EXPONENTIATIONS OVER THE QUANTUM ALGEBRA $U_q(sl_2(\mathbb{C}))$

Abstract. We define and compare, by model-theoretical methods, some exponentiations over the quantum algebra $U_q(sl_2(\mathbb{C}))$, for any parameter q. We discuss two cases, according to whether the parameter q is a root of unity.

1. Introduction

Quantum algebras are very interesting objects which are beginning to be investigated under a model theoretic point of view. This is witnessed, for instance, by [10], where one attempts to associate with a quantum algebra whose parameter of deformation is a root of unity a geometrical object, namely a Zariski geometry [3] (we will see this in more details below) , and by [2], where a model theoretic investigation of simple representations of quantum algebras is developed when the deformation parameter is not a root of unity.

In this paper we will construct exponential maps on the quantum algebra $U_q(sl_2(\mathbb{C}))$ for any parameter q, adopting the same method as in [8] for the universal enveloping $U(sl_2(\mathbb{C}))$ of the Lie algebra $sl_2(\mathbb{C})$ of 2x2 traceless matrices with complex entries.

The model theory of exponentiation has been considered not only in the classical frameworks of the real and complex field, but also over larger settings such as Lie algebras. Macintyre's paper [9] sketches a general picture of exponentiations over finite dimensional Lie algebras over both the real and the complex field. This led in [8] to the idea of approaching exponentiation over an infinite dimensional algebra, namely the universal enveloping $U(sl_2(\mathbb{C}))$.

The quantum algebras have occurred in the work of Boris Zilber in two ways. First, as *new* examples of Zariski geometries which are not interpretable in an algebraically closed field and second in the attempt to associate with a quantum algebra a geometrical object. There are one-dimensional Zariski geometries ([3]) which are finite coverings of algebraic curves but not algebraic curves (and whose automorphism group contains a subgroup generated by two elements $\tau_1,\ \tau_2$ whose commutator $[\tau_1,\tau_2]^n = 1$. Boris Zilber in [10] calls such object *a non classical Zariski geometry* and explores a method which associates a geometrical object, a Zariski geometry, to a typical quantum algebra (when the parameter of deformation is a root of unity). He begins with the *simplest* algebra, namely $U_q(sl_2(\mathbb{C}))$ to which he associates a many-sorted structure $\tilde{V}(U_q(sl_2(\mathbb{C})))$ consisting of a base field F, a variety V and a bundle of $U_q(sl_2(\mathbb{C}))$-modules of fixed finite dimension (the order of the root of unity) parametrized by V. He shows the theory of finite-dimensional $U_q(sl_2(\mathbb{C}))$-modules is $\aleph_1$-categorical and model-complete. Moreover, he shows that $\tilde{V}(U_q(sl_2(\mathbb{C})))$ is a Zariski geometry and that it is not definable in an algebraically closed field.

In this paper, we will consider the full algebra structure of $U_q(sl_2(\mathbb{C}))$, where q is arbitrary but $q^2 \neq 1$, and construct, using its finite-dimensional representations, an exponentiation

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function. When q is a root of unity, we will show that U embeds in a non-principal ultraproduct of the U_{q_ℓ}, where the order ℓ of the root of unity increases.

We will discuss two cases, according to whether the parameter q is a root of unity, or not. When q is not a root of unity, it is known that all finite dimensional representations of $U_q(sl_2(\mathbb{C}))$ are semisimple, the simple ones are classified in terms of highest weight and so are very similar to those of the classical case. Consequently various exponentiations over $U_q(sl_2(\mathbb{C}))$ can be defined just by strategies similar to the ones used in the classical case in [8]. We consider the Lie algebra $M_{\lambda+1}(\mathbb{C})$ of $(\lambda+1)\times(\lambda+1)$ matrices with entries in the complex field (λ a positive integer) and the relative matrix exponential map, introduced in terms of infinite power series from $M_{\lambda+1}(\mathbb{C})$ to the linear group $GL_{\lambda+1}(\mathbb{C})$. By connecting these exponentials to the simple finite dimensional $U_q(sl_2(\mathbb{C}))$-modules $V_{\lambda,\epsilon}$ (where $\epsilon=\pm1$), we first define *exponential maps* indexed by λ,ϵ from $U_q(sl_2(\mathbb{C}))$ to $GL_{\lambda+1}(\mathbb{C})$. By using the same technique as in the classical case, we will describe some properties of these maps and show that $U_q(sl_2(\mathbb{C}))$ embeds into any non-principal ultraproduct of the $M_{\lambda+1}(\mathbb{C})$. Then, we will define an exponential map EXP from $U_q(sl_2(\mathbb{C}))$ to any non-principal ultraproduct of the groups $GL_{\lambda+1}(\mathbb{C})$ and we will investigate some of its properties.

What is more interesting is the other case, when the parameter q is a root of unity. In fact the finite-dimensional representations of $U_q(sl_2(\mathbb{C}))$ are not semisimple and there are further finite-dimensional representations in addition to the highest weight ones. Anyway, using the characterization of its simple finite dimensional $U_q(sl_2(\mathbb{C}))$-modules, we will define exponential maps from $U_q(sl_2(\mathbb{C}))$ to certain ultrapowers of the linear group $GL_\ell(\mathbb{C})$, where ℓ is the order of the root q. In this case, we have to carefully choose appropriate ultrafilters.

2. Preliminaries.

Let k be a field; recall that the universal enveloping algebra U of $sl_2(k)$ can be presented as the associative algebra generated by X, Y, H, subject to the relations: $X.Y - Y.X = H$. It also be viewed as an iterated skew polynomial ring. Namely, set $A_0 = k[H]$, let σ_1 be an automorphism of A_0 which is the identity on k and which sends H to $H+2$.

Then let A_1 be the skew polynomial ring $A_0[Y;\sigma_1]$ and σ_2 be an automorphism of A_1 sending Y to Y and H to $H-2$. Then U is isomorphic to $A_2 := A_1[X;\sigma_2,\delta]$, where δ is a σ_2-derivation sending H to 0 and Y to H.

Fix an element $q \in k-\{0\}$ such that $q^2 \neq 1$. Now we want to present the *quantum algebra* $U_q(sl_2(k))$ in a similar way. Namely, $U_q(sl_2(k))$ is the associative k-algebra with generators K, K^{-1}, E, F and relations:

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ \ [E,F] = \frac{K-K^{-1}}{q-q^{-1}}. \tag{1}$$

The relations (1) imply by induction for every integers s and t, $s,\ t \geq 2$, the formulas:

$$[E,F^t] = [t]F^{t-1}\,\frac{Kq^{1-t} - K^{-1}q^{t-1}}{q-q^{-1}}, \tag{2}$$

$$[E^s,F] = [s]E^{s-1}\,\frac{Kq^{s-1} - K^{-1}q^{1-s}}{q-q^{-1}}, \tag{3}$$

where $[a]$ denotes the q-number that is defined for every $a \in \mathbb{Z}$ as:

$$[a] := \frac{q^a - q^{-a}}{q-q^{-1}}.$$

When q is not a root of unity, the representation theory of $U_q(sl_2(k))$ has similar properties as that of U, the universal enveloping algebra of $sl_2(k)$. If q is a primitive p^{th} root of unity, then the representation theory of $U_q(sl_2(k))$ looks like the representation theory of $sl_2(k)$ over an algebraically closed field of characteristic p.

Another (useful) way to present the algebra $U_q := U_q(sl_2(k))$, for any q, is to construct it as an iterated skew polynomial ring ([6]). Namely, let $A_0 := k[K, K^{-1}]$ with the automorphism α_1 sending K to $q^2.K$, then $A_1 := A_0[F; \alpha_1]$ with the commutation rule $a.F = F.a^{\alpha_1}$ with $a \in A_0$. Then extend α_1 to A_1 by $\alpha_1(F^j.K^\ell) = q^{2.\ell}.F^j.K^\ell$ and define an α_1-derivation on A_1 by $\delta(F) := \frac{K-K^{-1}}{q-q^{-1}}$, $\delta(K) = 0$ (see Lemma VI.1.5 in [6]). Finally, set $A_2 := A_1[E; \alpha_1, \delta]$ with the commutation rule $a.E = E.a^{\alpha_1} + \delta(a)$ with $a \in A_1$. Then A_2 is isomorphic to U_q and the set $\{E^i.F^j.K^z : \ i, j \in \mathbb{N};\ z \in \mathbb{Z}\}$ is a basis of U_q over k ([6] proof of Proposition VI.1.4, [5] Theorem 1.5).

Lemma 2.1. ([1] Theorem 1.12) *A_0 is a Euclidean commutative ring, A_1 (respectively A_2) is right and left Noetherian (and so right and left Ore with no zero-divisors).* □

The algebra U_q is a $\mathbb{Z}$-graded k-algebra with grading $deg(E) = 1$, $deg(F) = -1$ and $deg(K) = deg(K^{-1}) = 0$. (Note that the relations (1) preserve that grading.)

Let $U_{q,\ell}$ be the k-vector subspace generated by $\{E^i.K^z.F^j : i - j = \ell,\ \ i, j \in \mathbb{N}, z \in \mathbb{Z}\}$.

Lemma 2.2. $U_q = \oplus_{m\in\mathbb{Z}} U_{q,m}$.

