6.3] G is abstractly simple as a group. Since G' is a nontrivial normal subgroup of G, we then get $G = G'$ by abstract simplicity of G. Now G is the countable union of the definable sets $[G,G]_k$, so the ω-saturation of G implies that $G = [G,G]_k$ for some k. Hence in this case we are done.

Consider now the case G definably compact. As commented after Fact 6.2, G is definably isomorphic to a semialgebraic group over a real closed field $\mathcal{R}$ which is definable in our original o-minimal structure (but may be different from it). Hence we can see G as a definably compact group definable over an o-minimal expansion of an ordered field, and thus in particular over an o-minimal expansion of an ordered group. Then we can get $G' = [G,G]_k$ for some k as in the proof of [11, Corollary 6.4 (i)], and in this case we are done also. $\square$

Question 6.4 *Let G be a definably simple group definable in an o-minimal structure. Then is it the case that $[G,G] = [G,G]_{\dim(G)}$?*

We strongly believe in a positive answer to Question 6.4 but unfortunately we haven't found any model-theoretic proof. Actually, it seems even expected that the commutator width should be 1 in Question 6.4, or in other words that every element is a commutator. This corresponds to a conjecture of Ore in the case of *finite* simple groups (proved recently, by cases inspection [15]). As commented after Fact 6.2, there are two ways to see a definably simple group as in Question 6.4, either as the semialgebraic connected component of a real algebraic group or as (elementarily equivalent to) a simple Lie group. When G is definably compact, then it is known that every element is a commutator by an

older theorem of Gotô about semisimple Lie groups [9] (see also [10, Theorem 6.55]). Hence the question reduces to abstractly simple groups, as seen in the proof of Fact 6.3. In general it remains open but it is explicitly conjectured, from the simple Lie groups point of view, that $G = [G, G]_1$ in [18, Conjecture A]. See also [2, Page 15] for the same question and links with the Cartan decomposition. The reader can also consult [8] for a proof of the conjecture of Ore for groups of algebraic or Chevalley type.

We finish this section with another general remark about the structure of a group definable in an o-minimal structure. It is not used in the rest of the paper, but it is certainly worth to mention it at this point.

Remark 6.5 *Let G be any group definable in an o-minimal structure. By lifting of torsion, Fact 4.5, any definable subgroup without torsion must be definably connected, and the product of two such groups A and B which normalize each other is also torsion-free: if AB contains a torsion element, then the same occurs in $AB/A \simeq B/(A \cap B)$. Now Lemma 5.2 shows the existence of a unique maximal definable normal torsion free subgroup of G, the group generated by all of them. This gives a somewhat conceptual proof of the existence of such a subgroup, noticed recently in [5] using more specific machinery involving Euler characteristic.*

7 Definably connected solvable groups

In the present section we consider the structure of *solvable* definably connected groups definable in o-minimal structures. We first note that there is an o-minimal version of the Lie-Kolchin-Mal'cev theorem.

Fact 7.1 [7, Theorem 6.9] *Let G be a definably connected solvable group definable in an o-minimal structure. Then G' is nilpotent.*

As in [17] in the finite Morley rank case, the proof of Fact 7.1 is based ultimately on linearization. It is a question whether a version of Fact 7.1 can be proved under more general assumptions such as those used in Sections 3, 4, or 5. We also note that our main theorem implies that G' is definable in Fact 7.1 (Corollary 8.8 below), but that Fact 7.1 was proved before knowing this.

Recall that the Fitting subgroup is nilpotent and definable for any group definable in an o-minimal structure by Fact 3.5.

Proposition 7.2 *Let G be a definably connected solvable group definable in an o-minimal structure. Then $G' \leq F^\circ(G)$. In particular $G/F^\circ(G)$ and $G/F(G)$ are divisible abelian groups.*

Proof. Notice that both quotients $G/F^\circ(G)$ and $G/F(G)$ can then be considered as definable by Fact 6.1, and are both definably connected as G is.

