INFINITELY DEFINABLE STRUCTURES IN SMALL THEORIES

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ABSTRACT. We observe simple links between preorders, semi-groups, rings and categories (and between equivalence relations, groups, fields and groupoids), which are infinitely definable in an arbitrary structure, and apply these observations to small structures. Recall that a structure is *small* if it has countably many pure n-types for each integer n. A category defined by a pure n-type in a small structure is the conjunction of definable categories. For a group G_A defined by an n-type over some arbitrary set A in a small and simple structure, we deduce that

1) if G_A is included in some definable set such that boundedly many translates of G_A cover X, then G_A is the conjunction of definable groups.
2) for any finite tuple g in G_A, there is a definable group containing g.

In a universe $\mathfrak{M}$, an *infinitely A-definable* set, instead of being defined by a formula, is the conjunction of infinitely many formulae with parameters in some set A. An *infinitely A-definable* structure in $\mathfrak{M}$ is any structure whose domain, functions and relations are infinitely A-definable in $\mathfrak{M}$.

Definition. Let $\mathfrak{L}$ be a language, S a set of $\mathfrak{L}$-structures, and A an element of S which is infinitely definable in $\mathfrak{M}$. We say that $\mathfrak{M}$ *loosely envelopes A with respect to S* if A is contained in some definable structure belonging to S. We say that $\mathfrak{M}$ *envelopes A with respect to S* if A is the conjunction of definable structures in S.

In the sequel, the set S will consist either of groups, semi-groups, fields, rings, preorders, equivalence relations, categories or groupoids and will be obvious from the context. For instance, we shall say that a structure envelopes an infinitely definable group G to say that G is the conjunction of definable groups.

Note that being enveloped is strictly stronger that being loosely enveloped. A stable structure is known to envelope infinitely definable groups and fields [1, Hrushovski]. Consequently, in an omega-stable structure, an infinitely definable group is definable, as is an infinitely definable field in a superstable structure. Pillay and Poizat proved that an infinitely $\emptyset$-definable equivalence relation on a small structure is enveloped, provided that it be coarser than the equality of pure types [8]. Kim generalised Pillay and Poizat's result to arbitrary infinitely $\emptyset$-definable equivalence relations on a small structure [3]. In [10], Wagner deduces from Kim's result that if a small structure loosely envelopes an infinitely $\emptyset$-definable group of finite arity, it must envelope it. He asked whether an infinitely $\emptyset$-definable group in a small structure should be enveloped [10, Problem 6.1.14]. We shall show

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Theorem. *An infinitely $\emptyset$-definable category of finite arity in a small structure is the conjunction of definable categories.*

As the notion of category both generalises preorders and semi-groups, the latter includes Kim's result and gives a positive answer to Wagner's question. It also gives a similar conclusion for an infinitely $\emptyset$-definable *groupoid*, which is a category where every morphism is invertible. As infinitely definable groupoids arise naturally in some structures [2], this result might have an interest in itself. If we want to look at infinitely definable categories over infinitely many parameters, we have to assume additional conditions, in our case that the ambiant theory be simple. We obtain :

Theorem. *Let G_A be an infinitely A-definable group of finite arity in a small and simple structure.*

i) *If G_A is included in some definable set such that boundedly many translates of G_A cover X, then G_A is the conjunction of definable groups.*
ii) *For any finite tuple g in G_A, there is a definable group containing g.*

For an infinitely A-definable field K_A of finite arity in a small and simple structure, the latter statement provides definable fields around every point, which give information about the structure of K_A : it must be algebraically closed, and in positive characteristic, commutative.

1. A few words on structures enveloping algebraic structures

In the sequel, everything is inside some arbitrary universe $\mathfrak{M}$, who may have additional properties in the following sections.

Definition 1.1. A set X is *infinitely A-definable* in $\mathfrak{M}$ if it is a subset of $\mathfrak{M}^\alpha$ for some ordinal α, which is defined by a partial type with parameters in A. We call α the *arity* of X in $\mathfrak{M}$.

An *infinitely A-definable* structure is any structure whose domain, functions and relations are infinitely A-definable. When considering infinitely definable groups in $\mathfrak{M}$ (and more generally, infinitely definable structures satisfying a given set of axioms T), we suppose that its type still defines a group (respectively still satisfies the axioms of T) in any elementary extension of $\mathfrak{M}$. In this section, every infinitely definable set considered will have finite arity in $\mathfrak{M}$. As we make no assumption on the ambiant structure, we may also asssume in this section that every infinitely definable set considered is definable without parameters, by expanding the langage with possible parameters.

