One-basedness and groups of the form G/G^{00}

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ABSTRACT

We initiate a geometric stability study of groups of the form G/G^{00}, where G is a 1-dimensional definably compact, definably connected, definable group in a real closed field M. We consider an enriched structure M' with a predicate for G^{00} and prove 1-basedness for additive truncations of M, multiplicative truncations, $SO_2(M)$ and its truncations; such groups are now interpretable in M'. We prove that the only 1-based groups are sufficiently "big" multiplicative truncations, and we relate the results obtained to valuation theory. In the last section we extend our results to ind-definable groups constructed from these above.

1 Introduction

This paper is based on the positive solution of Pillay's conjecture for definably compact, definably connected, definable, 1-dimensional groups G in a real closed field M in [7] and on the Madden and Stanton classification of 1-dimensional Nash groups over the reals in [15]. Our aim is, considering some of the groups described in [15], to understand the geometric complexity (in the sense of Zil'ber trichotomy) of the groups G/G^{00}. This model theoretic analysis, though, is not directly possible in our ambient structure M, and a few changes of category will be required to extract the "theory of G/G^{00}" and to be able to study stability on it. Since the group G/G^{00} is an o-minimal definable set in a structure M with a predicate for G^{00}, we can apply the generalisation to o-minimal theories of the notion of 1-basedness, defined in [8], to such groups.

In section 2 we recall the notions of hyperdefinable set, externally definable set and logic topology. We recall a theorem from [9] that proves o-minimality as sets for hyperdefinable groups (which become definable in the appropriate expansion of our base structure); afterwards we generalise this theorem to ind-hyperdefinable groups.

In section 3 we recall the conditions we are imposing on our groups, Pillay's conjecture and the notion of a Nash group.

Section 4 provides the definition of 1-basedness for a stably embedded set; and it has a case study of additive truncations, multiplicative truncations, $SO_2(M)$ and its truncations.

Section 5 is a case study of certain ind-definable sets arising from those from section 4.

Notation is standard: T denotes a complete theory in a language L, M a model of a theory, usually $x, y, z, \dots$ denote finite tuples of variables, though occasionally we shall highlight if they are tuples, by putting $\overline{x}, \overline{y}, \overline{z}, \dots$ Formulae are denoted by Greek letters $\varphi, \psi, \chi, \dots$, and sets by capital letters: $X, Y, Z, \dots$.

We assume a basic knowledge of o-minimality and of the related notion of dimension. Details can be found in the first chapters of [18].

We also assume basic knowledge of valuation theory (see for example [2] for references), but we recall here briefly some notions. We view a real closed valued field as a structure $M_w = (M, R, \Gamma, 0, 1, +, -, \cdot, <)$, where M is a real closed field, R a valuation ring, Γ the value group, and w denotes the valuation $w : M \to \Gamma \cup \{\infty\}$. The unique maximal ideal of R is denoted by I, and the residue field by k. We recall that $\Gamma = M/(R \setminus I)$.

We can define the sets above in terms of the valuation:

- Valuation ring: $R = \{x | w(x) \geq 0\}$.
- Valuation ideal: $I = \{x | w(x) > 0\}$.
- Residue field: $k = R/I$.

We shall denote the standard valuation (with $R = Fin$ the convex hull of $\mathbb{Q}$, and I the infinitesimal neighbourhood of the identity) by v.

Sometimes we indicate M_w simply as (M, R), or as $(M, R, \dots)$, where the dots stand for the usual signature of the ordered fields.

2 Uniform o-minimality for bounded hyperdefinable sets

In this section we recall a fundamental tool in the proof of Pillay's conjecture: the logic topology, originally defined in [7]. We define o-minimality of a set, and related notions, and prove directly a special case of Theorem 8.6 [9]. We conclude with a generalisation to bounded ind-hyperdefinable sets.

Let M be a saturated model of a theory T. Given X a definable subset of M^n and an equivalence relation $E \subset X \times X$, we say that E is type definable if its graph is a type definable subset of $X \times X$. The quotient X/E is called a hyperdefinable set in M. We say that E is a bounded equivalence relation if $|X/E| < |M|$, and in this case we shall call X/E a bounded hyperdefinable set.

Let X/E be a bounded hyperdefinable set, and denote by π the canonical projection $\pi : X \to X/E$. We call a set $Y \subset X/E$ closed if $\pi^{-1}(Y)$ is type definable in M.

It is an easy exercise to show that these closed sets induce a topology on X/E, called logic topology.

The following theorem is from [7], and shows some interesting properties of the logic topology.

Theorem 2.1. *The bounded hyperdefinable set X/E equipped with the logic topology is a compact Hausdorff space. If, moreover, E is defined by a countable number of formulae, X/E is separable.*

Sketch proof: Compactness follows easily from saturation of M.

To prove that X/E is Hausdorff we need to find, for any given $y_\sim, z_\sim \in X/E$ such that $y_\sim \neq z_\sim$, two disjoint neighbourhoods. Taking $y \in \pi^{-1}(y_\sim)$ and $z \in \pi^{-1}(z_\sim)$ we clearly have $\neg E(y, z)$, which is expressed by an infinite disjunction of formulae. By compactness there is a formula φ such that $\varphi(y)$ and $\neg\varphi(z)$ and $\pi^{-1}(y_\sim) \subset \varphi(x)$, $\pi^{-1}(z_\sim) \subset \neg\varphi(x)$. We consider $\overline{\varphi(x)} = \{x \in X | \exists t \in y_\sim, E(x,t)\}$ and $\overline{\neg\varphi(x)} = \{x \in X | \exists t \in z_\sim, E(x,t)\}$: type definable sets in X (by saturation). Observe now that their images under π are closed sets, overlapping and covering X/E; so their complements are disjoint open sets. Moreover $z \in \overline{\varphi(x)}^c$ and $y \in \overline{\neg\varphi(x)}^c$. These are the neighbourhoods we are looking for.