By Fact 7.1, $G' \leq F(G)$. It follows that $G'F^\circ(G)/F^\circ(G)$ is finite. Now $(G/F^\circ(G))' = G'F^\circ(G)/F^\circ(G)$ is finite, and then it follows from Fact 3.1 (b)

that the definably connected group $G/F^\circ(G)$ must be abelian. Now it follows that $G' \leq F^\circ(G)$, and $G/F^\circ(G)$ is abelian, as well as $G/F(G)$.

The divisibility of $G/F^\circ(G)$ and $G/F(G)$ is true for any definably connected abelian group definable in an o-minimal structure, as mentioned before Lemma 4.6. $\qquad\square$

Corollary 7.3 *Let G be a nontrivial definably connected solvable group definable in an o-minimal structure. Then $F^\circ(G)$ is nontrivial. In particular G has an infinite abelian characteristic definable subgroup.*

Proof. Our statement is clear if G is abelian, so suppose G nonabelian. Fact 3.1 (b) implies that G' is then infinite. Now Proposition 7.2 shows that $F^\circ(G)$ is infinite as well. For the last claim we can use the fact that $F(G)$ has an infinite center, which follows from Fact 3.5 and Lemma 4.1. $\qquad\square$

We are not going to use the following lemma, but it is very similar to an analog fact about connected solvable groups of finite Morley rank and which is crucial for a finer inductive analysis of such groups. We expect it should have similar consequences for definably connected solvable groups definable in o-minimal structures.

If H and G are two subgroups of a group with G normalizing H, then a G-*minimal* subgroup of H is an infinite G-normal definable subgroup of H, which is minimal with respect to these properties (and where definability refers to a fixed underlying structure). If H is definable and satisfies the *dcc* on definable subgroups, then G-minimal subgroups of H always exist. As the definably connected component of a definable subgroup is a definably characteristic subgroup, we get also in this case that any G-minimal subgroup of H should be definably connected.

Lemma 7.4 *Let G be a definably connected solvable group definable in an o-minimal structure, and A a G-minimal subgroup of G. Then $A \leq Z^\circ(F(G))$, and $C_G(a) = C_G(A)$ for every nontrivial element a in A.*

Proof. By Corollary 7.3, A has an infinite characteristic abelian definable subgroup. Therefore the G-minimality of A forces A to be abelian. In particular, $A \leq F(G)$. Since A is normal in $F(G)$, Lemma 4.1 and the G-minimality of A now force that $A \leq Z(F(G))$. Since A is definably connected, we have indeed $A \leq Z^\circ(F(G))$.

Now $F(G) \leq C_G(A)$, and $G/C_G(A)$ is definably isomorphic to a quotient of $G/F(G)$. In particular $G/C_G(A)$ is abelian by Proposition 7.2. If $A \leq Z(G)$, then clearly $C_G(a) = C_G(A)$ $(= G)$ for every a in A, and thus we may assume $G/C_G(A)$ infinite. Consider the semidirect product $A \rtimes (G/C_G(A))$. Since A is G-minimal, A is also $G/C_G(A)$-minimal. Now an o-minimal version of Zilber's Field Interpretation Theorem for groups of finite Morley rank [21, Theorem 2.6] applies directly to $A \rtimes (G/C_G(A))$. It says that there is an infinite interpretable

field K, with $A \simeq K_+$ and $G/C_G(A)$ an infinite subgroup of $K^\times$, and such that the action of $G/C_G(A)$ on A corresponds to scalar multiplication. In particular, $G/C_G(A)$ acts *freely* (or *semiregularly* in another commonly used terminology) on $A \setminus \{1\}$. This means exactly that for any nontrivial element a in A, $C_G(a) \leq C_G(A)$, i.e., $C_G(a) = C_G(A)$. □

The following lemma will be used in a reduction to the solvable case in our proof of the main theorem. Of course, it becomes trivial once we now that G' is definable and definably connected in definably connected solvable groups (Corollary 8.8 below).

