

Tame structures and open cores

Version 3.2

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Abstract

We study various notions of “tameness” for definably complete expansions of ordered fields. We mainly study structures with locally o-minimal open core, d-minimal structures, and dense pairs of d-minimal structures.

Key words: o-minimal, open core, ordered field, definably complete, Baire, tame, dense pair, d-minimal, locally o-minimal.

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Contents

1	Introduction	2
2	Conventions, basic definitions, and notation	3
3	Locally o-minimal open core	4
4	Preliminaries	6
4.1	Dimension	7
4.2	Pseudo-finite sets	8
4.3	Bad and locally closed sets	10
4.3.1	Bad set	11
4.3.2	Locally closed sets	12
5	Proof of Thm. 3.3	13
5.1	Further results	15
6	Locally o-minimal structures	17
6.1	Examples	18
6.2	The monotonicity theorem	18
6.3	Constructibility and partition into cells	19
6.4	Additional results on locally o-minimal structures	23
7	Definably complete structures	24
7.1	Functions	24
7.2	Pseudo-enumerable sets	26
7.3	Uniform families	30
7.4	Is every definably complete structure Baire?	33
7.5	Fixed point theorems	35
7.6	Baire structures	36
8	I-minimal and constructible structures	36
8.1	I-minimal structures	36
8.2	Constructible structures	40
8.3	Proof of Thm. 8.4	42
9	Definable choice	45
9.1	Sard’s Lemma and dimension inequalities	46

10 D-minimal structures	47
11 Dense pairs	50
11.1 Cauchy completion	50
11.1.1 Polish structures and theories	52
11.2 The Z-closure	54
11.3 Dense pairs of d-minimal structures	59
11.3.1 The open core	60
12 Types in locally o-minimal structures	60
13 D-minimal open core	61

1 Introduction

We will study various notion of “tameness”, which generalize the notion of o-minimality. We will be interested only in definably complete structure: “tame” but not definable complete structures (e.g., weakly o-minimal structures), while important and interesting, are outside the scope of this article.

The first natural generalization of o-minimality (for definably complete structures) is asking for o-minimality only around each point of the structure (see Def. 3.2). Much of the theory of o-minimal structures can be generalized without difficulty to locally o-minimal ones: see §6, and from §8 to the end.

There is a dichotomy in further generalizing local o-minimality; let \mathbb{K} be a definably complete expansion of an ordered field:

1. Either we ask that the open core of \mathbb{K} is locally o-minimal, obtaining a-minimal structures; a stronger version is the requirement that the open core of \mathbb{K} is o-minimal (see §3).
2. Or we ask that every definable subset of \mathbb{K}' is a union of an open set and finitely many discrete sets, for every $\mathbb{K}' \succeq \mathbb{K}$, obtaining d-minimal structures (see from §10 to the end).

Definably complete structures were explicitly defined and studied in [Mil01]. The open core of \mathbb{K} was defined already in [MS99], where they study the case when \mathbb{K} is an expansion of \mathbb{R} . Structures with o-minimal open core are one of the main topics of [DMS10]; here, instead, they are only a side remark, because we show that many of the results (and some of the techniques) of [DMS10] can be generalized to a-minimal structures; moreover, we answer some questions left open there. One of the natural examples of structures with o-minimal open core is given by elementary pairs of o-minimal structures $A \preceq B$, studied in [vdD98a]; we show that the main results of [vdD98a] can be generalized to elementary pairs of d-minimal structures, with a very similar proof (see §11).

D-minimal structures were the main theme of [Mil05], where he also examines other notion of tameness, in the case when \mathbb{K} expands \mathbb{R} : many of the definitions and proofs of this article are either inspired by [Mil05], or a direct reference to it.

While o-minimal structures are geometric (that is, the algebraic closure acl satisfies the Exchange Principle), and therefore they rosy of p-rank 1, no such result is true for d-minimal structures: more precisely, if \mathbb{K} is a sufficiently saturated d-minimal non o-minimal structure, then acl does *not* satisfy the Exchange Principle, and \mathbb{K} is not rosy (Lemma 11.35). However, we have a

notion of dimension for d-minimal structures, given by the topology, (which for o-minimal structures coincides with the usual o-minimal dimension), which can be usefully employed in the study of d-minimal structures, and in particular of dense elementary pairs of such structures, in the same way as the o-minimal dimension is used in o-minimal structures.

All structure considered will moreover be definably Baire (see [FS09]): this will not be explicit in the definitions, but will follow quite easily from them. Also, many of the proofs will rely on consequences of this Baire property (much in the same way as in [Mil05] many theorems relied on Baire's category theorem). Therefore, a preliminary study of definably complete and of definably Baire structures is essential in order to understand tame structures, and it will be carried out Sections 4 and 7.

Some general results can be obtained under even weaker conditions: see Sections 8 and 9.

2 Conventions, basic definitions, and notation

Definable will always mean “definable with parameters”.

$\mathbb{R} = \langle \mathbb{R}, 0, 1, +, \cdot, < \rangle$ is the ordered field of real numbers. \mathbb{R}^{alg} is the subset of \mathbb{R} given by the real algebraic numbers.

A linearly ordered structure $\langle \mathbb{K}, < \rangle$ is **definably complete** if every definable subset of \mathbb{K} has a supremum in $\mathbb{K} \sqcup \{\pm\infty\}$.

Proviso. \mathbb{K} will always be a definably complete structure expanding an ordered field.

$d : \mathbb{K}^n \times \mathbb{K}^n \rightarrow \mathbb{K}$ is the **distance** function $d(x, y) := |x - y|$. For every $\bar{x} \in \mathbb{K}^n$ and $0 < r \in \mathbb{K}$, $B(\bar{x}, r)$ is the **open ball** of center \bar{x} and radius r , while $\bar{B}(\bar{x}, r)$ is the **closed ball**.

Let $X \subseteq \mathbb{K}^n$. \bar{X} , also denoted by $\text{cl}(X)$, is the topological **closure** of X inside \mathbb{K}^n , while $\overset{\circ}{X}$, also denoted by $\text{int}(X)$, is the **interior** of X ; $\partial X := \bar{X} \setminus \overset{\circ}{X}$ is the **frontier** of X ; $\text{bd}(X) := \bar{X} \setminus \overset{\circ}{X}$ is the **boundary** of X . X is **nowhere dense** if $\overset{\circ}{\bar{X}}$ is empty.

\mathbb{K} is **definably Baire** (or simply “Baire” for short) if \mathbb{K} is not the union of a definable increasing family of nowhere dense subsets [FS09].

X is an \mathcal{F}_σ -**set** if X is definable and is the union of a definable increasing family of closed subsets of \mathbb{K}^n , and is a \mathcal{G}_δ -set if its complement is an \mathcal{F}_σ -set. X is **meager** if is the union of a definable increasing family of nowhere dense sets. X is **almost open** (or a.o. for short) if there exists a definable open sets U such that $X \Delta U$ is meager [FS09], where Δ is the **symmetric difference** of sets.

X is **constructible** if it is a finite Boolean combination of open sets. X is **locally closed** if for every $x \in X$ there exists a neighbourhood $U \ni x$ such that $X \cap U$ is closed in U .

Fact. *If X is definable, then X is constructible iff it is a finite Boolean combination of definable open sets [All96, DM01]. X is locally closed iff it is of the form $C \cap U$, for some closed set C and some open set U . X is constructible iff it is a finite union of locally closed sets.*

Definition 2.1. Let $X \subseteq \mathbb{K}^n$ be definable. X is **d-compact** if it is closed and bounded. X is **pseudo-finite** if it is d-compact and discrete.

Definition 2.2. The **open core** of \mathbb{K} is the reduct of \mathbb{K} generated by all definable open subsets of \mathbb{K}^n , for every $n \in \mathbb{N}$.

Definition 2.3. \mathbb{K} is **locally o-minimal** if, for every definable function $f : \mathbb{K} \rightarrow \mathbb{K}$, the sign of f is eventually constant.

See also §6 for more details on local o-minimality.

Given $d \leq n \in \mathbb{N}$, we denote by $\Pi_d^n : \mathbb{K}^n \rightarrow \mathbb{K}^d$ the projection onto the first d coordinates. Given $X \subseteq \mathbb{K}^{n+m}$ and $\bar{c} \in \mathbb{K}^n$, we define $X_{\bar{c}} := \{\bar{y} \in \mathbb{K}^m : \langle \bar{c}, \bar{y} \rangle \in X\}$, and $X_{[\bar{c}]} := \{\bar{c}\} \times X_{\bar{c}} = X \cap \{\bar{c}\} \times \mathbb{K}^m$.

Remark 2.4. The open core of \mathbb{K} includes all definable constructible sets and, more generally, all \mathcal{F}_σ -sets. In fact, if X is an \mathcal{F}_σ -set, then X is the projection of a closed definable set: if $X = \bigcup_{t \in \mathbb{K}} X_t$, where $(X_t : t \in \mathbb{K})$ is a definable increasing family of closed subsets of \mathbb{K}^n , then $X = \Pi_n^{n+1} \text{cl}(\bigcup_{t \in \mathbb{K}} X_t \times \{t\})$.

I will now discuss briefly the proviso that \mathbb{K} expands a field. This assumption is often convenient for notational purposes and to simplify the statements of the theorems (compare e.g. our definition of \mathcal{F}_σ -sets with the corresponding definition of D_Σ sets in [DMS10]); in those cases, a reader that is interested in definably complete structures that may not expand a field can easily modify definitions, proofs, and statements to his situation. However, sometimes the field assumption is used in an essential way (e.g., in §7.4 and §11.3), and the reader assumed above should be more careful.

3 Structures with o-minimal and locally o-minimal open core

Definition 3.1. Let P be a property of definable sets. We say that P is **definable** (for \mathbb{K}), if for every definable family $(X_y)_{y \in A}$, the set $d_P(X) := \{y \in A : P(X_y)\}$ is definable. If T is a theory, we say that P is definable for T if P is definable for every model of T .

For instance, “being closed” and “being pseudo-finite” are definable properties. A type p over \mathbb{K} is definable iff the corresponding property “ $X \in p$ ” is definable [Poi85, §11.b]. We do not know if “being constructible” is definable (because a constructible set X is a finite union of locally closed sets; if we do not have a bound on the number of locally closed sets C_i such that $X = \bigcup_i C_i$, we are not able to express the constructibility of X in a definable way). Notice also that a property might be definable for \mathbb{K} without being definable for the theory of \mathbb{K} : for instance, if \mathbb{K} is *not* ω -saturated, then “being finite” might be definable for \mathbb{K} , without being definable for some $\mathbb{K}' \succ \mathbb{K}$.

Definition 3.2. K is **locally o-minimal** if, for every definable function $f : \mathbb{K} \rightarrow \mathbb{K}$ and every $x \in \mathbb{K}$, there exists $y > x$ such that either $f > 0$ on (x, y) , or $f = 0$ on (x, y) , or $f < 0$ on (x, y) .

Theorem 3.3. *The following are equivalent:*

1. Every definable discrete closed subset of \mathbb{K} is pseudo-finite;
2. Every definable discrete closed subset of \mathbb{K}^n is pseudo-finite for every $n \in \mathbb{N}$;
3. Every definable discrete subset of \mathbb{K} is pseudo-finite;
4. Every definable discrete subset of \mathbb{K}^n is pseudo-finite for every $n \in \mathbb{N}$;
5. Every definable nowhere-dense subset of \mathbb{K} is pseudo-finite;
6. Every definable meager subset of \mathbb{K} is pseudo-finite;
7. Every definable nowhere-dense subset of \mathbb{K} is discrete;
8. The **open core** of \mathbb{K} is **locally o-minimal**;
9. The union of any definable increasing family of pseudo-finite subsets of \mathbb{K} is pseudo-finite;
10. The union of any definable increasing family of pseudo-finite subsets of \mathbb{K}^n is pseudo-finite, for every $n \in \mathbb{N}$.

Moreover, if any of the above equivalent conditions is satisfied, then every \mathcal{F}_σ subset of \mathbb{K}^n is constructible, \mathbb{K} is Baire, and every meager subset of \mathbb{K}^n is nowhere dense.

Definition 3.4. We say that \mathbb{K} is **a-minimal** if it satisfies any of the equivalent conditions in Thm. 3.3 (that is, if it has locally o-minimal open core).

Definition 3.5. Let $X \subseteq \mathbb{K}^{n+m}$. Define $\text{Fin}_n(X) := \{y \in \mathbb{K}^n : X_Y \text{ is finite}\}$.

“Being finite” is not definable a definable property in general. The following lemma characterises when it is a definable property.

Lemma 3.6. *The following are equivalent:*

1. Every pseudo-finite set is finite;
2. For every $X \subseteq \mathbb{K}^2$ definable, $\text{Fin}_1(X)$ is also definable;
3. “Being finite” is a definable property.

Proof. (3 \Rightarrow 2) and (1 \Rightarrow 3) are clear. Assume (2). Let $Z \subset \mathbb{K}^n$ be pseudo-finite; we have to prove that Z is finite; for simplicity, assume $n = 1$. Let $r := \sup\{r \in \mathbb{K} : Z \cap [-r, r] \text{ is finite}\}$. Clearly, $r = +\infty$, and therefore, since Z is bounded, Z is finite. \square

Corollary 3.7. *The following are equivalent:*

1. \mathbb{K} is a-minimal, and every pseudo-finite subset of \mathbb{K} is finite;
2. \mathbb{K} is a-minimal, and every pseudo-finite subset of \mathbb{K}^n is finite, for every $n \in \mathbb{N}$;
3. \mathbb{K} is a-minimal, and for every $X \subseteq \mathbb{K}^2$ definable, $\text{Fin}_1(X)$ is definable;
4. \mathbb{K} is a-minimal, and “being finite” is a definable property for \mathbb{K} ;

5. \mathbb{K} has *o-minimal open core*;
6. Every definable closed discrete subset of \mathbb{K} is finite;
7. Every definable closed discrete subset of \mathbb{K}^n is finite, for every $n \in \mathbb{N}$;
8. Every definable discrete subset of \mathbb{K} is finite;
9. Every definable discrete subset of \mathbb{K}^n is finite, for every $n \in \mathbb{N}$.

The above corollary is a converse of [DMS10, Theorem A].

Definition 3.8. We say that \mathbb{K} has *o-minimal open core* if it satisfies any of the equivalent conditions in Corollary 3.7.

Notice that if \mathbb{K} expands \mathbb{R} , then “being finite” is a definable property. Notice also that “Uniform Finiteness” (UF), as defined in [DMS10] (also known as “elimination of the quantifier \exists^∞ ”) is a stronger property than “being finite is definable in \mathbb{K} ”, because the former says that “being finite” is definable in the theory of \mathbb{K} (for instance, if \mathbb{K} expands \mathbb{R} , then “being finite” is definable for \mathbb{K} , but \mathbb{K} does not necessarily satisfy UF).

Corollary 3.9 ([DMS10]). *If \mathbb{K} satisfies UF, then \mathbb{K} has o-minimal open core.*

Proof. Condition 3.7-6 follows easily from the hypothesis. \square

Remark 3.10. If \mathbb{K} is an a-minimal expansion of \mathbb{R} , then it has o-minimal open core.

4 Preliminaries

Let $X \subseteq \mathbb{K}^n$ be definable. Let $\partial A := \bar{A} \setminus A$, and $\text{bd}(A) := \bar{A} \setminus \overset{\circ}{A}$.

Lemma 4.1. *If $Y \subseteq \mathbb{K}^n$ is definable and definably connected (e.g., $n = 1$ and Y is an interval), then the following are equivalent:*

1. $Y \cap \text{bd}(X) = \emptyset$;
2. either $Y \subseteq \overset{\circ}{X}$, or $Y \cap \bar{X} = \emptyset$.

Remark 4.2. • ∂X has empty interior.

- $\text{bd}(X) = \partial X \sqcup \partial(\mathbb{K}^n \setminus X)$.
- $\text{bd}(X)$ is closed.
- $\text{bd}(X \cup X') \subseteq \text{bd}(X) \cup \text{bd}(X')$.
- If X and X' are nowhere dense, then $X \cup X'$ is also nowhere dense.
- If X is locally closed, then $\text{bd}(X)$ is nowhere dense. Therefore, if X is constructible, then $\text{bd}(X)$ is nowhere dense.

Corollary 4.3. *If X is constructible and $\overset{\circ}{X} = \emptyset$, then X is nowhere dense.*

Lemma 4.4. *X is locally closed iff ∂X is closed. If $\text{bd}(X)$ is discrete, then X is locally closed.*

Proof. For the first part, if ∂X is closed, then X is open in \overline{X} , and hence locally closed. Conversely, if $X = \overline{X} \cap U$ for some open set U , then $\partial X = X \setminus U$.

For the second part, if $\text{bd}(X)$ is discrete, then, since $\text{bd}(X)$ is also closed, $\text{bd}(X)$ has no accumulation points in \mathbb{K}^n . Since $\partial X \subseteq \text{bd}(X)$, we have that ∂X is also closed (in \mathbb{K}^n). \square

Lemma 4.5. *Let $f : [0, 1] \rightarrow \mathbb{K}$ be definable and continuous. Then, f is uniformly continuous.*

Proof. Assume not; then, there exists $\varepsilon > 0$ such that, for every $\delta > 0$, the set

$$X(\delta) := \{(x, y) \in [0, 1]^2 : |x - y| \leq \delta \ \& \ |f(x) - f(y)| \geq \varepsilon\}$$

is non-empty. Since $[0, 1]^2$ is d -compact, we have $X := \bigcap_{\delta > 0} X(\delta) \neq \emptyset$. If $(x, y) \in X$, then $x = y$ and f is not continuous at x , absurd. \square

Definition 4.6. Let P be a property of definable sets. We say that P is **monotone** if $X \subseteq Y$ and $P(Y)$ imply $P(X)$. We say that P is **additive** if $P(X)$ and $P(Y)$ imply $P(X \cup Y)$.

Lemma 4.7. *Let $C \subset \mathbb{K}^n$ be d -compact, $f : C \rightarrow \mathbb{K}^m$ be definable, and P be a property of definable sets. Assume that P is definable, monotone, and additive, and that, for every $c \in C$, there exists U_c definable neighbourhood of c , such that $P(f(U_c \cap C))$. Then, $P(f(C))$.*

Proof. W.l.o.g., f is the inclusion function. Proceed as in [FS09]. \square

For instance, we can apply the above lemma to the property “being nowhere dense”.

4.1 Dimension

Definition 4.8. Let $X \subseteq \mathbb{K}^n$ be definable and non-empty. The **dimension** of X is

$$\dim X := \max\{d \leq n : \text{there exists a coordinate space } L \text{ of dimension } d, \\ \text{s.t. } \Pi_L^n(X) \text{ has non-empty interior}\},$$

where Π_L^n is the projection from \mathbb{K}^n onto L . By convention, we say that $\dim \emptyset = -1$. The **full dimension** of X is the pair $\langle d, k \rangle$, where $d = \dim X$ and $1 \leq k$ is the number of coordinate spaces L of dimension d , s.t. $\Pi_L^n(X)$ has non-empty interior.

The set of full dimensions is ordered lexicographically, with the dimension component more important. Therefore, by induction on the full dimension we mean induction first on d and then on k . Dimension and full dimension were already defined in [DMS10].

Lemma 4.9. *Let $X \subseteq \mathbb{K}^{n+m}$ be a definable, of dimension n . Let $A := \{a \in \mathbb{K}^n : \dim(X_a) > 0\}$. If X is an \mathcal{F}_σ , then A is an \mathcal{F}_σ . If moreover \mathbb{K} is Baire, then $\overset{\circ}{A} = \emptyset$ (i.e., A has dimension less than n).*

Proof. Same as [DMS10, 2.8(3) and 3.4]. Since

$$A = \bigcup_{\rho \in \Pi(n,1)} \bigcup_{s>0} \{a \in \mathbb{K}^n : \rho(X_a) \text{ contains a closed interval of length } s\},$$

and X is an \mathcal{F}_σ , A is also an \mathcal{F}_σ . If, for contradiction, A has non-empty interior, then, w.l.o.g., forevery $a \in A$, $\Pi_1^n(X_a)$ contains an open interval. Thus, by Kuratowski-Ulam's Theorem [FS09], $\Pi_{n+1}^{n+m}(X)$ is non-meager, and thus has non-empty interior, contradicting $\dim X = n$. \square

In the above lemma we cannot drop the assumption that X is an \mathcal{F}_σ ; for instance, let X be the set of pairs $\langle x, y \rangle \in \mathbb{R}^2$ such that:

$$x \in \mathbb{Q} \ \& \ 0 < y < 1 \ \vee \ x \notin \mathbb{Q} \ \& \ 1 < y < 2,$$

in the structure $\mathbb{R}(\mathbb{N}) := \langle \mathbb{R}, +, \cdot, \mathbb{N} \rangle$.