For the "moreover" part we prove it is second countable, this always implies separability. Remark 1.6 from [5] says that X/E has a basis of cardinality at most $|M_0| + |L|$, for some submodel M_0 over which E is type definable. Now Löwenheim-Skolem allows us to find a model of countable cardinality, so the remark implies that X/E has a countable basis and is therefore second countable. □

A similar theorem can be obtained for the class of ind-definable sets: a set $\tilde{X}$ is said to be *ind-definable* if there is a chain of definable sets $X_0 \subset X_1 \subset X_2 \subset \dots$ such that $\tilde{X} = \bigcup_{i \in \omega} X_i$.

The following is the definition required for Remark 7.6 [9].

Definition 2.2. Given $\tilde{X} = \cup_{i \in \omega} X_i$ ind-definable, an equivalence relation $E \subseteq \tilde{X} \times \tilde{X}$ is *type definable* if for each $n \in \omega$, $E \upharpoonright (X_n \times X_n)$ is type definable and for each $i \in \omega$ exists $j \in \omega$ such that all the classes meeting X_i are contained in X_j. The set $\tilde{X}/E$ is called an *ind-hyperdefinable set.*

If each $E \upharpoonright (X_n \times X_n)$ has a bounded number of classes we call it a bounded type definable equivalence relation, and $\tilde{X}/E$ is called a *bounded ind-hyperdefinable* set.

The logic topology on $\tilde{X}/E$ is given by: $Y \subseteq \tilde{X}/E$ is closed if $\pi^{-1}(Y) \cap X_n$ is type definable for all $n \in \omega$.

It is easy to see that a compact set is a closed set whose preimage is contained in some X_i.

Theorem 2.3. *A bounded ind-hyperdefinable set $\tilde{X}/E$ with the logic topology is a locally compact Hausdorff space.*

Proof. We need to find a compact neighbourhood of any given point $c_\sim \in \tilde{X}/E$. By definition $\pi^{-1}(c_\sim) \subseteq X_i$ for some i.

We consider the set $O \subset \tilde{X}/E$ of the classes whose preimage is completely contained in X_i:

$$\pi^{-1}(O) = \left\{ x \in \tilde{X} | \neg \exists t(\neg X_i(t)) \wedge E(x,t) \right\}$$

It is open, since it is the complement of a closed set (it easy to see that the preimage of its complement is type-definable using saturation of M).

This set is contained in the set $C \subset \tilde{X}/E$ of all classes meeting X_i:

$$\pi^{-1}(C) = \left\{ x \in \tilde{X} | \exists t(X_i(t)) \wedge E(x,t) \right\}$$

It is closed, again by saturation; moreover, it is contained in some X_j by definition of the equivalence relation E. Therefore $c_\sim \in O \subset C$, and C is the required compact neighbourhood.

The proof that $\tilde{X}/E$ is Hausdorff is similar to the one in Theorem 2.1. □

Here we recall a theorem due to Baisalov and Poizat, which was then generalised by Shelah to a wider class of theories.

Let N be any structure. A subset $X \subseteq N^n$ is externally definable if, given $N' \succ N$ a $|N|^+$-saturated model of $Th(N)$, we can find parameters $c \in N'$ and a formula $\varphi(x,y) \in L_N$ such that $\varphi(x,c)$ defines X when restricted to N.

Equivalently, X is externally definable in N if there is a structure $N' \vDash Th(N)$, $N' \succ N$, and a definable set X' in N' such that $X = X' \cap N$.

We construct from N a new theory $Th(N^{\mathrm{Sh}})$ in this way: for each externally definable set, by an $L_{N'}$-formula φ say, we add a relation symbol R_φ in the language. We call the new language L^{Sh}; the model will be denoted by N^{Sh} and this gives rise to a new theory $Th(N^{\mathrm{Sh}})$. This new theory is called Shelah's expansion of the structure N.

Baisalov and Poizat proved in [11] the following:

Theorem 2.4. *If N is an o-minimal structure, $Th(N^{\mathrm{Sh}})$ admits quantifier elimination.*

In order to study quotients of the form X/E, where E is externally definable, and therefore definable in M^{Sh}, we need to localise the notions related to o-minimality to definable sets in a structure.

Let N be any structure and X be a definable set in N with a dense total order on it. We say that:

- X is *o-minimal* if, given any formula $\varphi(\overline{x}) \in \mathcal{L}_N$, with $\overline{x}$ a tuple of the correct length, it defines on X a finite union of intervals and points (with respect to the total order).
- X is *uniformly o-minimal* if it is o-minimal and, given any formula $\varphi(\overline{x}, \overline{y}) \in \mathcal{L}_N$, we can find a number $\alpha_\varphi \in \omega$ such that for each choice of parameters $\overline{b} \in N$, $\varphi(\overline{x}, \overline{b})$ defines on X no more than α_φ intervals and points.
- X is *weakly-o-minimal* if, given any formula $\varphi(\overline{x}) \in \mathcal{L}_N$, with $\overline{x}$ of the correct length, it defines on X a finite union of convex sets.
- X is *uniformly weakly o-minimal* if it is weakly-o-minimal and, given any formula $\varphi(\overline{x}, \overline{y}) \in \mathcal{L}_N$, we can find a number $\alpha_\varphi \in \omega$ such that for each choice of parameters $\overline{b} \in N$, $\varphi(\overline{x}, \overline{b})$ defines on X no more than α_φ convex sets.

An immediate consequence of theorem 2.4 is that if N is an *o*-minimal structure, then N^{Sh} is uniformly weakly-o-minimal.

By Shelah's generalisation to NIP theories in [14] we obtain also that if N is uniformly weakly-o-minimal, then N^{Sh} is uniformly weakly-o-minimal.