Lemma 7.5 *Let G be a definably connected solvable group definable in an o-minimal structure. If G is not abelian, then $G' \leq H < G$ for some definably connected and definably characteristic proper subgroup H definable in G.*

Proof. If $F^\circ(G) < G$, then it suffices to take $H = F^\circ(G)$ by Fact 3.5 and Proposition 7.2.

Suppose now $G = F^\circ(G)$, i.e., G nilpotent, and assume G nonabelian. Then G is nilpotent of class n for some $n \geq 2$. But now by Fact 2.4 $G = Z_n(G)$ and $Z_{n-1}(G) < G$, and we also have $G' \leq Z_{n-1}(G)$. Clearly, all groups $Z_i(G)$ are definable. Now one sees as in the proof of Proposition 7.2 that $G/Z_{n-1}^\circ(G)$ has a finite derived subgroup, which must then be trivial by Fact 3.1 (*b*). Hence $G' \leq Z_{n-1}^\circ(G) < G$, and we may take $H = Z_{n-1}^\circ(G)$. □

Corollary 7.6 *Let G be a group definable in an o-minimal structure, with G° solvable. Then G satisfies the assumption $(*)$ of Definition 1.2.*

Proof. By Lemma 7.5, G° cannot have nonabelian definably simple definable sections. □

8 Main theorem

We have now enough background to pass to the proof of our main result. We first prove a version of Theorem 1.3 for A and B definably connected.

Theorem 8.1 *Let G be a group definable in an o-minimal structure, A and B be two definably connected definable subgroups which normalize each other and such that AB satisfies the assumption $(*)$. Then $[A, B]$ is a definably connected definable subgroup, and moreover $[A, B] = [A, B]_{\dim([A,B])}$ whenever A or B is solvable.*

Proof. Notice that $[A, B] \leq A \cap B$ since A and B normalize each other . We proceed by induction on

$$d := \min(\dim(A), \dim(B)).$$

Notice that when $d = 0$ we have A or B trivial, by definable connectedness, and in particular $[A, B]$ is trivial as well. So the induction starts with $d = 0$. Assume from now on that G is a potential counterexample to our statement, with $d \geq 1$ minimal. We start with a series of reductions. Notice that since AB satisfies the assumption $(*)$, every definable section of AB does also.

Claim 8.2 *Let C be a definable subgroup of $A \cap B$ normalized by A and B and with $\dim(C) < d$. If $C \not\leq Z(A) \cap Z(B)$, then $[A, B]$ is definable and definably connected, and $[A, B] = [A, B]_{\dim([A,B])}$ whenever A or B is solvable.*

Proof. Suppose that C does not centralize one of the two groups A or B. By symmetry we may assume, for instance, $C \not\leq Z(B)$. We now have $C^\circ \not\leq Z(B)$ by Lemma 4.9. In particular $[C^\circ, B]$ is nontrivial. Since $\dim(C) < d$, our inductive assumption implies that $[C^\circ, B]$ is definable and definably connected, and that $[C^\circ, B] = [C^\circ, B]_{\dim([C^\circ, B])}$ whenever A or B is solvable (since A or B solvable implies C° solvable as $C \leq A \cap B$).

Notice that $[C^\circ, B]$ is normal in both A and B. We now work in $N(U)/U$, where $U = [C^\circ, B]$, and denote by the notation " $\overline{}$ " the quotients by U. Since $\dim(U) \geq 1$, $\overline{A}$ and $\overline{B}$ have dimensions strictly smaller than $\dim(A)$ and $\dim(B)$ respectively. Applying our inductive assumption in $N(U)/U$, we get $[\overline{A}, \overline{B}]$ definable and definably connected, and $[\overline{A}, \overline{B}] = [\overline{A}, \overline{B}]_{\dim([\overline{A}, \overline{B}])}$ whenever A or B is solvable.

But clearly

$$[\overline{A}, \overline{B}] = \overline{[A, B]} = [A, B]/U$$

since $U \leq [A, B]$, and it follows that $[A, B]$ is definable and definably connected.