Lemma 4.10. *Assume that \mathbb{K} is Baire. Let $X_1, X_2 \subseteq \mathbb{K}^n$ be definable. If X_1 and X_2 are both \mathcal{F}_σ , then $\text{fdim}(X_1 \cup X_2) = \max(\text{fdim}(X_1), \text{fdim}(X_2))$ and in particular $\dim(X_1 \cup X_2) = \max(\dim(X_1), \dim(X_2))$.*

Proof. Assume, for contradiction, that, for some $m \leq n$, $\Pi_m^n(X_1 \cup X_2)$ has non-empty interior, while $\Pi_m^n(X_i)$ has empty interior, for $i = 1, 2$. However, since $\Pi_m^n(X_i)$ is an \mathcal{F}_σ , this means that $\Pi_m^n(X_i)$ is meager, for $i = 1, 2$, and therefore $\Pi_m^n(X_1 \cup X_2) = \Pi_m^n(X_1) \cup \Pi_m^n(X_2)$ is also meager, absurd. \square

In the above lemma we cannot drop the assumptions that the X_i are \mathcal{F}_σ : for instance, let $X_1 = \mathbb{Q}$ and $X_2 = \mathbb{R} \setminus \mathbb{Q}$ in the structure $\mathbb{R}(\mathbb{N})$.

Example 4.11. It is not true that, if $X \subseteq \mathbb{K}^n$ is definable and constructible, then $\dim \bar{X} = \dim X$; cf. Thm. 8.4(5). In fact, let $\mathbb{K} := \mathbb{R}(\mathbb{N})$, and $X \subset \mathbb{R}^2$ defined by:

$$X := \{\langle x, y \rangle : x = p/q \in \mathbb{Q} \ \& \ 0 < p < q \in \mathbb{N} \ \& \ (p, q) = 1 \ \& \ y = 1/q\}.$$

Notice that X is locally closed (and *a fortiori* constructible), $\bar{X} = X \cup ([0, 1] \times \{0\})$, and $\dim X = 0$, while $\dim \bar{X} = 1$.

4.2 Pseudo-finite sets

Define $\delta(X) := \inf\{d(x, x') : x, x' \in X \ \& \ x \neq x'\}$.

Remark 4.12. If X is discrete, then it is nowhere dense in \mathbb{K}^n .

Proof. Clear. \square

Lemma 4.13. *If X and X' are pseudo-finite, then $X \times X'$ is also pseudo-finite. Moreover, if X is pseudo-finite, then every definable subset of X is also pseudo-finite.*

Proof. The first part is clear from the definition.

Let $Y \subseteq X$ be definable. It suffices to prove that Y is closed in \mathbb{K}^n , to conclude that Y is pseudo-finite. Let $x \in X \setminus Y$. Since X is discrete, $\{x\}$ is open in X , and therefore Y is closed in X ; since X is closed in \mathbb{K}^n , Y is also closed in \mathbb{K}^n . \square

Lemma 4.14. *The following are equivalent:*

1. X is pseudo-finite;
2. X is bounded and has no accumulation points in \mathbb{K}^n ;
3. X is bounded and $\delta(X) > 0$.

Proof. (1 \Leftrightarrow 2) follows from the definition of pseudo-finite.

(3 \Rightarrow 2) is clear.

(1 \Rightarrow 3). Assume that X is pseudo-finite. We want to prove that $\delta(X) > 0$. Let $Y := X \times X$, and $\Delta(Y)$ be its diagonal. Consider the map $d : Y \setminus \Delta(Y) \rightarrow \mathbb{K}$, mapping (x, x') to $d(x, x')$. Note that $Y \setminus \Delta(Y)$ is pseudo-finite, and that $\delta(X) = \inf_{Y \setminus \Delta(Y)} d(y)$. Thus, d attains a minimum on $Y \setminus \Delta(Y)$, and therefore $\delta(X) > 0$. \square

Lemma 4.15. $X \subseteq \mathbb{K}^n$ is discrete and closed iff, for every $r > 0$, $X \cap B(0, r)$ is pseudo-finite.

Proof. (\Rightarrow) is clear, because if X is discrete and closed, then $X \cap B(0, r)$ is discrete, closed and bounded. (\Leftarrow) follows from the fact that X has no accumulation points in \mathbb{K}^n . \square

Remark 4.16. If \mathbb{K} defines an unbounded discrete subset, then it defines an unbounded discrete closed subset.

The above remark answers a question in [Mil05, §5].

Proof. Let $D \subset \mathbb{K}$ be discrete and not closed. W.l.o.g., we can assume that D is unbounded (if a is an accumulation point for D , then ∞ is an accumulation point for $1/(D - a)$). For every $r > 0$, let

$$D(r) := \{x \in D : D \cap B(x, r) = \{x\}\},$$

the set of points in D at distance at least r from the other points of D . Each $D(r)$ is discrete and closed. If $D(r)$ is unbounded for some r , we are done. Otherwise, each $D(r)$ pseudo-finite; let $z(r) := \max(D(r))$, and $Z := \{z(r) : r > 0\}$. Since D is unbounded, Z is also unbounded.

Claim 1. Z is closed and discrete.

Otherwise, Z would have an accumulation point a . For every $r > 0$, let $Z(r) := \{z(r') : r' \leq r\}$. Notice that $Z(r)$ is bounded and $\delta(Z(r)) \geq r$; thus, $Z(r)$ is pseudo-finite. Moreover, since $z(r)$ is an increasing function of r , there exists $r_0 > 0$ such that $z(r) > a + 1$ for every $r > r_0$. Hence, a cannot be an accumulation point of Z , absurd. \square

Lemma 4.17. *If X is discrete, then it is the union of a definable increasing family of pseudo-finite sets. In particular, X is an \mathcal{F}_σ .*

Proof. After a change of coordinates, we can assume that X is bounded. For every $r > 0$, define $X(r) := \{x \in X : X \cap B(x, r) = \{x\}\}$. Since X is discrete, $X = \bigcup_r X(r)$. Therefore, it suffices to prove that, for each r , $X(r)$ is pseudo-finite. Fix $r > 0$. It is clear that $\delta(X(r)) \geq r > 0$, and therefore $X(r)$ is pseudo-finite. \square

Lemma 4.18. *Let X be pseudo-finite. If $f : X \rightarrow \mathbb{K}^m$ is definable (not necessarily continuous), then $f(X)$ is also pseudo-finite.*

Proof. Given $f : X \rightarrow \mathbb{K}^m$, we want to prove that $f(X)$ is d-compact and discrete. By a change of coordinates in \mathbb{K}^m , we can always assume that $f(X)$ is bounded. Assume, for contradiction, that $f(X)$ is not pseudo-finite. Let $y \in \mathbb{K}^m$ be an accumulation point for $f(X)$. For every $r > 0$, let $U(r) := f^{-1}(B(y, r) \setminus \{y\})$; note that each $U(r)$ is non-empty. By Lemma 4.13, each $U(r)$ is closed in X . Since X is d-compact, $\bigcap_r U(r) \neq \emptyset$, which is absurd. \square

Corollary 4.19. *Let $X \subseteq \mathbb{K}^n$ and $X' \subseteq \mathbb{K}^{n'}$ be definable. Then, $X \times X'$ is pseudo-finite iff X and X' are pseudo-finite.*

Corollary 4.20. *X is pseudo-finite iff every projection of X on the coordinate axes is pseudo-finite.*

Definition 4.21. A **pseudo-finite family** of sets is a definable family $(X_a : a \in A)$, such that A is pseudo-finite.

Lemma 4.22. *Let P be an additive definable property. Let $(X_y : y \in A)$ be a pseudo-finite family, such that, for every $y \in A$, $P(X_y)$. Then, $P(\bigcup_{y \in A} X_y)$.*

Proof. For simplicity, we assume that $A \subseteq \mathbb{K}$. Let

$$B := \{y \in A : P(\bigcup_{\substack{z \in A, \\ z \leq y}} A_z)\}.$$

Since P is definable, B is a definable subset of A . Hence, B is pseudo-finite, and therefore it has a maximum b . It is now easy to see that b is also the maximum of A . \square

Since “being closed” is an additive definable property, we see that the union of a pseudo-finite family of closed sets is closed, and the intersection of a pseudo-finite family of open sets is open. Similarly, the union of a pseudo-finite family of pseudo-finite sets is pseudo-finite, and the union of a pseudo-finite family of nowhere dense sets is nowhere dense.

Conjecture 4.23 (Pigeon Hole Principle). *Let $X \subseteq \mathbb{K}^n$ be pseudo-finite and $f : X \rightarrow X$ be definable. If f is injective, then it is surjective.*

4.3 Bad and locally closed sets

Lemma 4.24. *Let $d \leq n$, $A \subseteq \mathbb{K}^n$ be definable, $\pi := \Pi_d^n$, and*

$$Z := Z(A) := \{a \in A : \exists U \text{ neighbourhood of } A : \pi(A \cap U) \text{ is nowhere dense}\}.$$

Then, Z is definable and open subset in A , and $\pi(Z)$ is meager.

Proof. The fact that Z is definable and open in A is trivial. Let $Z' := Z(\overline{A})$.

Claim 2. $Z = Z' \cap A$.

Let $a \in Z$. Then, $\pi(A \cap U)$ is nowhere dense, for some U neighbourhood of a . Moreover, $\pi(\overline{A \cap U}) \subseteq \pi(A \cap U)$, because U is open. Hence, $\pi(\overline{A \cap U})$ is nowhere dense, and therefore $a \in Z'$.

Therefore, it suffices to prove the lemma in the case when A is closed. Since Z is closed in A , Z is locally closed. Moreover, for every $a \in Z$ there exists U neighbourhood of a , such that $\pi(U \cap Z)$ is nowhere dense; thus, by [FS09, Cor. 3.7], $\pi(Z)$ is meager. \square

4.3.1 Bad set

Definition 4.25. Let $A \subseteq \mathbb{K}^{n+m}$. The set of “bad points” for A is

$$\mathfrak{B}_n(A) := \{x \in \mathbb{K}^n : \text{cl}(A)_x \setminus \text{cl}(A_x) \neq \emptyset\}.$$

Notice that $\mathfrak{B}_n(A) = \{x \in \mathbb{K}^n : \text{cl}(A)_x \neq \text{cl}(A_x)\}$.

In the following, it will often be necessary to prove that $\mathfrak{B}_n(A)$, the set of bad points of A , is “small” (in some suitable sense).

Remark 4.26. Assume that $A \subseteq C \subseteq \overline{A} \subseteq \mathbb{K}^{n+m}$. Then, $\mathfrak{B}_n(A) \supseteq \mathfrak{B}_n(C)$.

Proof.

$$\text{cl}(A)_x \setminus \text{cl}(A_x) = \text{cl}(C)_x \setminus \text{cl}(A_x) \supseteq \text{cl}(C)_x \setminus \text{cl}(C_x). \quad \square$$

Lemma 4.27. If A is an \mathcal{F}_σ , then $\mathfrak{B}_n(A)$ is the projection of a \mathcal{G}_δ set. If A is open, then $\mathfrak{B}_n(A)$ is a meager \mathcal{F}_σ .

Proof. Let

$$F(A) := \{\langle x, y, r, y' \rangle \in \mathbb{K}^n \times \mathbb{K}^m \times \mathbb{K} \times \mathbb{K}^m : r > 0 \ \& \ \langle x, y' \rangle \in A \ \& \ |y - y'| < r\}.$$

Claim 3.

$$\mathfrak{B}_n(A) = \pi(\overline{A \cap \pi'(\{r > 0\}} \setminus \pi''(F(A)))),$$

where π' and π'' are suitable projections.

If A is an \mathcal{F}_σ , then $F(A)$ is also an \mathcal{F}_σ , and therefore $\mathfrak{B}_n(A)$ is the projection of a \mathcal{G}_δ set.

If A is open, then $F(A)$ is also open, and therefore $\mathfrak{B}_n(A)$ is an \mathcal{F}_σ .

For every $r > 0$, define $C(r) := \{\langle x, y \rangle \in \overline{A} : |x, y| \leq 1/r \ \& \ d(y, A_x) \geq r\}$. Notice that $\mathfrak{B}_n(A) = \bigcup_{r>0} \pi(C(r))$. Assume that A is open. Then, each $C(r)$ is d-compact. Hence, to prove that $\mathfrak{B}_n(A)$ is meager, it suffices to prove that, for every $r > 0$, $\pi(C(r))$ has empty interior. W.l.o.g., \mathbb{K} is Baire. Assume, for contradiction, that $\pi(C(r_0))$ contains a non-empty open box W , for some $r_0 > 0$. To simplify the notation, assume that $m = 1$. Define $f : W \rightarrow \mathbb{K}$, $f(x) := \min(C(r_0)_x)$. By [FS09], there exists a non-empty open box $W' \subseteq W$, such that $f \upharpoonright W'$ is continuous; w.l.o.g., $W = W'$. Fix $x_0 \in W$, call $y_0 := f(x_0)$, and let V_{x_0} be an open box around x_0 contained in W , and such that, for every $x \in V_{x_0}$, $d(f(x), y_0) < r_0/4$. Since $\langle x_0, y_0 \rangle \in \overline{U}$, there exists $\langle x, y' \rangle \in U$, such that $x \in V_{x_0}$, and $d(y_0, y') < r_0/4$. Let $y := f(x)$. Since $\langle x, y \rangle \in C(r)$, $d(y, U_x) \geq r$; in particular, $d(y, y') \geq r$. However, this contradicts $d(y, y_0) < r_0/4$ and $d(y_0, y') < r_0/4$. \square

Lemma 4.28. $\mathfrak{B}_n(A \cup B) \subseteq \mathfrak{B}_n(A) \cup \mathfrak{B}_n(B)$.

4.3.2 Locally closed sets

Definition 4.29. Let $X \subseteq \mathbb{K}^n$. Define $\text{lc}(X) := \{x \in X : X \text{ is locally closed at } x\}$, (that is, $x \in \text{lc}(X)$ iff there exists an open ball U of center x , such that $X \cap B = \overline{X} \cap B$), and $\ulcorner X \urcorner := X \setminus \text{lc}(X)$.

Define $X^{\ulcorner 0 \urcorner} := X$, and, for each $k \in \mathbb{N}$, $X^{\ulcorner k+1 \urcorner} := \ulcorner X^{\ulcorner k \urcorner} \urcorner$.

Notice that $\text{lc}(X)$ is locally closed, and therefore constructible. Notice also that, if X is definable, then also $\ulcorner X \urcorner$ and each $X^{\ulcorner k \urcorner}$ are definable. Therefore, if X is an \mathcal{F}_σ , then $\ulcorner X \urcorner$ is also an \mathcal{F}_σ .

Proposition 4.30. $\ulcorner A \urcorner = A \cap \partial(\partial A)$. A is the union of m locally closed sets if and only if $A^{\ulcorner m+1 \urcorner}$ is empty.

Proof. See [All96], where ∂A is denoted by \check{A} , and $\ulcorner A \urcorner$ by either $\mathcal{B}(A)$ or $H(A)$. \square

Remark 4.31. If $U \subseteq \mathbb{K}^n$ is open, then $\text{lc}(A) \cap U = \text{lc}(A \cap U)$, and $\ulcorner A \cap U \urcorner = \ulcorner A \urcorner \cap U$.

Remark 4.32. Let $U \subseteq \mathbb{K}^n$ be open, and $A \subseteq U$ be closed in U . Let $E := \overline{A}$. Then, $E \cap \overline{U} = E$. Therefore, $E \cap \overline{U} = \overline{E \cap U}$.

Proof. The \subseteq inclusion is obvious. The opposite inclusion follows immediately from $E = \overline{E \cap U}$. \square

Theorem 4.33. Let $A \subseteq \mathbb{K}^n$ be locally definable closed, and $d \leq n$. Let $U \subseteq \mathbb{K}^n$ be open, such that $A = \overline{A} \cap U$. Then, for every $x \in \mathbb{K}^d$,

$$A_x = \text{cl}(A_x) \cap U_x,$$

and in particular A_x is locally closed. Moreover, $\mathfrak{B}_d(A) \subseteq \mathfrak{B}_d(U)$, and therefore $\mathfrak{B}_d(A)$ is meager.

Proof. $A_x \subseteq \text{cl}(A_x) \cap U_x$ is obvious. For the opposite inclusion,

$$\text{cl}(A_x) \cap U_x \subseteq (\overline{A})_x \cap U_x = (\overline{A} \cap U)_x = A_x.$$

Assume, for contradiction, that $x \in \mathfrak{B}_d(A) \setminus \mathfrak{B}_d(U)$. Let $E := \overline{A}$; notice that $A = E \cap U$. Since $x \notin \mathfrak{B}_d(U)$, $\text{cl}(U_x) = (\overline{U})_x$. By the above Remark, applied to $E_x \cap U_x$ and to $E \cap U$,

$$\text{cl}(A_x) = \text{cl}(E_x \cap U_x) = E_x \cap \text{cl}(U_x) = E_x \cap (\overline{U})_x = (E \cap \overline{U})_x = E_x = (\overline{A})_x,$$

contradicting $x \notin \mathfrak{B}_d(A)$.

By Lemma 4.27, $\mathfrak{B}_d(U)$ is meager, and we are done. \square

Corollary 4.34. Let $A \subseteq \mathbb{K}^n$ be definable and constructible, and $d \leq n$. Then, $\mathfrak{B}_d(A)$ is meager.

5 Proof of Thm. 3.3

(3 \Rightarrow 1) is obvious.

(1 \Rightarrow 9). Let $(X(r))_{r \in \mathbb{K}}$ be a definable increasing family of pseudo-finite subsets of \mathbb{K} , and $X := \bigcup_r X(r)$. Assume, for contradiction, that X is not pseudo-finite. W.l.o.g., we can assume that $X \subseteq (0, 1)$, and that 0 is an accumulation point of X . For every $r \in \mathbb{K}$, let $z(r) := \min(X(r))$, $Z := \{z(r) : r \in \mathbb{K}\}$, $Y := \{1/z(r) : r \in \mathbb{K}\}$.

Claim 4. The only accumulation point of Z in \mathbb{K} is 0.

In fact, suppose, for contradiction, that $c > 0$ is an accumulation point of Z . Since 0 is an accumulation point for X ; there exists $r_0 \in \mathbb{K}$ such that $X(r_0) \cap (0, c/2) \neq \emptyset$. Thus, $z(r_0) < c/2$, and, since $z(x)$ is a decreasing function, $z(r) < c/2$ for every $r \geq r_0$. Let $Z(r_0) := \{z(r) : r < r_0\}$, and $Z' := Z \setminus Z(r_0)$. Since $Z(r_0) \subseteq X(r_0)$, $Z(r_0)$ is pseudo-finite. Moreover, since $Z' \subseteq (0, c/2)$, c is not an accumulation point of Z' , and thus it is not an accumulation point of Z , absurd.

By the claim, Y is discrete and closed, and therefore, by hypothesis, it is pseudo-finite. Hence, Z is also pseudo-finite; therefore, 0 cannot be an accumulation point for Z , absurd.

(9 \Leftrightarrow 10) is clear from Corollary 4.19.

(9 \Rightarrow 3). Follows from Lemma 4.17.

Hence, we have the equivalence (1 \Leftrightarrow 3 \Leftrightarrow 9 \Leftrightarrow 10).

We now prove that (1 \Rightarrow 6). Let $X \subseteq \mathbb{K}$ be meager; thus, X is the union of a definable increasing family $(X(r))_{r \in K}$ of nowhere dense subsets of \mathbb{K} . We want to prove that X is pseudo-finite. W.l.o.g., we can assume that each $X(r)$ is d-compact. By condition 9, it suffices to prove that each $X(r)$ is pseudo-finite. Thus, we fix $r \in \mathbb{K}$, and prove that $Y := X(r)$ is pseudo-finite, knowing that it is d-compact and has empty interior. Assume, for contradiction, that Y has an accumulation point in \mathbb{K} . W.l.o.g., we can assume that $Y \subset (0, 1)$, and that 0 is an accumulation point of Y . Since \mathbb{K} is definably complete and Y is closed, $(0, 1) \setminus Y$ is a union of disjoint open intervals; let D be the set of centres of those intervals, that is:

$$D := \{z \in (0, 1) : \exists r > 0, z - r \in Y, z + r \in Y, (z - r, z + r) \cap Y = \emptyset\}.$$

Note that D is discrete. By condition 3, D is pseudo-finite; let $a := \min(D)$, and $r > 0$ such that $a - r, a + r \in Y$ and $(a - r, a + r) \cap Y = \emptyset$. Thus, $(0, a - r) \subseteq Y$. Since Y has empty interior, $a - r = 0$. However, this contradicts the fact that 0 is an accumulation point for Y .

(6 \Rightarrow 5 \Rightarrow 1) is clear.

(7 \Rightarrow 1). Let $X \subseteq \mathbb{K}$ be discrete and close. Assume, for contradiction, that X is not bounded; let $Y := \{1/x : 0 \neq x \in X\} \cup \{0\}$. Since Y is nowhere dense, Y is also discrete, contradicting the fact that X is unbounded. Hence, we have the equivalence (1 \Leftrightarrow 3 \Leftrightarrow 5 \Leftrightarrow 6 \Leftrightarrow 7 \Leftrightarrow 9 \Leftrightarrow 10).

Now, we prove that condition 6 implies that every \mathcal{F}_σ subset of \mathbb{K}^n is constructible. This in turns imply that the open core of \mathbb{K} is locally o-minimal, since every set definable in the open core will be constructible, and constructible sets with empty interior are meager, and thus pseudo-finite.