We can now prove directly a special case of Theorem 8.6 [9], but firstly an easy observation:

Observation 2.5. A definable, definably densly linearly ordered set X in a uniformly weakly-o-minimal structure M, is a uniformly weakly-o-minimal set in M

The proof is standard and uses the weak form of cell decomposition in theorem 4.6 of [16].

Theorem 2.6. *Given a dense o-minimal theory T, a saturated model M, a definable, definably densly linearly ordered $X \subset M^n$ and a type-definable, externally definable, convex equivalence relation E (each class is convex with respect to the order on X) with a bounded number of classes; then the hyperdefinable set X/E, definable in M^{Sh}, is uniformly o-minimal in M^{Sh}.*

(Actually the convex assumption is not necessary, but it simplifies the definition of the induced order on X/E.)

Proof. Note that, if E is convex, the order on X/E can be trivially defined by $[x]_\sim \leq' [y]_\sim$ if $x \leq y$. We shall drop the $'$ in the future. (If it is not convex it is nevertheless possible to define the order using o-minimality of the structure, see Prop 8.6 [9].)

Firstly we work in M^{Sh} and we consider a formula $\varphi(\overline{x}, \overline{y}, \overline{c})$ in $\mathcal{L}$ with $\overline{c} \in M' \succ M$. Since M^{Sh} is uniformly weakly-o-minimal, so is X/E, and for each choice of $\overline{b} \in M$, $\varphi(\overline{x}, \overline{b}, \overline{c})$ defining a set Y say, is realized only by $\leq \alpha_\varphi$ convex sets. Without loss of generality we can suppose $Y \subset X$, in fact Y will determine $\leq \alpha_\varphi$ cuts also in X. Let $\psi(\overline{x}, \overline{b}, \overline{c})$ be the formula in M^{Sh} that defines the quotient $Y/E \subseteq X/E$, it will define $\leq \alpha_\varphi$ convex subsets of X/E. Since $\overline{b}$ was arbitrarily chosen we get that X/E is uniformly weakly-o-minimal.

To prove now that X/E is uniformly o-minimal it is sufficient to prove that it is complete (all the convex sets in one-space will then have a supremum and an infimum, and therefore they will be intervals).

Now we forget about Shelah's expansion and we regard E as a bounded type definable equivalence relation in a model M. Consider a Dedekind cut (A, B) of X/E. For each $a \in A$ we can define the set $A_a = \{x \in X/E | a \leq x\}$; the preimage of this set is type definable in M and therefore it is a closed set. Analogously for each $b \in B$ we define $B_b = \{x \in X/E | b \geq x\}$, which has type definable preimage, and is therefore closed. Consider now the family of closed sets $\{B_b, A_a\}_{a,b}$; it has the finite intersection property by density of the order, so, since X/E with the logic topology is compact by 2.1, there is an element in the intersection. Thus X/E is complete.

□

Observation: Note that the theorem above holds also if T is uniformly weakly-*o*-minimal.

We can easily generalise the result to ind-hyperdefinable sets. Firstly notice that the definitions of induced order and those related to *o*-minimality can be extended to this class of sets in the obvious way.

Theorem 2.7. *Given an o-minimal theory T, a saturated model M, an ind-definable, externally definable, convex set $\tilde{X} = \bigcup_i X_i$, densly linearly ordered and such that the restriction of the ordering onto X_i is definable, and a bounded type-definable, externally definable, convex equivalence relation E on $\tilde{X}$. Then the ind-hyperdefinable set $\tilde{X}/E$, definable in M^{Sh}, is uniformly o-minimal in M^{Sh}.*

Proof. The proof of uniform weak-o-minimality of $\tilde{X}/E$ goes through as in the previous theorem.

We now need to prove completeness: consider a Dedekind cut (A, B) of $\tilde{X}/E$. Denote $\tilde{X}/E \restriction X_i$ by X_i/E. Then there exists $X_i \subset \tilde{X}$ such that $A \cap X_i/E \neq \emptyset$ and $B \cap X_i/E \neq \emptyset$. As in the previous theorem we define for each $a \in A$, $A_a = \{x \in X_i/E : a \leq x\}$, and for each $b \in B$, $B_b = \{x \in X_i/E : b \geq x\}$. Again the family of closed sets $\{B_b, A_a\}_{a,b}$ has the finite intersection property. Theorem 2.3 states that $\tilde{X}/E$ is locally compact, so X_i/E is compact and we can find an element in the intersection of $\{B_b, A_a\}_{a,b}$. This gives us completeness of $\tilde{X}/E$ and ends the proof.

□

3 Pillay's conjecture and Nash groups

In [6] Pillay proved that a group definable in an *o*-minimal structure can be equipped with the structure of a definable manifold where the group operation is continuous. Here we present a brief review in the real closed field case, and recall Pillay's conjecture stated in [7] and completely proved in [10].

By a definable group G in a structure M we mean a group whose underlying set $G \subset M^n$ is definable, and the graphs of the operations $* : G \times G \to G$ and $^{-1} : G \to G$ are definable.

Definition 3.1. Given a set A containing the parameters defining G, we define G^0_A as the intersection of all A-definable subgroup of finite index. If G^0_A does not depend on A we call it G^0 and we say that G^0 exists.

Given a set A containing the parameters defining G, we define G^{00}_A as the smallest A-type-definable subgroup of bounded index. If G^{00}_A does not depend on A we call it G^{00} and we say that G^{00} exists.

For the rest of the section suppose M is a saturated real closed field. Then G^0 exists, has finite index and is definable.

Definition 3.2. Given a set X, a *definable atlas* is a **finite** collection

$$\{(U_1, \psi_1), (U_2, \psi_2), \ldots, (U_n, \psi_n)\}$$

such that each U_j is a subset of X, and for each of them there is a bijective homeomorphism $\psi_j : U_j \to V_j$, where V_j is a definable open subset of M^{n_j}, and whenever $U_j \cap U_k \neq \emptyset$ then the map $\psi_j \circ \psi_k^{-1}$ is a definable bijective homeomorphism $\psi_k(U_j \cap U_k) \to \psi_j(U_j \cap U_k)$.