Proof: let $L_1,\ L_2$ be two disjoint subsets of integers, then $\sum_{m\in L_1}\sum_{i,z}\alpha_{i,z}.E^i.F^{i+m}.K^z = \sum_{m'\in L_2}\sum_{j,z'}\beta_{j,z'}.E^j.F^{j+m'}.K^{z'}$ implies that $\sum_{m\in L_1}\sum_{i,z}\alpha_{i,z}.E^i.F^{i+m}.K^z = 0$ and $0 = \sum_{m'\in L_2}\sum_{j,z}\alpha_{j,z}.E^i.F^{i+m'}.K^z$. It suffices to note that if $E^i.F^{i+m_1}$, $m_1 \in L_1$, occurs in the right hand side with a nontrivial coefficient, then it doesn't occur in the left hand side. Indeed assume that for some $m_2 \in L_2$, $E^j.F^{j+m_2} = E^i.F^{i+m_2}$, then $i = j$ and so $m_1 = m_2$ a contradiction.

For $u \in U_{q,i}$, we have (see [5], 1.9):

$$K.u.K^{-1} = q^{2i}.u. \tag{4}$$

So, if q is not a root of unity, then $U_{q,0}$ is equal to the centralizer of K.

In the general case, we have the following Lemma. Let

$$C_q := \frac{q^{-1}.K + q.K^{-1}}{(q-q^{-1})^2} + E.F = F.E + \frac{q.K + q^{-1}.K^{-1}}{(q-q^{-1})^2} \tag{5}$$

be the quantized Casimir element of U_q.

Lemma 2.3. ([6] Proposition VI.4.1, Lemma VI.4.2) *For any q, $U_{q,0}$ is equal to the polynomial ring $k[C_q, K, K^{-1}]$.*

Proof: The fact that C_q commutes with K follows from the above relation. Further, using relations (1), we show that C_q belongs to the center of U_q.

First, by relation (4), $E^i.K^n.F^i{=}E^i.K^n.F^i.K^{-n}.K^n = q^{2.n.i}.E^i.F^i.K^n$. Then one proceeds by induction on i, to show that $E^i.F^i \in k[C_q, K, K^{-1}]$. This holds for $i = 1$. If $u \in k[C_q, K, K^{-1}]$, it remains to show that $E.u.F \in k[C_q, K, K^{-1}]$. By definition of C_q, this holds for $E.F$. Now, note that any element of $k[C_q, K, K^{-1}]$ can be represented as $K^{-d}.p[C_q, K]$ with $p[x_1, x_2] \in k[x_1, x_2]$ and $d \in \omega$. If $u = K^n$, $n \in \mathbb{Z}$, then $E.K^n.F = q^{2.n}.E.F.K^n$. □

We will use later the fact that any element of $U_{q,m}$, for any q, can be written as $E^m.u$, for $m \geq 0$, and $u.F^{-m}$, for $m < 0$, with $u \in U_{q,0}$ and also that for any $u \in U_{q,0}$, there exist u', $u'' \in U_{q,0}$ such that $E.u = u'.E$ (respectively $F.u = u''.F$).

If q is not a root of unity, then the center of U_q has dimension 1 over k and is generated by C_q (see Proposition 2.18 in [5], or Theorem VI.4.8 in [6]).

If q is a ℓ^{th} root of unity, the center of U_q is generated by E^ℓ, F^ℓ, K^ℓ, $K^{-\ell}$ and C_q (see Proposition 2.20 in [5]).

3. Finite-dimensional representations of U_q, for q not a root of unity.

In this section q is not a root of unity, and k is an algebraically closed field of characteristic different from 2.

Every finite-dimensional representation of U_q admits a direct sum decomposition by simple U_q-modules ([5] Theorem 2.9 and Proposition 2.3). For every positive integer λ, there exist (up to isomorphism) exactly two simple modules of dimension $\lambda+1$ as k-vector spaces. They will be denoted by $V_{\epsilon,\lambda}$, where $\epsilon \in \{-1, 1\}$. First, let us describe the U_q-module $V_{1,\lambda}$; it has a basis $\{v_0, v_1, \dots, v_\lambda\}$ for which the actions of the generators E, F, K can be described as follows:

(6)

$$Ev_j = \begin{cases} [n-j+1]v_{j-1}, & \text{if } j = 1, \dots, \lambda \\ 0, & \text{if } j = 0, \end{cases} \qquad Fm_j = \begin{cases} [j+1]v_{j+1}, & \text{if } j = 0, \dots, \lambda-1, \\ 0, & \text{if } j = \lambda, \end{cases}$$

$$K\, v_j \;=\; q^{\lambda-2j}\, v_j \qquad\qquad j = 0, \dots, \lambda. \tag{7}$$

In particular, E annihilates v_0 and F the vector v_λ, and up to the scalar multiplication these are the only vectors with these properties. So, $V_{1,\lambda}$ is an irreducible representation of U_q. Furthermore, on $V_{1,\lambda}$, the quantized Casimir element C_q acts by scalar multiplication of $\frac{q^{\lambda-1}+q^{1-\lambda}}{(q-q^{-1})^2}$.

The other simple representation $V_{-1,\lambda}$ of dimension $\lambda+1$ is obtained by composing the action of U_q on $V_{1,\lambda}$ with the automorphism σ (see [5, §5.2]) of U_q determined by

$$\sigma(E) = -E, \quad \sigma(F) = F, \quad \sigma(K) = -K.$$

Furthermore, σ maps C_q to $-C_q$. We will also refer to the module $V_{-1,\lambda}$ as $V^\sigma_{1,\lambda}$. Denote by $V_{\epsilon,\lambda}$ (for every $\epsilon = \pm 1$), any simple representation of U_q (of dimension $\lambda+1$) and by $V^j_{\epsilon,\lambda}$ the eigenspace of K with eigenvalue $\epsilon q^{\lambda-2j}$, namely $\{v \in V_{\epsilon,\lambda} : Kv = \epsilon q^{\lambda-2j} v\}$. So, we have that $V_{\epsilon,\lambda} = \oplus_{0\leq j\leq\lambda} V^j_{\epsilon,\lambda}$. For every $\epsilon = \pm 1$, the actions of the generators E, F, K and the central element C_q, according to the representation map $\Theta_{\epsilon,\lambda} : U_q \to \mathrm{End}(V_{\epsilon,\lambda+1})$, are described by the matrices denoted respectively as $E_{\epsilon,\lambda} := \Theta_{\epsilon,\lambda}(E)$, $F_{\epsilon,\lambda} := \Theta_{\epsilon,\lambda}(F)$, $K_{\epsilon,\lambda} := \Theta_{\epsilon,\lambda}(K)$ and $C_{q,\epsilon,\lambda} := \Theta_{\epsilon,\lambda}(C_q)$:

$$E_{\epsilon,\lambda} = \epsilon \begin{pmatrix} 0 & [\lambda] & 0\dots & 0 \\ 0 & 0 & [\lambda-1]\dots & 0 \\ \vdots & \vdots & & [1] \\ 0 & 0 & 0\dots & 0 \end{pmatrix}, \quad F_{\epsilon,\lambda} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & [2] & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & [\lambda] & 0 \end{pmatrix} \tag{8}$$

$$K_{\epsilon,\lambda} = \epsilon\,\mathrm{diag}(q^\lambda,\, q^{\lambda-2}, \dots, q^{-\lambda+2} q^{-\lambda}), \quad C_{q,\epsilon,\lambda} = \epsilon\,\mathrm{diag}\Big(\frac{q^{\lambda-1}+q^{1-\lambda}}{(q-q^{-1})^2}, \dots, \frac{q^{\lambda-1}+q^{1-\lambda}}{(q-q^{-1})^2}\Big).$$

In the next proposition, we translate in our notations the property that for any $r \in U_{q,0}$, the formula $\phi(v) := r.v = 0$ is uniformly bounded as defined in ([2] Lemma 4.2).

Proposition 3.1. *Let $r \in U_{q,0}$, then the dimension of the kernel of $\Theta_{\epsilon,\lambda}(r)$ in $V_{\epsilon,\lambda}$ is bounded independently of λ.*

Proof: We have that $r = K^{-n}.p(C_q, K)$ with $p(x_1, x_2) \in k[x_1, x_2]$ and $n \in \mathbb{N}$. Let $v \in V^i_{\epsilon,\lambda}$ with $\lambda \in \mathbb{N} - \{0\}$, then $r.v_i = \epsilon^{-n}.q^{-n.(\lambda-2.i)}.p(\frac{q^{-1}(\epsilon.q^\lambda)+q.(\epsilon.q^\lambda)^{-1}}{(q-q^{-1})^2}, \epsilon.q^{\lambda-2.i}).v_i$. Now $r.v_i = 0$ iff $p(\frac{q^{-1}(\epsilon.q^\lambda)+q.(\epsilon.q^\lambda)^{-1}}{(q-q^{-1})^2}, \epsilon.q^{\lambda-2.i}) = 0$.

We write $p(x_1, x_2) = \sum_{j=0}^m p_j(x_1).x_2^j$. Since q is not a root of unity, for all $n \in \mathbb{N} - \{0\}$ $q^n \neq 1$ and the map sending n to q^n is a monomorphism from $(\mathbb{Z}, +, 0)$ to $(k - \{0\}, ., 1)$.