1.1. Equivalence relations, groups and fields.

We begin by recalling from [7] that if f is an infinitely definable map between two infinitely definable sets A and B of finite arity, by compactness, f is the trace of a definable map, so we may assume all the laws to be definable in this section. We go on by stating simple, but new observations.

Proposition 1.2. *If $\mathfrak{M}$ envelopes every equivalence relation, it also envelopes every group.*

Proof. Let G be an infinitely definable group in $\mathfrak{M}$. By compactness, there exists a definable set X_0 such that the group law be associative on X_0, and such that each element of X_0 have a unique inverse in G_0. Let $X_1, X_2, \dots$ be a chain of definable subsets of X_0 whose intersection equals G. As G is stable by multiplication, by compactness there is some X_j, say X_1, such that $X_1 \cdot X_1$ be a subset of X_0. We consider the equivalence relation E on X_0 saying that x and y are related if $xy^{-1} \in G$. By hypothesis, E is the conjunction of definable equivalence relations E_i. Note that an element g in X_0 belongs to G if and only if $1Eg$. By compactness, there exists some index j such that the set $\{x \in X_0 : xE_j1\}$ be included in X_1. Let J be the set $\{x \in X_1 : xE_j1\}$. The set $G \cdot J$ is included in J. For instance, if $g \in G$ and if $y \in J$, then g equals gyy^{-1} and gy belongs to X_0, so $gyEy$, hence gyE_j1 by transitivity. Thus, if H denotes the left stabiliser of J, that is, the set $\{g \in X_0 : gJ \subset J\}$, then the set $H \cap H^{-1}$ is a group in X_0 containing G. Note that if the ambient structure envelopes every equivalence relation, so does every X_i. It follows that every X_i contains some definable group H_i around G, and G is the conjunction of every H_i. □

Remark 1.3. Note that the converse fails, as there are superstable structures that do not envelope every equivalence relation [8, Exemple 2], but next section will show that we just need to remove the symmetry assumption to make the equivalence true.

Remark 1.4. Let E be an infinitely definable equivalence relation, and let E^* the infinitely definable equivalence relation defined by

$$xE^*y \text{ if and only if there exists some } b \text{ such that } tp(b) = tp(x) \text{ and } bEz$$

Recall from [8, Pillay Poizat] that E is enveloped if and only if so is every restriction of E to a complete type, and so is E^*. Actually, replacing in the previous proof the equivalence relation $xy^{-1} \in G$ by $\exists b \models tp(x) \wedge by^{-1} \in G$, one shows that the structure $\mathfrak{M}$ needs only envelope the equivalence relations coarser than the equality of types to envelope every group.

Proposition 1.5. *If $\mathfrak{M}$ envelopes every group, it also envelopes commutative fields, and envelopes every (possibly skew) field with respect to integral rings.*

Proof. Let K be an infinitely definable field in this structure. By compactness, there is a definable set X containing K such that addition and multiplication be associative on X, and such that multiplication be distributive over addition. We may also assume that every element in X have an additive and multiplicative inverse, and set 0^{-1} equal to 0. Replacing X by $X \cap -X \cap X^{-1} \cap -X^{-1}$, we may assume that X equals $-X$ and X^{-1}. So X is integral. By hypothesis there exists a definable additive group H inside X and around K^+, and also a definable multiplicative group M inside H and around $K^\times$. Let S be the set $\{h \in H : M \cdot h \subset H\}$. This is an additive subgroup of H stabilised by left multiplication by M. Let L be the set $\{h \in H : h \cdot S \subset S\}$. This is a definable integral ring containing M. If multiplication is commutative, the product $L \cdot L^{-1}$ is a field containing K : for every a, b, c, d in L, one has the equality

$$ab^{-1} + cd^{-1} = (bd)^{-1}(ad + cb)$$

□

1.2. **Preorders, semi-groups, rings and categories.** We call a *preorder* any binary relation which is reflexive and transitive. A *semi-group* is any set with an associative binary operation. A semi-group might have no identity element.