It is clear that condition 6 implies that \mathbb{K} is Baire. Let $X \subseteq \mathbb{K}^n$ be an \mathcal{F}_σ .

If $n = 1$, then $X = \overset{\circ}{X} \cup (X \setminus \overset{\circ}{X})$, and therefore X is the union of an open set and a meager set; thus, X is constructible.

If $n > 1$, we proceed, as in the proof of [DMS10, 3.4], by induction on n and $(d, k) = \text{fdim } X$ (the full dimension of F). Note that if $d = 0$, then X is pseudo-finite, and hence constructible.

If $d = n$, then $X = \overset{\circ}{X} \cup (X \setminus \overset{\circ}{X})$. $\overset{\circ}{X}$ is open, and hence constructible, while $X \setminus \overset{\circ}{X}$ has dimension less than n , and therefore it is also constructible; thus, X is constructible.

If $0 < d < n$, w.l.o.g. we can assume that $\pi(X)$ has non-empty interior, where $\pi := \Pi_d^n$. Let $A := \{a \in \mathbb{K}^d : \dim(X_a) > 0\}$. By Lemma 4.9, A is an \mathcal{F}_σ of dimension $< d$; therefore, by inductive hypothesis, A is constructible. Let $Y := \pi^{-1}(A) \cap X$. Since $\text{fdim}(Y) < \text{fdim}(X)$, by induction, Y is constructible. Thus, it suffices to prove that $X \setminus Y$ is constructible, and hence we can reduce to the case when $Y = \emptyset$.

Thus, we can reduce to the case when $d < n$, $B := \pi(X)$ has non-empty interior, and, for every $y \in B$, X_y is pseudo-finite.

Here the proof will diverge from [DMS10], because we do not have the hypothesis UF, and therefore we cannot proceed by induction on a uniform bound N on the cardinality of the fibers X_y .

We claim that $\text{fdim}(\ulcorner X \urcorner) < \text{fdim}(X)$. If the claim is true, then, by inductive hypothesis, $\ulcorner X \urcorner$ is constructible, and therefore X is constructible.

If, for contradiction, $\text{fdim}(\ulcorner X \urcorner) = \text{fdim}(X)$, then $\pi(\ulcorner X \urcorner)$ contains a non-empty open ball B' ; by shrinking X , we can assume that $B = B'$.

By hypothesis, there exists a definable increasing family $(X(t))_{t \in K}$ of d-compact subsets of \mathbb{K}^n , such that $X = \bigcup_t X(t)$. Let $Y(t) := \pi(X(t))$; note that each $Y(t)$ is d-compact, and $B = \bigcup_t Y(t)$. Since \mathbb{K} is Baire, by [FS09, Prop. 2.7], \mathbb{K}^d is also Baire, and therefore there exists $t_0 \in K$ such that $Y(t_0)$ has non-empty interior. Let $B' \subseteq Y(t_0)$ be a non-empty open box; by shrinking X , we can assume that $B = B'$, and, by re-defining the family $(Y(t))_{t \in \mathbb{K}}$, that $B = \pi(Y(t))$ for every t .

For every $y \in B$, define $X_{[y]} := \{y\} \times X_y \subseteq X$. The fact that $X_{[y]}$ is pseudo-finite easily implies the following:

Remark 5.1. For every $y \in B$ there exists $t \in \mathbb{K}$ such that $X_{[y]} \subseteq X(t)$.

Lemma 5.2. *There exists $B' \subseteq B$ non-empty open box, and $t_0 \in \mathbb{K}$, such that $X \cap (B' \times \mathbb{K}^{n-d}) \subseteq X(t_0)$.*

Proof. For every $t \in \mathbb{K}$, define $Z(t) := X \setminus X(t)$, and $Y(t) := B \setminus \pi(Z(t))$. Note that each $Z(t)$ is an \mathcal{F}_σ , and therefore $Y(t)$ is a \mathcal{G}_δ . By induction on n , $Y(t)$ is constructible. Moreover, by the remark, $B = \bigcup_t Y(t)$. Since B is not meager, there exists $t_0 \in \mathbb{K}$ such that $\overline{Y(t_0)}$ has non-empty interior. However, since $Y(t_0)$ is constructible, this implies that $Y(t_0)$ itself has non-empty interior, and therefore contains an open box B' . \square

Hence, $X \cap (B' \times \mathbb{K}^{n-d}) = X(t_0) \cap (B' \times \mathbb{K}^{n-d})$, and therefore $X \cap (B' \times \mathbb{K}^{n-d}) \subseteq \text{lc}(X)$. Since the same reasoning can be made for every open box $\tilde{B} \subseteq B$ instead of B , we conclude that $\pi(\ulcorner X \urcorner)$ has empty interior, absurd.

To conclude, it remains to prove $(1 \Rightarrow 2)$. Let $X \subseteq \mathbb{K}^n$ be discrete and definable. Since we have seen that \mathbb{K} is Baire, this implies that every projection

$\mu(X)$ of X on a coordinate axis has empty interior; however, $\mu(X)$ is an \mathcal{F}_σ , and thus $\mu(X)$ is pseudo-finite for every coordinate axis. Thus, X is pseudo-finite.

The fact that every nowhere dense definable subset of \mathbb{K}^n is meager follows from the fact that, if X is nowhere dense, then it is contained in a nowhere dense \mathcal{F}_σ set Y ; we have seen that Y is constructible; hence, Y is nowhere dense. \square

5.1 Further results on a-minimal structures

From the above proof, we can extract the following results.

Proviso. *For the remainder of this subsection, \mathbb{K} is a-minimal.*

Lemma 5.3. *Let $C \subseteq \mathbb{K}^n$ be definable and constructible (or, equivalently, an \mathcal{F}_σ) set of dimension d . Then, $\dim(\ulcorner C \urcorner) < d$. Therefore, C is the union of $d + 1$ locally closed definable sets.*

As a corollary, we have that, for an a-minimal structure, “being constructible” is a definable property. In fact (for \mathbb{K} a-minimal) $X \subseteq \mathbb{K}^n$ is constructible iff $X^{\ulcorner n+1 \urcorner}$ is empty. Thus, Lemma 4.22 implies the following:

Corollary 5.4. *Let A be pseudo-finite, and $(X_a)_{a \in A}$ be a definable family of constructible subsets of \mathbb{K}^n . Then, $\bigcup_{a \in A} X_a$ is also constructible.*

Moreover, the dimension is well-behaved for *constructible* sets definable in an a-minimal structure. We have already seen that $\dim(C \cup C') = \max(\dim C, \dim C')$ for constructible definable sets (Lemma 4.10).

Lemma 5.5. *Let C be a definable constructible subset of \mathbb{K}^n , where \mathbb{K} is a-minimal. $\dim C \leq 0$ iff C is pseudo-finite.*

Moreover, in Lemma 6.14, we will show that, for C definable, constructible, and non-empty subset of an a-minimal structure, we have $\dim(\partial C) < \dim C$ if C is non-empty.

Lemma 5.6. *Let $C(t)$ be a definable increasing family of subsets of \mathbb{K}^n , such that each $C(t)$ is constructible (or, equivalently, an \mathcal{F}_σ), and $\dim(C(t)) \leq d$. Let $C := \bigcup_t C(t)$. Then, $\dim C \leq d$.*

Proof. Assume, for contradiction, that $\pi(C)$ contains a non-empty open set U , where $\pi := \Pi_d^n$. For every t , let $D(t) := \pi(C(t))$. By hypothesis, each $D(t)$ has empty interior and is an \mathcal{F}_σ , and therefore it is meager. Thus, by a-minimality, each $D(t)$ is nowhere dense. However, $\bigcup_t D(t) = U$, and thus U is meager, which is absurd. \square

For a similar result, see also 8.4(II).

We can prove a different version of Corollary 5.4, albeit with a longer proof.

Lemma 5.7. *Let $(A(t))_{t \in \mathbb{K}}$ be a definable increasing family of constructible subsets of \mathbb{K}^n . Then, $X := \bigcup_{t \in \mathbb{K}} A_t$ is also constructible.*

Proof. We will proceed by induction on $(d, k) := \text{fdim } X$. If $d = 0$, then each $A(t)$ is pseudo-finite (by Lemma 5.5); therefore, X is pseudo-finite, and hence constructible. If $d = n$, then $X := \dot{X} \sqcup (X \setminus \dot{X})$. Define $B(t) := A(t) \setminus \dot{X}$; notice that $X \setminus \dot{X} = \bigcup_t B(t)$, and $(B(t))_{t \in \mathbb{K}}$ is a definable increasing family of

constructible sets. Thus, by inductive hypothesis, $X \setminus \overset{\circ}{X}$ is constructible, and therefore X is constructible.

It remains to treat the case $0 < d < n$. W.l.o.g., $\pi(X)$ has non-empty interior, where $\pi := \Pi_d^n$. By inductive hypothesis, $\pi(X)$ is constructible. For each $t \in \mathbb{K}$, define $C(t) := \{y \in \mathbb{K}^d : \dim(X(t)_y) > 0\}$. Each $C(t)$ is definable in the open core of \mathbb{K} , and thus constructible.

Claim 5. $C(t)$ has empty interior, for every t .

If, for some $t \in \mathbb{K}$, $C(t)$ has non-empty interior, then, since $\dim(A(t)) \leq \dim X = d$, $\pi(A(t))$ has empty interior, and thus it is nowhere dense. Hence, $\pi(X)$ is meager, a contradiction.

Let $Y := \bigcup_t C(t)$. By inductive hypothesis, C is constructible; moreover, by Claim 5, it is meager, and therefore $\dim Y < d$. Let $X_1 := X \setminus (Y \times \mathbb{K}^{n-d})$ and $X_2 := X \cap (Y \times \mathbb{K}^{n-d})$. We have seen that $\text{fdim}(X_2) < \text{fdim} X$, and thus, by inductive hypothesis, X_2 is constructible. Thus, w.l.o.g. we can assume that $X = X_1$, that is, $A(t)_y$ is pseudo-finite for every $t \in \mathbb{K}$ and $y \in \mathbb{K}^d$. By Lemma 5.5, this means that X_y is pseudo-finite for every $y \in \mathbb{K}^d$.

Notice that, for every $t \in \mathbb{K}$, $X \setminus A(t) = \bigcup_s (A(s) \setminus A(t))$, and therefore, by induction on n , $\pi(X \setminus A(t))$ is constructible. Thus, reasoning as in Lemma 5.2, we can prove the following Claim.

Claim 6. For every non-empty open box B there exists $t \in \mathbb{K}$ and $B' \subseteq B$ non-empty open box, such that $X \cap (B' \times \mathbb{K}^{n-d}) = A(t) \cap (B' \times \mathbb{K}^{n-d})$.

Fix B' as in the above claim, and let $\tilde{A}(t) := A(t) \cap (B' \times \mathbb{K}^{n-d})$. By Lemma 5.3 $\dim(\tilde{A}(t) \setminus \text{lc}(\tilde{A}(t))) < \dim(\tilde{A}(t)) \leq d$. Moreover, $\ulcorner X \urcorner \cap (B' \times \mathbb{K}^{n-d}) = \ulcorner \tilde{A}(t) \urcorner$. Therefore, by Claim 6, $\pi(\ulcorner X \urcorner)$ has empty interior. Thus, by inductive hypothesis, $\ulcorner X \urcorner$ is constructible, and hence X is constructible. \square

Lemma 5.8. *Let $C \subseteq \mathbb{K}^n$ be definable. T.f.a.e.:*

1. C is a.o.;
2. $C \setminus \overset{\circ}{C}$ is nowhere dense;
3. ∂C is nowhere dense;
4. $\text{bd} C$ is nowhere dense;

Proof. $(4 \Rightarrow 3 \Rightarrow 1)$ and $(4 \Rightarrow 2 \Rightarrow 1)$ are clear, and are true even without the a-minimality hypothesis. For $(1 \Rightarrow 2)$, assume C a.o.. Then, $C = U\Delta F$, for some definable meager set F , and some definable open set U . Since \mathbb{K} is a-minimal, F is nowhere dense; let $D := \overline{F}$.

Claim 7. $U \setminus D \subseteq \overset{\circ}{C}$.

In fact,

$$\overset{\circ}{C} = \text{int}(U\Delta F) \supseteq \text{int}(U \setminus F) = U \setminus D.$$

Thus,

$$C \setminus \overset{\circ}{C} \subseteq C \setminus (U \setminus D) = (U\Delta F) \setminus (U \setminus D) \subseteq (U \setminus (U \setminus D)) \cup F \subseteq D \cup F = D,$$

and D is nowhere dense.

The above proof applied to $\mathbb{K}^n \setminus C$ shows that $(1 \Rightarrow 3)$.

Finally, for $1 \Rightarrow 4$, observe that $\text{bd } C = \partial C \cup (C \setminus \overset{\circ}{C})$. Hence, if C is a.o., then, by (2) and (3), ∂C and $C \setminus \overset{\circ}{C}$ are nowhere dense, and thus $\text{bd } C$ is nowhere dense. \square

Lemma 5.9. *Let $(A(t))_{t \in \mathbb{K}}$ be a definable increasing family of subsets of \mathbb{K}^n , and $C := \bigcup_t A(t)$. If each $A(t)$ is meager, then C is meager (and thus nowhere dense). If each $A(t)$ is a.o., then C is a.o..*

Proof. Since \mathbb{K} is a-minimal, if each $A(t)$ is meager, then each $A(t)$ is nowhere dense, and therefore C is meager.

If each $A(t)$ is a.o., let $U := \overset{\circ}{C}$, and $A'(t) := A(t) \setminus U$. Notice that $A'(t) \subseteq A'(t) \setminus \text{interior}(A'(t))$; thus, Lemma 5.8, each $A'(t)$ is nowhere dense. Therefore, $C \setminus U = \bigcup_t A'(t)$ is meager, and hence C is a.o.. \square

6 Locally o-minimal structures

Lemma 6.1. *The following are equivalent:*

1. \mathbb{K} is locally o-minimal;
2. for every $X \subset \mathbb{K}$ definable, and for every $x \in \mathbb{K}$, there exists $y > x$ such that, either $(x, y) \subseteq X$, or $(x, y) \subseteq \mathbb{K} \setminus X$;
3. for every X definable subset of \mathbb{K} , either X is pseudo-finite, or it has non-empty interior.

See [DMS10, 2.11] for other equivalent formulations of local o-minimality (for definably complete structures, not necessarily expanding a field).

Proof. (1 \Rightarrow 2). Apply the definition of locally o-minimal to the characteristic function of X . (2 \Rightarrow 1). Let $f : \mathbb{K} \rightarrow \mathbb{K}$ be definable, and $x \in \mathbb{K}$. Let $X := f^{-1}(0)$. By hypothesis, there exists $y > x$ such that (x, y) is a subset either of X , or of $\mathbb{K} \setminus X$. In the first case, we are done; in the second, let $Y := f^{-1}((-\infty, 0))$. By decreasing y if necessary, either $(x, y) \subseteq Y$, or $(x, y) \subseteq \mathbb{K} \setminus (X \cap Y)$.

(2 \Rightarrow 3). Let $X \subset \mathbb{K}$ be definable with empty interior. We must prove that X is pseudo-finite. W.l.o.g., X is bounded. Let $x \in \mathbb{K}$. By condition 2, since X has empty interior, there exists $a < x < b$ such that both (a, x) and (x, b) are disjoint from X . Thus, x cannot be an accumulation point for X , and therefore X is pseudo-finite.

(3 \Rightarrow 2). Let $X \subseteq \mathbb{K}$ be definable, $x \in \mathbb{K}$, $Y := (x, +\infty) \cap X$, and $Z := \overset{\circ}{Y}$. Since Z is open, $\text{bd}(Z) := \overline{Z} \setminus \overset{\circ}{Z}$ has empty interior, and therefore it is pseudo-finite. Thus, there exists $y > x$ such that (x, y) is contained either in Z or in $\mathbb{K} \setminus Z$. In the first case, we have the conclusion. In the second case, we have that $W := X \cap (x, y)$ has empty interior. Thus, W is pseudo-finite, and therefore, after possibly decreasing $y > x$, we have $(x, y) \subseteq \mathbb{K} \setminus X$. \square

Proviso. *In the remainder of this section, \mathbb{K} is a definably complete and locally o-minimal expansion of a field.*

Since locally o-minimal structures are a-minimal, Thm. 3.3 applies to them. Moreover, as shown in Section 5, every definable constructible subset of \mathbb{K}^n has a well-behaved dimension. We will show later that every definable set is constructible.

6.1 Examples

Every o-minimal structure is locally o-minimal. Since “local o-minimality” is a first-order property, an ultra-product of locally o-minimal structures also locally o-minimal. The above observation leads us to the following example.

Example 6.2. Let M be a fixed o-minimal structure, in a language \mathcal{L} . Let P be a new unary predicate. For every $n \in \mathbb{N}$, let $P_n := \{1, 2, \dots, n\} \subset M$. Let $M_n := (M, P_n)$ be the $L(P)$ -expansion of M , where P is interpreted by P_n . Let $N := (M^*, P^*)$ be a non-principal ultra-product of the M_n . Then, N is locally o-minimal, but not o-minimal (because P^* is pseudo-finite, but not finite).

If M is an expansion of \mathbb{R} , then the above structure can be considered a restriction of an ultra-product of (\mathbb{R}, \mathbb{N}) (the expansion of \mathbb{R} with a predicate for the natural numbers). However, we can take for M the counter-example of Hrushovski and Peterzil: M^* is an elementary extension of M , and therefore also satisfies a formula that cannot be true in any expansion of the reals, and the same holds for N .

Since every locally o-minimal structure is also Baire (we are assuming that everything is definably complete), then the above is also a non-trivial example of definably complete and Baire structure.

Along the same lines, we can also consider the following example: let M be as before, and M' be an elementary extension of M , such that M is dense in M' and different from M' . Define P_n as before, let $M_n := (M', M, P_n)$, and $N := (M'^*, M^*, P^*)$ be a non-principal ultra-product of the M_n . Each M_n has o-minimal open core, hence it is a-minimal; thus, N is a-minimal (and hence Baire), but does not have o-minimal open core. Again, if M is the counter-example of Hrushovski and Peterzil, we see that N is not the restriction of an elementary extension of an expansion of \mathbb{R} .

Regarding structures with o-minimal open core, [DMS10, §8] ask the following question: Suppose that \mathbb{K} is Archimedean and has o-minimal open core; does $\text{Th}(\mathbb{K})$ have a model over \mathbb{R} that is unique up to isomorphism? While we do not have an answer regarding the existence, it is easy to see that uniqueness might fail. In fact, consider the case when \mathbb{K} is given by the dense pair $(\bar{\mathbb{R}}, \mathbb{R}^{\text{alg}})$. Let S be an uncountable real-closed subfield of \mathbb{R} , which is not all of \mathbb{R} . Then, $(\bar{\mathbb{R}}, S)$ is elementarily equivalent to \mathbb{K} , but it is not isomorphic to \mathbb{K} . A more interesting question is: with the above hypothesis, does \mathbb{K} have an elementary extension expanding $\bar{\mathbb{R}}$? Again, uniqueness cannot be expected. For instance, let U be a real-closed subfield of \mathbb{R} , such that $U \cap S = \mathbb{R}^{\text{alg}}$, $\mathbb{R}^{\text{alg}} \neq U \neq \mathbb{R}$, and U and S are free over \mathbb{R}^{alg} ; then, both $(\bar{\mathbb{R}}, S)$ and $(\bar{\mathbb{R}}, \mathbb{R}^{\text{alg}})$ are elementary extensions of $(U, \mathbb{R}^{\text{alg}})$ (see [vdD98a, Corollary 2.7]).

6.2 The monotonicity theorem

Let $A \subseteq (a, b)$ be pseudo-finite. For every $x \in A$, we denote by $s(x)$ the smallest element of A strictly greater than x , if x is not the maximum of A , and b otherwise. Similarly, we denote by $p(x)$ the greatest element of A strictly smaller than x , or a if x is the minimum of A . We denote by $s(a)$ the minimum of A .

Theorem 6.3 (Monotonicity theorem). *Let $f : (a, b) \rightarrow \mathbb{K}$ be a definable function. Then, there exists a pseudo-finite set $A \subseteq (a, b)$, such that on each sub-*

interval $(x, s(x))$, with $x \in A \cup \{a\}$, the function is either constant, or strictly monotone and continuous.

The proof proceeds as in [vdD98b, Thm. 3.1.2]: we derive it from the three lemmata below. Let $I \subseteq \mathbb{K}$ be an open interval, and $f : I \rightarrow \mathbb{K}$ be definable.

Lemma 6.4. *There is a subinterval of I on which f is constant or injective.*

Lemma 6.5. *If f is injective, then f is strictly monotone on a subinterval of I .*

Lemma 6.6. *If f is strictly monotone, then f is continuous on a subinterval of I .*

The three lemmata imply the monotonicity theorem as follows. Let

$$X := \{x \in (a, b) : \text{on some subinterval of } (a, b) \text{ containing } x, \\ f \text{ is either constant, or strictly monotone and continuous}\}.$$

By the three lemmata, $A := (a, b) \setminus X$ has empty interior, and therefore it is pseudo-finite. It is easy to see that A satisfies the conclusion of the theorem.