We write $\frac{q^{-1}(\epsilon.q^\lambda)+q.(\epsilon.q^\lambda)^{-1}}{(q-q^{-1})^2} = q^{-1}.\epsilon.q^\lambda.\frac{(1+q^{-2\lambda})}{(q-q^{-1})^2}$. Suppose that we have that more than $\deg(p_j)$ values of λ such that $p_j(\frac{q^{-1}(\epsilon.q^\lambda)+q.(\epsilon.q^\lambda)^{-1}}{(q-q^{-1})^2}) = 0$. This entails that

$$q^{-1}.\epsilon.q^{\lambda_1}.\frac{(1+q^{-2\lambda_1})}{(q-q^{-1})^2} = q^{-1}.\epsilon.q^{\lambda_2}.\frac{(1+q^{-2\lambda_2})}{(q-q^{-1})^2}.$$

Therefore, $(q^{\lambda_1} + q^{-\lambda_1}) = (q^{\lambda_2} + q^{-\lambda_2})$, so $q^{\lambda_1+\lambda_2}(q^{\lambda_1} - q^{\lambda_2}) = (q^{\lambda_1} - q^{\lambda_2})$. So, $q^{\lambda_1+\lambda_2} = 1$, which implies that $\lambda_1+\lambda_2 = 0$ and since these are positive numbers, a contradiction. Denote by Z_j the (finite) set of values of λ such that $p_j(\frac{q^{-1}(\epsilon.q^\lambda)+q.(\epsilon.q^\lambda)^{-1}}{(q-q^{-1})^2}) = 0$. Suppose that $\lambda \notin \bigcap_{j=0}^m Z_j$, then there are finitely many $i \leq m$ such that $p(\frac{q^{-1}(\epsilon.q^\lambda)+q.(\epsilon.q^\lambda)}{(q-q^{-1})^2}, \epsilon.q^{\lambda-2.i}) = 0$.
$\square$

A uniform way of presenting these representations is to introduce the quantum plane ([2]).

The quantum plane $k[x_1, x_2]_q$ is the quotient of the free k-algebra generated by x_1 and x_2 by the ideal generated by $x_1.x_2 - q.x_2.x_1$. A basis is $\{x_1^i.x_2^j\}_{i,j\in\mathbb{N}}$ with the commutation relation $x_2^j.x_1^i = q^{i.j}.x_1^i.x_2^j$. Let $k[x_1, x_2]_{q,\lambda}$ be the k-vector space generated by the homogeneous elements of degree λ. (We have $k[x_1, x_2]_q = \oplus_{\lambda\in\mathbb{N}} k[x_1, x_2]_{q,\lambda}$.) It is an U_q-module with the actions of E, F and K defined as follows: $K.x_1^i.x_2^j = q^{i-j}.x_1^i.x_2^j$, $E.x_1^i.x_2^j = [i].x_1^{i-1}.x_2^{j+1}$, $F.x_1^i.x - 2^j = [j].x_1^{i+1}.x_2^{j-1}$. We could also have defined the action of U_q as follows: first send U_q to $\sigma(U_q)$ and then let it act on $k[x_1, x_2]_q$ as before. In the second case, we will denote the quantum plane by $k[x_1, x_2]_{q,\sigma}$.

The simple finite dimensional modules $V_{\epsilon,\lambda}$ are, as U_q-modules, either isomorphic to $k[x_1, x_2]_{q,\lambda}$ $(\epsilon = 1)$ or $k[x_1, x_2]_{q,\sigma}$ $(\epsilon = -1)$.

4. The exponential maps on U_q, q not a root of unity.

In this section we set $k = \mathbb{C}$ (in fact we just need a field endowed with a norm and complete for the induced topology). We endow $M_\ell(\mathbb{C})$ with the Hermitian sesquilinear form $(\cdot, \cdot)$, defined by $(A, B) := \mathrm{tr}(B^* \cdot A) = \sum_{i,j} A_{ij} \cdot \bar{B}_{ij}$, where A, $B \in M_\ell(\mathbb{C})$ and B^* is the conjugate of the transpose of B. Let $\| \cdot \|_\ell$ be the norm induced by this form (usually called the Frobenius norm) hence for every A, we have $\|A\|_\ell^2 := (A, A)$.

We denote by *exp* the matrix exponential map from the algebra of matrices $M_{\lambda+1}(\mathbb{C})$ to the algebra of invertible matrices $GL_{\lambda+1}(\mathbb{C})$, which sends any matrix $A \in M_{\lambda+1}(\mathbb{C})$ to the matrix exponential $\exp(A)$, defined as the power series

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad . \tag{9}$$

If A is a 1×1 matrix, that is, a scalar a of the field $\mathbb{C}$), then $exp(A) = e^a$ where e^a denotes the ordinary exponential of the element $a \in \mathbb{C}$.

Actually, there exists a q-variant of the exponential map defined as an element of the formal power series ring $\mathbb{C}[[X]]$(see [6], pag. 76). The q-exponential is defined as the formal series

$$e_q(X) = \sum_{n=0}^{\infty} \frac{X^n}{[n]!} \quad ,$$

where $[n]! = [1] \dots [n]$, (note that $[n]!$ for $q = 1$ is equal to the usual factorial $n!$). Observe that the series is well-defined (provides q is not a root of unity).The q-exponential is any invertible series, but in contrast with the ordinary exponential (that is, for $q = 1$), we have $e_q(X)^{-1} \neq e_q(-X)$. Anyway, for any variable X and Y such that $XY = qYX$, the fundamental property of the exponentials $e_q(X+Y) = e_q(X)e_q(Y)$ is satisfied.

Anyway, we will work with the matrix exponential defined by (9) in order to introduce a new exponential map over U_q by using its representation theory. We will compose this map with $\Theta_{\epsilon,\lambda}$ in order to get exponential maps from U_q to $GL_{\lambda+1}(\mathbb{C})$. Since for each λ, the kernels of these maps $\Theta_{\epsilon,\lambda}$ are non-trivial, we will consider non-principal ultraproducts of the $GL_{\lambda+1}(\mathbb{C})$.

In [8] (see Definition 4.1), we defined the notion of (non-commutative) exponential rings (respectively non-commutative exponential $\mathbb{C}$-algebras).

For each λ, we define the exponential map $EXP_{\epsilon,\lambda}$ on U_q as follows. Let $u \in U_q$, then

$$EXP_{\epsilon,\lambda}(u) := exp(\Theta_{\epsilon,\lambda}(u)), \quad \text{for } \epsilon = \pm 1.$$

For instance,

(1) $EXP_{\epsilon,\lambda}(E) = exp(\Theta_{\epsilon,\lambda}(E))$=$exp(E_{\epsilon,\lambda})$,
(2) $EXP_{\epsilon,\lambda}(F) = exp(\Theta_{\epsilon,\lambda}(F)) = exp(F_\lambda)$,
(3) $EXP_{\epsilon,\lambda}(K) = exp(\Theta_{\epsilon,\lambda}(K)) = \text{ diag}\left(e^{\epsilon.q^{\lambda}}, e^{\epsilon.q^{\lambda-2}}, \dots, e^{\epsilon.q^{-\lambda+2}}, e^{\epsilon.q^{-\lambda}}\right)$.,
(4) $EXP_{\epsilon,\lambda}(C_q) = exp(\Theta_{\epsilon,\lambda}(C_q)) = e^{\frac{q^{-1}(\epsilon.q^{\lambda})+q.(\epsilon.q^{\lambda})^{-1}}{(q-q^{-1})^2}}.1_{\lambda+1}$.

We get a transfer of the properties of the classical matrix exponential to this new exponential map, as follows.

Proposition 4.1. *If u, $v \in U_q$ and a, $b \in \mathbb{C}$, then $\forall \lambda \in \mathbb{N} - \{0\}$:*

(i) $EXP_{\epsilon,\lambda}(0_{U_q}) = I_\lambda$, *where 0_{U_q} denotes the identity element (with respect to the addition) in U_q and I_λ is the identity matrix in $GL_{\lambda+1}(\mathbb{C})$.*
(ii) $EXP_{\epsilon,\lambda}(a.u).EXP_{\epsilon,\lambda}(b.u) = EXP_{\epsilon,\lambda}((a+b).u)$;
(iii) $EXP_{\epsilon,\lambda}(u).EXP_{\epsilon,\lambda}(-u) = I_\lambda$;
(iv) *for u and v commuting, $EXP_{\epsilon,\lambda}(u+v) = EXP_{\epsilon,\lambda}(u).EXP_{\epsilon,\lambda}(v)$;*
(v) *for an invertible element v in U_q, $EXP_{\epsilon,\lambda}(vuv^{-1}) = \Theta_{\epsilon,\lambda}(v).EXP_{\epsilon,\lambda}(u).\Theta_{\epsilon,\lambda}(v)^{-1}$;*

So, $(U_q, \mathbb{C}, EXP_{\epsilon,\lambda}, GL_{\lambda+1}(\mathbb{C}))$ is an exponential $\mathbb{C}$-algebra.

As in [8] Proposition 7.2, we obtain the following result.

Proposition 4.2. *$\forall \lambda \in \mathbb{N} - \{0\}$, the map $EXP_{\epsilon,\lambda}$ is surjective.*

Proof. Since exp is surjective from $M_{\lambda+1}(\mathbb{C})$ to $GL_{\lambda+1}(\mathbb{C})$, it suffices to prove that $\Theta_{\epsilon,\lambda} : U_q \to M_{\lambda+1}(\mathbb{C})$ is surjective. The latter is deduced directly by Jacobson density theorem [4, Section 2.2]. $\square$

Let $\mathcal{U}$ be a non principal ultrafilter on ω. Recall that the ring $(\prod_{\mathcal{U}} M_{\lambda+1}(\mathbb{C}), exp, \prod_{\mathcal{U}} GL_{\lambda+1}(\mathbb{C}))$ is an exponential ring ([8] Proposition 5.1). We will view U_q as an exponential sub-ring of that ring.