Proposition 1.6. *If $\mathfrak{M}$ envelopes every preorder, it also envelopes every semi-group.*

Proof. Let M be an infinitely definable semi-group. We may add a new constant 1 to M, and the set $\{(1,x,x),(x,1,x) : x \in M \cup 1\}$ to the graph of multiplication, and assume that M has an identity. Let X_0 be a definable set containing M where the law is associative. Let X_1 be a definable set containing M such that $X_1 \cdot X_1 \subset X_0$. We consider the preorder R on X_0 defined by xRy if and only if $x \in My$, and finish as in the group case. □

We shall show in the sequel that the converse is also true (see Propositions 1.9 and 1.10). As in the field case, and with a similar proof, we have

Proposition 1.7. *If $\mathfrak{M}$ envelopes every semi-group, it also envelopes every (possibly non-commutative) ring.*

Definition 1.8. A *category* is a two-sorted structure, the objects O, and morphisms M, together with applications i_0 and i_1 from M to O (saying that the morphism m from M goes from $i_0(m)$ to $i_1(m)$), a partial associative composition map $\circ$ from $M \times_{i_0,i_1} M$ to M ($m \circ n$ is defined when $i_0(m)$ equals $i_1(n)$), and an identity map Id from O to M (such that $Id(x)$ be the identity morphism from x to x).

On the sorts of objects of a given category, one can define a preorder by setting $a \leq b$ if there is a morphism from a to b, as well as semi-groups M_a whose elements are morphisms from a to a for any object a. Conversely, on the one hand, a preorder $\leq$ is a category with trivial semi-groups, and with one morphism for every couple a, b satisfying $a \leq b$. On the other hand, a semi-group is a category with one single object and morphisms given by right multiplication by any element. Hence, the notion of category generalises both preorders and semi-groups.

Proposition 1.9. *If $\mathfrak{M}$ envelopes any semi-group, it also envelopes any category.*

Proof. Let C be an infinitely definable category, with objects O and morphisms M. The set M has a partial structure of semi-group with law $\circ$, which can be extended to the whole of M : let o be a new object and 0 a new morphism from o to o. Let $\overline{O}$ equal $O \cup \{o\}$, and $\overline{M}$ equal $M \cup \{0\}$. We extend i_0, i_1 and $\circ$ respectively to $\bar{i}_0$, $\bar{i}_1$ and $\bar{\circ}$ by setting $\bar{i}_0(0) = \bar{i}_1(0) = o$ and $0 \bar{\circ} 0 = 0 \bar{\circ} m = m \bar{\circ} 0 = m \bar{\circ} n = 0$ for all morphisms m, n such that $i_0(m) \neq i_1(n)$; the law $\bar{\circ}$ is still infinitely definable (as O has finite arity), and associative over $\overline{M}$. By hypothesis, $\overline{M}$ is the conjunction of definable semi-groups $\overline{M}_i$. By compactness, i_0 and i_1 are defined on $\overline{M}_i$ for sufficiently large i. Let M_i equal $\overline{M}_i$ minus 0 and let O_i equal $i_0(M_i) \cup i_1(M_i)$. (O_i, M_i) is not a category yet as the map Id need not be defined on O_i. But the equalities $Id(i_1(m)) \circ m = m$ and $n \circ Id(i_0(n)) = n$ hold for all m, n in M. By compactness, they must still hold for every m, n in M_i for some sufficiently large i. In particular, Id is defined on O_i. □

Proposition 1.10. *If $\mathfrak{M}$ envelopes any category, it also envelopes any preorder.*

Proof. A preorder $\leq$ on some set X is a category C with objects X, morphisms $\{(x,y) : x \leq y\}$, and maps i_0, i_1, $\circ$ and Id defined by $i_0(x,y) = x$, $i_1(x,y) = y$, $(x,y) \circ (y,z) = (x,z)$, and $Id(x) = (x,x)$. By hypothesis, if $\leq$ is infinitely definable, it is the conjunction of definable categories C_i. By compactness, for sufficiently large i, the category C_i is a preorder, ie there is at most one morphism between every ordered pair of objects. □

Definition 1.11. A *groupoid* is a category whose morphisms are invertible.

Note that this generalises both the notions of groups and equivalence relations.

Remark 1.12. Similarly to the proof of Proposition 1.10, a structure which envelopes any groupoid also envelopes any equivalence relation.

Proposition 1.13. *$\mathfrak{M}$ envelopes any equivalence relation if and only if it envelopes any groupoid.*