Two definable atlases are compatible if their union is a definable atlas. For fixed X compatibility of definable atlases is an equivalence relation.

We call $n = sup\{n_j | j < n\}$ the dimension of the atlas.

A *definable manifold* of dimension n is a definable set X with an equivalence class of definable atlases of dimension n.

Such a definable manifold has an induced topology, called the t-topology: $Y \subset X$ is open if and only if each $\psi_i(Y \cap U_i)$ is open in M^n.

Moreover from [6] we get:

Fact 3.3. *A group G definable in M can be given the structure of a definable manifold over M in which multiplication and inverse are continuous operations with respect to the t-topology.*

There are other conditions we can impose on G in order to get good behaviour:

Definition 3.4. G is *definably connected* if there are no proper definable subgroups of finite index. Equivalently $G = G^0$.

From [6] this is equivalent to G not being the disjoint union of two nonempty definable t-open subsets.

Definition 3.5. G is *definably compact* if given an interval $I = [a, b)$ in M and a definable and continuous function $f : I \to G$, then $\lim_{x \to b} f(x)$ exists in G.

In the case $M \vDash RCF$ definably compact equals closed and bounded.

We state now Pillay's conjecture, completely proved in [10]:

Theorem 3.6. *Given a definably connected, definably compact definable group G in M, then G^{00} exists, and G/G^{00} is a compact connected Lie group of (Lie) dimension equal to the o-minimal one of G. If moreover G is commutative, then G^{00} is divisible and torsion free.*

4 1-basedness and case studies

The notion we shall deal with is a variant of 1-basedness for o-minimal theories, adapted to definable sets; but firstly we recall the classical definition from [8].

Let M be o-minimal. Given $f(x, \overline{y})$ a $\emptyset$-definable function, and $a \in M$, we define an equivalence relation $\sim_a$ on tuples of the same length as $\overline{y}$ by $\overline{c} \sim_a \overline{c}'$ if neither of $f(-, \overline{c})$, $f(-, \overline{c}')$ is defined in an open neighbourhood of a or if there is an open neighbourhood U of a such that $f(-, \overline{c}) = f(-, \overline{c}')$ in U. We call the equivalence class of $\overline{c}$ the *germ of* $f(-, \overline{c})$ *at* a, and denote it by $\overline{c}/\sim_a$.

Definition 4.1. Given an *o*-minimal theory T, we say that T is 1-*based* if in any saturated model $M \models T$, for any $a \in M$, for all definable functions $f(x, \overline{y}) : M \times M^n \to M$, and for any $\overline{c} \in M^n$ such that $a \notin dcl(\overline{c})$, we have $\overline{c}/\sim_a \in dcl(a, f(a, \overline{c}))$.

Definition 4.2. Given a theory T and $\varphi(x) \in L^{eq}$, φ is *stably embedded* in T if for any saturated model N of T and $X = \varphi(N)$, any subset of X^n definable (with parameters) in N is definable with parameters from X.

We localise 1-basedness to stably embedded formulae considering the following construction:

Let $\varphi(x)$ be an L^{eq}-formula, M any saturated model of T, and suppose $X = \varphi(M)$ is a uniformly o-minimal set in M. Then it is stably embedded by Theorem 2 of [4].

We define $\mathcal{X}$ to be the set X equipped with all $\emptyset$-definable (in M) subsets of X^n. This is sufficient to preserve the structure induced on X by M, in fact each definable subset Y of X in $\mathcal{X}$ is definable in M, and viceversa, by stable embeddability. We still have to show that given $\mathcal{X}' \succ \mathcal{X}$ there is $M' \succ M$ for which each definable set Y in $\mathcal{X}'$ is a definable subset $Y \subseteq X'$ in M', where $X = X' \cap M$, and viceversa. The first implication is obvious. For the latter let Y be defined by $\varphi(\overline{x}, \overline{a})$, by stable embeddiness we can suppose $\overline{a} \in X'$, and let $\mathcal{X}'$ be such that X' is its universe. Now Y is clearly definable in $\mathcal{X}'$.

Definition 4.3. We say that a stably embedded formula (or the set X it defines) is 1-*based* in T if the theory $T_X = Th(\mathcal{X})$ is 1-based.

4.1 The cases

From now on M will be a saturated real closed field, by o-minimality we have a notion of dimension for definable sets (see [18] for more details).

We shall proceed in a case study of G/G^{00} for G a 1-dimensional, definable, definably connected, definably compact group. We put the previous facts and theorems together: By the results in [12], G can be definably circularly ordered; by fixing a point we suppose that the order is linear. We consider the structure $(M, G^{00}, a, \dots)^{eq}$, obtained by adding a predicate for G^{00} to M and parameters a defining G, if needed. Let M' be a saturated model of $Th\big((M, G^{00}, a, \dots)^{eq}\big)$. In M', G/G^{00} becomes $\emptyset$-definable and is, by theorem 2.6, uniformly o-minimal in M'. (G^{00} is convex, so externally definable. We apply the theorem, obtaining uniform o-minimality in M^{Sh}, this clearly implies uniform o-minimality in M' since the latter is a reduct of the former).

Definition 4.4. Given a 1-dimensional, definable, definably linearly ordered group $(H, *)$ in M, a *truncation* of H is a group whose universe is $[\alpha^{-1}, \alpha)$, for some $\alpha \in H$, and whose operation is $*\ mod\ [2]\alpha$, defined as

$$\beta *_{mod\ [2]\alpha} \gamma = \begin{cases} \beta * \gamma & if\ \alpha^{-1} < \beta * \gamma < \alpha \\ \beta * \gamma * \alpha^{-1} & if\ \beta * \gamma > \alpha \\ \beta * \gamma * \alpha & if\ \beta * \gamma < \alpha^{-1} \end{cases}$$

A truncation of the additive group will be called an additive truncation, and one of the multiplicative group a multiplicative truncation. In addition to these we shall consider $SO_2(M)$ and its truncations.