Proposition 4.3. *For every non-principal ultrafilter $\mathcal{U}$ on ω, the map $[\Theta_{\epsilon,\lambda}]$ is injective from U_q to $\prod_{\mathcal{U}} M_{\lambda+1}(\mathbb{C})$.*

Proof: We proceed as in [8], using Proposition 3.1. Any element u of U_q can be written as, with $m \geq 0$, $\sum_{z=-m}^{-1} F^{-z}.u_z + \sum_{z=0}^{m} u_z.E^z$. Then, for $\lambda \geq m$, whenever $\Theta_{\epsilon,\lambda}(u_z) \neq 0$, for some z, $\Theta_{\epsilon,\lambda}(u) \neq 0$. Then, by Proposition 3.1, if $u_z \neq 0$, for all λ but finitely many of them, $\Theta_{\epsilon,\lambda}(u_z) \neq 0$. $\square$

Define EXP from U_q to $\prod_{\mathcal{U}} GL_{\lambda+1}(\mathbb{C})$ by

$$EXP(u) = [EXP_{\epsilon,\lambda}(u)],$$

(for $\epsilon \pm 1$).

It follows by Łos theorem that $(U_q, EXP, \prod_{\mathcal{U}} GL_{\lambda+1}(\mathbb{C}))$ is an exponential $\mathbb{C}$-algebra.

5. Finite-dimensional representations of U_q, for q a root of unity.

In this section, we will assume that q is a primitive ℓ^{th} root of unity for $\ell \geq 3$ and that k is algebraically closed. In this case, the dimension of a finite-dimensional simple U_q-module is bounded by ℓ; in dimension ℓ, there are more simple U_q-modules (than for q not a root of unity).

First a simple U_q-module of dimension $\lambda < \ell$ is isomorphic to a module of the form $V_{\pm,\,\lambda}$ (see Proposition VI.5.1 in [6]). Then there are no simple finite-dimensional U_q-module of dimension $> \ell$ (see Proposition VI.5.2 in [6]).

Now let us describe the module $V_{a,b,c}(\ell)$ of dimension ℓ, $a, b, c \in k$, $c \neq 0$.

For ease of notation, we will set $e_i := a.b + [i].\frac{c.q^{-i+1} - c^{-1}.q^{i-1}}{q - q^{-1}}, 1 \leq i \leq \ell - 1$, $e_\ell := a$ and $e = \prod_{i=1}^{\ell} e_i$. Note that the e_i's and e depend on a, b, c, and when we want to stress it, we denote e_i (respectively e) by $e_i(a, b, c)$ (respectively $e(a, b, c)$). Also, we will always assume that $c^2 \neq 1$.

For $z \in \mathbb{C}$, let $\bar{z}$ be the complex conjugate of z, since $\overline{q^i} = q^{\ell-i}$, we have that $\overline{[i]} = [\ell - i]$.

The actions of E, F and K are represented by the following three $\ell \times \ell$ matrices $E_{a,b,c}$, $F_{b,c}$, K_c:

$$E_{a,b,c} = \begin{pmatrix} 0 & e_1 & 0\dots & 0 \\ 0 & 0 & e_2\dots & 0 \\ \vdots & \vdots & & e_{\ell-1} \\ e_\ell & 0 & 0\dots & 0 \end{pmatrix}, \tag{10}$$

$$F_b = \begin{pmatrix} 0 & 0 & \dots & b \\ 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{11}$$

$$K_c = c.\mathrm{diag}\left(1, q^{-2}, \ldots, q^{-2.\ell+4}, q^{-2\ell+2}\right),$$

$$C_{a,b,c} = \mathrm{diag}\left(ab + \tfrac{c.q+c^{-1}.q^{-1}}{(q-q^{-1})^2}\right).$$

Note that the actions of respectively E, F, K and C on an ℓ-dimensional space that these matrices represent either are cyclic permutations of one-dimensional subspaces, or leave these subspaces invariant.

We will denote by $\Theta_{a,b,c}$ the maps from U_q to $M_\ell(k)$ sending E to $E_{a,b,c}$, F to F_b and K to K_c.

Now let us describe the module $\tilde{V}_{d,f}(\ell)$, $d, f \in k$, $d, f \neq 0$, it is a ℓ-dimensional k-vector space.

For ease of notation, we will set $f_i := [i].\frac{f^{-1}.q^{-i+1}-f.q^{i-1}}{q-q^{-1}}$ and we will always assume that $f^2 \neq 1$. The actions of E, F and K are represented by the following three $\ell \times \ell$ matrices E_d, F_f, K_f:

$$F_f = \begin{pmatrix} 0 & f_1 & 0\ldots & 0 \\ 0 & 0 & f_2\ldots & 0 \\ \vdots & \vdots & & f_{\ell-1} \\ 0 & 0 & 0\ldots & 0 \end{pmatrix},$$

$$E_d = \begin{pmatrix} 0 & 0 & \ldots & d \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{12}$$

$$K_f = f.\mathrm{diag}\left(1, q^2, \ldots, q^{2.\ell-4}, q^{2.\ell-2}\right).$$

$$C_{q,f} = \mathrm{diag}\left(\tfrac{f.q^{-1}+f^{-1}.q}{(q-q^{-1})^2}\right).$$

Note that the action of E_d on an ℓ-dimensional space is a cyclic permutation of one-dimensional subspaces, whereas the action of F_f is nilpotent.

We will denote by $\Theta_{d,f}$ the maps from U_q to $M_\ell(k)$ sending E to $E_{d,f}$, F to F_f and K to K_f.

Any simple U_q-module of dimension ℓ is isomorphic to a module of the form (see Theorem VI.5.5 in [6]): $(c \neq 0)$

(1) $V_{a,b,c}(\ell)$ with $b \neq 0$,
(2) $V_{a,0,c}(\ell)$, whenever $c \notin \{\pm 1, \pm q, \cdots, \pm q^{\ell-2}\}$ or
(3) $\tilde{V}_{d,\pm q^{1-j}}(\ell)$, for $j \in \{1, \cdots, \ell-1\}$ and $d \neq 0$.

In the following we will use on one hand the family of representations $\Theta_{a,b,c} := \Theta_{\ell,a,b,c}$ with a, b, c all non-zero and the family $\Theta_{f,d} := \Theta_{\ell,f,d}$ with f, d all non-zero.

6. The exponential maps on U_q, q a root of unity.

Let $k = \mathbb{C}$, let q be a primitive ℓ^{th}-root of unity. We denote by $\mathbb{R}^+$ the set of strictly positive real numbers and by $\mathbb{N}^+$ the set of strictly positive natural numbers.

We denote by exp the matrix exponential map from $M_\ell(\mathbb{C})$ to $GL_\ell(\mathbb{C})$. As in section 4, we will compose this map with $\Theta_{a,b,c}$ (respectively $\tilde{\Theta}_{d,f}$) in order to get exponential maps from U_q to $GL_\ell(\mathbb{C})$.

For each (a, b, c) (respectively (d, f)), we define the exponential map $EXP_{(a,b,c)}$ (respectively $EXP_{(c,d)}$) on U_q as follows. Let $u \in U_q$, then $EXP_{(a,b,c)}(u) := exp(\Theta_{(a,b,c)}(u))$ (respectively $EXP_{(d,f)}(u) := exp(\Theta_{(d,f)}(u))$.)

Similarly to Proposition 4.1, we obtain that $(U_q, \mathbb{C}, EXP_{(a,b,c)}, GL_\ell(\mathbb{C}))$ (respectively $(U_q, \mathbb{C}, EXP_{(d,f)}, GL_\ell(\mathbb{C})))$ are exponential $\mathbb{C}$-algebras. Moreover, if the parameters (a, b, c) (respectively (d, f)) are chosen such that the corresponding module $V_{a,b,c}(\ell)$ (respectively $\tilde{V}_{d,f}(\ell)$) is simple, then the map $EXP_{(a,b,c)}$ (respectively $EXP_{(d,f)}$) is surjective (the argument is the same as the one used in Proposition 4.2).

Now, we will vary these maps along certain non principal ultrafilters $\mathcal{W}$ on ω^2 in order to embed U_q in the corresponding non-principal ultraproduct of the $M_\ell(\mathbb{C})$.

We want to find necessary conditions on a domain of variation for a, b, c (respectively $d,\ f$) in order to get for $u \neq 0$ that $\Theta_{a,b,c}(u) \neq 0$ (respectively $\Theta_{d,\ f}(u) \neq 0$), for *sufficiently many* $a,\ b\ c$ (respectively $d,\ f$).

First let us consider the case of an element $u \in U_{q,0}$. So, u is of the form $K^{-n}.p(C_q, K)$ with $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ and $n \in \mathbb{N}$. Let us write $p(x_1, x_2) = \sum_{j=0}^{d} s_j(x_1).x_2^j$ and we may further assume that $s_0(x_1) \in \mathbb{C}[x_1] - \{0\}$, varying $n \in \mathbb{Z}$.

For ease of notation, let us denote in both representations $\Theta_{a,b,c}$ and $\Theta_{d,f}$ the coefficients occurring in the matrix representation of the Casimir element by the same letter $c(q)$ (even though, it varies according to the chosen representation).

So if $u \in U_{q,0}$, we have that both matrices $\Theta_{d,f}(u)$ and $\Theta_{a,b,c}(u)$ are diagonal matrices whose $(i + 1)^{th}$ entry on the diagonal, with $0 \leq i \leq \ell - 1$, either is equal to $f^{-n}.q^{-2.n.i}p(c(q), f.q^{2.i})$ or to $c^{-n}.q^{2.n.i}p(c(q), c.q^{-2.i})$ and $p(c(q), f.q^{2.i}) = \sum_{j=0}^{d} s_j c(q).(f.q^{2.i})^j$, respectively $p(c(q), c.q^{-2.i}) = \sum_{j=0}^{d} s_j(c(q)).(c.q^{-2.i})^j$.