The proof of the first lemma is as in [vdD98b, ¶3.1.5, Lemma 1], substituting “finite” with “pseudo-finite”, and “infinite” with “non pseudo-finite”, and using the results in Section 4.2.

The proof of the second lemma is as in [vdD98b, ¶3.1.5, Lemma 2]. The “difficult case”, that is proving that the set of points in \mathbb{K} satisfying the formula

$$\Phi_{++}(x) := x \text{ is a strict local maximum for } f$$

has empty interior, is immediate from Lemma 7.3.

The proof of the third lemma in [vdD98b, ¶3.1.5, Lemma 3] works also here, with the same modifications as for the first lemma.

We also have the following consequence:

Corollary 6.7. *Let $f : (a, b) \rightarrow \mathbb{K}$ be definable, and $c \in (a, b)$. The limits $\lim_{x \rightarrow c^-} f(x)$ and $\lim_{x \rightarrow c^+} f(x)$ exist in \mathbb{K}_∞ . Also the limits $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} f(x)$ exist in \mathbb{K}_∞ .*

6.3 Constructibility and partition into cells

In this sub-section we will prove the following theorem.

Theorem 6.8. *If \mathbb{K} is locally o-minimal, then every definable subset of \mathbb{K}^n is constructible. Therefore, \mathbb{K} coincides with its open core.*

Let $X \subseteq \mathbb{K}^n$ be definable.

If $n = 1$, then $\partial(X)$ has empty interior (by Remark 4.2) and therefore is pseudo-finite. The same is true for $\mathbb{K} \setminus X$, and therefore, (also by Remark 4.2) $\text{bd}(X)$ is pseudo-finite. Thus, by Lemma 4.4, X is locally closed, and, *a fortiori*, constructible.

If $n = 2$, we consider first the case when, for every $a \in \mathbb{K}$, X_a is pseudo-finite. W.l.o.g., we can assume that $X \subseteq (0, 1) \times (0, 1)$. Following [vdD98b, 3.1.7], we call $(a, b) \in \mathbb{K}^2$ *normal* if there is an open box $I \times J$ around (a, b) such that

- either $(I \times J) \cap X = \emptyset$ (and hence $(a, b) \notin X$,

- or $(a, b) \in X$ and $(I \times J) \cap X = \Gamma(f)$ for some continuous function $f : I \rightarrow J$.⁽¹⁾

Note that the set $\mathcal{N} := \{(a, b) \in \mathbb{K}^2 : (a, b) \text{ is normal}\}$ is definable. Note also that $Z := \mathbb{K}^2 \setminus \mathcal{N}$ is contained in $[0, 1] \times [0, 1]$.

We call $\mathcal{B} := \Pi_1^2(Z) \subseteq \mathbb{K}$ the set of “bad” points, and $\mathcal{G} := \mathbb{K} \setminus \mathcal{B}$ the set of “good” points.

Claim 8. Z is d-compact.

In fact, it is bounded, and, by definition, its complement Y is open.

Hence, \mathcal{B} is d-compact.

Claim 9. \mathcal{B} is pseudo-finite.

Proof. If not, let $I \subseteq \mathcal{B}$ be a non-empty open interval. For every $a \in I$; let $\beta(a) := \min(Z_a)$. Conclude as in the proof of [vdD98b, 3.1.7]. \square

Note that the set of normal points \mathcal{N} is locally closed by definition. Moreover, by Lemma 4.22, Z is pseudo-finite, and hence closed. Thus, $X = \mathcal{N} \cup Z$ is constructible.

We treat now the general case, when there exists some $a \in \mathbb{K}$ such that X_a is not pseudo-finite.

Let $A := \{a \in \mathbb{K} : X_a \text{ is not pseudo-finite}\}$.

Lemma 6.9. *If A is not meager, then X has non-empty interior.*

Proof. If A is not meager, then it contains an open interval I . So, w.l.o.g. we can assume $A = I$. For every $a \in A$, there exist $b \in \mathbb{K}$ accumulation point of X_a . Let $f(a) \in \mathbb{K}$ be the smallest such b . Note that there exists $b' > f(a)$ such that $(f(a), b') \subseteq X_a$; let $g(a)$ the greatest such b' . Let $I' \subseteq I$ open interval, such that f and g are continuous on I' . Then, the set $\{(a, b) : a \in I, f(a) < a < g(a)\}$ is an open set contained in X . \square

Note that $X = \overset{\circ}{X} \cup (X \setminus \overset{\circ}{X})$. Since $X \setminus \overset{\circ}{X}$ has empty interior, we can assume that X has empty interior. Thus, A is pseudo-finite. By the case $n = 1$, we have that, for every $a \in \mathbb{K}$, $X_{[a]}$ is constructible. Thus, by Corollary 5.4, $X \cap \pi^{-1}(A) = \bigcup_{a \in A} X_{[a]}$ is constructible. Since we have seen that $X \setminus \pi^{-1}(A)$ is also constructible, we are done.

The rest of the proof proceeds by enumerating all the possible values of n .

For the general case, let us prove, by induction on n , the following 2 statements:

I_n Every definable $X \subseteq \mathbb{K}^n$ is constructible;

II_n If $A \subseteq \mathbb{K}^n$ is open and definable, and $f : A \rightarrow \mathbb{K}$ is a definable function, then there exists a non-empty open box $B \subseteq A$ such that $f \upharpoonright B$ is continuous.

Note that II_n implies that the set of discontinuity points of f is nowhere dense.

We have already proved I_1 and II_1 . So, assume that we have already proved I_m and II_m , for every $m \leq n$, and let us prove them for $n + 1$.

⁽¹⁾Since we assumed that $X \subseteq (0, 1) \times (0, 1)$, we do not need to worry about the behaviour at infinity.

We will prove I_{n+1} by induction on $(d, k) := \text{fdim}(X)$, the full dimension of X . Assume we have already proved the statement for any X' of full dimension less than (d, k) .

If $d = n$, note that $X = \overset{\circ}{X} \sqcup (X \setminus \overset{\circ}{X})$. $\overset{\circ}{X}$ is open, and therefore constructible, and $(X \setminus \overset{\circ}{X})$ has empty interior, and thus dimension less than n ; therefore, by the inductive hypothesis, $X \setminus \overset{\circ}{X}$ is constructible, and we are done.

If $d = 0$, then X is pseudo-finite, and we are done.

If $0 < d < n$, w.l.o.g. we can assume that $Y := \pi(X)$ has non-empty interior, where $\pi := \Pi_d^n$. Let $X' := X \cap \pi^{-1}(Y \setminus \overset{\circ}{Y})$. Note that $\text{fdim}(X') < \text{fdim}(X)$, and therefore X' is constructible. Thus, we can assume that Y is open.

Moreover, after a change of coordinates, we can assume that X is bounded.

We call $(a, b) \in \mathbb{K}^n \times \mathbb{K}$ normal if there is an open box $B \times J$ around (a, b) such that

- either $(B \times J) \cap X = \emptyset$ (and hence $(a, b) \notin X$),
- or $(a, b) \in X$ and $(B \times J) \cap X = \Gamma(f)$ for some continuous function $f : B \rightarrow J$.

Note that the set $\mathcal{N} := \{(a, b) \in \mathbb{K}^n \times \mathbb{K} : (a, b) \text{ is normal}\}$ is definable. Note also that $Z := \mathbb{K}^n \times \mathbb{K} \setminus \mathcal{N}$ is contained in $[0, 1] \times [0, 1]$.

We call $\mathcal{B} := \pi(Z) \subseteq \mathbb{K}^n$ the set of “bad” points, and $\mathcal{G} := \mathbb{K}^n \setminus \mathcal{B}$ the set of “good” points.

Again, \mathcal{N} is open by definition, and thus \mathcal{B} is d-compact. Thus, it suffices to prove that $\text{fdim}(Z) < \text{fdim}(X)$ to obtain that X is constructible. If, for contradiction, $\text{fdim}(Z) = \text{fdim}(X)$, then $\pi(Z)$ has non-empty interior; let $B \subseteq \pi(Z)$ be a non-empty open box. For every $a \in B$, let $\beta(a) := \min Z_a$. Proceeding as in the case $n = 2$, and using the inductive hypothesis II_n , we get a contradiction.

Let us prove now II_{n+1} . We use the same technique in [vdD98b, 3.2.17]. Let $f : A \rightarrow \mathbb{K}$ be a definable function, with $A \subseteq \mathbb{K}^n \times \mathbb{K}$ open and definable. Define the set A^* of well-behaved points for f as in [vdD98b, 3.2.17]. By the same proof as in [vdD98b], A^* is dense in A . Since A is definable, by I_{n+1} A is constructible, and therefore it contains a non-empty open box B . Moreover, by [vdD98b, Lemma 3.2.16], f is continuous on B . \square

We shall now prove a stronger version of Thm. 6.8, where we give some more details on the structure of definable sets.

Definition 6.10. Let $X \subseteq \mathbb{K}^n$ be definable (and hence constructible) and of dimension d , and let $\pi : \mathbb{K}^n \rightarrow L$ be a projection onto a coordinate space of dimension d . For notational convenience, assume that $\pi = \Pi_d^n$, and that X is bounded. A point $a \in \mathbb{K}^d$ is (X, π) -**good** if there exists a neighbourhood A of a , such that, for every $b \in \mathbb{K}^{n-d}$ there exists a neighbourhood B of b , with either $A \times B$ is disjoint from X , or $(A \times B) \cap X = \Gamma(f)$ for some (unique and definable) continuous function $f : A \rightarrow B$. A point $x \in \mathbb{K}^d$ is (X, π) -**bad** if it is not good.

If X is unbounded, let $\phi : \mathbb{K} \rightarrow (0, 1)$ be a definable order-preserving homeomorphism, and $\phi^n : \mathbb{K}^n \rightarrow (0, 1)^n$ be the corresponding map. Then, we say that a is (X, π) -**good** if $\phi^d(a)$ is $(\phi^n(X), \pi)$ -good: the definition does not depend on the particular choice of ϕ , and it is equivalent to the one in [vdD98b].

Note that if $L = \mathbb{K}^n$, then no point of ∂X is (X, π) -good (where π is the identity map). Note also that if $y \in L$ is (X, π) -good, then X_y is pseudo-finite (because it is discrete).

Definition 6.11. A **multi-cell** of dimension d in \mathbb{K}^n , with respect to a coordinate plane L , is a definable subset $X \subseteq \mathbb{K}^n$ of dimension equal to $\dim L$, such that, calling π the orthogonal projection onto L :

1. $U := \pi(X)$ is open in L ;
2. every point of U is (X, π) -good.

Note that the multi-cells of dimension 0 are the pseudo-finite sets, and the multi-cells of dimension n in \mathbb{K}^n are the definable open subsets of \mathbb{K}^n .

Note also that if X is a multi-cell, then, locally around every point of X , X is a cell. However, X might not be a cell around points not of X . For instance, let $D \subseteq (0, 1)$ be an infinite pseudo-finite set, and let

$$X := \{(x, y) \in \mathbb{K}^2 : x > 0, y > 0, \exists m \in D, y = mx\}.$$

Then, X is a multi-cell, but, for every neighbourhood V of 0, $X \cap V$ has infinitely many definably connected components, and thus cannot be a cell, nor a finite union of cells.

Theorem 6.12. *Let $X \subseteq \mathbb{K}^n$ be definable. Then, there exists a finite partition of X into multi-cells. The number of multi-cells is bounded by a function of n .*

Proof. We will prove the statement by induction on n and $(d, k) := \text{fdim}(X)$. Assume that we have already proved the statement for every $n' < n$ and for every X' of full dimension less than (d, k) .

If $d = n$, then $X = \overset{\circ}{X} \sqcup (X \setminus \overset{\circ}{X})$. $\overset{\circ}{X}$ is open, and thus a multi-cell. $(X \setminus \overset{\circ}{X})$ has dimension less than n , and thus it is a finite disjoint union of multi-cells. Therefore, X is a finite disjoint union of multi-cells.

If $d = 0$, then X is pseudo-finite, and hence a multi-cell.

If $0 < d < n$, let L be a coordinate space of dimension d , such that $\pi(X)$ has non-empty interior in L , where π is the projection on L . As in the proof of Thm. 6.8, the set \mathcal{B} of (X, π) -bad points is nowhere dense and closed, and \mathcal{G} , the set of (π, X) -good points, is open. Thus, let $Y := X \cap \pi^{-1}(\mathcal{B})$ and $Z := X \cap \pi^{-1}(\mathcal{G})$. By definition, Z is a multi-cell, and, since \mathcal{B} is nowhere dense, Y has full dimension less than (k, d) , and therefore it is a finite disjoint union of multi-cells. Thus, X is a finite union of multi-cells, and we are done.

From the above argument, we can see that the number N of multi-cells partitioning X is bounded by a function of n , d , and k . However, d and k are bounded by functions of n , and thus N is bounded by a function of n .⁽²⁾ \square

We would like to obtain a structure theorem for open definable sets. Such a theorem should also give a further refinement of the above theorem.

Remark 6.13. Let $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be definable. Since $\Gamma(f)$ is constructible, then, by Lemma 7.2, $\mathcal{D}(f)$ is meager, and thus nowhere dense and of dimension strictly less than n .

⁽²⁾Something like n^2 .

6.4 Additional results on locally o-minimal structures

Proviso. *In this subsection, \mathbb{K} is a locally o-minimal structure.*

Therefore, \mathbb{K} is a-minimal, and every definable subset of \mathbb{K}^n is constructible. Hence, we can apply lemmata 4.10, 5.5, and 5.6, and obtain the following lemma (cf. Example 4.11).

Lemma 6.14. *Let C and C' be definable subsets of \mathbb{K}^n . Then, $\dim(C \cup C') = \max(\dim C, \dim C')$, and $\dim(\partial C) < \dim C$ if C is non-empty; besides, $\dim C = 0$ iff C is pseudo-finite.*

Proof. We only need to show that $\dim(\partial C) < \dim C$.

First, we claim that $\text{fdim}(\overline{C}) = \text{fdim} C$. In fact, $\Pi_m^n(\overline{C}) \subseteq \overline{\Pi_m^n C}$, and therefore, by local o-minimality, if the former has non-empty interior, also the latter has non-empty interior. Hence, $\text{fdim}(\partial C) \leq \text{fdim} C$.

Assume, for contradiction, that $d := \dim C = \dim(\partial C)$; w.l.o.g., $\pi(\partial C)$ contains a non-empty open box U , where $\pi := \Pi_d^n$. Let \mathcal{B} be the set of $\langle C, \pi \rangle$ -bad points: \mathcal{B} is nowhere dense, and therefore there exists a non empty open box $V \subseteq U \setminus \mathcal{B}$. However, $\pi(\partial(C \cap \pi^{-1}(V))) \subseteq \partial V$, contradicting the definition of U . \square

Lemma 6.15. *Let $C(t)$ be a definable increasing family of subsets of \mathbb{K}^n , such that $\dim(C(t)) \leq d$ for every t . Let $C := \bigcup_t C(t)$. Then, $\dim C \leq d$.*

Lemma 6.16. *Every locally o-minimal structure has definable Skolem functions.*

Proof. Same as in [vdD98b, Prop. 6.1.2]. \square

Lemma 6.17 (Curve selection). *If $a \in \partial X$, where $X \subseteq \mathbb{K}^n$ is definable, then there exists a definable continuous injective map $\gamma : (0, 1) \rightarrow X$, such that $\lim_{t \rightarrow 0} \gamma(t) = a$.*

Proof. Same as in [vdD98b, Corollary 6.1.5]. \square

Given $A \subseteq \mathbb{K}$, denote by $\text{dcl}(A)$ the definable closure of A in \mathbb{K} .

Lemma 6.18. *Assume that $\text{dcl}(\emptyset)$ is dense in \mathbb{K} . Then, \mathbb{K} has the Exchange Property (EP).*

Proof. Let $A \subseteq \mathbb{K}$, and $a, b \in \mathbb{K}$. We have to prove that, if $b \in \text{dcl}(A \cup \{a\}) \setminus \text{dcl}(A)$, then $a \in \text{dcl}(A \cup \{b\})$. Let $f : \mathbb{K} \rightarrow \mathbb{K}$ be A -definable, such that $f(a) = b$. By the monotonicity theorem, there exists $D \subset \mathbb{K}$ pseudo-finite, such that f is continuous and either constant or strictly monotone on each subinterval of $\mathbb{K} \setminus D$. Note that D can be taken A -definable.

If $a \in D$, let $y', y'' \in \text{dcl}(\emptyset)$, such that $y' < a < y''$, and $D \cap (y', y'') = \{a\}$. Then, $a \in \text{dcl}(A)$, a contradiction.

If instead $a \notin D$, let $y', y'' \in \text{dcl}(\emptyset)$, such that $y' < a < y''$ and f is continuous and either constant or strictly monotone on $I := (y', y'')$. If f is constant on I , then $b \in \text{dcl}(A)$, absurd. If instead f is continuous and strictly monotone on I , then f has a compositional inverse g on I , and g is A -definable; thus, $a = g(b)$, and therefore $a \in \text{dcl}(A \cup \{b\})$. \square

Example 6.19. A locally o-minimal structure does not necessarily satisfy the NIP. Equivalently, there exists a locally o-minimal structure that does have the Independence Property. In fact, fix an o-minimal structure M (expanding a field) in a language L . Let P a new binary predicate symbol. For every $0 < n \in \mathbb{N}$, let $f_n : 1, \dots, 2^n \rightarrow \mathcal{P}(n)$ be an enumeration of the subsets of $1, 2, \dots, n$, such that f_n extends f_m for every $m \leq n$. Let $P_n := \bigcup_{k=1}^{2^n} \{k\} \times f_n(k) \subseteq M^2$, and $N_n := (M, P_n)$ be the $L(P)$ -expansion of M , where P is interpreted by P_n . Let $N^* = (M^*, P^*)$ be a non-principal ultra-product of then N_m . Since M is o-minimal, M^* and each N_m are o-minimal, and N^* is locally o-minimal. However, it is clear that N^* is not o-minimal, and does have the Independence Property. Moreover, if L is countable, then N^* is ω -saturated, and therefore, by the proof [DMS10, 1.17], N^* does not have the Exchange Property.

The definitions and proof in [Mil96] work almost verbatim for locally o-minimal structures. Hence, we have the following theorem.

Theorem 6.20. *If \mathbb{K} is a locally o-minimal structure, then it is either **power-bounded**, or it is **exponential** (that is defines an exponential function). If \mathbb{K} is power bounded, then for every ultimately non-zero definable function $f : \mathbb{K} \rightarrow \mathbb{K}$ there exists $c \in \mathbb{K}^*$ and r in the field of exponents of \mathbb{K} , such that $f \sim cx^r$.*

7 More results on definably complete structures

7.1 Functions

Let \mathbb{K} be a definably complete. In this sub-section, we will prove that certain subsets of \mathbb{K}^n are meager. Thus, if \mathbb{K} is Baire, those subsets cannot be all of \mathbb{K}^n . However, by inspecting the proofs, one will see that in some cases (marked by (*)), we actually prove that those sets are pseudo-enumerable, and thus, even without assuming that \mathbb{K} is Baire, they cannot be all of \mathbb{K}^n (see subsection 7.4).

Let $f : A \rightarrow \mathbb{K}^n$ be a definable function. For every $s > 0$, define

$$\mathcal{D}(f, s) := \{x \in \mathbb{K} : \forall t > 0, \\ f(A \cap B(x, t)) \text{ is not contained in any open ball of radius } s\}.$$
⁽³⁾

Then, each $\mathcal{D}(f, s)$ is closed in A , and $\mathcal{D}(f)$, the set of discontinuity points of f , is equal to $\bigcup_s \mathcal{D}(f, s)$, and, therefore, it is an \mathcal{F}_σ subset of A .

Lemma 7.1 (*). *Let $f : \mathbb{K} \rightarrow \mathbb{K}$ be definable and monotone. Then, $\mathcal{D}(f)$ is meager.*

Proof. If $\mathcal{D}(f)$ were not meager, then there would exist $\varepsilon > 0$ such that $\mathcal{D}(f, \varepsilon)$ has non-empty interior; let $(a, b) \subseteq \mathcal{D}(f, \varepsilon)$, with $a < b$. W.l.o.g., f is increasing. Define $m := \sup_{t < b} f(t) = \lim_{t \rightarrow b^-} f(t) \in \mathbb{K}$. Let $c \in (a, b)$ such that $f(c) > m - \varepsilon/2$. Since f is monotone, and $(a, b) \subseteq \mathcal{D}(f, \varepsilon)$, there exists $d \in (c, b)$ such that $f(d) \geq f(c) + \varepsilon > m + \varepsilon/2$, contradicting the definition of m . \square

The open question is whether, with the same hypothesis as in the above lemma, there exists a point $x \in \mathbb{K}$ such that f is differentiable at x .

⁽³⁾Note that there is a misprint in the definition of $\mathcal{D}(f, s)$ in [DMS10, 1.8].