By additivity of the rank, we now get that

$$\dim([A, B]) = \dim(\overline{[A, B]}) + \dim(U).$$

Hence, whenever A or B is solvable, we get also that $[A, B] = [A, B]_{\dim([A,B])}$ by the additivity of the commutator width provided by Lemma 2.1. □

Claim 8.3 *We may assume $A \cap B \not\leq Z(A) \cap Z(B)$, and $\dim(A \cap B) = d$.*

Proof. If $(A \cap B) \leq Z(A) \cap Z(B)$, then in particular $[A, B] \leq Z(A) \cap Z(B)$ and Corollary 5.4 would give all the conclusions of Theorem 8.1 for $[A, B]$. Since we are dealing with a potential counterexample, we may thus assume $A \cap B \not\leq Z(A) \cap Z(B)$.

Suppose now $\dim(A \cap B) < d$. Then we may apply Claim 8.2 with $C = A \cap B$. Since we are now assuming $C \not\leq Z(A) \cap Z(B)$, it would give all the conclusions of Theorem 8.1 for $[A, B]$. As above, we may thus assume $\dim(A \cap B) = d$. □

By Claim 8.3, we now have $\dim(A \cap B) = \min(\dim(A), \dim(B))$ and since A and B are definably connected it follows that $A \cap B$ equals A or B. By symmetry, we may assume $A \cap B = A$, or in other words

$$A \leq B.$$

Notice that the first part of Claim 8.3 now says that $A \not\leq Z(A) \cap Z(B)$.

Claim 8.4 *We may assume $A = B$.*

Proof. We have to show that if the conclusions of Theorem 8.1 are valid for $[A, A] = A'$, then they are also valid for $[A, B]$. So suppose $[A, A]$ definable and definably connected, and $[A, A] = [A, A]_{\dim(A')}$ whenever A is solvable.

Working modulo A' (which is normal in B), we may assume A abelian. Then $[A, B] \leq A = Z(A)$ and Corollary 5.4 gives that $[A, B] = [A, B]_{\dim([A,B])}$ is definable and definably connected.

We can now come back to the original groups A and B, i.e., not divided by A'. Since $A' \leq [A, B]$, we deduce as in the proof of Claim 8.2 the definability and definable connectedness of $[A, B]$. Since $A \leq B$, A is solvable whenever A or B is. Hence we also deduce as in the proof of Claim 8.2, using the additivity of the dimension and of the commutator width given by Lemma 2.1, that $[A, B] = [A, B]_{\dim([A,B])}$ whenever A or B is solvable. $\square$

After the preceding series of reductions we are now in the situation

$$A = B$$

and Claim 8.2, which of course is still valid, now takes the following form. Any proper definable normal subgroup C of A is central in A, as otherwise we have finished the proof of Theorem 8.1. We split our final analysis in two cases.

Case A solvable. Suppose first A solvable. If A is abelian we have of course nothing to do, hence we may suppose A nonabelian. By Lemma 7.5, $A' \leq C$ for some proper normal definably connected definable subgroup C, and since we may assume $C \leq Z(A)$ we have $A' \leq Z(A)$ (i.e., A is 2-nilpotent). Now Corollary 5.4 (applied with $H = X = A$) gives all the desired conclusions again in this case, including the control of the commutator width: $[A, A] = [A, A]_{\dim([A,A])}$.

Case A nonsolvable. Suppose now A non-solvable, i.e., $R(A) < A$. Notice that in this case we just want to conclude to the definability and the definable connectedness of A', and we don't care about its commutator width.

Notice that we actually may suppose with Claim 8.2 that A is indecomposable in the following sense: it cannot be the product $R_1 R_2$ of two proper definable normal subgroups R_1 and R_2. Otherwise, since we may assume R_1, $R_2 \leq Z(A)$, we would get $A = R_1 R_2$ abelian, which is excluded at this stage.

By the structure of semisimple groups definable in o-minimal structures, Fact 6.2, we get that $A/R(A)$ consists of a single definably simple (and nonabelian) factor.