Let us first consider a very special case where $u = p(C_q, K)$ with $s_0 \in \mathbb{C}[X] - \{0\}$.

Assume that all these entries of $\Theta_{d,f}(u)$ (respectively $\Theta_{a,b,c}(u)$) are zero, taking their sum, we get that for each $j > 0$, $\sum_{i=0}^{\ell-1} s_j(c(q)).f^j.q^{2i.j} = 0$, respectively $\sum_{i=0}^{\ell-1} s_j(c(q)).c^j.q^{-2.i.j} = 0$. But, $\sum_{i=0}^{\ell-1} q^{2.j.i} = 0$, for any $j \neq 0$. So, what remains is the case when $j = 0$, and we get in both cases that $\ell.s_0(c(q)) = 0$. So, $s_0(c(q)) = 0$, which cannot happen for infinitely many values of $c(q)$ since s_0 is a non-zero polynomial.

In the general case, we have to proceed as follows. We first consider the representation $\Theta_{d,f}$.

We will assume that the elements f are chosen such that:
$\overline{f.q^{-1} + f^{-1}.q} = f.q^{-1} + f^{-1}.q$, so $\bar{f}.q + \bar{f}^{-1}.q^{-1} = f.q^{-1} + f^{-1}.q$, or equivalently $(\bar{f} - f^{-1}).q = (f - \bar{f}^{-1}).q^{-1}$. Multiplying both sides by $f.\bar{f}$, we get $\bar{f}.(f.\bar{f} - 1).q = f.(\bar{f}.f - 1).q^{-1}$. So, the above condition is equivalent to $\bar{f}.q = f.q^{-1}$, or to $\bar{f} = f.q^{-2}$.

Let $u = K^{-n}.p(C_q, K)$, $n \in \mathbb{N}^+$, where $p(x_1, x_2) = \sum_{j=0}^{d} s_j(x_1).x_2^j$, $s_j(x_1) \in \mathbb{C}[x_1]$. Further we may assume that $s_0 \in \mathbb{C}[x_1] - \{0\}$, letting $n \in \mathbb{Z}$.

First assume that $p(x_1, x_2) \notin \mathbb{R}[x_1, x_2]$ and that for all $0 \leq i \leq \ell-1$, $p(\frac{f.q^{-1}+f^{-1}.q}{(q-q^{-1})^2}, f.q^{2i}) = 0$. This implies that $\bar{p}(\overline{\frac{f.q^{-1}+f^{-1}.q}{(q-q^{-1})^2}}, \overline{f.q^{2i}}) = 0$. By assumption on f, $\bar{f} = f.q^{-2}$ and so $\bar{p}(\frac{f.q^{-1}+f^{-1}.q}{(q-q^{-1})^2}, f.q^{-2i-2}) = 0$.

In other words, if $\Theta_{f,d}(u) = 0$, then we found a common root of $p(x_1, x_2)$ and $\bar{p}(x_1, x_2)$.

Second, assume that $p(x_1, x_2)$ and $\bar{p}(x_1, x_2)$ have no common irreducible factors, then by Bezout theorem, they have finitely many common zeroes.

Thirdly, assume that $p(x_1, x_2)$ and $\bar{p}(x_1, x_2)$ have a common irreducible factor. Therefore, we get that $p(x_1, x_2)$ has a factor with real coefficients.

Let us assume that $p(x_1, x_2) \in \mathbb{R}[x_1, x_2]$ and that $p(\frac{f.q^{-1}+f^{-1}.q}{(q-q^{-1})^2}, f.q^{2i}) = 0$. Since the squares of the roots of a polynomial with real coefficients belong to $\mathbb{R}$ and if for some j, $s_j(\frac{f.q^{-1}+f^{-1}.q}{(q-q^{-1})^2}) \neq 0$, then $f.q^{2i} \in \mathbb{R}$, or $f^2.q^{4i} \in \mathbb{R}$. Suppose $f.q^{2i} \in \mathbb{R}$, then since $f.q^{-2} = \bar{f}$ and $\bar{f}.q^{-2i} \in \mathbb{R}$, we get that $f.q^{-2}.q^{-2i}.q^{4i+2} \in \mathbb{R}$, so $q^{4i+2} \in \mathbb{R}$, which implies that ℓ divides $4i + 2$. Suppose $f^2.q^{4i} \in \mathbb{R}$, then $f.f.q^{-2}.q^{4i+2} \in \mathbb{R}$ i.e. ℓ divides $4i + 2$ i.e. $i = (\ell - 1)/2$. So, at most one entry of the matrix is equal to zero.

Now let us consider the representation $\Theta_{a,b,c}$.

We will assume that the coefficients a, b, c satisfy the following conditions: $a.b \in \mathbb{R}$, and $\overline{c.q + c^{-1}.q^{-1}} = c.q + c^{-1}.q^{-1}$, equivalently $\overline{c^{-1}} = c^{-1}.q^{-2}$, or $\bar{c} = c.q^2$.

So, $\overline{[i].\frac{c.q^{-i+1}-c^{-1}.q^{i-1}}{q-q^{-1}}} = [\ell - i].\frac{c.q^{-\ell+i-1}-c^{-1}.q^{\ell-i+1}}{q-q^{-1}}$

Note that if $a.b \in \mathbb{R}$, then $\bar{e}_i = e_{\ell-i}$, $1 \leq i \leq \ell - 1$, and so $e = \prod_{i=1}^{\ell-1} e_i.a = \prod_{i=1}^{\frac{\ell}{2}} |e_i|^2.a \in \mathbb{R}$. So, $e \in \mathbb{R}$ iff $a \in \mathbb{R}$.

As for the other representation, we make the following case distinctions.

First assume that $p(X, Y) \notin \mathbb{R}[X, Y]$ and for $0 \leq i \leq \ell - 1$, $p(a.b + \frac{c.q+c^{-1}.q^{-1}}{(q-q^{-1})^2}, c.q^{-2i}) = 0$. This implies that $\bar{p}(a.b + \overline{\frac{c.q+c^{-1}.q^{-1}}{(q-q^{-1})^2}}, \overline{c.q^{2i}}) = 0$. By assumption on c, $\bar{c} = c.q^2$ and so $\bar{p}(a.b + \frac{c.q+c^{-1}.q}{(q-q^{-1})^2}, c.q^{-2i+2}) = 0$.

In other words, if $\Theta_{a,b,c}(u) = 0$, then we found a common root of $p(x_1, x_2)$ and $\bar{p}(x_1, x_2)$.

Second, assume that $p(x_1, x_2)$ and $\bar{p}(x_1, x_2)$ have no common irreducible factors, then by Bezout theorem, they have finitely many common zeroes.

Thirdly, assume that $p(x_1, x_2)$ and $\bar{p}(x_1, x_2)$ have a common irreducible factor. Therefore, we get that $p(x_1, x_2)$ has a factor with real coefficients.

Let us assume that $p(x_1, x_2) \in \mathbb{R}[x_1, x_2]$ and that $p(a.b + \frac{c.q+c^{-1}.q^{-1}}{(q-q^{-1})^2}, c.q^{-2i}) = 0$. Since the squares of the roots of a polynomial with real coefficients belong to $\mathbb{R}$ and if for some j, $s_j(a.b + \frac{c.q+c^{-1}.q^{-1}}{(q-q^{-1})^2}) \neq 0$, then $c.q^{-2i} \in \mathbb{R}$, or $c^2.q^{-4i} \in \mathbb{R}$. Suppose $c.q^{-2i} \in \mathbb{R}$, then since $c.q^2 = \bar{c}$ and $\bar{c}.q^{2i} \in \mathbb{R}$, we get that $c.q^2.q^{2i} \in \mathbb{R}$, so $q^{4i+2} \in \mathbb{R}$, which implies that ℓ divides $2(2i + 1)$. Suppose $c^2.q^{-4i} \in \mathbb{R}$, then $c^2.q^4.q^{4i} \in \mathbb{R}$ i.e. ℓ divides $4(2i + 1)$ i.e. $i = (\ell - 1)/2$. So, at most one entry of the matrix is equal to zero.

Now we will show that under some conditions on where the elements f vary in $\mathbb{C}$ (respectively a, b, c), we get that the coefficients $c(q)$ take infinitely many values.

Let us check when different values of f give us the same values for $\frac{f.q^{-1}+f^{-1}.q}{(q-q^{-1})^2}$. Assume that $\frac{f.q^{-1}+f^{-1}.q}{(q-q^{-1})^2} = \frac{g.q^{-1}+g^{-1}.q}{(q-q^{-1})^2}$, so $g.f = q^2$, in particular $|f| = |g|^{-1}$. So, it suffices to let f vary over an infinite subset of elements of $\mathbb{C}$ of modulus bigger than 1, to get infinitely may different values for the coefficients $c(q)$.

Now let us do the same reasoning for the other representation. For sake of simplicity, we will assume that the product $a.b$ is constant and belongs to $\mathbb{R}$. So, now if $a'.b'+\frac{c'.q+(c'.q)^{-1}}{(q-q^{-1})^2}=a.b+\frac{c.q+(c.q)^{-1}}{(q-q^{-1})^2}$, then $\frac{c'.q+(c'.q)^{-1}}{(q-q^{-1})^2}=\frac{c.q+(c.q)^{-1}}{(q-q^{-1})^2}$, equivalently $c.c'=q^{-2}$, whenever $c\neq c'$. So, it suffices to let c vary over a subset of $\mathbb{C}$ of elements of modulus strictly bigger than 1, to get infinitely may different values for the coefficients $c(q)$.