Proof. We adapt the proof from the group case. Let G be a groupoid, and let O and M be its objects and morphisms. By compactness, there are definable sets X_O and X_M containing O and M, such that i_0 and i_1 be defined over X_M, and such that Id be defined over X_O, and $\circ$ associative and defined over X_M, with in addition the equality $Id(i_1(m)) \circ m = m \circ Id(i_0(m)) = m$ holding for every m in X_M. We may assume that X_M equal X_M^{-1}. By compactness, there is some definable Z_M containing M with $Z_M \circ Z_M$ included in X_M. Let E be the equivalence relation over X_M defined by

$$xEy \iff i_0(x) = i_0(y) \wedge x \circ y^{-1} \in M$$

By hypothesis, E is the conjunction of definable equivalence relations E_i. Any element x of X_M belongs to M if and only if $xE\ Id(i_0(x))$; by compactness, there is some index j such that the inclusion $\{x \in X_M : xE_jId(i_0(x))\} \subset Z_M$ holds. Let J equal $\{x \in Z_M : xE_jId(i_0(x))\}$: it is stabilised by left multiplication by M. Namely, if g is in M and y in J, and if $i_0(g)$ equals $i_1(y)$ then

$$g = g \circ Id(i_0(g)) = g \circ y \circ y^{-1}$$

so $g \circ y$ is in X hence $g \circ yEy$, thus $g \circ yE_jId(i_0(y))$. Let H be the set $\{x \in J : x \circ J \subset J\}$. H is closed under composition. $(X_O, H \cap H^{-1} \cup Id(X_O))$ is a groupoid containing G. □

2. Application to small structures

Definition 1. A structure is *small* if it has countably many n-types without parameters for every integer n.

In this section, we assume the ambiant structure $\mathfrak{M}$ to be small. We recall a theorem of Kim, using a result of Pillay and Poizat :

Fact 2.1. (Kim-Pillay-Poizat [8, 3]) *A small structure $\mathfrak{M}$ envelopes every infinitely $\emptyset$-definable equivalence relation over $\mathfrak{M}$.*

Note that [5, Krupiński, Newelski] gives an analytic proof of the previous theorem.

Remark 2.2. As $\mathfrak{M}$ is small, every finite cartesian power of $\mathfrak{M}$ is again small. The result fails for an infinitely definable equivalence relation over some infinite cartesian power of $\mathfrak{M}$, even in a $\aleph_0$-categorical structure : if $\mathfrak{M}$ is a dense linear order without end points, take the relation E over $\mathfrak{M}^{\mathbf{Q}}$ saying that xEy if and only if $x_i < y_j$ and $y_i < x_j$ for every $i < j$.

According to our previous observations, this answers Wagner's problem 6.1.14 in [10], and shows that a small structure envelopes any infinitely $\emptyset$-definable group, field and groupoid of finite arity. Recall that a definable small field is either finite or algebraically closed [9], and that a small field of positive characteristic cannot be skew [6].

Corollary 2.3. *In a small structure, an infinitely $\emptyset$-definable field of finite arity is finite or algebraically closed, and in positive characteristic, commutativity need not be assumed but follows.*

Three main questions arise : what happens for infinitely $\emptyset$-definable groups of infinite arity? For infinitely $\emptyset$-definable semi-groups (even of finite arity)? And for infinitely A-definable groups, where the set of parameters A is allowed to be infinite? We tackle the two first questions in the next paragraphs, and give a partial answer to the third question in section 3.

2.1. **Preorders and semi-groups of finite arity.** The following proposition is inspired from [8] and [3].

Proposition 2.4. *A closed preorder on a denumerable Hausdorff compact space is the conjunction of clopen preorders.*

Proof. Let X be this Hausdorff compact space, and R a closed preorder over X. Let S^c stand for the complement of any subset S of X. The space X has a clopen basis, and R is a closed set of tuples in $X \times X$. If (x, y) is not in R, there exists a basic open set $O_1 \times O_2$ outside R containing the tuple (x, y) ; the set $O_1 \cap O_2$ is empty as R is reflexive. We choose O_1 and O_2 such that $(O_1 \cup O_2)^c$ have minimal Cantor-Bendixson rank and degree, and write Y for $(O_1 \cup O_2)^c$. We show that Y is empty ; otherwise, let y be in Y with maximal rank. If $(O_1 \times \{y\}) \cap R$ and $(\{y\} \times O_2) \cap R$ are both non-empty, as R is transitive, $(O_1 \times O_2) \cap R$ is also non-empty, a contradiction. We may assume $(O_1 \times \{y\}) \cap R$ to be empty. The set $O_1 \times \{y\}$ is contained in the open set R^c. So we can choose a basic open set Q_2 containing y with $O_1 \times Q_2 \subset R$. But $O_1 \times (Q_2 \cup O_2)$ is outside R. So $(O_1 \cup O_2 \cup Q_2)^c$ equals $Y^c \cap Q_2^c$, which misses y, a contradiction with the degree of Y being minimal. So Y is empty, X equals $O_1 \cup O_2$, and $O_1 \times O_1^c \subset R^c$. Therefore, $R \subset (X \times O_1) \cup (O_1^c \times X)$, and $(a, b) \in R$ implies $(a, b) \in R_{x,y}$ where $R_{x,y}$ is the preorder defined by

$$(a, b) \in R_{x,y} \iff \big(a \in O_1 \Rightarrow b \in O_1\big)$$

We have shown that $(a, b) \in R$ is equivalent to $\bigwedge_{(x,y)\in R^c} \big((a, b) \in R_{x,y}\big)$. □