These are the only possible groups G as above when $M = \mathbb{R}$, due to Madden and Stanton classification of 1-dimensional Nash groups over the reals in [15]; in a subsequent paper we shall prove that these are the only possible groups with the above properties in a generic real closed field M.

Moreover by definable connectedness and corollary 2.15 of [6], G is commutative, so Pillay's conjecture implies that G^{00} is torsion-free and it is the subgroup bounded by the torsion points.

4.2 Additive truncation

We begin our analysis with the easiest case. The method used here to prove non-1-basedness is the standard one, to which we shall refer throughout the rest of the paper.

It is an easy observation that every additive truncation is definably isomorphic to the group $G = ([-1, 1), +\ mod\ 2)$; it will suffice to prove non-1-based for this case.

In this case G^{00} corresponds to the additive group of infinitesimal elements of M.

We add now a predicate for G^{00} to M, obtaining $M' = (M, G^{00}, \dots)^{eq}$. The hyperdefinable group G/G^{00} in M is therefore definable, and clearly stably embedded, in M'. Let us consider $M'' \succ M'$ saturated. In this structure we can check 1-basedness of G/G^{00} as we defined it. Therefore let $\mathcal{G}$ be G/G^{00} with predicates for every $\emptyset$-definable set of M'', as in the construction above. We consider the theory $T_G = Th(\mathcal{G})$. Let $\mathcal{G}'$ be a saturated model of T_G.

Let G' be a group such that G'/G'^{00} equals $\mathcal{G}'$ as a set, let $g, h \in \mathcal{G}'$, which we identify with elements of G'/G'^{00}, and let $\hat{g}, \hat{h}$ be elements of G' such that $\hat{g}/G'^{00} = g$ and $\hat{h}/G'^{00} = h$. We can define in $\mathcal{G}'$ the operations $+$ and $\cdot$ as follows: $g + h = (\hat{g} +_M \hat{h})/G^{00}$, where $+_M$ is the usual addition in M; and $g \cdot h = (\hat{g} \cdot_M \hat{h})/G^{00}$, again where $\cdot_M$ is the usual multiplication in M. Observe that these are well defined in a neighbourhood of $0_{\mathcal{G}'}$.

By saturation of $\mathcal{G}'$ we can find algebraically independent elements $a, b, c \in \mathcal{G}'$. Let $d = a \cdot b + c$; clearly $dim(a, b, c, d) = 3$.

Let us now define a function $f_{bc}(x) = x \cdot b + c$; the germ of this function is exactly $bc/ \sim = bc$: in fact $f_{bc} = f_{b'c'}$ at a neighbourhood of a point if and only if $bc = b'c'$. If G/G^{00} were 1-based we would have then $bc/ \sim = bc \in acl\,(a, f_{bc}(a)) = acl(a, d)$, so $dim(a, b, c, d) = 2$, that contradicts what we have previously observed.

This proves non-1-basedness of G/G^{00}.

Observation 4.5. Any group in definable bijection with a non-1-based group is non-1-based

This is trivial: for example consider any group H/H^{00} in definable bijection with G/G^{00} as above, in a structure where G^{00} and H^{00} are definable. If $f : G/G^{00} \to H/H^{00}$ is the bijection, we define the operations on H/H^{00} as follows: let $+$ be $a/H^{00} + b/H^{00} = f\left(f^{-1}\left(a/H^{00}\right) +_{G/G^{00}} f^{-1}\left(b/H^{00}\right)\right)$, and $\cdot$ be defined in an analogous way. The proof above goes through with these new operations, showing that H/H^{00} is non-1-based in the theory with a language enriched by the parameters defining H.

We have therefore proved:

Theorem 4.6. *Given G any additive truncation of M, the group G/G^{00} is non-1-based in $Th(M, G^{00}, a, \dots)$, where a is a tuple of parameters defining G.*

4.3 Multiplicative truncation

The second case is $G = \left([b^{-1}, b), \cdot\ mod\ b^2\right)$, a multiplicative truncation.

We recall that G^{00} is the neighbourhood of the identity bounded by the torsion points, so $G^{00} = \bigwedge_{n\in\omega}\{x | b^{-1/n} < x < b^{1/n}\}$.

The operation on G^{00} coincides with the multiplication on M and is closed, so G^{00} is also a multiplicative subgroup of M.

The fundamental point in this section is the following valuation theoretical observation, see [17] Definition 3.4 and Proposition 3.5:

Observation 4.7. Given a convex multiplicative subgroup S of M

- If $2 \in S$, then S is the set of positive units of the convex valuation ring $R = \{a | |a| < g \text{ for some } g \in S\}$

- If $2 \notin S$, then $S - 1$ is the underlying set of a (convex) additive subgroup of M (note: also the converse holds: for every additive convex subgroup S of M s.t. $1 \notin S$ then $1 + S$ is the set of a multiplicative group of M). Moreover $S - 1$ is the maximal ideal of a convex valuation ring.

We shall apply this to G^{00} and show how the behaviour of G/G^{00} depends entirely on whether the parameter b defining G is finite or infinite.

4.3.1 Small multiplicative trunctions

Let $G = ([b^{-1}, b), \cdot \ mod\ b^2)$, with b a finite element (i.e. $v(b) = 0$). Then $2 \notin G^{00}$, so $G^{00} - 1$ is a convex additive subgroup. In this case we can follow the method of the additive case, with a little variation. As before we consider $M' = (M, G^{00}, b, \dots)^{eq}$ and construct $\mathcal{G}'$, saturated model of the theory of G/G^{00}. Observe that addition is not well defined in a neighbouhood of $0_{\mathcal{G}'}$, but we can define an operation $\oplus : G \times G \to G$: $a \oplus b = a + b - 1$; this operation is preserved passing to the quotient and is well defined for an interval (h^{-1}, h), say, with $G^{00} < h < b$. We denote the operation on the quotient in the same way, $\oplus : \mathcal{G}' \to \mathcal{G}'$, and let U be the neighbourhood of the identity over which it is defined.