Lemma 7.2. *Let $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ be definable. If $\Gamma(f)$ is an \mathcal{F}_σ set, then $\mathcal{D}(f)$ is meager.*

Proof. If, for contradiction, $\mathcal{D}(f)$ is not meager, then, since it is an \mathcal{F}_σ , it contains a non-empty open box B . Therefore, w.l.o.g. we can assume that $\mathcal{D}(f) = \mathbb{K}^n$. Let $\Gamma(f) = \bigcup_t X(t)$, where $(X(t))_{t \in \mathbb{K}}$ is a definable increasing family of d-compact sets. Let $Y(t) := \Pi_n^{n+m}(X(t))$. Note that each $Y(t)$ is d-compact, and $\mathbb{K}^n = \bigcup_t Y(t)$. Since \mathbb{K}^n is Baire, there exists t_0 such that $Y(t)$ contains a non-empty open box B' . Let $B'' \subseteq B'$ be a closed box with non-empty interior, and $g := f \upharpoonright B''$. Note that $\Gamma(g) = X(t_0) \cap (B'' \times \mathbb{K}^m)$; therefore, $\Gamma(g)$ is compact, and so g is continuous, contradicting the fact that $B'' \subseteq \mathcal{D}(f)$. \square

Lemma 7.3 (*). *Let $f : \mathbb{K}^n \rightarrow \mathbb{K}$ be definable. Define $N_f := \{x \in \mathbb{K}^n : x \text{ is a strict local minimum for } f\}$. Then, N_f is meager in \mathbb{K}^n .*

Proof. For every $r > 0$, let

$$N(r) := \{x \in \mathbb{K}^n : |x| \leq r \text{ \& } x \text{ is a strict minimum for } f \text{ in the ball } B(x, 1/r)\}.$$

Note that $N_f = \bigcup_r N(r)$. Moreover, each $N(r)$ is closed and discrete, and hence nowhere dense. Thus, N_f is meager. \square

Lemma 7.4 (*). *Let $f : \mathbb{K}^n \rightarrow \mathbb{K}$ be definable. Define $M_f := \{x \in \mathbb{K}^n : x \text{ is a local minimum for } f\}$. Then, $f(M_f)$ is meager in \mathbb{K} .*

Proof. For every $r > 0$, let

$$M(r) := \{x \in \mathbb{K}^n : |x| \leq r \text{ \& } x \text{ is a minimum for } f \text{ in the ball } B(x, 1/r)\}.$$

Note that $f(M_f) = \bigcup_r f(M(r))$. Fix $r > 0$. It suffices to prove that $Y := f(M(r))$ is nowhere dense. We will prove the stronger statement that Y is pseudo-finite. Assume, for contradiction, that Y has an accumulation point $y \in \mathbb{K}$. For every $\delta > 0$, let $U(\delta) := f^{-1}(B(y, \delta) \cap Y \setminus \{y\})$, that is: $x \in U(\delta)$ iff $f(x) \in B(y, \delta)$, $f(x) \neq y$, $|f(x)| \leq r$, and x is a minimum of f on $B(x, 1/r)$.

Let $C(\delta)$ be the closure of $U(\delta)$ in \mathbb{K}^n . Since each $C(\delta)$ is a non-empty d-compact subset of $\overline{B(0, r)}$, the intersection of the $C(\delta)$ is non-empty; let $x \in \bigcap_\delta C(\delta)$. Let $x_1, x_2 \in B(x, 1/(2r)) \cap U(\delta)$, such that $f(x_1) < f(x_2)$ (they exist by definitions). However, $x_1 \in B(x_2, r)$, and $x_2 \in M(r)$; therefore, $f(x_1) \geq f(x_2)$, absurd. \square

Definition 7.5. A definable function $f : X \rightarrow Y$ is of **first class** if there exists a function $F : \mathbb{K} \times X \rightarrow Y$, such that, for every $t \in \mathbb{K}$ and $x \in X$,

1. $f_t(x) := F(t, x) : X \rightarrow Y$ is a continuous function of x ,
2. $\lim_{t \rightarrow +\infty} f_t(x) = f(x)$;

that is, f is a point-wise limit of a definable family of continuous functions $(f_t)_{t \in \mathbb{K}}$.

Lemma 7.6. *Let X be Baire, and $f : X \rightarrow \mathbb{K}^m$ be of first class. Then, $\mathcal{D}(f)$ is meager (in X).*

Proof. Minor variation of [Oxt71, Thm. 7.3]. It suffices to show that, for every $\varepsilon > 0$, $F_\varepsilon := \{x \in X : \omega(x) \geq 5\varepsilon\}$ is nowhere dense, where

$$\omega(x) := \limsup_{\delta \rightarrow 0^+} \{|f(x') - f(x'')| : x', x'' \in B(x; \delta)\}.$$

Fix an open definable subset $X' \subseteq X$. For every $i \in \mathbb{K}$, let

$$E_i := \bigcap_{s, t \geq i} \{x \in X' : |f_s(x) - f_t(x)| \leq \varepsilon\}.$$

Then, $(E_i)_{i \in \mathbb{K}}$ is an increasing family of closed subsets of X' , and $\bigcup_i E_i = X'$. Since X is Baire and X' is open, X' is not meager in itself, and therefore E_{i_0} must have non-empty interior (in X'), for some $i_0 \in \mathbb{K}$. Let V be a non-empty open subset in E_{i_0} . We have $|f_i(x) - f_s(x)| \leq \varepsilon$ for all $x \in V$ and $s, t \geq i_0$. Putting $t = i_0$ and letting $s \rightarrow \infty$, it follows that $|f_{i_0}(x) - f(x)| \leq \varepsilon$ for all $x \in V$. For all $x_0 \in V$ there exists a neighbourhood $U(x_0) \subseteq V$, such that $|f_{i_0}(x) - f_{i_0}(x_0)| \leq \varepsilon$ for all $x \in U(x_0)$. Hence $|f(x) - f_{i_0}(x_0)| \leq 2\varepsilon$ for all $x \in U(x_0)$. Therefore $\omega(x_0) \leq 4\varepsilon$, and so no point of V belongs to F . Thus, every non-empty open set X' contains a non-empty open subset V disjoint from F . This shows that F is nowhere dense. \square

7.2 Pseudo-enumerable sets

As usual, \mathbb{K} will be a definably complete structure, expanding an ordered field.

Definition 7.7. $N \subset \mathbb{K}$ is a pseudo- \mathbb{N} subset of \mathbb{K} if it is a definable closed, discrete and unbounded subset of $K^{\geq 0}$.

There are 2 cases: either every closed, definable and discrete subset of \mathbb{K} is pseudo-finite (and hence \mathbb{K} is a-minimal, and thus Baire), or not. Equivalently: either \mathbb{K} is a-minimal, or a pseudo- \mathbb{N} set exists. Suppose we are in the second case.

Definition 7.8. Let $X \subseteq \mathbb{K}^n$ be definable. We say that X is **pseudo-enumerable** if there exists N a pseudo- \mathbb{N} subset of \mathbb{K} , and a definable bijection $f : N \xrightarrow{\sim} X$. We say that X is **at most pseudo-enumerable** if X is either pseudo-finite or pseudo-enumerable.

We say that $(X(i))_{i \in I}$ is a **strongly uniform family** of at most pseudo-enumerable sets, if $(X(i))_{i \in I}$ is a definable family of subsets of \mathbb{K}^n (for some n), and there exist $Y \subset \mathbb{K} \times Y$ definable, and $F : Y \rightarrow \mathbb{K}^n$ definable, such that, for every $i \in I$, $F(Y_{[i]}) = X(i)$, and Y_i is a discrete definable closed subset of $\mathbb{K}^{\geq 0}$.

Note that we did *not* claim that 2 pseudo-enumerable sets are in a definable bijection (we do not know whether this is true or not, but we conjecture it *is* true).

Remark 7.9. Let X be at most pseudo-enumerable, and Y be a definable subset of X . Then, Y is at most pseudo-enumerable. In particular, a definable subset of a pseudo- \mathbb{N} set is at most pseudo-enumerable.

Proof. Assume that Y is not pseudo-finite. Therefore, X is not pseudo-finite, and hence it is pseudo-enumerable;

If X is a pseudo- \mathbb{N} set, then Y is not bounded (otherwise, it would be pseudo-finite), and hence it is a pseudo- \mathbb{N} set, and in particular pseudo-enumerable.

Otherwise, let $g : N \rightarrow X$ bijection, where N is a pseudo- \mathbb{N} subset of \mathbb{K} . Let $h := g \upharpoonright Y : Y \rightarrow N$, and $N' := h(N) \subseteq N$. Notice that $h : Y \rightarrow N'$ is a bijection; therefore, N' is not pseudo-finite, and Y is pseudo-enumerable. \square

Remark 7.10. If X is at most pseudo-enumerable, and $f : X \rightarrow \mathbb{K}^m$ is definable, then $f(X)$ is at most pseudo-enumerable. In particular, the image of a pseudo- \mathbb{N} subset of \mathbb{K} is either pseudo-finite, or pseudo-enumerable.

Proof. Assume that $Y := f(X)$ is not pseudo-finite. Thus, X is not pseudo-finite, and hence it is pseudo-enumerable; let $g : N \rightarrow X$ bijection, where N is a pseudo- \mathbb{N} subset of \mathbb{K} . For every $y \in Y$, define

$$h(y) := \min\{i \in N : f(g(i)) = y\},$$

and $N' := h(N) \subseteq N$. Notice that $h : Y \rightarrow N'$ is a bijection; therefore, N' is not pseudo-finite. Thus, N' is a pseudo- \mathbb{N} set, and $h^{-1} : N' \rightarrow Y$ is a bijection, proving that Y is pseudo-enumerable. \square

Lemma 7.11. *If X is at most pseudo-enumerable, then there exists $Y \subset \mathbb{K}^{\geq 0}$ definable, closed and discrete, and $f : Y \xrightarrow{\sim} X$ definable bijection.*

Proof. If X is pseudo-enumerable, the above is true by definition. If $X \subset \mathbb{K}^n$ is pseudo-finite, we proceed by induction on n .

If $n = 1$, $Y = X$ satisfies the conclusion. If $n = 2$, w.l.o.g. we can assume that $X \subset [0, 1]^2$. Let $X_1 := \Pi_1^2(X) \subset [0, 1]$, and $d := \frac{1}{2} \min\{x' - x : x < x' \in X_1\}$ (or $d := 1/2$ if X_1 has only one element). For every $\langle x, y \rangle \in X$, define $g(x, y) := x + dy$. Then, g is injective, and $Y := g(X)$ is a pseudo-finite subset of \mathbb{K} . Assume that we have already proved the conclusion for $n \geq 2$, we prove it for $n + 1$. Let $X_1 := \pi(X)$, where $\pi := \Pi_n^{n+1}$. By inductive hypothesis, there exists $Y_1 \subset \mathbb{K}$ pseudo-finite, and $g_1 : X_1 \xrightarrow{\sim} Y_1$ definable bijection. For every $\langle \bar{x}, y \rangle \in X$, let $f(\bar{x}, y) := \langle g_1(\bar{x}), y \rangle$, $f : X \rightarrow \mathbb{K}^2$, and $X' := f(X)$. Then, f is injective, and, by the case $n = 2$, we can conclude. \square

Remark 7.12. The union of 2 at most pseudo-enumerable sets is pseudo-enumerable.

Proof. Let X_1 and X_2 be at most pseudo-enumerable. We want to prove that $X_1 \cup X_2$ is at most pseudo-enumerable. W.l.o.g., X_1 and X_2 are disjoint. If both X_1 and X_2 are pseudo-finite, then we are done. Otherwise, w.l.o.g. X_2 is pseudo-enumerable. By Lemma 7.11, there exist N_i closed, definable, and discrete subsets of $\mathbb{K}^{\geq 0}$, and $f_i : N_i \xrightarrow{\sim} X_i$ definable bijections, $i = 1, 2$.

For every $i \in N_k$, let $s_k(i) := \min\{i' \in N_k : i' > i\}$ be the successor of i in N_k (or $s_1(m) = m + 1$ if $m = \max(S_1)$), $k = 1, 2$. Define $f : X_1 \cup X_2 \rightarrow \mathbb{K}$ as:

$$f(x) := \begin{cases} f_2(x) & \text{if } x \in X_2; \\ f_1(x) & \text{if } x \in X_1 \text{ \& } f_1(x) \notin N_2; \\ \frac{1}{2} \left(f_1(x) + \min(s_1(f_1(x)), s_2(f_1(x))) \right) & \text{if } x \in X_1 \text{ \& } f_1(x) \in N_2. \end{cases}$$

Then, f is injective, and $f(X_1 \cup X_2)$ is a pseudo- \mathbb{N} subset of \mathbb{K} . \square

Lemma 7.13. 1. *The product of 2 pseudo-enumerable sets is pseudo-enumerable.*

2. *The product of 2 at most pseudo-enumerable sets is at most pseudo-enumerable.*

Proof. It suffices to prove that if X and Y are at most pseudo-enumerable, then $Z := X \times Y$ is at most pseudo-enumerable. If X and Y are both pseudo-finite, then Z is also pseudo-finite, and we are done. By Lemma 7.11 and the above Remark, w.l.o.g. $X = Y$ is a pseudo- \mathbb{N} subset of \mathbb{K} , and $X > 1$. By applying the above remark again to $X \times \{0\}$ and $Y \times \{1\}$, w.l.o.g. X and Y are disjoint subsets of a pseudo- \mathbb{N} subset N of \mathbb{K} , and $N > 1$.

For every $i \in N$, let $s(i) := \min\{i' \in N : i' > i\}$ be the successor of i in N , and

$$d(i) := \min\{(s(i') - i') : i \geq i' \in N\} > 0.$$

Define $f : X \times Y \rightarrow \mathbb{K}$,

$$f(x, y) := \begin{cases} x + \frac{d(x)}{2x}y & \text{if } x > y; \\ y + \frac{d(y)}{2y}x & \text{if } x < y. \end{cases}$$

Notice that the definition is symmetric in X and Y (that is, if we exchange the rôles of X and Y , f remains the same).

Claim 10. f is injective.

Assume not. Let $\langle x, y \rangle \neq \langle x', y' \rangle \in X \times X$, such that $f(x, y) = f(x', y')$. Let $a := \max(x, y)$, $b := \min(x, y)$, $a' := \max(x', y')$, and $b' := \min(x', y')$. Then, $f(x, y) = a + \frac{d(a)}{2a}b$, and similarly for $f(x', y')$. Hence,

$$a + \frac{d(a)}{2a}b = a' + \frac{d(a')}{2a'}b'.$$

W.l.o.g., $a' \geq a$. If $a' = a$, then $b = b'$; since X and Y are disjoint, this implies that $\langle x, y \rangle = \langle x', y' \rangle$, a contradiction. Otherwise, $a' > a$, and

$$d(a)/2 < a' - a = \frac{d(a)}{2a}b - \frac{d(a')}{2a'}b' \leq d(a)/2,$$

absurd, proving the claim.

Claim 11. $f(Z)$ is pseudo- \mathbb{N} .

Since X and Y are not pseudo-finite, it suffices to show that $f(Z)$ is closed and discrete. Both closedness and discreteness follow if we show that, for every $r > 0$, $f(Z) \cap [0, r]$ is pseudo-finite. However, $f(x, y) > \max(x, y)$, and therefore $f^{-1}([0, r]) \subseteq (X \cap [0, r]) \times (Y \cap [0, r])$, and the latter is pseudo-finite. \square

Lemma 7.14. *The union of an at most pseudo-enumerable strongly uniform family of at most pseudo-enumerable sets is at most pseudo-enumerable.*

Proof. Let $(X(i))_{i \in N}$ be a pseudo-enumerable uniform family of at most pseudo-enumerable subsets of \mathbb{K}^n , and $Z := \bigcup_{i \in N} X(i)$. If \mathbb{K} is a-minimal, then Z is pseudo-finite. Otherwise, w.l.o.g. N is a pseudo- \mathbb{N} subset of \mathbb{K} . Let $Y \subset \mathbb{K} \times N$ and $F : Y \rightarrow \mathbb{K}^n$ be definable, such that, for every $i \in N$, $F(Y_{[i]}) = X(i)$, and Y_i is a discrete definable closed subset of $\mathbb{K}^{\geq 0}$. For every $i, j \in N$, let $Y(i, j) := Y_i \cap B(0, j)$. Notice that $Y(i, j)$ is pseudo-finite for every i, j , and

$$Z = \bigcup_{\langle i, j \rangle \in N^2} F(Y(i, j) \times \{i\}).$$

Since N^2 is pseudo-enumerable, by changing the family, w.l.o.g. we can assume that each $X(i)$ is pseudo-finite. Moreover, w.l.o.g. each $X(i)$ is of the form $Y(i) \times \{i\}$, where $Y(i)$ is a pseudo-finite subset of $\mathbb{K}^{\geq 0}$. As usual, for every $i \in N$, let $s(i)$ be the successor of i in N . Define $d(i) := s(i) - i$, and Moreover, let $u(i) := \max(Y(i))$. Define $g : Z \rightarrow \mathbb{K}$,

$$g(y, i) = i + \frac{d(i)}{2u(i)}y.$$

Then, g is injective, and $g(Z)$ is a closed discrete subset of $\mathbb{K}^{\geq 0}$. \square

Lemma 7.15. *Let $D \subseteq \mathbb{K}^n$ be pseudo-finite. Then, there exists $C \subseteq \mathbb{K}$ pseudo-finite, and a definable bijection $f : D \xrightarrow{\sim} C$. Moreover, C and f can be defined uniformly from C : that is, if $(D_i)_{i \in I}$ is a definable family of pseudo-finite subsets of \mathbb{K}^n , then there exists $C \subseteq \mathbb{K}^n \times I$ and $f : D \rightarrow C$ definable and injective, such that, for every $i \in I$, $f(C_{[i]}) = D_{[i]}$. Hence, every definable family of pseudo-finite sets is uniform.*

Proof. Induction on n . If $n = 1$, take $D = C$. If $n = 2$, let $X := \pi(D)$ and $Y := \theta(D)$, where π and θ are the projections onto the first and second coordinates, respectively. W.l.o.g., $0 = \min Y$; let $r := \max Y$. For every $x \in X$, let $s(x)$ be the successor of x in X (or $x + 1$ if $x = \max X$) and $d(x) := s(x) - x > 0$.

Define $f : D \rightarrow \mathbb{K}$,

$$f(x, y) := x + \frac{d(x)}{2r} * y,$$

and $C := f(D)$. It is clear that f is injective, and, since D is pseudo-finite, C is also pseudo-finite.

The case $n > 2$ follows easily by induction on n . \square

Missing: If \mathbb{K} is saturated, every def. family of pseudo-enumerable sets is uniform.

Lemma 7.16. *Every definable discrete subset X of \mathbb{K}^n is at most pseudo-enumerable.*

Proof. If \mathbb{K}^n is a -minimal, then X is pseudo-finite. Otherwise, let N be a pseudo- \mathbb{N} subset of \mathbb{K} . By Lemma 4.17, X is a union of a uniform increasing family of pseudo-finite sets. As usual, $X = \bigcup_{i \in N} Y(i)$, where each $Y(i)$ is pseudo-finite. Hence, X is the union of a pseudo-enumerable uniform family of pseudo-finite sets, and thus X is at most pseudo-enumerable. \square

Corollary 7.17. *Let $X \subseteq \mathbb{K}^n$ be definable. Then, X is pseudo-enumerable iff there exists $D \subseteq \mathbb{K}^{n+1}$ closed and discrete, such that $X = \Pi_n^{n+1}(X)$.*

Proof. Assume $X \subseteq \mathbb{K}^n$ is pseudo-enumerable. Let $N \subseteq \mathbb{K}$ be a pseudo- \mathbb{N} set, and $f : N \rightarrow X$ be a definable bijection. Let $D \subseteq \mathbb{K}^{n+1}$ be the graph of f . Then, D is discrete and $X = \pi(D)$. \square

Corollary 7.18. *Let $U \subseteq \mathbb{K}$ be definable and open. Then, U is the union of an at most pseudo-enumerable family of open intervals.*

Proof. Let D be the set of centers of the connected components of U . Then, D is discrete and definable, and hence at most pseudo-enumerable, and U is a union of open intervals indexed by D . \square

7.3 Uniform families

Proviso. *In this subsection, we assume that \mathbb{K} is definably complete and Baire.*

We do not know if, in general, definable increasing union of \mathcal{F}_σ (resp., meager) sets are \mathcal{F}_σ (resp., meager). We will give some sufficient conditions for this to be the case.

Remember that if we say that $(A_i)_{i \in I}$ is a definable family of subsets of \mathbb{K}^n , we mean that I is a definable subset of \mathbb{K}^m , for some m , and $\mathcal{A} := \bigcup_{i \in I} A_i \times \{i\}$ is a definable subset of \mathbb{K}^{n+m} . Moreover, remember that $A \subset \mathbb{K}^n$ is an \mathcal{F}_σ iff it is the projection of some definable closed subset of \mathbb{K}^{n+1} [FS09].

Definition 7.19. Let $(A_i)_{i \in I}$ be a definable family of subsets of \mathbb{K}^n . We say that \mathcal{A} is a **uniform family** of \mathcal{F}_σ sets if there exists a definable set $C \subseteq \mathbb{K}^{n+1} \times \mathbb{K}^{m'}$, for some m' , such that:

- each fiber $(C_t)_{t \in \mathbb{K}^{m'}}$ is closed;
- for each $i \in I$ there exists $t \in \mathbb{K}^{m'}$ such that $A_i = \Pi_n^{n+1}(C_t)$.