Notation 6.1. Let $\{f_m : m\in\omega\}$ (respectively $\{c_m : m\in\omega\}$) be a countable subset of $\mathbb{C}$ of modulus strictly bigger than 1, let $\{d_n : n\in\omega\}$ (respectively $\{a_n : n\in\omega\}$, $\{b_n : n\in\omega\}$) be countable sets of distinct complexes of bounded modulus and assume that $a_n.b_n$ a constant real number with modulus strictly bigger than 1 and each $|a_n|>1$ and that $e(m,n)$ (where $e(n,m)=a_n.\prod_{i=1}^{\ell} e_{i,m,n}$), which depends on a_n, b_n and c_m, has modulus strictly bigger than 1.

Let $\mathcal{W}$ be a non-principal ultrafilter on ω^2. Such ultrafilter will index subsets of complex numbers of the form (d_n,f_m) with $|f_m|>1$, or (b_n,c_m) with $a_n.b_n$ a real constant and $|c_m|>1$. The ultrafilter $\mathcal{W}$ will either contain subsets of the form $\{(d_n,f_m):\ m>m_0\ \ n\notin I_m, |I_m|<C\}$, or of the form $\{(b_n,c_m):\ m>m_0\ \ n\notin I_m, |I_m|<C\}$ where I_m is a finite subset of ω and $C\in\omega^+$.

For $u\in U_q$, we denote by $\tilde{\Theta}_{n,m}(u):=\Theta_{d_n,f_m}(u)$ and $\Theta_{n,m}(u):=\Theta_{a_n,b_n,c_m}(u)$.

From the above discussion, we deduce the following.

Lemma 6.1. *Let $\mathcal{W}$ and the elements d_n, f_m (respectively a_n, b_n, c_m) be chosen as above. Then for any $u\in U_{q,0}$, $[\tilde{\Theta}_{n,m}(u)]_{\mathcal{W}}\neq 0$ (respectively $[\Theta_{n,m}(u)]_{\mathcal{W}}\neq 0$) and its norm is bounded by an element of $\mathbb{R}^+$.* $\square$

Now we want to examine the general case.

Any element u of U_q can be written as a finite sum of the form $\sum_{z\in\mathbb{N}^+} F^z.u_{-z}+\sum_{z\in\mathbb{N}}^m u_z.E^z$ with $u_z\in U_{q,0}$.

Note that we have that $F_f^{\ell}=0$, and for $n\in\mathbb{N}^+$ and $0\le j\le \ell-1$ that $E_d^{n.\ell+j}=d^n.E_d^j$, $F_b^{n.\ell+j}=b^n.F_b^j$ and $E_{a,b,c}^{n.\ell+j}=e^n.E_{a,b,c}^j$.

Moreover, $E_{a,b,c}^{\ell-i}\sim F_b$, for $0\le i\le \ell$, where $\sim$ means both matrices induce the same permutation of the one-dimensional subspaces.

Re-write the element u as a finite sum of the form

$$\sum_{j=0}^{\ell-1}\sum_{z\in(j+\ell.\mathbb{N}^+)} F^z.u_{-z}+\sum_{j=0}^{\ell-1}\sum_{z\in(j+\ell.\mathbb{N})} E^z.u_z, \tag{13}$$

where $u_z\in U_{q,0}$.

First we examine the representation $\tilde{\Theta}_{n,m}$. As we already noted, we have, for $m\in\mathbb{N}^+$ and $0\le j\le\ell-1$, that $\tilde{\Theta}_{n,m}(E^{j+\ell.t})=E_{d_n}^{j+\ell.t}=d_n^t.E_{d_n}^j$, and $\tilde{\Theta}_{n,m}(F^{j+\ell.t})=0$ if $t\neq 0$.

Let $u\in U_q-\{0\}$ and calculate $\tilde{\Theta}_{n,m}(u)$. It is of the form:
$V_{-\ell+1}.\tilde{\Theta}_{n,m}(F^{\ell-1})+\cdots+V_{-1}.\tilde{\Theta}_{n,m}(F)+(V_0+V_\ell.d_n+\cdots+V_{\ell.i}.d_n^i)+\tilde{\Theta}_{n,m}(E).(V_1+V_{1+\ell}.d_n+\cdots+V_{1+i.\ell}.d_n^i)+\cdots+\tilde{\Theta}_{n,m}(E^{\ell-1}).(V_{\ell-1}+V_{\ell-1+\ell}.d_n+\cdots+V_{\ell-1+i.\ell}.d_n^i)$, where $V_z=\tilde{\Theta}_{n,m}(u_z)$ (see (13)).

Either there is $t\in\mathbb{N}$ such that the t^{th} component u_t is nonzero and so in order to show that $\tilde{\Theta}_{n,m}(u)\neq 0$, it suffices to examine the lower triangular part of the matrix. So, it suffices to show that $(V_0+V_\ell.d_n+\cdots+V_{\ell.i}.d_n^i)+\tilde{\Theta}_{n,m}(E).(V_1+V_{1+\ell}.d_n+\cdots+V_{1+i.\ell}.d_n^i)+\cdots+\tilde{\Theta}_{n,m}(E^{\ell-1}).(V_{\ell-1}+V_{\ell-1+\ell}.d_n+\cdots+V_{\ell-1+i.\ell}.d_n^i)\neq 0$. Then by the same reasoning as for $U_{q,0}$, we get that in the above expression the coefficients of the polynomial in d_n are non zero for cofinitely many values of f_m and for such coefficients, the polynomial in d_n is nonzero for cofinitely many values of d_n.

Or, all the positive components of u are zero and there is one negative component $u_{-t} \neq 0$ with $1 \leq t \leq \ell - 1$. Otherwise, if $t \geq \ell$, we have $\tilde{\Theta}_{n,m}(u) = 0$.

So we will only consider the case of $u \in U_{q,\geq 0} = \oplus_{m \geq 0} U_{q,m}$. Let $\mathcal{W}$ be as in Notation 6.1.

Proposition 6.2. *For any $u \in U_{q,\geq 0} - \{0\}$, there exists $W_u \in \mathcal{W}$ such that for all $(n,m) \in W_u$ we have $\tilde{\Theta}_{n,m}(u) \neq 0$. So, the map $[\tilde{\Theta}_{n,m}]_{\mathcal{W}} : U_{q,\geq 0} \to \prod_{\mathcal{W}} M_\ell(\mathbb{C})$ is injective.* $\square$

Then, we examine the other representation $\Theta_{m,n} := \Theta_{a_m,b_m,c_n}$, with $(m,n) \in \omega^2$. In that case, we get an analogous result, but for the whole algebra U_q.

Proposition 6.3. *Let $\Theta_{n,m}$ and $\mathcal{W}$ be as above. For any $u \in U_q - \{0\}$, there exists $W_u \in \mathcal{W}$ such that for all $(n,m) \in W_u$ we have $\Theta_{n,m}(u) \neq 0$. So, the map $[\Theta_{n,m}]_{\mathcal{W}} : U_q \to \prod_{\mathcal{W}} M_l(\mathbb{C})$ is injective.*

Proof:
We decompose $u \in U_q$ as in (13). Note that for $t \in \omega$ and $0 \leq j \leq \ell - 1$, $\Theta_{m,n}(E^{j+\ell.t}) = e^t_{m,n}.E^j_{a_m,b_m,c_n}$, $\Theta_{m,n}(F^{j+\ell.t}) = F^{j+\ell.t}_{b_m} = b^t_m.F^j_{b_m}$ and $\Theta_{m,n}(E^{j+\ell.t}) \sim \Theta_{m,n}(F^{\ell-j+\ell.t'})$, for $0 \leq j \leq \ell$, t, $t' \in \omega$.

Denote $\Theta_{m,n}(u_t) := K_{c_n}^{-s_t}.p_t(C_{q,a_m,b_m,c_n}, K_{c_n})$ by $V_{t,m,n} \in Diag_\ell(\mathbb{C})$.

Now, calculate $\Theta_{m,n}(u)$. It is of the form: $(\Theta_{m,n}(F^{\ell-1})+\Theta_{m,n}(E)).[(V_{\ell-1,m,n}+V_{\ell-1+\ell,m,n}.b_n+\cdots+V_{\ell-1+i.\ell,m,n}.b^i_n)+(V_{1,m,n}+V_{1+\ell,m,n}.e_{m,n}+\cdots+V_{1+i.\ell,m,n}.e^i_{m,n})]+\cdots+(\Theta_{m,n}(F)+\Theta_{m,n}(E^{\ell-1})).(V_1+V_{1+\ell}.b_n+\cdots+V_{1+i.\ell,m,n}.b^i_n)+(V_{\ell-1,m,n}+V_{\ell-1+\ell,m,n}.e_{m,n}+\cdots+V_{\ell-1+i.\ell,m,n}.e^i_{m,n})]+(V_{0,m,n}+V_{-\ell,m,n}.b_n+\cdots+V_{-\ell.i,m,n}.b^i_n+V_{\ell,m,n}.e_{m,n}+\cdots+V_{\ell.i,m,n}.e^i_{m,n})$,

Let us show that if $u \neq 0$, then there exists an element W_u of $\mathcal{W}$ such that if $(n,m) \in W_u$, then $\Theta_{m,n}(u) \neq 0$.