We can then find $s, t, u \in U \subset \mathcal{G}'$ that are independent and such that $(s \cdot t) \oplus u = d \in U$ by saturation.

The function $f(x) = s \cdot x \oplus u$ is similar to the one for the additive case and analogously witnesses non-1-basedness of G/G^{00} in $Th(M')$.

4.3.2 Big multiplicative trunctions

In this case b is an infinite element (i.e. $v(b) < 0$). Clearly $2 \in G^{00}$, so G^{00} is the set of units of a valuation w, say.

Instead of working in $M' = (M, G^{00}, b, \dots)^{eq}$ we can work with a well known structure: the real closed valued field $M_w = (M, \Gamma, w, b, \dots)^{eq}$.

Lemma 4.8. *If $v(b) < 0$ the structures $M' = (M, G^{00}, b, \dots)^{eq}$ and $M_w = (M, \Gamma, w, b, \dots)^{eq}$ are interdefinable.*

Proof. Given $M' = (M, G^{00}, b, \dots)^{eq}$, we know that G^{00} is the set of positive units of a valuation ring, therefore $\Gamma = M^{>0}/G^{00}$ is definable, and also the valuation $w : M \to \Gamma \cup \{\infty\}$ is, by "$w(x) = w(y)$ if and only if $xy^{-1} \in G^{00}$ and $yx^{-1} \in G^{00}$". Thus M_w is definable from M'.

On the other hand from M_w we can define $R = \{x | w(x) \geq 0\}$ and $I = \{x | w(x) > 0\}$. Then $G^{00} = \{x | x > 0 \wedge x \in R \setminus I\}$. So we obtain M' from M_w. □

Moreover, since $\Gamma = M/(R \setminus I)$, G/G^{00} is a truncation of Γ in M_w.

We recall a well known fact about value groups of real closed valued fields, the sketch proof we present below has been suggested to the author by Macpherson and is implicit in [3]:

Theorem 4.9. *The value group Γ of a real closed valued field is stably embedded*

Proof. In [1] it is proved that the theory of real closed valued fields in the two-sorted language with one sort for the real closed field M, with the usual signature for ordered fields, the other one for Γ with the usual ordered group signature plus constants for elements of $\mathbb{Q}$, and a function symbol v for the valuation $v : M \to \Gamma \cup \{\infty\}$, admits quantifier elimination.

Thus any definable set in the value group sort can be defined by a boolean combination of formulae of the form

$$t(\gamma_1, \ldots, \gamma_n, v\,(p(\overline{a}))) \geq t'(\gamma'_1, \ldots, \gamma'_m, v\,(p'(\overline{a}')))$$

where t, t' are terms in the signature of the value group, $\gamma_i, \gamma'_j \in \Gamma$, and $p(\overline{a}), p'(\overline{a}')$ are polyonomials in variables $\overline{a} = (a_1, \ldots, a_r), \overline{a}' = (a'_1, \ldots, a'_s) \in M$. By using the properties of valuation $v(p(\overline{a}))$ can be written as $\sum_i \alpha_i v(a_i)$, similarly $v(p'(\overline{a}'))$ becomes $\sum_i \alpha'_i v(a'_i)$, with $\alpha, \alpha' \in \mathbb{Q}$, and clearly $v(a_i), v(a'_i)$ are elements of Γ. Therefore any formula defining a set in Γ is equivalent to one with parameters only from Γ. □

Therefore $T_\Gamma = Th(\Gamma)$ is $Th(\mathbb{Q}, +, 0, <, \lambda_q), q \in \mathbb{Q}$, and it is clearly a 1-based theory.

We can consider a Γ as a model of T_Γ and G/G^{00} will be a definable group in this structure. By stable embeddability, all definable sets of G/G^{00} in M_w are definable with parameters from Γ. We can therefore use 1-basedness of T_Γ to obtain the result. Consider a saturated model $\mathcal{G}'$ of $T_{G/G^{00}}$; there is a saturated model Γ' of T_Γ for which $\mathcal{G}'$ is a truncation. If $\mathcal{G}'$ were non-1-based, then Γ' would not be 1-based, contradicting what we just observed.

We have therefore proved:

Theorem 4.10. *The group G/G^{00}, where $G = \left(\left[b^{-1}, b\right), \cdot \bmod b^2\right)$, is 1-based in $Th(M, G^{00}, b, \ldots)$ if and only if $v(b) < 0$.*

4.4 SO_2 and truncations

At last we consider the group $\left(SO_2(M), *\right)$ and its truncations. These are definable, definably compact, definably connected 1-dimensional groups, subsets of M^2, which can be definably circularly ordered anticlockwise (we recall that $SO_2(M)$ lies on the set defined by $x^2 + y^2 = 1$). Fixing the identity $O = (1, 0)$ this order becomes linear.

A truncation of $SO_2(M)$ is a group of the form $G = \left([-S, S), * \bmod [2]S\right)$, where $S = (x_S, y_S)$ and $-S = (x_S, -y_S)$ for some $x_S, y_S \in (-1, 1)$, the interval is in the sense of the order of $SO_2(M)$, and $[n]P$ means $P * \cdots * P$, n times. We denote the coordinates of a point P by x_P and y_P.

It is sufficient to consider truncations with $x_S \in [0, 1)$: in fact if such truncation is non-1-based then any other truncation (including $SO_2(M)$ itself) with the same G^{00} is non-1-based.

We need now to calculate G^{00}; the following lemma gives us a precise definition of G^{00} in terms of the standard valuation:

Lemma 4.11. *A group* $G = \big([-S,S), * \; mod \; [2]S\big)$ *has*

$$G^{00} = \{P | v(y_P) > v(y_S)\}$$

Proof. The lemma follows immediately from the claim: either $v(y_{[n]P}) = v(y_P)$ for any $n \in \omega$, or $v(y_P) = 0$; in fact G^{00} is bounded by its torsion points.