Notice also that since we may assume the proper normal definable solvable radical $R(A)$ central in A, we have also $R(A) \leq Z(A)$, so that $R(A) = Z(A)$.

To summarize, A is a central extension of a nonabelian definably simple group $A/R(A)$.

By Fact 6.3, $A/Z(A) = [A/Z(A), A/Z(A)]_k$ for some k, and lifting commutators one gets

$$A = [A, A]_k \cdot Z(A)$$

and in particular $A = A' \cdot Z(A)$. We distinguish two final subcases.

Subcase $R(A) = Z(A)$ **finite.** In this case we clearly have $[A, A] \cap Z(A) = [A, A]_s \cap Z(A)$ for some finite $s \geq 1$. Now $A' = [A, A] = [A, A]_{k+s}$ by Lemma 2.1, and in particular A' is definable. Since $A = A'Z(A)$ and $Z(A)$ is finite, A' has finite index in A. Now since A is definably connected we conclude that $[A, A] = A' = A$ is definable and definably connected, as desired.

Subcase $R(A) = Z(A)$ **infinite.** In this final case A is a strict central extension of a nonabelian definably simple group. Since our original group AB satisfied the assumption $(*)$, the same is true for A at this stage (recall that in our preparatory reductions we only took definable sections). Hence, the assumption $(*)$ now says that A' is a definable subgroup of A. Notice that A' cannot be finite, as otherwise A is abelian by Fact 3.1 (b). If $A' < A$, then our general application of Claim 8.2 implies that we may assume $A' \leq Z(A)$, which is excluded since A is not solvable. Remains only the case in which $[A, A] = A' = A$ is definable and definably connected, as desired.

This finishes the proof of Theorem 8.1 in all cases. $\square$

We now derive from Theorem 8.1 a version of Theorem 1.3 with one of the two groups A or B definably connected.

Corollary 8.5 *Let G be a group definable in an o-minimal structure, A and B be two definable subgroups which normalize each other with $A = A^\circ$ and such that AB° satisfies the assumption $(*)$. Then $[A, B]$ is a definably connected definable subgroup, and moreover $[A, B] = [A, B]_{\dim([A,B])}$ whenever A or B° is solvable.*

Proof. We have $[A, B]$ and $[A, B^\circ]$ normal in both A and B.

By Theorem 8.1 we have $[A, B^\circ]$ definable and definably connected, and $[A, B^\circ] = [A, B^\circ]_{\dim([A,B^\circ])}$ whenever A or B° is solvable.

Working in $N([A, B^\circ])/[A, B^\circ]$, and denoting by the notation "$\overline{}$" the quotients by $[A, B^\circ]$, we may assume that $[\overline{A}, \overline{B^\circ}] = 1$. Now the definably connected group $\overline{A}$ acts by conjugation on the finite quotient $\overline{B}/\overline{B^\circ}$, and by Fact 3.1 (a) this

action must be trivial. This shows that $[\overline{A}, \overline{B}] \leq \overline{B^\circ}$. Hence $[\overline{A}, \overline{B}] \leq \overline{A} \cap \overline{B^\circ}$, and since $\overline{A}$ and $\overline{B^\circ}$ commute we get that

$$[\overline{A}, \overline{B}] \leq Z(\overline{A}).$$

We may now apply Corollary 5.4 (with $H = \overline{A}$ and $X = \overline{B}$), and it gives that $[\overline{A}, \overline{B}]$ is a definably connected definable (abelian) subgroup of $Z(\overline{A})$, and furthermore that

$$[\overline{A}, \overline{B}] = [\overline{A}, \overline{B}]_{\dim([\overline{A}, \overline{B}])}.$$

Since $[A, B^\circ] \leq [A, B]$ and $[\overline{A}, \overline{B}] = [A, B]/[A, B^\circ]$ by lifting of commutators, the first point implies that $[A, B]$ is definable and definably connected.