Notice that in the above definition, up to taking a smaller definable family C' , we can always impose the additional condition that, for each $t \in \mathbb{K}^{m'}$, either C_t is empty, or $\Pi_n^{n+1}(C_t) = A_i$ for some $i \in I$.

Definition 7.20. Let $(A_i)_{i \in I}$ be a definable family of subsets of \mathbb{K}^n . We say that $(A_i)_{i \in I}$ is a **strongly uniform family** of \mathcal{F}_σ sets if and there exist $C \subset \mathbb{K}^{n+1} \times I$ definable, and $F : C \rightarrow \mathbb{K}^n$ definable, such that, for every $i \in I$, $F(C_{[i]}) = A_i$, and C_i is a closed subset of \mathbb{K}^{n+1} .

Remark 7.21. Let $\mathcal{A} := (A_i)_{i \in I}$ be a definable family of subsets of \mathbb{K}^n . Then, \mathcal{A} is a uniform family of \mathcal{F}_σ sets iff there exists $k, m' \in \mathbb{N}$ and $D \subseteq \mathbb{K}^{n+k} \times \mathbb{K}^{m'}$, such that:

- each fiber $(D_t)_{t \in \mathbb{K}^{m'}}$ is closed;
- for each $i \in I$ there exists $t \in \mathbb{K}^{m'}$ such that $A_i = \Pi_n^{n+k}(D_t)$.

Proof. The “only if” direction is obvious. For the “if” direction, let $(D_t)_{t \in \mathbb{K}^{m'}}$ as in the Remark. For each $t \in \mathbb{K}^{m'}$, define

$$C_t := \overline{\bigcup_{0 < r \in \mathbb{K}} \Pi_n^{n+k}(D_t \cap \overline{B}(0; r)) \times \{r\}}. \quad \square$$

Given $X \subseteq \mathbb{K}^n$ definable, and $\mathbb{K}' \succeq \mathbb{K}$, denote by $X(\mathbb{K}')$ the interpretation in \mathbb{K}' of (the formula defining) X ; similarly, we denote $\mathcal{A}(\mathbb{K}') := (A_i(\mathbb{K}'))_{i \in I(\mathbb{K}')}$.

Lemma 7.22. *Let $\mathcal{A} := (A_i)_{i \in I}$ be a definable family of subsets of \mathbb{K}^n . T.f.a.e.:*

1. \mathcal{A} is a uniform family of \mathcal{F}_σ sets;
2. for every $\mathbb{K}' \succeq \mathbb{K}$, and for each $i \in I(\mathbb{K}')$, $A_i(\mathbb{K}')$ is an \mathcal{F}_σ set;
3. there exists $\mathbb{K}' \succeq \mathbb{K}$ such that \mathbb{K}' is ω -saturated, and for each $i \in I(\mathbb{K}')$, $A_i(\mathbb{K}')$ is an \mathcal{F}_σ set.

Proof. Standard compactness argument. For instance, let us show that not (1) implies not (3). Assume that \mathcal{A} is not a uniform family of \mathcal{F}_σ sets. Let Ψ be the set of formulae $\psi(x, y)$ (without parameters), where x and y are of length $n+1$ and m' respectively (m' varies), such that, for every $c \in \mathbb{K}^{m'}$, $\psi(\mathbb{K}^{n+1}, c)$ is closed. By hypothesis, for every $\phi \in \Psi$, there exists $i \in I$ such that, for every c , $A_i \neq \Pi_n^{n+1}(\phi(\mathbb{K}^{n+1}, c))$. The above condition determine a partial type $q(i)$, with a finite number of parameters (that is, the parameters used to define \mathcal{A}). Let $\mathbb{K}' \succ \mathbb{K}$ be ω -saturated, and let $i_0 \in I(\mathbb{K}')$ satisfying q ; we claim that $A_{i_0}(\mathbb{K}')$ is not an \mathcal{F}_σ . If, for contradiction, $A_{i_0}(\mathbb{K}')$ were an \mathcal{F}_σ , there would exist a definable closed set $C \subset \mathbb{K}^{n+1}$ such that $A_{i_0}(\mathbb{K}') = \Pi_n^{n+1}(C)$. Let $c \in \mathbb{K}^{m'}$ and $\phi_0(x, c)$ be the formula defining C . Define $J := \{y \in \mathbb{K}^{m'} : \phi_0(\mathbb{K}^{n+1}, y) \text{ is closed}\}$, and $\phi(x, y) \equiv (\phi_0(x, y) \ \& \ y \in J)$. Then, $\phi \in \Psi$, but $A_{i_0}(\mathbb{K}') = \Pi_n^{n+1}(\phi(\mathbb{K}^{n+1}, c))$, a contradiction. \square

Lemma 7.23. *Let $\mathcal{A} := (A_i)_{i \in I}$ be a family of subsets of \mathbb{K}^n , definable with parameters \bar{a} . Then, t.f.a.e.:*

1. \mathcal{A} is a strongly uniform family of \mathcal{F}_σ sets;
2. There exists $\mathbb{K}' \succeq \mathbb{K}$ such that \mathbb{K}' is ω -saturated and, for every $i \in I(\mathbb{K}')$, there exists $C_i \subseteq \mathbb{K}^{n+1}$ which is closed and definable with parameters $\bar{a}i$, and such that $\Pi_n^{n+1}(C_i) = A_i$;
3. For every $\mathbb{K}' \succeq \mathbb{K}$ and for every $i \in I(\mathbb{K}')$, there exists $C_i \subseteq \mathbb{K}^{n+1}$ which is closed and definable with parameters $\bar{a}i$, and such that $\Pi_n^{n+1}(C_i) = A_i$.

Example 7.24. Let \mathbb{K} be a-minimal. Then, every definable family $(A_i)_{i \in I}$ of \mathcal{F}_σ subsets of \mathbb{K}^n is strongly uniform.

Proof. Since each \mathcal{F}_σ set X is a union of at most $n+1$ locally closed sets X_0, \dots, X_n , and each X_i is definable in a uniform way from X , w.l.o.g. we can assume that each A_i is locally closed. For each $x \in A_i$, let

$$r_i(x) := \sup\{s \in \mathbb{K}^{>0} : A_i \cap \bar{B}(x; s) \text{ is compact}\},$$

and $U_i := \bigcup_{x \in A_i} B(x; r_i(x)/2)$. Notice that each U_i is open, and A_i is closed in U_i . For each $i \in I$ and $r \in \mathbb{K}^{>0}$, define

$$C(i, r) := \bar{A}_i \cap \{x \in U_i : d(x, U_i^c) \geq r\}.$$

Notice that each $C(i, r)$ is closed, and $A_i := \bigcup_{r>0} C(i, r)$. Finally, let $D(i) := \overline{\bigcup_{r>0} C(i, r) \times \{r\}}$. Each $D(i)$ is closed, and $A_i = \Pi_n^{n+1}D(i)$. \square

Lemma 7.25. *Let $\mathcal{A} := (A_i)_{i \in I}$ be a strongly uniform family of \mathcal{F}_σ sets, and assume that either I is at most pseudo-enumerable, or that \mathcal{A} is increasing. Then, $D := \bigcup_{i \in I} A_i$ is \mathcal{F}_σ .*

Proof. We distinguish 2 cases: either \mathbb{K} is a-minimal, or not. If \mathbb{K} is a-minimal, the conclusion follows from Lemma 5.7 and Cor. 5.4.

Otherwise, let J be a pseudo- \mathbb{N} subset of \mathbb{K} . For every $i \in I$ and $r \in \mathbb{K}^{>0}$, let $E(i, r) := \{x \in C_i : |x| \leq r\}$. Notice that each $E(i, r)$ is compact, and that $A_i = \bigcup_r \theta(E(i, r))$, where $\theta := \Pi_n^{n+1+m'}$. Then, $D = \bigcup_{(i,j) \in I \times J} \theta(E(i, j))$. Since $I \times J$ is pseudo-enumerable, and each $\theta(E(i, j))$ is compact, D is an \mathcal{F}_σ set. \square

Remember that an \mathcal{F}_σ is meager iff it has empty interior.

Definition 7.26. Let $\mathcal{A} := (A_i)_{i \in I}$ be a definable family of subsets of \mathbb{K}^n , and $\pi := \prod_n^{n+1}$.

- \mathcal{A} is a **uniform family** of meager sets, if there exists a definable family $(C_t)_{t \in \mathbb{K}^m}$ of closed subsets of \mathbb{K}^{n+1} , such that
 1. for every $t \in \mathbb{K}^m$, $\pi(C_t)$ has empty interior;
 2. and for every $i \in I$ there exists $t \in \mathbb{K}^m$ such that $A_i \subseteq \pi(C_t)$.
- \mathcal{A} is a **strongly uniform family** of meager sets, if there exists a definable family $(C_i)_{i \in I}$ of closed subsets of \mathbb{K}^{n+1} , such that, for every $i \in I$, $\pi(C_i)$ has empty interior, and $A_i \subseteq \pi(C_i)$.
- \mathcal{A} is a **uniform family** of a.o. sets, if there exists a definable family $(C_t)_{t \in \mathbb{K}^m}$ of closed subsets of \mathbb{K}^{n+1} , and a definable family $(U_s)_{s \in \mathbb{K}^l}$ of open subsets of \mathbb{K}^n , such that
 1. for every $t \in \mathbb{K}^m$, $\pi(C_t)$ has empty interior;
 2. and for every $i \in I$ there exists $t \in \mathbb{K}^m$ and $s \in \mathbb{K}^l$, such that $A_i \Delta U_s \subseteq \pi(C_t)$.
- \mathcal{A} is a **strongly uniform family** of a.o. sets, if there exists a definable family $(U_i)_{i \in I}$ of open subsets of \mathbb{K}^n , and a definable family $(C_i)_{i \in I}$ of closed subsets of \mathbb{K}^{n+1} , such that, for every $i \in I$, $\pi(C_i)$ has empty interior, and $A_i \Delta U_i \subseteq \pi(C_i)$.

Question 7.27. Are uniform family of \mathcal{F}_σ sets (resp. meager, resp. a.o. sets) strongly uniform?

If \mathbb{K} has definable choice, the answer is positive.

Reasoning as for the \mathcal{F}_σ case, we can prove the following results:

Lemma 7.28. Let $\mathcal{A} := (A_i)_{i \in I}$ be a definable family of subsets of \mathbb{K}^n . T.f.a.e.:

1. \mathcal{A} is a uniform family of meager (resp., a.o.) sets;
2. for every $\mathbb{K}' \succeq \mathbb{K}$, and for each $i \in I(\mathbb{K}')$, $A_i(\mathbb{K}')$ is a meager (resp., a.o.) set;
3. there exists $\mathbb{K}' \succeq \mathbb{K}$ such that \mathbb{K}' is ω -saturated, and for each $i \in I(\mathbb{K}')$, $A_i(\mathbb{K}')$ is a meager (resp., a.o.) set.

Proof. Let us prove (3 \Rightarrow 1) for meager sets. Assume not. By adding a finite number of constants to the language if necessary, we can assume that either \mathbb{K} is a-minimal, or there exists N pseudo- \mathbb{N} subset of \mathbb{K} definable *without parameters*. Let Ψ be the set of formulae (without parameters) $\phi(\bar{x}, \bar{y})$, such that, for every \bar{y} , $\phi(\mathbb{K}, \bar{y})$ is a closed subset of \mathbb{K}^{n+1} , and $\pi(\phi(\mathbb{K}, \bar{y}))$ has empty interior, where $\pi := \prod_n^{n+1}$. By hypothesis, for every $\phi \in \Psi$, there exists $i \in I$, such that, for every \bar{c} , $A_i \not\subseteq \pi(\phi(\mathbb{K}^n, \bar{c}))$. Let $q(i)$ be the partial type $\{\forall \bar{c} A_i \not\subseteq \pi(\phi(\mathbb{K}^n, \bar{c})) : \phi \in \Psi\}$. q has a finite number of parameters (the parameters used in the definition of \mathcal{A}), and is consistent; therefore, q has a realization i_0 in \mathbb{K}' . Since A_{i_0} is an \mathcal{F}_σ , there exists $D \subseteq \mathbb{K}'^{n+1}$ closed and definable with parameters \bar{c} , such that $A_{i_0} \subseteq \pi(D)$, and $\text{int}(\pi(D)) = \emptyset$. Hence, $D = \phi(\mathbb{K}, \bar{c})$ for some $\phi \in \Psi$, contradiction. \square

Example 7.29. Let \mathbb{K} be a-minimal. Then, every definable family of meager (resp., a.o.) sets is strongly uniform.

Lemma 7.30. *Let $\mathcal{A} := (A_i)_{i \in I}$ be a strongly uniform family of meager sets (resp., a.o.), and assume that either I is at most pseudo-enumerable, or that \mathcal{A} is increasing. Then, $D := \bigcup_{i \in I} A_i$ is meager (resp., a.o.).*

Proof. If \mathbb{K} is a-minimal, use Lemma 5.9. Otherwise, it suffices to treat the case when I is pseudo-enumerable, and w.l.o.g. we can reduce to the case when I is a pseudo- \mathbb{N} subset of \mathbb{K} .

If \mathcal{A} is a strongly uniform family of meager set, Let $(C_i)_{i \in I}$ be a definable family of closed subsets of \mathbb{K}^{n+1} , such that, for every $i \in I$, $A_i \subseteq \pi(C_i)$, and $\pi(C_i)$ has empty interior, where $\pi := \Pi_n^{n+1}$. Then,

$$D \subseteq \bigcup_{i \in I} \pi(C_i) = \bigcup_{(i,j) \in I \times J} \Pi_n^{n+1}(C_i \cap \overline{B(0; j)}),$$

and the latter is the union of a pseudo-enumerable family of nowhere dense sets.

If \mathcal{A} is a strongly uniform family of a.o. sets, let $(U_i)_{i \in I}$ be a definable family of open sets, such that $(A_i \Delta U_i)_{i \in I}$ is a strongly uniform family of meager sets. Let $V := \bigcup_{i \in I} U_i$. Notice that

$$D \Delta V = \left(\bigcup_i A_i \right) \Delta \left(\bigcup_j U_j \right) \subseteq \bigcup_i (A_i \Delta U_i),$$

and, by the previous point, the latter is a meager set. \square

Corollary 7.31. *Assume that either \mathbb{K} be ω -saturated and has definable choice. or \mathbb{K} is a-minimal. Let $(A_i)_{i \in I}$ be a definable family. Assume that either I is at most pseudo-enumerable, or that $(A_i)_{i \in I}$ is increasing. Assume moreover that each A_i is an \mathcal{F}_σ (resp., meager, resp. a.o.). Then, $\bigcup_{i \in I} A_i$ is an \mathcal{F}_σ (resp., meager, resp. a.o.).*

Proof. The hypothesis on \mathbb{K} implies that definable families of \mathcal{F}_σ (resp., meager, resp. a.o.) sets are strongly uniform. \square

7.4 Is every definably complete structure Baire?

We are not able to answer the above question, even if we conjecture that the answer is positive.

However, we shall give a partial results, where we prove that a strong failure of the Baire property is not possible in a definably complete structure.

Note that if \mathbb{K} is pseudo-enumerable, then \mathbb{K} is not Baire.

Theorem 7.32. *\mathbb{K} is not pseudo-enumerable.*

The proof can be understood better if one considers the case when $\mathbb{K} = \mathbb{R}$, and considers it as a proof of the fact that \mathbb{R} is not countable.

We need some preliminary results and definitions.

For every $I = (a, b) \subseteq (0, 1)$ open interval, and $0 < \lambda < 1/2$, define $|I| := b - a$, and $C_\lambda(I) := [a + \lambda(b - a), b - \lambda(b - a)]$. Note that $C_\lambda(I)$ is a closed subset of I , and that $|C_\lambda(I)| = (1 - 2\lambda)|I| > 0$.

Fix such an open interval I , and let $J \subseteq I$ be a closed interval, and $0 < d < 1$, such that $|J| \geq d|I|$. Let $x \in I$, and consider the two intervals $I'_1 := (a, x)$ and $I'_2 := (x, b)$.

Remark 7.33. There exists $k \in \{1, 2\}$ such that

$$\frac{|C_\lambda(I'_k) \cap J|}{|I'_k|} \geq (d - 2\lambda).$$

Proof. Let call the elements of J “good points”. The fraction of good points in I is at least d . Hence, for at least one k , the fraction of good points in I'_k is at least d . Only a fraction of 2λ points are not in $C_\lambda(I'_k)$, and therefore at least a fraction of $d - 2\lambda$ points are good and in $C_\lambda(I'_k)$. \square

Now, suppose, for contradiction, that \mathbb{K} is pseudo-enumerable. Hence, $X := [0, 1]$ is also pseudo-enumerable; let N be a pseudo- \mathbb{N} subset of \mathbb{K} , and $f : N \rightarrow X$ be a definable surjective function. W.l.o.g., 1 is the minimum of N . We denote by $x_n := f(n)$, for every $n \in N$.

For every $n \in N$, let $s(n)$ be the successor of n in N .

Note that N must be unbounded (otherwise, it would be pseudo-finite). Let $\phi : \mathbb{K}^{\geq 0} \rightarrow (0, 1]$ be an order-reversing homeomorphism, and $D := \phi(N)$; denote by $d_n := \phi(n)$. Let $\lambda_1 := (1 - d_1)/4$, and, for every $n \in N$, let $\lambda_{s(n)} := (d_n - d_{s(n)})/4 > 0$.

For every $n \in N$, let $N_{\leq n} := \{n' \in N : n' \leq n\}$. Let $X(n) := f(N_{\leq n})$. Note that $X_{s(n)} = X(n) \cup \{x_{s(n)}\}$. Note also that N_n is pseudo-finite, and thus X is also pseudo-finite. Let $U(n) := X \setminus X(n)$: note that $U(n)$ is open, and, since $X(n)$ is pseudo-finite, $U(n)$ is a pseudo-finite union of disjoint open intervals:

$$U(n) = \bigsqcup_{i \in A(n)} I_{i,n},$$

where each $I_{i,n}$ is a non-empty open interval, and $A(n)$ is pseudo-finite. Define

$$F(n) := \bigcup_{i \in A(n)} C_{\lambda_n}(I_{i,n}).$$

$F(n)$ is a pseudo-finite union of closed intervals, and therefore it is definable and closed.

We claim that $F := \bigcap_n F(n) \neq \emptyset$. If it is so, since $F(n) \subset U(n)$, we have that $X \neq \bigcup_n X(n)$, and we have a contradiction.

For every $n \in N$, let $G(n) := \bigcap_{m \leq n} F(m)$. Note that $G(n)$ is a definable decreasing family of d -compact sets, and that $F = \bigcap_n G(n)$. Thus, to prove that F is non-empty, it suffices to prove that each $G(n)$ is non-empty. We shall prove this by induction on n ; however, we will need to prove a stronger statement.

Lemma 7.34. *For each $n \in N$,*

1. $G(n)$ is a pseudo-finite union of disjoint closed intervals, with non-empty interior, $(J_{n,j})_{j \in B(n)}$;
2. the intervals $J_{n,j}$ are the definably connected components of $G(n)$;
3. each $J_{n,j}$ is contained in a unique open interval $I_{n,i}$, where $i = i_{n,j}$ depends in a definable way from (n, j) ;
4. there exists at least one $j \in B(n)$ such that

$$\frac{|J_{n,j}|}{|I_{n,i_{n,j}}|} \geq d_n.$$

Note that the lemma implies that each $G(n)$ is non-empty.

Proof. If $n = 1$, then, by definition, $A(1) = B(1)$, and every interval $J_{1,i}$ satisfies $J_{1,i} = C_{\lambda_1}(I_{1,i_{1,j}})$, and $|J_{1,j}| = (1 - 2\lambda_1)|I_{1,i_{1,j}}|$.

Suppose that the statement is true for n ; we want to prove it for $s(n)$. Let $x := x_{s(n)}$, $\lambda := \lambda_{s(n)}$, and let $J := J_{n,j}$ and $(a, b) := I := I_{n,i_{n,j}}$ be the intervals satisfying the condition for n . If $x \notin I$, then, since $d_{s(n)} < d_n$, the same J and I satisfy the condition for $n + 1$.

If instead $x \in I$, let $I'_1 := (a, x)$ and $I'_2 := (x, b)$. Note that I is substituted by the pair (I'_1, I'_2) in the expression of $U_{s(n)}$ as disjoint union of open intervals. Let $L'_k := C_\lambda(I'_k)$, $k = 1, 2$. By Remark 7.33, at least one of the L'_k satisfies

$$\frac{|L'_k \cap J|}{|I'_k|} \geq d_n - 2\lambda > d_{n+1}.$$

Since $J'_k := L'_k \cap J$, $k = 1, 2$, are definably connected components of $G(n + 1)$, we are done. \square

Lemma 7.35. *Assume that \mathbb{K} is not Baire (but it is still definably complete and expands a field). Then, there exists a dense pseudo-enumerable subset $Q \subseteq \mathbb{K}$.*

Note that, by the previous theorem, Q has empty interior.