Corollary 2.5. *A small structure envelopes any infinitely $\emptyset$-definable preorder of finite arity n which is coarser than equality between n-types without parameters.*

Proof. Such a preorder $\leq$ induces a closed preorder $\lesssim$ on the space of n-types, defined by

$$tp(a) \lesssim tp(b) \iff a \leq b$$

By Proposition 2.4, the preorder $\lesssim$ is the conjunction of definable preorders. □

Proposition 2.6. *A small structure envelopes every infinitely $\emptyset$-definable semi-group of finite arity.*

Proof. Let M be this semi-group. Without loss of generality or smallness, we may assume that M have a unit, and add it to the language. There is a definable set X containing M such that the law be associative on X. Let R be a preorder on X defined by

$$xRy \iff \exists z \models tp(y)\ (x \in Mz)$$

Note that if x and y have the same type over $\emptyset$, then x and y are in relation by R. By Corollary 2.5, R is the conjunction of definable preorders R_i. Note that $m \in M$ if and only if $mR1$. By compactness, there is some j such that $\{x \in X : xR_j1\} \subset X$. Let J be the set $\{x \in X : xR_j1\}$. It is left stabilised by M : if m is in M and y in J, then $my \in My$, so $myRy$, thus myR_j1. Consider the left stabiliser of J in X : it is a semi-group containing M. □

Remark 2.7. Note that by compactness, an infinitely definable semi-group is the conjunction of infinitely definable groups defined by countable types. It follows from Proposition 2.6 that an ω-stable structure envelopes any infinitely definable semi-group with parameters in an arbitrary set.

From Propositions 1.9 and 1.10, it follows :

Corollary 2.8. *A small structure envelopes any infinitely $\emptyset$-definable preorder of finite arity.*

2.2. **Semi-groups of arbitrary arity.** A semi-group G with identity 1_G is said to *almost act on* a set X if there is a map $G \times X \to X$. It *acts on* X if in addition, for all g, h, x in $G \times G \times X$, the equalities $(gh) \cdot x = g \cdot (h \cdot x)$ and $1_G \cdot x = x$ hold.

Lemma 2.9. *In the small structure $\mathfrak{M}$, let p be a partial type of finite arity, and let X be the set $\{x \in \mathfrak{M}^\omega :\models p(x)\}$. Let G be a semi-group acting on X so that the action be infinitely $\emptyset$-definable in $\mathfrak{M}$. Then, there are formulae f_i, such that X be the intersection of sets of the form $\{x \in \mathfrak{M}^\omega :\models f_i(x)\}$ on which G almost acts (with the same map).*

Proof. Let f_0 be any formula in p, and let X_0 be the set $\{x \in \mathfrak{M}^\omega :\models f_0(x)\}$. By compactness, there is some formula f_1 in p such that $G \cdot X_1 \subset X_0$, where X_1 is the set $\{x \in \mathfrak{M}^\omega :\models f_1(x)\}$. Let $X_2, X_3 \ldots$ be a sequence of definable subsets of X_1 whose conjunction is X. Let E be the equivalence relation on X_0 defined by

$$xEy \iff \exists g \in G\ (g \cdot x = y)$$

E is the conjunction of definable equivalence relations E_i. Note that $x \in X$ if and only if there exists some $a \in X$ with aEx. So there is some index j such that $\{x \in \mathfrak{M}^\omega : \exists a \in X_j,\ aE_jx\} \subset X_1$. We show that G acts on $\{x \in \mathfrak{M}^\omega : \exists a \in X_j,\ aE_jx\}$. We call Y the latter set, and take some g in G and x in Y ; the product $g \cdot x$ is in X_0 so $xEg \cdot x$, hence $xE_jg \cdot x$ and $aE_jg \cdot x$. □

Remark 2.10. The point of the previous lemma is that the semigroup G may have infinite arity.

Remark 2.11. The result holds if the set X is infinitely A-definable (of finite arity), as E only involves parameters defining the semigroup G. Hence, if G_A is an infinitely A-definable group of finite arity, with an infinitely $\emptyset$-definable subgroup H, there exists a definable set X containing G_A stable under multiplication by H.