Now $\dim([A, B]) = \dim([\overline{A}, \overline{B}]) + \dim([A, B^\circ])$ by additivity of the dimension. If A or B° is solvable, then as in Claim 8.2 we get that

$$[A, B] = [A, B]_{\dim([\overline{A}, \overline{B}]) + \dim([A, B^\circ])} = [A, B]_{\dim([A, B])}$$

with Lemma 2.1. $\qquad\square$

Remark 8.6 *In Corollary 8.5 we see from the proof that $[A, B]/[A, B^\circ]$ is abelian. In general it need not be trivial, or in other words it is not necessarily the case that $[A, B] = [A, B^\circ]$. Consider for example the following semidirect product $B = A \rtimes \langle i \rangle$ where A is a divisible abelian group and i is an element of order 2 acting by inversion on A. One may of course realize a group definable in an o-minimal structure of this isomorphism type. Clearly $A = B^\circ$ is definably connected in such a group. We have $[A, B^\circ] = 1$, but $[A, B] = A$.*

We now prove Theorem 1.3 without assuming any of the two groups A or B definably connected.

Proof of Theorem 1.3. Since AB is a definable subgroup of G, we may assume $G = AB$, with A and B two normal subgroups. All subgroups $[A, B]$, $[A, B^\circ]$, and $[A^\circ, B]$ are normal, in A and B, and by Corollary 8.5 the two latter subgroups are definable and definably connected. By Fact 3.2, the normal definable subgroup

$$C := [A^\circ, B][A, B^\circ]$$

of $[A, B]$ is definably connected.

By Corollary 8.5 we also have, as far as the commutator width is concerned, $[A^\circ, B] = [A^\circ, B]_{\dim([A^\circ, B])}$ whenever A° or B° is solvable. Working modulo $[A^\circ, B]$, we also find similarly as in Claim 8.2 or Corollary 8.5 (essentially by the additivity of the dimension, Lemma 2.1, and using Corollary 8.5 modulo $[A^\circ, B]$) that the commutator width of C is bounded by $\dim(C)$ whenever A° or B° is solvable.

Working modulo C, we have $A^\circ \leq C_A(B)$ and $B^\circ \leq C_B(A)$, and Lemma 2.2 gives the finiteness of $[A, B]$ modulo C. In other words, $[A, B]/C$ is finite, and

since $[A, B]$ is a finite extension of the definably connected definable subgroup C, we have $[A, B]$ definable, and $[A, B]^\circ = C$.

This completes our proof of Theorem 1.3. $\qquad\square$

Remark 8.7 *Clearly, by the last paragraph of the proof of Theorem 1.3 and Lemma 2.1, we also have $[A, B]$ of finite commutator width in Theorem 1.3 whenever A° or B° is solvable. We do not provide explicit bounds here, since it depends on the commutator width of the finite section $[A, B]/[A, B]^\circ$.*

Applying inductively Theorem 1.3, we get the following corollary. Recall that it applies to groups G with G° solvable, by Corollary 7.6.

Corollary 8.8 *Let G be a group definable in an o-minimal structure such that G° satisfies the assumption $(*)$. Then, for each $n \geq 1$, G^n and $G^{(n)}$ are definable, and definably connected whenever G is.*

In principle, the commutator widths of the groups $[G^n]^\circ$ and $[G^{(n)}]^\circ$ in Corollary 8.8 are also controlled by their dimension whenever G° is solvable.

We finish this section with two types of questions.

Question 8.9 (The normalization assumption) *Can one prove versions of Theorem 8.1, Corollary 8.5, and Theorem 1.3, with A normalizing B only, instead of A and B normalizing each other?*

A reasonable approach to Question 8.9 would be to proceed as in the proof of Theorem 8.1, but now by induction on $\dim(B)$. Unfortunately, we haven't been able to make it work.

Other natural questions arising concern the control of the commutator width. The first one concerns the commutator width in the definably connected case of Theorem 8.1 (and similarly Corollary 8.5).