Proof. \mathbb{K} is not a-minimal, and therefore there exists D pseudo- \mathbb{N} subset of \mathbb{K} .

Let $(X(n))_{n \in D}$ be a definable increasing family of d -compact nowhere-dense subsets of $[0, 1]$, such that $[0, 1] = \bigcup_{n \in D} X(n)$. For every $n \in D$, let $Y(n) \subseteq X(n)$ be the set of isolated points of $X(n)$, and $Z(n) \subset [0, 1]$ be the set of centres of the definable connected components of $[0, 1] \setminus X(n)$. Note that both $Y(n)$ and $Z(n)$ are discrete; thus, $Y(n)$ and $Z(n)$ are (at most) pseudo-enumerable. Let $Q' := \bigcup_{n \in D} (Z(n) \cup Y(n))$. Since Q' is the union of a pseudo-enumerable family of pseudo-enumerable sets, Q' is pseudo-enumerable. Moreover, Q' is dense in $[0, 1]$. Using Q' , it is trivial to obtain $Q \subseteq \mathbb{K}$ pseudo-enumerable and dense in \mathbb{K} . \square

Corollary 7.36. *If \mathbb{K} is not Baire, then, for every $n \in \mathbb{N}$, \mathbb{K}^n has a pseudo-enumerable basis of open sets.*

Proof. Let I be a pseudo- \mathbb{N} subset of \mathbb{K} , such that $0 \notin I$, and $(x_i)_{i \in I}$ be a dense pseudo-enumerable subset of \mathbb{K}^n . Then, $\{B(x_i; 1/i) : i \in I\}$ is a pseudo-enumerable basis of open sets for \mathbb{K}^n . \square

7.5 Fixed point theorems

Lemma 7.37 (Banach fixed point theorem). *Let $X \subseteq \mathbb{K}^n$ be a definable non-empty closed set, and $f : X \rightarrow X$ be a definable map. Assume that:*

1. *either X is d -compact, and, for every $x \neq y \in X$, $|f(x) - f(y)| < |x - y|$;*
2. *or there exists $C \in \mathbb{K}$ such that $0 \leq C < 1$ and, for every $x, y \in Y$, $|f(x) - f(y)| \leq C|x - y|$. Then, f has a unique fixed point.*

Proof. (1) Let $g : X \rightarrow \mathbb{K}$, $g(x) := |x - f(x)|$, and let $d := \min(g)$. If f has no fixed point, then $d > 0$; let $x_1 \in X$ such that $g(x_1) = d$, and $x_2 := f(x_1)$. Then, $|f(x_2) - x_2| < |x_2 - x_1| = d$, absurd.

(2) Choose $x_1 \in X$; let $x_2 := f(x_1)$, $r_0 := |x_1 - x_2|$, $r := r_0/(1 - c)$, and $Y := X \cap \overline{B}(x_0, r)$. Then, Y is d -compact and $f(Y) \subseteq Y$. Hence, by (1), f has a fixed point in Y . \square

Conjecture 7.38. *Brower fixed point theorem. Kakutani fixed point theorem.*

7.6 Baire structures

Let \mathbb{K} be definably complete and **Baire**.

If X and Y are subsets of \mathbb{K}^n , then $X - Y := \{x - y : x \in X, y \in Y\}$.

Lemma 7.39 (Pettis' Theorem). *Let $A \subseteq \mathbb{K}^n$ be definable and a.o.. If A is non-meager, then $A - A$ contains a non-empty open neighbourhood of 0. If \mathbb{K} is i -minimal with DSF (see §8) and $A - A$ is non-meager, then A is non-meager.*

Proof. Minor variation of [Oxt71, Thm. 4.8]. Let $A = U \Delta P$, where U is open and definable, and P is meager. A is non-meager iff U is non-empty. If A is non-meager, let $B \subseteq U$ be a non-empty open ball, of radius $\delta > 0$. For any $x \in \mathbb{K}^n$, we have

$$(x + A) \cap A = ((x + U) \Delta (x + P)) \cap (U \Delta P) = ((x + U) \cap U) \Delta ((x + U) \cap P) \Delta ((x + P) \cap U) \Delta ((x + P) \cap P) \supseteq [(x + B) \cap B] - [P \cup (x + P)].$$

If $|x| < \delta$, the right member represents a non-empty open set, minus a meager set; it is therefore non-empty. Thus, for every $x \in B(0, \delta)$, $(x + A) \cap A$ is non-empty, and therefore $x \in A - A$.

Conversely, if $A - A$ is non-meager, then, by Thm. 8.4, $\dim(A - A) = 1$; therefore, by Lemma 9.5, $\dim A = 0$, and thus A is meager. \square

Define $F : \mathbb{K}^4 \rightarrow \mathbb{K}$ as $F(x_1, x_2, y_1, y_2) := (x_1 - x_2)/(y_1 - y_2)$ if $y_1 \neq y_2$, and 0 otherwise.

Corollary 7.40. *Let $A \subseteq \mathbb{K}$ be definable and a.o.. If A is non-meager, then $F(A^4) = \mathbb{K}$. If \mathbb{K} is i -minimal with DSF (see §8) and $F(A^4)$ is non-meager, then A is non-meager.*

Proof. If A is non-meager, then $A - A$ contains an open neighbourhood of 0, and therefore $F(A^4) = \mathbb{K}$. The converse is proved as in the previous lemma. \square

8 I-minimal and constructible structures

As usual, \mathbb{K} is a definably complete structure, expanding a field.

8.1 I-minimal structures

Definition 8.1. \mathbb{K} is *i -minimal* if, for every unary definable set X , if X has empty interior, then X is nowhere dense.

See also [Mil05] for the case when \mathbb{K} is an expansion of \mathbb{R} .

Lemma 8.2. *If \mathbb{K} is i -minimal, then it is Baire*

Proof. By Lemma 7.35, if \mathbb{K} were not Baire, it would contain a definable dense and co-dense subset Q . However, Q would have empty interior, but it would not be nowhere dense. \square

Examples 8.3. If \mathbb{K} is locally o-minimal, then it is i -minimal (trivial).

If \mathbb{K} is d-minimal, then it is i -minimal. In fact, by definition of d-minimality if $X \subseteq \mathbb{K}$ is definable and with empty interior, then X is a finite union of discrete sets X_1, \dots, X_n . Every discrete subset of a definably complete structure is nowhere dense. Thus, X is a finite union of nowhere dense sets, and thus it is nowhere dense.

Theorem 8.4. *The following are equivalent:*

1. \mathbb{K} is i -minimal;
2. for every $n \in \mathbb{N}$, if X is a definable subset of \mathbb{K}^n with empty interior, then X is nowhere dense;
3. for every definable set X , $\text{bd}(X)$ has empty interior;
4. for every definable $X \subseteq \mathbb{K}$, $\dim X = \dim \overline{X}$;
5. for every definable X , $\dim X = \dim \overline{X}$;
6. if $U \subseteq \mathbb{K}^n$ is definable and open, and $f : U \rightarrow \mathbb{K}$ is definable, then $\mathcal{D}(f)$ is nowhere dense;
7. for every $n, m \in \mathbb{N}$, if $A \subseteq \mathbb{K}^{n+m}$ is definable, then $\mathfrak{B}_n(A)$ is nowhere dense, where

$$\mathfrak{B}_n(A) := \{x \in \mathbb{K}^n : \text{cl}(A)_x \setminus \text{cl}(A_x) \neq \emptyset\}.$$

8. for every $n, m \in \mathbb{N}$, if $A \subseteq \mathbb{K}^{n+m}$ is definable, then the set

$$\{x \in \mathbb{K}^n : (\partial A)_x \neq \partial(A_x)\}$$

is nowhere dense;

9. \mathbb{K} is Baire, and, for every definable $X \subseteq \mathbb{K}$, either X has interior, or it is meager;
10. \mathbb{K} is Baire, and every definable set either has interior, or it is meager;
11. for all definable $A, B \subseteq \mathbb{K}$, $\dim(A \cup B) = \max\{\dim A, \dim B\}$;
12. for all definable $A, B \subseteq \mathbb{K}^n$, $\dim(A \cup B) = \max\{\dim A, \dim B\}$;
13. for all definable $A \subseteq \mathbb{K}^n$, if $\dim A = d$, then $\{x \in \mathbb{K}^d : \dim A_x > 0\}$ is nowhere dense;
14. let $d, k, m, n \in \mathbb{N}$, with $k \leq n$ and $d \leq m$; let $A \subseteq \mathbb{K}^{n+m}$ be definable, and $\dim A \leq d + k$; define $C := \{x \in \mathbb{K}^n : \dim A_x \geq d\}$; then, $\dim(C) \leq k$;
15. any pseudo-enumerable union of subsets of \mathbb{K} with empty interior has empty interior;

16. for every $d \leq n \in \mathbb{N}$, any pseudo-enumerable union of subsets of \mathbb{K}^n of dimension less or equal to d has dimension less or equal to d ;

Moreover, if \mathbb{K} is *i*-minimal, then:

- I every meager set is nowhere dense;
- II for every $d \leq n \in \mathbb{N}$, any increasing definable union of subsets of \mathbb{K}^n of dimension less or equal to d has dimension less or equal to d ;
- III if $U \subseteq \mathbb{K}$ is open and definable, and $f : U \rightarrow \mathbb{K}$ is definable, then there exists $D \subseteq \mathbb{K}$ definable, closed and with empty interior, such that, for every definably connected component I of $U \setminus D$, $f \upharpoonright I$ is continuous, and either constant or strictly monotone;
- IV if $U \subseteq \mathbb{K}^n$ is open and definable, and $f : U \rightarrow \mathbb{K}$ is definable, then there exists $D \subseteq \mathbb{K}$ definable, closed and with empty interior, such that the restriction $f \upharpoonright U \setminus D$ is continuous.

The proof is postponed to Section 8.3; cf. [Mil05, Main Lemma].

Example 8.5. Let (M', M) be *o*-minimal structures (expanding a field), such that M is an elementary substructure of M' and it is dense in M' . Thus, the structure $N := (M, M')$ has *o*-minimal open core. Therefore, if $X \subseteq N$ is meager, then X is nowhere dense. However, N is not *i*-minimal, because M is a definable dense subset of N such that $\overline{M} = N$ (thus, clause I in Thm. 8.4 does not imply *i*-minimality).

Lemma 8.6. \mathbb{K} is locally *o*-minimal iff it is *a*-minimal and *i*-minimal.

Proof. The “only if” direction is clear. Let us prove the “if” direction. Let $X \subseteq \mathbb{K}$ be definable and with empty interior. By *i*-minimality, X is nowhere dense. By *a*-minimality, X is pseudo-finite. Thus, \mathbb{K} is locally *o*-minimal. \square

Corollary 8.7. Assume that \mathbb{K} is *i*-minimal. Then, \mathbb{K} is *o*-minimal iff every definable discrete set is finite.

Lemma 8.8. The following are equivalent:

1. \mathbb{K} is locally *o*-minimal;
2. for every $X \subseteq \mathbb{K}$ definable and non-empty $\dim(\partial X) < \dim X$;
3. for every $X \subseteq \mathbb{K}^n$ definable and non-empty $\dim(\partial X) < \dim X$.

Proof. (3 \Rightarrow 2) is obvious.

(2 \Rightarrow 1). First, let us prove that \mathbb{K} is *a*-minimal. Let $X \subseteq [0, 1)$ definable, discrete and closed in $[0, 1)$. We have to prove that X is pseudo-finite. If not, 1 is an accumulation point for X . Thus, $\partial X = \{1\}$, and therefore $\dim(\partial X) = \dim X = 0$, absurd.

Then, we prove that X is locally *o*-minimal. It suffices to prove that if $X \subseteq \mathbb{K}$ has dimension 0, then it is pseudo-finite. Since $\dim X = 0$, ∂X is empty, and therefore X is closed; thus, X is nowhere dense, and hence, by *a*-minimality, X is pseudo-finite.

(1 \Rightarrow 2) is obvious, because, if $A \subseteq \mathbb{K}$ is definable, then, if $\dim A = 0$, then A is pseudo-finite, and hence closed, and hence $\partial A = \emptyset$; if instead $\dim A = 1$, then ∂A has empty interior, and thus $\dim(\partial A) \leq 0$.

(1 \Rightarrow 3). Let $A \subseteq \mathbb{K}^n$ be definable, with $\dim A = d \geq 0$. We have to prove that $\dim(\partial A) < d$. If $n = 1$, we have already proved it above. We proceed by induction on n : we assume we have proved the conclusion for every $n' < n$, and we want to prove it for n . If $d = n$, then ∂A has empty interior, and thus $\dim(\partial A) < n$, and we are done. Thus, we can assume $0 \leq d < n$. Assume, for contradiction, that $\dim(\partial A) \geq d$. W.l.o.g., $U := \Pi_d^n(\partial A)$ has non-empty interior. Since \mathbb{K} is i -minimal, the set

$$\{u \in \mathbb{K}^d : (\partial A)_u \neq \partial(A_u)\}$$

is nowhere dense; thus, there exists $V \subseteq U$ open, definable and non-empty, such that, for every $v \in V$, $(\partial A)_v = \partial(A_v)$. By i -minimality, and since $\dim A = d$, $\dim(A_u) = 0$ outside a nowhere dense set; thus, after shrinking V , we can assume that $\dim(A_u) = 0$ for every $u \in V$. Thus, by the inductive hypothesis, $\partial(A_u)$ is empty for every $u \in V$, contradicting the fact that $V \subseteq \Pi_d^n(\partial A)$. \square

Since there do exist i -minimal structures that are not locally o-minimal (e.g., d -minimal not o-minimal expansions of the real field), we have that for some i -minimal structure there is some definable non-empty set X such that $\dim(\partial X) = \dim X$ (notice that if \mathbb{K} is i -minimal, then $\dim(\partial X) \leq \dim X$, because $\dim X = \dim(\bar{X}) = \max\{\dim(X), \dim(\partial X)\}$).

Proviso. For the remainder of this subsection, we will assume that \mathbb{K} is i -minimal.

Lemma 8.9. Let $f : U \rightarrow \mathbb{K}$ be definable, where $U \subseteq \mathbb{K}^n$ is open and definable. Then, for every $p \in \mathbb{N}$, there exists $D \subset U$ closed, definable and nowhere dense, such that f is C^p on $U \setminus D$.

Proof. If $p = 0$, we have already proved it in Thm. 8.4(IV). If one proves the case $p = 1$, then an easy induction on p proves the conclusion for every p .

The case $n = 1$ can be done as for o-minimal structures [vdD98b, Proposition 7.2.5].

The case $n > 1$ can be done as in [Mil05, Thm. 3.3] (note that Miller uses instead Lebesgue differentiability theorem for the case $n = 1$). \square

Lemma 8.10. Let $d \leq n$, $A \subseteq \mathbb{K}^n$ be definable, $\pi := \Pi_d^n$, and

$$Z := Z(A) := \{a \in A : \exists U \text{ neighbourhood of } a : \pi(A \cap U) \text{ is nowhere dense}\}.$$

Then, Z is a definable open subset of A , and $\pi(Z(A))$ is nowhere dense.

Proof. Follows immediately from Lemma 4.24. \square

Definition 8.11. We define Π -good sets as in [Mil05, §7]. That is, we say that a definable set $A \subseteq \mathbb{K}^{n+m}$ is π -good (where $\pi := \Pi_m^{n+m}$) if:

- $\dim A = m$;
- πA is open;

- $\pi(A \cap U)$ has interior for every $a \in A$ and open neighbourhood U of a ;
- for all $x \in \pi A$, $\dim(A_x) = 0$ and $\text{cl}(A_x) = \text{cl}(A)_x$.

More generally, A is μ -good (where μ is a projection from \mathbb{K}^{n+m} to an m -dimensional coordinate space) if there is a permutation of coordinates σ such that $\mu = \pi \circ \sigma$, and σA is π -good. Finally, A is Π -good, if it is μ -good for some μ as above.

Lemma 8.12 (Partition Lemma). *Let A be a finite collection of definable subsets of \mathbb{K}^n . Then, there exists a Π -good partition of \mathbb{K}^n compatible with A .*

Proof. The proof proceeds as in [Mil05, §7, Partition Lemma], using the previous lemma. \square

Lemma 8.13. *Let $A \subseteq \mathbb{K}^{n+m}$ be definable. Then,*

1. $\{x \in \mathbb{K}^m : \text{lc}(A_x) \neq (\text{lc } A)_x\}$ is nowhere dense;
2. for each $k \in \mathbb{N}$, $\{x \in \mathbb{K}^m : (A^{(k)})_x \neq (A_x)^{(k)}\}$ is nowhere dense;
3. if $\{x \in \mathbb{K}^m : \text{lc}(A_x) \neq \emptyset\}$ is somewhere dense, then $\text{lc}(A) \neq \emptyset$.

Proof. The same as [Mil05, Lemma 8.1]. \square

8.2 Constructible structures

Definition 8.14. \mathbb{K} is a **constructible** structure if, for every $\mathbb{K}' \equiv \mathbb{K}$, every 0-dimensional set definable in \mathbb{K}' is constructible. T is a constructible theory if every model of T is constructible.

Theorem 8.15. *The following are equivalent:*

1. \mathbb{K} is constructible;
2. for every \emptyset -definable $A \subseteq \mathbb{K}^{n+m}$ there exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{K}^m$, if $\dim A_x = 0$, then $(A_x)^{(N)} = \emptyset$;
3. every \emptyset -definable set is a finite union of \emptyset -definable locally closed sets;
4. every definable set is constructible;
5. every definable subset of \mathbb{K}^n is a finite union of sets of the form

$$\{x \in \mathbb{K}^m : f(b, x) = 0 \ \& \ g(b, x) = 0\},$$

where f and g are \emptyset -definable and continuous, and $b \in \mathbb{K}^m$.

Moreover, if \mathbb{K} is constructible, then it is *i-minimal*.

Proof. The equivalence of the first 4 points is proved in the same way as [Mil05, Thm. 3.2], using Lemma 8.13. (5 \Rightarrow 4) is obvious. (4 \Rightarrow 5) is proved in the same way as [vdD98a, Lemma 2.10]. The “moreover” clause follows from the fact that a constructible set with empty interior is nowhere dense. \square

Remark 8.16. *I-minimality is not equivalent to constructibility. In fact, it is not difficult to build an ultra-product of constructible structures which is not constructible.*

[Pil87], extending the work in [Rob74], studies topological structures M satisfying a weaker version of constructibility; that is, Pillay's condition (A) asks that every definable subset of M is constructible. In this context, he defines the **dimension rank** of closed definable subsets of M , which we will denote by rk^P , in the following way:

1. If X is non-empty, then $\text{rk}^P(X) \geq 0$.
2. $\text{rk}^P(X) \geq \lambda$ iff $\text{rk}^P(X) \geq \alpha$ for all $\alpha < \lambda$, where λ is limit.
3. $\text{rk}^P(X) \geq \alpha + 1$ iff X contains subset Y which is closed, nowhere dense (in X), and with $\text{rk}^P(Y) \geq \alpha$.

Notice that rk^P might depend on the ambient space M .

Let M be a Hausdorff topological structure, such that every M' elementarily equivalent to M satisfies condition (A) (e.g., M is constructible), and $X \subseteq M$ be definable and closed.

Lemma 8.17 (Pillay). *1. $\text{rk}^P(X) = 0$ iff X is discrete and non-empty;*

2. $Y \subseteq X \Rightarrow \text{rk}^P(Y) \leq \text{rk}^P(X)$;
3. *If $X = X_1 \cup \dots \cup X_n$, where the X_i are closed and definable, then $\text{rk}^P(X) = \max_{1 \leq i \leq n} \text{rk}^P(X_i)$.*

Lemma 8.18 (Pillay). *T.f.a.e.:*

1. $\text{rk}^P(X) = \infty$;
2. *there is a decreasing sequence $(X_i)_{i < \omega}$ of definable closed subsets of X , such that X_{i+1} is definable, closed, and nowhere dense in X_i , for all $i < \omega$;*
3. *X contains a definable definable closed nowhere dense subset Y , such that $\text{rk}^P(Y) = \infty$;*
4. $\text{rk}^P(X) > 2^{|M|}$.

Lemma 8.19. *Let \mathbb{K} be constructible, and $C \subseteq \mathbb{K}$ be non-empty, definable, closed, and with empty interior. Then, C has at least one isolated point; in particular, no non-empty closed perfect subset of \mathbb{K} with empty interior is definable.*

Proof. Assume, for contradiction, that C is perfect. Let $A := \mathbb{K} \setminus C$; A is an open set; let C^L be the set of left end-points of the connected components of A and C^R be the set of right end-points. Notice that C^L and C^R are definable subsets of C . Since \mathbb{K} is constructible, there exists an open interval I such that $I \cap C^L$ and $I \cap C^R$ are closed in I and $I \cap C^L \neq \emptyset$. Let $a \in C^L \cap I$. Since C is perfect and has empty interior, a is an accumulation point for C^R ; hence, $a \in C^L \cap C^R$, implying that a is isolated in C , absurd. \square

In the above Lemma, the hypothesis “ \mathbb{K} constructible” can be relaxed to “for every non-empty definable subset A of \mathbb{K} there exists an open interval I , such that $I \cap A$ is non-empty and closed in I ”.