Proposition 2.12. *In a small structure, an infinitely definable group is the intersection of definable sets each one equiped with an infinitely definable binary operation whose conjunction of graphs gives the group law.*

Proof. Let G be this group. As G is the intersection of infinitely definable groups defined by countable types, we may assume that $G \subset \mathfrak{M}^\omega$ and that G is the conjunction of countably many sets of the form $X_i = \{x \in \mathfrak{M}^\omega :\models f_i(x)\}$ where f_i are formulae. By compactness, we may assume that $G \cdot X_1 \subset X_0$. For every integer n, let E_n be the equivalence relation "to have the same n first coordinates". On X_0, we set

$$xR_ny \iff \exists g, h \in G\ (g \cdot xE_nh \cdot y)$$

Note that $x \in G$ if and only if xR_n1 for all n. By compactness, there is an integer n such that $R_n1 \subset X_1$. Then, R_n1 is stabilised left multiplicatively by G. As the type defining R_n1 constrains only finitely many variables, by Lemma 2.9, we may assume that G almost acts on every X_i. By compactness, X_i is stable under multiplication for sufficiently large i. □

3. Infinitely definable groups and fields in a small and simple structure

3.1. **Groups.** In [3], Kim shows that the notion of strong type and Lascar strong type coincide in a small and simple theory, a necessary condition to eliminate hyperimaginaries. He proceeds in two steps, considering in the first one equivalence relations with boundedly many classes. We give an analogue of the first step for infinitely definable groups of finite arity.

In this last section, all infinitely definable groups and field considered will have finite arity.

Let $\mathfrak{M}$ be a κ-satured model of some theory T. For a set, *bounded* will mean strictly smaller than κ. An *hyperimaginary* is a class a/E of some a in $\mathfrak{M}^\alpha$ modulo an infinitely definable equivalence relation E on $\mathfrak{M}^\alpha$, where α is a bounded ordinal. We write $Aut(\mathfrak{M}/A)$ for the group of automorphisms of $\mathfrak{M}$ fixing A setwise. The action of $Aut(\mathfrak{M}/A)$ over $\mathfrak{M}$ naturally extends to hyperimaginaries. The *bounded closure* of some set A, written $bdd(A)$, is the set of hyperimaginaries whose orbit under $Aut(\mathfrak{M}/A)$ is bounded. We will not define here what a simple theory is, but refer the reader to [10] for more details. If T is simple, two elements a and b have the same *Lascar strong type* over A, which we write "$Lstp(a/A) = Lstp(b/A)$", if and only if they have the same type over $bdd(A)$ (see [10, Lemma 3.2.13]). Let us recall the independence Theorem for Lascar strong types in simple theories.

Fact 3.1. (Kim-Pillay [4]) *In a simple theory, let A, B, C, b and c satisfy*

1) $A \subset B$, $A \subset C$ *and* $B \perp_A C$,

2) *Neither* $tp(b/B)$, *nor* $tp(c/C)$ *fork over* A,
3) $Lstp(b/A) = Lstp(c/A)$.

Then there exists some a *such that* $tp(a/BC)$ *extends both* $tp(b/B)$ *and* $tp(c/C)$, *such that* $tp(a/BC)$ *does not fork over* A, *and such that* a, b *and* c *have the same Lascar strong type over* A.

Two subgroups G and H of some group F are *commensurable* if the indices $[G : G \cap H]$ and $[H : G \cap H]$ are bounded. The A-*connected component* of a group G is the smallest infinitely A-definable group of bounded index in G. Every infinitely definable group in a simple theory has an A-connected component (see [10, Lemma 4.1.11]), which we will denote by G^0_A. When it exists, G^0_A is always a normal subgroup of G.

Lemma 3.2. *Let* X *be a definable set in any structure with an infinitely definable composition law (the product of two elements of* X *may be outside* X*) such that the product of every six elements of* X *be defined and associative. Let* G_A *be an infinitely* A*-definable group inside* X*. If the* A*-connected component of* G_A *is contained in some definable group* H *in* X*, then* G_A *is contained in a definable group included in* $H \cdot G_A$*.*

Proof. The group $G_A \cap H$ has finite index in G_A, so $H \cdot G_A$ is definable, being a finite union of cosets of H. The group $\bigcap_{h \in H \cdot G_A} H^h$ is thus also definable. Let us call it N : this is a subgroup invariant under conjugation by elements of G_A. As N is the intersection of conjugates of H under G_A, and as the connected component G^0_A is normal in G_A, the group N contains G^0_A. The product $N \cdot G_A$ is a definable group containing G_A. □