Either $u_{\ell.t} \neq 0$ for some $t \in \omega$, so first for all but finitely many c_m, $\Theta(u_{\ell.t}) \neq 0$, then we fix such a c_m we get a bound on the norm of the matrix $V_{\ell,m,n}.e_{m,n}+\cdots+V_{\ell.i,m,n}.e^i_{m,n}$ and then for all but finitely b_n of modulus bigger than 1, we get a non zero sum $V_{0,m,n}+V_{-\ell,m,n}.b_n+\cdots+V_{-\ell.i,m,n}.b^i_n+(V_{\ell,m,n}.e_{m,n}+\cdots+V_{\ell.i,m,n}.e^i_{m,n})$.

Or $u_{\ell.t} = 0$ for all $t \in \mathbb{Z}$, and for some $z < 0$, $z \notin \ell.\mathbb{Z}$ and $u_z \neq 0$ and the reasoning is similar, or for all $z < 0$, $u_z = 0$, but for some $z > 0$, $z \notin \ell.\mathbb{Z}$ and $u_z \neq 0$ and so we choose a_m of modulus strictly bigger than 1 and we apply the above reasoning with a_m playing the role of b_m. $\square$

Choose an ultrafilter $\mathcal{W}$ on ω^2 as in Notation 6.1. First, we define a map Exp from $\prod_{\mathcal{W}} M_\ell(\mathbb{C})$ to $\prod_{\mathcal{W}} GL_\ell(\mathbb{C})$, simply as $Exp([A_i]_{\mathcal{W}}) := [exp(A_i)]_{\mathcal{W}}$, where $A_i \in M_\ell(\mathbb{C})$, $i \in \omega^2$. Note that $\prod_{\mathcal{W}} M_\ell(\mathbb{C}) \cong M_\ell(\mathbb{C}^{\omega^2}/\mathcal{W})$ (respectively $\prod_{\mathcal{W}} GL_\ell(\mathbb{C}) \cong GL_\ell(\mathbb{C}^{\omega^2}/\mathcal{W})$), so Exp also defines a map from $M_\ell(\mathbb{C}^{\omega^2}/\mathcal{W})$ to $GL_\ell(\mathbb{C}^{\omega^2}/\mathcal{W})$.

Note that if the norm $\|.\|_\ell$ of $(A_i)_{i\in\omega^2}$ is bounded on an element of $\mathcal{W}$, then $Exp([A_i]) = [exp(A_i)] = [\sum_{n=0}^{\infty} \frac{A_i^n}{n!}]$ can be viewed as a limit up to an infinitesimal element of $M_\ell(\mathbb{C})$ of the sequence $(\sum_{n=0}^{m} \frac{[A_i]^n}{n!})_{m\in\omega}$.

Indeed, the sequence in $M_\ell(\mathbb{C}^{\omega^2}/\mathcal{W})$ of matrices $([\sum_{n=0}^{m} \frac{A_i^n}{n!}])_{m\in\omega}$ is bounded and Cauchy. Indeed, the norm $\|\sum_{n=0}^{m_1} \frac{A_i^n}{n!}\| \leq \sum_{n=0}^{m} \frac{\|A_i\|^n}{n!} \leq e^{\|A_i\|}$ and for any $\epsilon \in \mathbb{R}^+$, there exists

$m \in \mathbb{N}^+$ such that for any $m_1 > m_2 > m$, $\| \sum_{n=0}^{m_1} \frac{[A_i]^n}{n!} - \sum_{n=0}^{m_2} \frac{[A_i]^n}{n!} \| \leq \sum_{n=m_2+1}^{m_1-1} \frac{[A_i]^n}{n!} \leq \frac{\|A_i\|^{m_2+1}}{(m_2+1)!}.e^{\|A_i\|} \leq \frac{\|A_i\|^{m+1}}{(m+1)!}.e^{\|A_i\|}$.

Finally $\|[\sum_{n=0}^{m} \frac{A_i^n}{n!}] - [exp(A_i)]\| = [\| \sum_{n=0}^{m} \frac{A_i^n}{n!} - exp(A_i)\|] = [\| \sum_{n=m+1}^{\infty} \frac{A_i^n}{n!} \|] \leq [\frac{\|A_i\|^{m+1}}{(m+1)!}.e^{\|A_i\|}]$.

Let $A_i \in M_\ell(\mathbb{C})$. Following the discussion of [9] Theorem 3.1, we calculate $exp(A_i)$ (for the reader convenience, we reproduce it below). We use the Jordan form of A_i, A_i can be written uniquely as a sum $B_i + C_i$, where B_i is diagonalizable and C_i is nilpotent of class $\leq \ell - 1$ and B_i commutes with C_i. So, we can explicitely calculate $exp(A_i) = exp(B_i).exp(C_i) = exp(B_i).(I + C_i + \cdots + \frac{C_i^{\ell-1}}{(\ell-1)!})$. Since B_i is diagonalizable, there exists an invertible matrix D_i such that $D_i^{-1}.B_i.D_i = (b_{i1}, \cdots, b_{i\ell})$, where $b_{ij} \in \mathbb{C}$, $1 \leq j \leq \ell$, are the eigenvalues of B_i. So, $exp(B_i) = D_i.(e^{b_{i1}}, \cdots, e^{b_{i\ell}}).D_i^{-1}$. Now, $[exp(A_i)] = [D_i].(e^{[b_{i1}]}, \cdots, e^{[b_{i\ell}]}).[D_i]^{-1}.(I + [C_i] + \cdots + \frac{[C_i]^{\ell-1}}{(\ell-1)!})$. In particular, $(M_\ell(\mathbb{C}^{\omega^2}/\mathcal{W}), Exp, GL_\ell(\mathbb{C}^{\omega^2}/\mathcal{W}))$ is interpretable in the structure $(\mathbb{C}^{\omega^2}/\mathcal{W}, e^x)$.

As previously, we define EXP from U_q to $\prod_{\mathcal{W}} GL_\ell(\mathbb{C}))$ by

$$EXP(u) = [exp \circ \Theta_{a,b,c}(u)]_{\mathcal{W}}$$

and similarly by

$$E\tilde{X}P(u) := [exp \circ \tilde{\Theta}_{d,f}(u)]_{\mathcal{W}}.$$

Then $(U_q, EXP, GL_\ell(\mathbb{C}^{\omega^2}/\mathcal{W}))$ (respectively $(U_q, E\tilde{X}P, GL_\ell(\mathbb{C}^{\omega^2}/\mathcal{W})))$ is an exponential $\mathbb{C}$-algebra and as such embeds in $(M_\ell(\mathbb{C}^{\omega^2}/\mathcal{W}), Exp, GL_\ell(\mathbb{C}^{\omega^2}/\mathcal{W}))$.

On the image of U_q in $M_\ell(\mathbb{C}^{\omega^2}/\mathcal{W}$, we can say the following. Note that the trace of K_c (respectively K_f) is equal to $c.(1 + q^{-2} + \cdots + q^{-2\ell+2}) = c.\frac{1-q^{-2.\ell}}{1-q^{-2}} = 0$ (respectively $f.(1 + q^2 + \cdots + q^{2\ell-2}) = f.\frac{1-q^{2.\ell}}{1-q^2} = 0$) and so the image of K by $exp \circ \Theta_{a,b,c}$ (respectively $exp \circ \Theta_{d,f}$) will belong to $SL_\ell(\mathbb{C})$, as well as the images of E^i, F^j, for $i, j \in \mathbb{Z} - \ell.\mathbb{Z}$.

7. Approximation

In this section, using ultraproducts and the representations of U_q, we will relate U and the quantum algebras U_q, for q a root of unity.

One known way to view U as a limit of the U_q's is to use another presentation of U_q by adding one more generator, which will allow us to set $q = 1$. Let $\tilde{U}_q$ be this new isomorphic presentation of U_q and then one gets U as a quotient of $\tilde{U}_1/ < K - 1 >$ (see [7] page 58 and [6] chapter VI.2.2).

For $k = \mathbb{C}$, a heuristic way to see U as the limit of U_q for $q \to 1$, is to proceed as follows ([7] pages 6, 57). Recall that U as an associative $\mathbb{C}$-algebra is generated by X, Y, H and defining relations $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$.

Formally write $q = e^\theta$ with $\theta \in \mathbb{C}$, make the change of variables $K := e^{\theta H}$ with H a new variable. Consider the limit $\theta \to 0$. First, differentiate with respect to θ the relation $[K, E] = K.E - E.K = (K.E.K^{-1} - E).K = (q^2 - 1).E.K$. We get $2.e^\theta.E.e^{\theta.H} + (e^{2\theta} - 1).E.H.e^{\theta.H}$ and take the value at $\theta = 0$. We obtain $2.E$ on one hand; on the hand this is also equal to $[H, E]$ since H is equal to the derivative of K with respect to θ, evaluated at $\theta = 0$. The calculation is similar for $[H, F] = -2F$. Then, if we take the value at $\theta = 0$ of the relation $[E,F]=\frac{K-K^{-1}}{q-q^{-1}}$, using L'Hospital rule, we get the relation $[E,F]=H$. These are the relations of U.

Let us consider $q_\ell=e^{\theta_\ell}$, where $\theta_\ell=\frac{2\pi.i}{\ell}$ and a non-principal ultraproduct of U_{q_ℓ}, $\ell\in\omega$, over a non principal ultrafilter $\mathcal{U}$ over ω. Denote the generators of U_{q_ℓ} by E_ℓ, F_ℓ and K_ℓ. Consider the $\mathbb{C}$-algebra homomorphism τ_ℓ from U to U_{q_ℓ} sending X to E_ℓ, Y to F_ℓ (and so H to $\frac{K_\ell-K_\ell^{-1}}{q_\ell-q_\ell^{-1}}$). Define the map $\tau:=[\tau_\ell]_{\mathcal{U}}$ from U to $\prod_{\mathcal{U}}U_{q_\ell}$.