We take P such that $v(y_P) > 0$, and prove it by induction considering just the powers of 2. Observe that $x_{[2]P} = x_P^2 - y_P^2$ and $y_{[2]P} = 2x_P y_P$. It is clear that if $v(y_P) = 0$, P cannot be in G^{00}, so suppose $v(y_P) > 0$, then $v(x_P) = 0$, so $v(y_{[2]P}) = v(2x_P y_P) = v(y_P)$. Suppose now $v(y_{[2^n]P}) = v(y_P)$, then $v(y_{[2^{n+1}]P}) = v(2x_{[2^n]P} y_{[2^n]P}) = v(y_{[2^n]P}) = v(y_P)$. □

To prove non-1-basedness using observation 4.5 we construct a definable bijection with a non-1-based group, namely an additive truncation.

Lemma 4.12. *The group* $G/G^{00} = ([-S,S), * \; mod \; [2]S)/G^{00}$ *is in definable bijection with the quotient of an additive truncation* A *by its* A^{00}.

Proof. We define the function $l : G \to M$ that sends a point P to the second coordinate of the intersection of the line through P and the origin with the line $x = 1$. Namely $l(P) = \frac{y_P}{x_P}$. We then define $A = \Big([-l(S), l(S)), + \; mod \; 2 \; l(S)\Big)$. Observe that $v(l(S)) = v(y_S)$, and therefore $A^{00} = \big\{x | v(x) > v(l(S))\big\} = l(G^{00})$ by lemma 4.11. It is then sufficient to prove that $l : G/G^{00} \to A/A^{00}$ is well defined and injective passing to the quotient, and by construction it will then be a bijection. So it sufficies to show that, given $\tilde{P}, \tilde{Q} \in G \setminus G^{00}$, and $P, Q \in G$ representatives of the respective equivalence classes, $v(l(P * -Q)) > v(l(S))$ if and only if $v(l(P) - l(Q)) > v(l(S))$. But by assumption $v(y_P) = v(y_Q) = v(y_S)$ so $v(l(P)) = v(l(Q)) = v(l(S))$, and then $v\big(l(P * -Q)\big) = v\left(\frac{x_Q y_P - x_P y_Q}{x_P x_Q - y_P y_Q}\right) = v\left(x_Q y_P - x_P y_Q\right) = v\left(x_P x_Q \left(\frac{y_P}{x_P} - \frac{y_Q}{x_Q}\right)\right) = v\left(\frac{y_P}{x_P} - \frac{y_Q}{x_Q}\right) = v(l(P) - l(Q))$. This proves the statement. □

We then get:

Theorem 4.13. *Given* $G = SO_2(M)$ *or* G *a truncation of* $SO_2(M)$, G/G^{00} *is non-1-based in* $Th\big((M, G^{00}, \dots)^{eq}\big)$ *(resp.* $Th\big((M, G^{00}, x_S, y_S, \dots)^{eq}\big)$*).*

5 Ind-definable sets

In this section we generalise the results of section 4 to certain *ind*-definable groups. We take a truncation $(G, *)$ of a group H, whose operation is $\cdot$, and we say that $\tilde{G}$ is the group *ind*-defined by G if $\tilde{G} = \bigcup_{i \in \omega} G_i$ where $G_0 = G$ and $G_{i+1} = < G_i >$ the closure of the set G_i under the operation $\cdot$. Clearly $\tilde{G}$ is an *ind*-definable group in M.

It is an easy observation that $\tilde{G}^{00} = G^{00}$.

Our aim is to recover 1-basedness (or non-1-basedness) from the compact cases:

Theorem 5.1. *A group $\tilde{G}/\tilde{G}^{00}$, where $\tilde{G}$ is ind-defined by a group G, either an additive or a multiplicative or an $SO_2(M)$-truncation, is 1-based if and only if G/G^{00} is.*

The first step to prove this theorem is:

Lemma 5.2. *If G is an additive truncation, a "small" multiplicative truncation or a truncation of $SO_2(M)$, the structure $(M, \tilde{G}, \tilde{G^{00}}, \bar{a}, \dots)^{eq}$, in which $\tilde{G}/\tilde{G}^{00}$ is definable, is interdefinable with $(M, G^{00}, \bar{a}, \dots)^{eq}$.*

Proof. Since $G^{00} = \tilde{G}^{00}$ we only have to show that $\tilde{G}$ is definable in a structure with a predicate for G^{00}; we proceed by cases:

- Additive truncation: if $\tilde{G}$ is constructed from an additive truncation G, we can rescale both of them and suppose $G = [-1, 1)$. Then G^{00} is the set of infinitesimal elements, and $\tilde{G}$ the set of finite elements, so we can simply define $\tilde{G}$ as $\{x | x \notin G^{00} \wedge x^{-1} \notin G^{00}\}$.
- Small multiplicative truncation: if $\tilde{G}$ is constructed from a multiplicative truncation $G = ([b^{-1}, b), \cdot\ mod\ b^2)$, with $v(b) \geq 0$, then we know that $G^{00} - 1 = A^{00}$: the minimal bounded index type-definable subgroup of the additive truncation $A = \Big(\big[-(b-1), b-1\big), +\ mod\ 2(b-1)\Big)$. We just proved that $\tilde{A}$ is definable in the structure $(M, G^{00}, b, \dots)^{eq}$ using the predicate for A^{00} and this latter is definable using G^{00}. We now define $\tilde{G}$ using $\tilde{A}$: let α be the upper cut of $\tilde{A}$; it is sufficient to prove that $\alpha + 1$ is the upper cut of $\tilde{G}$. In fact α is definable from G^{00} and $\tilde{G}$ would be defined by $\{x | x > 0 \wedge x < (\alpha + 1) \wedge x^{-1} < (\alpha + 1)\}$.