A family $\mathfrak{H}$ of infinitely definable subsets of $\mathfrak{M}^\alpha$ is *uniformly infinitely definable* if there are two partial types $p(x, y)$ and $q(z)$ such that

$$\mathfrak{H} = \{\{x \in \mathfrak{M}^\alpha :\models p(x, a)\} :\models q(a)\}$$

If q and p are types over A, the family $\mathfrak{H}$ is *uniformly infinitely* A*-definable*. Let us now point out a result from Wagner's proof of [10, Theorem 4.5.13] and the earlier [10, Remark 4.1.20] :

Fact 3.3. *In a simple structure, let* X *be an infinitely* A*-definable set with an infinitely* A*-definable composition law (the product of two elements of* X *may be outside* X*). Let* $\mathfrak{H}$ *be a uniformly infinitely* A*-definable family of pairwise commensurable groups in* X*. If* X *contains* $\mathfrak{H} \cdot \mathfrak{H} \cdot \mathfrak{H} \cdot \mathfrak{H}$*, there exists an infinitely* A*-definable group* N *inside* $\mathfrak{H} \cdot \mathfrak{H} \cdot \mathfrak{H} \cdot \mathfrak{H}$ *which is commensurable with every* H *in* $\mathfrak{H}$*.*

Proposition 3.4. *In a small and simple structure, let* Z *be a definable set, and* G_A *be an infinitely* A*-definable group inside* Z*, such that boundedly many translates of* G_A *cover* Z*. Then* Z *envelopes* G_A*.*

Proof. We may restrict Z and suppose that the group law be defined and associative on Z. By compactness there is some definable set Y containing G_A such that $Y \cdot Y \cdot Y \cdot Y \subset Z$. We may suppose that Z, Y and the group law are definable without parameters. Let $\mathfrak{H}$ be the set $\{G_B : B \models tp(A/\emptyset)\}$. The elements in $\mathfrak{H}$ are pairwise commensurable. According to Fact 3.3, there exists an infinitely $\emptyset$-definable group N which is commensurable with G_A ; hence, N is contained in a definable group included in Y by Proposition 1.2 and Fact 2.1. But N contains the connected component of G_A, so G_A is contained in some definable group included in $N \cdot G_A$ according to Lemma 3.2. $\square$

If we could not do better, at least can we state local results. Recall that in an infinitely definable group G with simple theory, an element g is *generic* over A if for every $h \mathop{\smile\!\!\!\!\!\mid}_A g$ in G, we have $hg \mathop{\smile\!\!\!\!\!\mid} A, h$. Recall [10, Lemma 4.1.19] and [10, Remark 4.1.20], which together give :

Fact 3.5. (Wagner [10]) *In a simple structure, let G be a definable set with a definable composition law having an identity, and such that the product of every three elements of G be defined and associative, and such that any element have a right and left inverse in G. In G, let X be an infinitely definable set such that for all x and y independent in X, the product $x^{-1}y$ be in X. Then $X \cdot X$ is an infinitely definable group and X is generic in $X \cdot X$. Actually, X contains every generic type of $X \cdot X$.*

Lemma 3.6. *In a simple structure, let G_A be an infinitely A-definable group, and let g be a finite tuple of elements in G_A. There is a finite set B and an infinitely B-definable group containing $acl(g) \cap G_A$.*

Proof. The group law is defined and associative on a definable set X containing G_A. By compactness, there is a definable set Y inside X such that $Y \cdot Y \subset X$. We may assume that X and Y are definable without parameters. Let Γ be the bounded closure of g. Let N_Γ be the set

$$\{x \in Y : \exists A' \models tp(A/\Gamma)\ (A' \mathop{\smile\!\!\!\!\!\mid}_{\Gamma} x \wedge x \in G_{A'})\}$$

N_Γ is an infinitely Γ-definable set containing $acl(g) \cap G_A$. Let x and y be two elements of N_Γ such that $x \mathop{\smile\!\!\!\!\!\mid}_\Gamma y$. We show that $x^{-1}y$ is inside N_Γ : there are elements A' and A'' realising $tp(A/\Gamma)$ such that $A' \mathop{\smile\!\!\!\!\!\mid}_\Gamma x$, $A'' \mathop{\smile\!\!\!\!\!\mid}_\Gamma y$, and such that x belong to $G_{A'}$ and y to $G_{A''}$. According to the Independence Theorem 3.1, there exists some A''' realising $tp(A'/x\Gamma) \cup tp(A''/y\Gamma)$ such that $A''' \mathop{\smile\!\!\!\!\!\mid}_\Gamma x, y$. Thus $A''' \mathop{\smile\!\!\!\!\!\mid}_\Gamma x^{-1}y$. But x and y are in $G_{A'''}$ so $x^{-1}y$ is in $G_{A'''}$ too ; a fortiori, $x^{-1}y$ is in Y. After Fact 3.5, the product $N_\Gamma \cdot N_\Gamma$ is an infinitely Γ-definable group. Let us consider the group

$$\bigcap_{\sigma \in Aut(\mathfrak{C}/g)} \sigma(N_\Gamma \cdot N_\Gamma)$$

This is a bounded, infinitely g-definable intersection containing $acl(g) \cap G_A$. $\square$