Proposition 7.1. *The map $\tau:U\to\prod_{\mathcal{U}}U_{q_\ell}$ is injective.*

Proof: It is useful to remind that U, as a $\mathbb{Z}$-graded algebra, can be written as a infinite sum of m-homogenous components, $m\in\mathbb{Z}$, namely $U=\sum_{m\in\mathbb{Z}}U_m$; furthermore note that if m is positive $U_m=X^m.U_0$, if m is negative $U_m=Y^m.U_0$, and the 0-component U_0 coincides with the ring of polynomials $\mathbb{C}[C,H]$ where C is the (classical) Casimir element $C=2XY+2YX+H^2$ (which generates the center of U).

Recall that we defined, for each root of unity q_ℓ, representations maps $\Theta_{a,b,c}$ from U_q to $M_\ell(\mathbb{C})$, where ℓ is the order of q.

We will compose the map τ with the representation maps $[\Theta_{a,b,c}]_{\mathcal{U}}$ from $\prod_{\mathcal{U}}U_q$ to $\prod_{\mathcal{U}}M_\ell(\mathbb{C})$ and we will show that we can choose $a,b,c\in\mathbb{C}$ such that this composition is injective on U.

First, we will assume that $u\in U_0$.

Let $u=p(C,H)$ be a nonzero element of U_0, where $p(x_1,x_2)\in\mathbb{C}[x_1,x_2]-\{0\}$. Write $p(x_1,x_2)=\sum_{j=0}^d s_j(x_1).x_2^j$, where $s_j\in\mathbb{C}[x_1]$. So the image $\tau(p(C,H))=p(\tau(C),\tau(H))=\sum_{j=0}^d s_j(\tau(C)).\tau(H)^j$ in the ultraproduct is a polynomial in the image of H and its coefficients are polynomials in the image of C. We evaluate the polynomials $s_j(x_1)$ at $[2E_\ell F_\ell+2F_\ell E_\ell+(\frac{K_\ell-K_\ell^{-1}}{q_\ell-q_\ell^{-1}})^2]_{\mathcal{U}}$ on one hand and the polynomial $\sum_{j=0}^d s_j([2E_\ell F_\ell+2F_\ell E_\ell+\frac{K_\ell-K_\ell^{-1}}{q_\ell-q_\ell^{-1}}^2]_{\mathcal{U}}).x_2^j$ at $[\frac{K_\ell-K_\ell^{-1}}{q_\ell-q_\ell^{-1}}]_{\mathcal{U}}$ on the other hand.

Now, let us show that if $p(C,H)\neq 0$, then $[\Theta_{a,b,c}]_{\mathcal{U}}(p([\tau(C)]_{\mathcal{U}},[\tau(H)]_{\mathcal{U}}))\neq 0$. So, we will have that $\tau(p(C,H))=p([\tau(C)]_{\mathcal{U}},[\tau(H)]_{\mathcal{U}})\neq 0$.
Let $[\Theta_{a,b,c}]_{\mathcal{U}}(\tau(p(C,H)))$=

$$\begin{aligned}
&= [\Theta_{a,b,c}(\tau_\ell(p(C,H)))]_{\mathcal{U}} = \\
&= \left[\Theta_{a,b,c}\left(p\left(2E_\ell.F_\ell+2F_\ell.E_\ell+\frac{(K_\ell-K_\ell^{-1})^2}{(q_\ell-q_\ell^{-1})^2},\frac{K_\ell-K_\ell^{-1}}{q_\ell-q_\ell^{-1}}\right)\right)\right]_{\mathcal{U}} 0 \\
&= \Big[p\Big(2\Theta_{a,b,c}(E_\ell).\Theta_{a,b,c}(F_\ell)+2\Theta_{a,b,c}(F_\ell).\Theta_{a,b,c}(E_\ell)+ \\
&+ \left.\left.\frac{(\Theta_{a,b,c}(K_\ell-K_\ell^{-1}))^2}{(q_\ell-q_\ell^{-1}))^2},\frac{\Theta_{a,b,c}(K_\ell-K_\ell^{-1})}{q_\ell-q_\ell^{-1}}\right)\right]_{\mathcal{U}}
\end{aligned}$$

Now if we fix ℓ, the entries of the diagonal matrix $\Theta_{a,b,c}(\tau_\ell(p(C,H)))$ are of the form

$$p\left(2(e_{s+1}+e_s)+\left(\frac{cq_\ell^{-2s}-c^{-1}q_\ell^{2s}}{q_\ell-q_\ell^{-1}}\right)^2,\frac{cq_\ell^{-2s}-c^{-1}q_\ell^{2s}}{q_\ell-q_\ell^{-1}}\right)=$$

$$\sum_{j=0}^{d} s_l \left(2(e_{s+1}+e_s) + \left(\frac{cq_\ell^{-2s} - c^{-1}q_\ell^{2s}}{q_\ell - q_\ell^{-1}} \right)^2 \right) \cdot \left(\frac{cq_\ell^{-2s} - c^{-1}q_\ell^{2s}}{q_\ell - q_\ell^{-1}} \right)^j,$$

with $0 \leq s \leq \ell - 1$, setting that $e_0 = e_\ell = a.b$. In order to ensure that for cofinitely many values of ℓ, the entries of this matrix is non-zero, we will choose a, b, c in a certain way.

First assume that $p(x_1, x_2)$ and its complex conjugate $\bar{p}$ have no irreducible factors in common. Then, as the proof of Proposition 6.1, in order to get a contradiction, it suffices to show that p and $\bar{p}$ have infinitely many common distinct roots. We will use the fact that if $c \in i.\mathbb{R}$, then $\bar{c} = -c$. If $s = 0$, then the first diagonal entry of the matrix is of the form $p(2.(e_1 + ab) + (\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}})^2, \frac{c-c^{-1}}{q_\ell - q_\ell^{-1}})$ and $e_1 = ab + \frac{c^{-1}-c}{q_\ell - q_\ell^{-1}}$ (since $[1] = 1$). So $\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}} \in \mathbb{R}$ and $2.(e_1 + ab) + (\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}})^2$ belongs to $\mathbb{R}$, whenever $ab \in \mathbb{R}$. So we have a common root of p and $\bar{p}$. Varying q_ℓ over a set of primitive roots of unity with distinct imaginary parts and assuming that $c \in i\mathbb{R}$ and $(c^{-1} - c)^{-1} \notin \{\frac{1}{q_{\ell_1} - q_{\ell_1}^{-1}} + \frac{1}{q_{\ell_2} - q_{\ell_2}^{-1}} : \ \ell_1 \neq \ell_2\}$, we get infinitely many distinct common roots.

Now suppose that $p(x_1, x_2)$ and its complex conjugate $\bar{p}$ have an irreducible factor in common. So, they have a common factor with real coefficients.

Assume that $p(x_1, x_2)$ has real coefficients. We write it now as a polynomial in x_1 with as coefficients polynomials in x_2. So if we choose $c \in i\mathbb{R}$, its coefficients belong to $\mathbb{R}$. Set $r := 2.\frac{c^{-1}-c}{q_\ell - q_\ell^{-1}} + (\frac{c-c^{-1}}{q_\ell - q_\ell^{-1}})^2$ (note that $r \in \mathbb{R}$) and if we choose $a.b$ such that $(r + 4a.b)^2 \notin \mathbb{R}$ (equivalently $r.ab + 2.(ab)^2 \notin \mathbb{R}$), we arrive to a contradiction. Again, we get that $a.b$ has to avoid a certain subset depending on q_ℓ; for instance, we can choose $ab \in i\mathbb{R}$.

Suppose now that $u \notin U_0$. So there exists $m \neq 0$ such that $u_m \neq 0$. Let m be maximal in absolute value such that $u_m \neq 0$. If $m > 0$, write $u_m = X^m.p_m(C, H)$ and if $m < 0$, write $u_m = Y^m.p_m(C, H)$, with $p_m(C, H) \in U_0 - \{0\}$. Set $\Theta_{a,b,c}(F_\ell) = F_b$ and $\Theta_{a,b,c}(E_\ell) = E_{a,b,c}$). Then for $\ell > 2m$, we have that F_b^m and $E_{a,b,c}^m$ have no entries in common.

If u has a non-zero components u_m with $m > 0$ (respectively $m < 0$), then we consider the product of the two matrices $E_{a,b,c}^m$ and $p_m(\Theta_{a,b,c}(C), \Theta_{a,b,c}(H))$ (respectively F_b^m and $p_m(\Theta_{a,b,c}(C), \Theta_{a,b,c}(H))$). The non-zeroes entries of the corresponding permutation matrix are of the form $e_s. \cdots .e_{s+m}.p(2(e_{s+1} + e_s) + (\frac{cq_\ell^{-2s} - c^{-1}q_\ell^{2s}}{q_\ell - q_\ell^{-1}})^2, \frac{cq_\ell^{-2s} - c^{-1}q_\ell^{2s}}{q_\ell - q_\ell^{-1}})$ (respectively $b.p(2(e_{s+1} + e_s) + (\frac{cq_\ell^{-2s} - c^{-1}q_\ell^{2s}}{q_\ell - q_\ell^{-1}})^2, \frac{cq_\ell^{-2s} - c^{-1}q_\ell^{2s}}{q_\ell - q_\ell^{-1}}))$ with $p_m(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ and $1 \leq s \leq \ell$ with the convention that $\ell + s$ is calculated modulo ℓ. So, it suffices to evaluate the coefficient corresponding to the case when $s = \ell$. $\square$

Note that by composing the map τ with the exponential maps on U_{q_ℓ}, we get possibly new exponential maps on U.

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