Question 8.10 (The commutator width, I) *Can one get rid of the solvability assumption on A or B for the control of the commutator width of $[A, B]$ in Theorem 8.1.*

We see from the proof of Theorem 8.1 that a positive answer to Question 8.10 would only require a uniform control on the commutator width of A' in the final case "A nonsolvable" of that proof. More precisely, the question boils down to the following situation. If A is a definably connected group definable in an o-minimal structure, with $R(A) = Z(A)$ and $A/R(A)$ (nonabelian) definably simple, and such that A' is definable, then can one bound the commutator width of $[A, A]$? This would a priori require a control of the commutator width of the definably simple group $A/R(A)$, which is also perfect and of finite commutator width by Fact 6.3. Here, the problem relates to Question 6.4 and has probably a positive answer, possibly with a very small commutator width, as commented

after Question 6.4. But even if this turned out to be true, one might then need to apply Lemma 2.1 to relate the commutator width of A' to that of $(A/Z(A))'$, and thus to bound the number s such that $A' \cap Z(A) = [A, A]_s \cap Z(A)$. Even when $Z(A)$ is finite, it is not clear whether the are such uniform bounds (as A varies in the class of such central extensions).

The second natural question about the commutator width concerns the "finite bits" modulo the connected components. It can be formulated as follows.

Question 8.11 (The commutator width, II) *Fix G a group as in Theorem 1.3. Is there a uniform bound on the commutator width of the full group $[A, B]$, "modulo" the commutator width of $[A, B]^\circ$, as A and B vary as in that theorem?*

A positive answer to Question 8.11 would only require to know that, for G fixed, the commutator width of the finite section $[A, B]/[A, B]^\circ$ is uniformly bounded as long as A and B vary in the set of all definable subgroups of G satisfying the assumption $(*)$ (Remark 8.7). By the specific structure of G following Fact 6.2 ($G^\circ/R(G^\circ)$ is a direct product of definably simple groups) and a closer look at finite sections of the solvable group $R(G^\circ)$, it might be true.

9 Further questions and remarks

In any group G of finite Morley rank we have the following: if H is any definable and definably connected subgroup, and X an arbitrary subset of G, then $[X, H]$ is definable and definably connected [3, Corollary 5.29]. Of course the proof of this general statement uses the definition of the Morley rank in a crucial way via Zilber's stabilizer argument, but one may wonder when such stronger versions of Corollary 8.5 remain valid in the context of groups definable in o-minimal structures.

Question 9.1 *In a group definable in an o-minimal structure, let H be a definable and definably connected subgroup and X an arbitrary subset. When is it true that $[H, X]$ is definable and definably connected?*

Corollary 5.4 remains our most general answer to Question 9.1. On the other hand, Question 9.1 may fail even if X is a normal subgroup of H, as the following example shows.

Example 9.2 *Let H be a definably compact definably connected definably simple nonabelian group definable in an o-minimal structure. Taking a sufficiently saturated model, we know that the smallest type-definable subgroup $X = H^{\circ\circ}$ of bounded index in H is normal, proper, and nontrivial (by [24, 2.12], and [24, 3.6] or [1]). Then $[H, X]$ is not definable. Otherwise, the proper normal definable subgroup $[H, X] \le X < H$ would be trivial by definable simplicity of H, implying the existence of a nontrivial center in H.*

We manage however to treat the case $H \subseteq X$ as a corollary of Theorem 8.1 as follows.

Corollary 9.3 *Let G be a group definable in an o-minimal structure, H a definably connected definable subgroup which satisfies the assumption $(*)$, and X any subset such that $H \subseteq X \subseteq N(H)$. Then $[H, X]$ is a definable and definably connected subgroup of H.*

Proof. The group H' is definable and definably connected by Theorem 8.1. Since H' is normalized by H and X, we may replace G by $N(H')$, and working modulo H' we may assume H abelian. Now, since $[H, X] \leq H = Z(H)$, Corollary 5.4 gives the definability and definable connectedness of $[H, X]$. $\square$

Of course, we can also obtain $[H, X] = [H, X]_{\dim([H,X])}$ as in the proof of Claim 8.4 when H is solvable in Corollary 9.3.

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