Corollary 3.7. *In a small and simple structure, let g be a finite tuple of an infinitely A-definable group G_A. There is a definable group containing $acl(g) \cap G_A$.*

3.2. **Fields.**

Lemma 3.8. *In a simple structure, let K be a definable set with two definable composition laws (the sum and product of two elements of K may be outside K), each having an identity 0 and 1 respectively (which are a constant in the language), and such that the sum and product of every three elements of G be defined and associative. We assume that multiplication is distributive over addition and that any element in K has a right and left inverse in K for each law. In K, let X be an* *infinitely definable set containing* 0 *and* 1*, and such that for all* x *and* y *independent in* X*, the sum* $x-y$ *and product* $x^{-1}y$ *be in* X*. Then* $X+X$ *is an infinitely definable field.*

Proof. Note first that X equals $-X$ and X^{-1} as X contains 0 and 1. According to Fact 3.5, $X+X$ is an additive group ; we need just show that $X \cdot X$ is included in $X+X$, for we shall have

$$(X+X)\cdot(X+X) \subset X\cdot X + X\cdot X + X\cdot X + X\cdot X \subset X+X$$

Let p be an additive generic type in of $X+X$. Then p is in X. Let g and g' be in X, and let h be in p such that $h \mathop{\smile\!\!\!\!\!\!|}\, g, g'$. Then, $h \mathop{\smile\!\!\!\!\!\!|}_{g'} g$ and $h+g' \mathop{\smile\!\!\!\!\!\!|}_{g'} g$. Moreover, we have $g'+h \mathop{\smile\!\!\!\!\!\!|}\, g'$, so $g'+h \mathop{\smile\!\!\!\!\!\!|}\, g$ by transitivity. Hence, $gg'+gh$ is in X. As h^{-1} is in X and $g \mathop{\smile\!\!\!\!\!\!|}\, h^{-1}$, the product gh belongs to X and gg' to $X+X$. □

Lemma 3.9. *In a simple structure, let* g *be finite tuple of an infinitely* A*-definable field* K_A*. There is a finite set* B *and an infinitely* B*-definable field containing* $acl(g) \cap G_A$*.*

Proof. Let X be a definable set where addition and multiplication are defined and associative, and where multiplication is distributive over addition, and let Y be definable subset of X such that $Y \cdot Y$ and $Y+Y$ are in X. Let B be the bounded closure of g, and let L_B be the set

$$\{x \in Y : \exists C \models tp(A/B)\ (C \mathop{\smile\!\!\!\!\!\!|}_{B} x \wedge x \in K_C)\}$$

L_B is an infinitely B-definable set containing $acl(g) \cap G_A$. If x and y are two elements of L_B independent over B, then $x^{-1}y$ and $x-y$ also lie in L_B. According to Fact 3.5, the set $\bigcap_{\sigma \in Aut(\mathfrak{C}/g)} \sigma(L_B + L_B)$ has the required properties. □

Corollary 3.10. *In a small and simple structure, let* g *be a finite tuple of an infinitely* A*-definable field* K_A*. Then there is a definable field containing* $acl(g) \cap K_A$*.*

Corollary 3.11. *In a small and simple structure, an infinitely* A*-definable field of finite arity is finite or algebraically closed, and in positive characteristic, commutativity need not be assumed but follows.*

Proof. Let K_A be commutative field in a small and simple structure. If it is infinite, by compactness, there is an element x of infinite order in K_A. Let P be a polynomial with coefficients in K_A. According to Corollary 3.10, for every definable set X containing K_A, there is a definable field L_X in X which contains x and the coefficients of P. By [9], L_X is algebraically closed. The field $\bigcap_{X \supset K_A} L_X$ is an algebraically closed subfield of K_A which contains every coefficient of P.

If K_A has positive characteristic and is not assumed to be commutative, let x and y be in K_A. By Corollary 3.10, there is a definable field containing x and y, so x and y commute after [6]. □

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