  Consider then $g \in \tilde{G}$, and $g > 1$. To prove that $g - 1 < \alpha$, it is sufficient to show that $v\left(\frac{g-1}{b-1}\right) \geq 0$, since $\frac{1}{b-1}\alpha$ is the upper cut of Fin: the convex hull of $\mathbb{Q}$. But $g \in \tilde{G}$ implies that $g < b^n$ for some n, and so $\frac{g-1}{b-1} < \frac{b^n-1}{b-1}$. Using the valuation: $v\left(\frac{g-1}{b-1}\right) \geq v\left(\frac{b^n-1}{b-1}\right) = v(b^{n-1} + b^{n-2} + \dots + 1) = 0$.

  On the other hand if $a < \alpha$, $a < n(b-1)$, then for some n, $a + 1 < n(b-1) + 1 < (b-1)^n + \dots + n(b-1) + 1 < (b - 1 + 1)^n < b^n$, therefore $a + 1 \in \tilde{G}$.

  Thus $\tilde{G}$ is definable in $(M, G^{00}, b, \dots)^{eq}$.
- Truncations of $SO_2(M)$. If $\tilde{G}$ is constructed from a truncation $G = [-S, S)$ of $SO_2(M)$, then either $v(y_S) = 0$, and in this case $\tilde{G}$ is $SO_2(M)$ itself and we are done, or $v(y_S) > 0$. If this is the case, we want to construct a definable bijection (in the structure $M' = (M, \tilde{G}^{00}, x_S, y_S, \dots)^{eq}$) between $\tilde{G}/\tilde{G}^{00}$ and the quotient of an ind-definable group $\tilde{A}$ from an additive truncation A, by its own $\tilde{A}^{00}$.

  We consider again the function $l : \tilde{G} \to M$, $l(P) = \frac{y_P}{x_P}$. Let $A = \big([-l(S), l(S)) +\ \ mod\ 2 \cdot l(S)\big)$, then $A^{00} = \tilde{A}^{00} = \{x | v(x) > v(y_S)\}$, in fact $v(l(S)) = v(y_S)$. Then $\tilde{A} = \{x | v(x) \geq v(l(S))\}$. Firstly we prove that $l(\tilde{G}^{00}) = \tilde{A}^{00}$ (this would also imply interdefinability of the structures M' and $(M, \tilde{A}^{00}, l(S), \dots)^{eq}$). We can of course simply consider G^{00} and A^{00}.

Following the proof of lemma 4.12 we obtain that $l(G^{00}) = A^{00}$. The same argument proves also that $l(\tilde{G}) = \tilde{A} = \{x | v(x) \geq v(y_S)\}$.

The second step is to prove that the function on the quotient $l : \tilde{G}/\tilde{G}^{00} \to \tilde{A}/\tilde{A}^{00}$ is well defined and it is therefore a bijection. We need to prove that $P/\tilde{G}^{00} = Q/\tilde{G}^{00}$ if and only if $l(P)/\tilde{A}^{00} = l(Q)/\tilde{A}^{00}$, i.e. that $v(l(P * -Q)) > v(y_S)$ if and only if $v(l(P) - l(Q)) > v(y_S)$.

This is exactly what we already proved in lemma 4.12, we therefore obtain the statement.

□

By lemma 2.7, $\tilde{G}/\tilde{G}^{00}$ is uniformly-*o*-minimal, so it makes sense to talk about 1-basedness of the theory of $\tilde{G}/\tilde{G}^{00}$. It is clear in these cases that also $\tilde{G}/\tilde{G}^{00}$ is non-1-based: in fact we can witness it in a neighbourhood of the identity with the same function used to prove non-1-basedness of G/G^{00} in the corresponding cases of the previous section.

So we only need to deal with the group *ind*-defined by the multiplicative truncation $G = ([b^{-1}, b), \cdot \ mod \ b^2)$, with $v(b) < 0$.

We consider again the structure M_w^{eq} of the valued field interdefinable with $(M, G^{00}, \dots)^{eq}$. $\tilde{G}$ is only $\bigvee$-definable in this structure, by $\bigvee_{n \in \omega} \{x | x > 0 \wedge v(x) > nv(b) \wedge -v(x) > nv(b)\}$. We restrict our attention to the sort Γ of the value group, in which G/G^{00} is definable. We recall that $Th(\Gamma) = Th(\mathbb{Q}, +, 0, <, q)_{q \in \mathbb{Q}}$, and we add a predicate for $\tilde{G}/\tilde{G}^{00}$, that is a cut in Γ, obtaining T'.

We are now able finish the proof of theorem 5.1 by showing by contradiction that this last $\tilde{G}/\tilde{G}^{00}$ is 1-based.

Let then $\tilde{\mathcal{G}}$ be a saturated model of $T_{\tilde{G}/\tilde{G}^{00}}$, and $f(x, \overline{y})$ be a function $\tilde{\mathcal{G}} \times \tilde{\mathcal{G}}^m \to \tilde{\mathcal{G}}$ definable in an open set S of $\tilde{\mathcal{G}}^{1+m}$, and a, $\overline{c}$ contained in that open subset, and for which $\overline{c}/ \sim \notin dcl(a, f(a, \overline{c}))$.

We can suppose $S \subset \tilde{\mathcal{G}}_i^{1+m}$, for some i, such that $\tilde{\mathcal{G}}_i \cap \tilde{G}/\tilde{G}^{00} = G_i/\tilde{G}^{00}$.

We consider then the truncation $(G_i, *)$, definable in $(M, G^{00}, \dots)$. The function above is then definable in $\mathcal{G}_i = T_{G_i/\tilde{G}^{00}}$, and proves 1-basedness of $G_i/\tilde{G}^{00}$, contradicting theorem 4.10.

We have therefore proved the theorem 5.1.

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