Lemma 8.20. *Let \mathbb{K} be constructible, $C \subseteq \mathbb{K}$ be definable with empty interior, and D be the set of isolated points of C . Then, D is discrete, definable, and dense in C . Moreover, $C' := C \setminus D$ is nowhere dense in C .*

Proof. That D is discrete and definable is clear.

Claim 12. It suffices to prove the conclusion for \overline{C} .

In fact, the isolated points of \overline{C} are isolated points of C . Thus, w.l.o.g. C is closed. Let $A := \mathbb{K} \setminus C$, C^L be the set of left end-points of the connected components of A and C^R be the set of right end-points. C^L is dense in C . If, for contradiction, D is not dense in C , let I be a closed interval, such that $C \cap I$ has no isolated points and is non empty: but this contradicts the previous lemma.

The fact that C' is nowhere dense in C follows immediately from the first part. \square

From the above lemmata, it is easy to deduce the following.

Lemma 8.21. *Assume that \mathbb{K} is constructible, and $X \subseteq \mathbb{K}$ is definable and closed.*

1. *If X is a finite union of discrete sets, then $\text{rk}^{CB}(X) = \text{rk}^P(X)$.*
2. *\mathbb{K} is locally o-minimal iff $\text{rk}^P(\mathbb{K}) = 1$.*
3. *If \mathbb{K} is not locally o-minimal, then $\text{rk}^P(\mathbb{K}) \geq \omega$.*
4. *If \mathbb{K} is d-minimal but not locally o-minimal, then $\text{rk}^P(\mathbb{K}) = \omega$.*
5. *If \mathbb{K} is ω -saturated, constructible, but not d-minimal, then $\text{rk}^P(\mathbb{K}) = \infty$.*

Proof. The last point follows from the fact that, since \mathbb{K} is not d-minimal, then, by saturation, we can find $X \subset \mathbb{K}$ definable, closed, with empty interior, and such that $\text{rk}^{CB}(X) \geq \omega$. Hence, by the previous lemma, $X \supset X^{(1)} \supset X^{(2)} \supset \dots$ is an infinite descending chain of definable sets, such that $X^{(i+1)}$ is closed and nowhere-dense in $X^{(i)}$. \square

8.3 Proof of Thm. 8.4

(1 \Leftrightarrow 4) and (5 \Rightarrow 4) are clear.

For every $0 < n \in \mathbb{N}$, and $K = 2$, let (K_n) be the instantiation at n of the K th statement. For instance, (2_n) is equal to (1). We will prove that $(2_n) \Rightarrow (7_n) \Rightarrow (6_n) \Rightarrow (2_n)$, that $(2_n) \Rightarrow (2_{n+1})$, that $(2)_n \Leftrightarrow (3)_n$, and that (2_n) implies that every meager $X \subset \mathbb{K}^n$ is nowhere dense.

By induction on n , the above would imply $(1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 6 \Leftrightarrow 7)$.

$((2)_n \Leftrightarrow (3)_n)$ is clear.

$((6_n) \Rightarrow (3_n))$. Let $X \subseteq \mathbb{K}^n$ be definable and with empty interior. Let $f := 1_X$ be the characteristic function of X ; then, $\mathcal{D}(f) = \text{bd}(X)$; thus, $\text{bd}(X)$ is nowhere dense.

$((7_n) \Rightarrow (6_n))$. Let $f : U \rightarrow \mathbb{K}$ be definable, with $U \subseteq \mathbb{K}^n$ open. We want to prove that $\mathcal{D}(f)$ is nowhere dense; w.l.o.g., f is bounded. Let $A := \Gamma(f)$; then, $\mathcal{D}(f) = \mathfrak{B}_n(A)$. Thus, $\mathcal{D}(A)$ is nowhere dense.

Assume now that we have (2_n) .

Note that if $Y \subseteq \mathbb{K}^n$ is meager and definable, then, since \mathbb{K} is Baire, Y has empty interior, and thus Y is nowhere dense; therefore, we have proved (I_n) .

We prove now (7_n) . Let $A \subseteq \mathbb{K}^{n+m}$ be definable, and $B := \mathfrak{B}_n(A)$.

If we prove that B is meager, then, since \mathbb{K} is Baire, B has empty interior, and thus, by inductive hypothesis, B is nowhere dense. W.l.o.g., A is bounded

(because, after using a definable homeomorphism from \mathbb{K} to $(0, 1)$, B can only become larger).

Let $\pi := \Pi_n^{n+m}$, and $U := \pi(A)$. If U has empty interior, then, by (2_n) , U is nowhere dense; thus, B is also nowhere dense, because $B \subseteq \overline{\pi(A)}$. Therefore, we can assume that U has non-empty interior, and hence, w.l.o.g., that U is open.

Thus, for every $r > 0$, let

$$C(r) := \{(x, y) \in \overline{A} : d(y, A_x) \geq r\},$$

and $B(r) := \pi(C(r))$. Since $B = \bigcup_r B(r)$, and by (2_n) , it suffices to prove that each $B(r)$ has empty interior.

Fix $r > 0$, and assume, for contradiction, that $U' \subseteq B(r)$ is a non-empty open definable set. W.l.o.g., $U' = U$. Let $C' := \overline{C(r)}$: notice that C' is d-compact. Define $g : U \rightarrow \mathbb{K}^m$, $x \mapsto \text{lexmin}(C'_x)$. Since g is lower semi-continuous, there exists $U' \subseteq U$ open and non-empty, such that g is continuous on U' ; w.l.o.g., $U = U'$. Define also $f : U \rightarrow \mathbb{K}^m$; $x \mapsto \text{lexinf}(C(r)_x)$. Note that $f(x) \geq g(x)$ for every $x \in U$.

Claim 13. The set $D := \{x \in U : f(x) > g(x)\}$ is nowhere dense.

We will do only the case $m = 1$. It suffices to prove that D is meager. For every $s > 0$, let $D(s) := \{x \in U : f(x) \geq g(x) + s\}$. If we prove that each $D(s)$ is nowhere dense, we have the claim. By (2_n) , it suffices to prove that $D(s)$ has empty interior. Assume, for contradiction, that V is a non-empty subset of U , and let $x \in V$. Since g is continuous, we can assume that $d(g(x'), g(x)) < s/2$ for every $x' \in V$. By definition of f and g , there exists $x' \in V$ such that $d(f(x'), g(x)) < s/2$. Hence, $d(f(x'), g(x')) < s$, absurd.

Thus, after shrinking U , we can assume that $f = g$. Fix $x \in U$, and let $y := f(x)$. Since g is continuous on U , after shrinking U we can assume that $d(f(x'), y) < r/3$ for every $x' \in U$. Since $\Gamma(f) \subseteq \overline{A}$, there exists $(x', y') \in A$, such that (x', y') is near (x, y) ; that is, $x' \in U$ and $d(y', y) < r/3$. Moreover, by definition of f , there exists $y'' \in C_{x'}$, such that $d(f(x'), y'') < r/3$. However, this imply that $d(y', f(x')) < r$, contradicting the definition of $C(r)$.

We prove now (2_{n+1}) : let $A \subseteq \mathbb{K}^{n+1}$ is definable and has empty interior; we want to show that A is nowhere dense. If not, let

$$E' := \{x \in \mathbb{K}^n : \text{cl}(A_x) \text{ has non-empty interior}\}.$$

By assumption, E' has non-empty interior. W.l.o.g., A is bounded. Let $E := \{x \in \mathbb{K}^n : \text{cl}(A_x) \text{ has non-empty interior}\}$. Since, by (6_n) , $E \Delta E'$ is nowhere dense, E has non-empty interior. Since \mathbb{K} is Baire, there exists $0 < r \in \mathbb{K}$ such that

$$E(r) := \{x \in \mathbb{K}^n : A_x \text{ contains an interval of length } r\}$$

has non-empty interior; let $U \subseteq E(r)$ be a non-empty open set.

After shrinking A if necessary, we can assume that, for every $x \in U$, A is bounded and A_x is an open interval of length r . For every $x \in U$, let $h(x)$ be the centre of A_x .

By (6_n) , the set of points where h is continuous has non-empty interior. Thus, after shrinking U , we can assume that h is continuous. But then the set $\{(x, y) \in U \times \mathbb{K} : h(x) - r < y < h(x) + r\}$ is open and contained in A , absurd.

(7 \Leftrightarrow 8) is clear.

(2 \Rightarrow 10 \Rightarrow 9) are also clear.

(9 \Rightarrow 1). Let $X \subseteq \mathbb{K}$ be definable and have empty interior. By hypothesis, X is meager. Moreover, $\partial X = \overline{X} \setminus X$ has also empty interior, and thus it is meager. Therefore, \overline{X} is meager. Since \mathbb{K} is Baire, \overline{X} has empty interior.

(7 \Rightarrow 5). Let $X \subseteq \mathbb{K}^n$ be definable, and let $d := \dim \overline{X}$. We want to prove that $\dim X = d$. W.l.o.g., $\pi(\overline{X})$ contains an open subset of \mathbb{K}^d , where $\pi := \Pi_d^n$. If, for contradiction, $\dim X < d$, then, by (2), $\pi(X)$ is nowhere dense. Notice that $\pi(\overline{X}) \setminus \pi(X) \subseteq \mathfrak{B}_d(X)$. Since $\mathfrak{B}_d(X)$ is nowhere dense, we get a contradiction.

(2 \Rightarrow 13). Let $X := \{x \in \mathbb{K}^d : \dim A_x > 0\}$, where $d := \dim A$, and assume, for contradiction, that $\dim X > 0$. If $d = 0$, then $X = \emptyset \subset \mathbb{K}^0 = \{0\}$, and we have a contradiction. Thus, w.l.o.g., A is closed (because $\dim \overline{A} = \dim A$), and $Y := \Pi_{d+1}^n(A)$ satisfies

$$\forall x \in X \quad \dim(Y_x) > 0.$$

By Kuratowski-Ulam's theorem, this implies that Y is not meager, and thus has non-empty interior, contradicting $\dim A = d$.

(2 \Rightarrow 12). Let $A_1, A_2 \subseteq \mathbb{K}$ be definable, such that $\dim A_i < d$, $i = 1, 2$. We have to prove that $\dim(A_1 \cup A_2) < d$. Assume, for contradiction, that $B := \Pi_d^n(A_1 \cup A_2)$ has non-empty interior. Let $B_i := \Pi_d^n(A_i)$; notice that $\dim B_i < d$, and $B_1 \cup B_2 = B$. By (2), the B_i are nowhere dense in \mathbb{K}^d ; thus, B is nowhere dense, absurd.

(12 \Rightarrow 11) is obvious.

(11 \Rightarrow 4). Let $A \subseteq \mathbb{K}$ be definable. We have to prove that $\dim(\overline{A}) = \dim(A)$. However, $\overline{A} = A \cup \partial A$. Since ∂A has empty interior, $\dim \partial A = 0$, and we are done.

(13 \Rightarrow 11). Let A, B be definable subsets of \mathbb{K} with empty interior. We have to prove that $A \cup B$ has also empty interior. Define $X := ((0, 1) \times A) \cup (\partial B \times (0, 1)) \subset \mathbb{K}^2$. Notice that $\dim X = 1$. By 13, the set $Y := \{y \in \mathbb{K} : \dim(X_y) > 0\}$ is nowhere dense. However, $Y = A \cup B$; thus, $\dim(A \cup B) = 0$.

(2 \Rightarrow II). Let $(A_x)_{x \in \mathbb{K}}$ be an increasing definable family of subsets of \mathbb{K}^n , each of them of dimension less or equal to d . Let $A := \bigcup_x A_x$. Assume, for contradiction, that $\dim A > d$; w.l.o.g., $U := \Pi_{d+1}^n(A)$ has non-empty interior. However, $U = \bigcup_x \Pi_{d+1}^n(A_x)$. Since $\dim A_x \leq d$, each $\Pi_{d+1}^n(A_x)$ is nowhere dense, and thus U is meager, contradicting the fact that \mathbb{K} is Baire.

(14 \Rightarrow 13) is obvious.

(13 \Rightarrow 14). Assume, for contradiction, that $\dim C > k$; w.l.o.g., $U := \Pi_{k+1}^n(C)$ has non-empty interior. Moreover, since $\dim(A) = \dim(\overline{A})$, by property (II), w.l.o.g. A is d-compact. By 12, w.l.o.g. the set

$$C' := \{x \in \mathbb{K}^n : \dim(\Pi_d^n(A_x)) \geq d\}$$

has dimension greater than k , and $D' := \Pi_{k+1}^n(C')$ has non-empty interior. Let $B := \Pi_{d+k+1}^{n+m}(A)$; by assumption, B is nowhere dense. Hence, by Kuratowski-Ulam's theorem, the set

$$D := \{u \in \mathbb{K}^{k+1} : \dim(B_u) \geq d\}$$

has empty interior. However, for every $u \in \mathbb{K}^{k+1}$, $B_u = \Pi_d(A_u)$, and thus $D' \subseteq D$, absurd.

(1 \Rightarrow III) and (1 \Rightarrow IV) have the same proof as [Mil05, Thm. 3.3]. \square

9 Definable choice

As usual, \mathbb{K} is some definably complete expansion of an ordered field.

Some version of the following lemma has been proved by C. Miller.

Lemma 9.1 (Definable Choice). *1. Let $P \subseteq \mathbb{K}$ be a set of parameters, and $A \subset \mathbb{K}^n$ be P -definable, non-empty and constructible. Then, there exists a P -definable point $a \in A$.*

2. Let $X \subseteq \mathbb{K}^{n+m}$ be definable and such that X_b is constructible for every $b \in \mathbb{K}^m$. Then, X has a definable n -choice function, that is a definable function $f : \Pi_m^{n+m}(X) \rightarrow X$, such that $f(a) \in X_a$ for every $a \in \Pi_m^{n+m}(X)$.

3. Let $X \subseteq \mathbb{K}^{n+m}$ be definable and constructible. Then, X has a definable n -choice function.

*4. Suppose that every unary definable set contains a locally closed point. Then, \mathbb{K} has definable Skolem functions (**DSF**).*

Proof. (1). Since A is constructible, $A' := \text{lc}(A)$ is non-empty; thus, since A' is also P -definable, it suffices to prove the conclusion for A' ; therefore, w.l.o.g. A is locally closed.

First, we will prove the case when A is closed in \mathbb{K}^n . For every $r > 0$, let $A(r) := \{a \in A : |a| \leq r\}$, let $r_0 := \inf\{r \in \mathbb{K} : A(r) \neq \emptyset\}$, and let $A' := A(2r_0)$. Notice that A' is d-compact, non-empty and P -definable. Let $a := \text{lexmin}(A')$: notice that a is also P -definable, and in A .

Next, we will treat the case when A is discrete. For every $r > 0$, let $A(r) := \{a \in A : B(a, r) \cap A = \{a\}\}$, let $r_0 := \inf\{r \in \mathbb{K} : A(r) \neq \emptyset\}$, and let $A' := A(2r_0)$. Notice that A' is closed, non-empty, and P -definable, and apply the previous case.

We will prove the general case by induction on n . If $n = 1$, for every $a \in A$ let $r(a) := \sup\{r > 0 : A \cap B(a, r) = \overline{A} \cap B(a, r)\}$. Since A is locally closed, $r(a) > 0$ for every $a \in A$. Let $U := \bigcup_{a \in A} B(a, r(a)/2)$. Notice that U is open, P -definable, and $A = \overline{A} \cap U$. By the “open set” case, there exists $u \in U$ P -definable. Let I be the definably connected component of U containing u : I exists because $n = 1$, and it is a P -definable open interval; let $A' := A \cap I$. Therefore, A' is closed in I , P -definable and non-empty; since I is definably homeomorphic to \mathbb{K} , via a P -definable map, we can apply the “closed set” case to find a P -definable point in A' .

If $n > 1$, we proceed by induction, and we assume we have already proved the conclusion for $n - 1$. let $A' := \Pi_{n-1}^n(A)$. By the inductive hypothesis, there exists $a' \in A'$ P -definable. By the case $n = 1$, there exists $a'' \in A_{a'}$ also P -definable. Let $A := (a', a'')$.

(2). The construction in (1) gives a definable way to choose $x_b \in X_b$ for every $b \in \Pi_m^{n+m}(X)$.

(3) is immediate from (2).

(4). Let $A \subseteq \mathbb{K}^{n+m}$ be definable. We have to prove that there exists $f : B \rightarrow A$ definable, such that $f(b) \in A_b$ for every $b \in B$, where $B := \Pi_m^{n+m}(A)$. We proceed by induction on n .

If $n = 0$, f is the identity. If $n = 1$, for every $b \in B$ let $C_b := \text{lc}(A_b)$. By hypothesis, $C_b \neq \emptyset$ for every $b \in B$, and it is constructible. Thus, by (2), C has a definable 1-choice function, and the same function will work for A .

Assume that $n > 1$ and we have already proved the conclusion for every $n' < n$. Let $C := \Pi_{m+1}^{n+m}(A)$. By inductive hypothesis, there exists a definable $(n-1)$ -choice function $g : C \rightarrow A$. By the case $n = 1$, there exists a definable 1-choice function $h : \Pi_m^{m+1}(C) \rightarrow C$. Let $f := g \circ h : \Pi_m^{n+m}(A) \rightarrow A$: f is an n -choice function for A . \square

Therefore, locally o-minimal, d-minimal and constructible structures have DSF. On the other hand, a-minimal structures might not have DSF: for instance, $(\mathbb{R}, \mathbb{R}^{\text{alg}})$ does not have DSF [DMS10, 5.4].

For the same reason, constructible structure have elimination of imaginaries.

Lemma 9.2. *Let $X \subseteq \mathbb{K}^n$ be definable. Assume that X is both and \mathcal{F}_σ and a \mathcal{G}_δ , and that \overline{X} is Baire. Then, $\text{lc}(X) \neq \emptyset$.*

Cf. [Kur66, §34.VI].

Proof. Let $Y := \overline{X}$. We have to prove that the interior of X inside Y is non-empty. Otherwise, X is both dense and co-dense in Y . However, since X is an \mathcal{F}_σ , this implies that X is meager in Y . For the same reason, $Y \setminus X$ is meager in Y , contradicting the fact that Y is Baire. \square

9.1 Sard's Lemma and dimension inequalities

For every \mathcal{C}^1 function $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$, define $\Lambda_f(k) := \{x \in \mathbb{K}^n : \text{rk}(Df(x)) \leq k\}$, and $\Sigma_f(k) := f(\Lambda_f(k))$. The set of singular values of f is $\Sigma_f := \bigcup_{k=0}^{n-1} \Sigma_f(k)$.

Lemma 9.3 (Sard's Lemma). *Let $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$ be definable and \mathcal{C}^1 . If \mathbb{K} is i -minimal and has DSF (and, in particular, if \mathbb{K} is constructible), then $\dim(\Sigma_f(d)) \leq d$.*

Proof. If $d \geq n$, the conclusion is trivial. Let $d < n$, and assume, for contradiction, that $\Pi_d^n(\Sigma_f(d))$ contains a non-empty open box B . By DSF, there exists $g : B \rightarrow \Lambda_f(d)$, such that $\Pi_d^n \circ f \circ g = 1_B$. Since \mathbb{K} is i -minimal, we can apply Lemma 8.9, and, by shrinking B if necessary, we can assume that g is \mathcal{C}^1 . Hence, by differentiation, we have that, for every $x \in B$, $\Pi_d^n(f(g(x))) \cdot (Df)(g(x)) \cdot Dg(x) = \mathbf{1}_d$. Therefore, $\text{rk}((Df)(g(x))) \geq d$, a contradiction. \square

Lemma 9.4. *Assume that \mathbb{K} is i -minimal with DSF. Let l, m, n, d be natural numbers. Let $X \subseteq \mathbb{K}^n$ be definable, such that $\dim X \geq m + d$, $Y \subseteq \mathbb{K}^l$ be definable of dimension m , and $f : X \rightarrow Y$ definable, Then there exists $a \in Y$ such that $\dim(f^{-1}(a)) \geq d$.*

Proof. By induction on (l, m, n, d) .

- If $m + d < n$, then w.l.o.g. $\pi(X)$ contains a non-empty open box B , where $\pi := \Pi_{m+d}^n$. By DSF, there exists a definable function $g : B \rightarrow X$ such that $g \circ \pi = \text{id}_B$. Let $\tilde{f} := f \circ g : B \rightarrow Y$. By induction on n , there exists $a \in Y$ such that $\dim(\tilde{f}^{-1}(a)) \geq d$. Therefore, w.l.o.g. $m + d = n$ and $X = \mathbb{K}^n$. Moreover, by Theorem 8.4(IV), w.l.o.g. f is continuous.

- If $m = 0$, since f is continuous, $f(\mathbb{K}^n)$ is a connected subset of the 0-dimensional set Y , and therefore $f(\mathbb{K}^n) = \{a\}$ for some $a \in Y$.
- If $d = 0$, there is nothing to prove. Thus, w.l.o.g. $m > 0$ and $d > 0$.
- If $l > m$, then, by the Partition Lemma, w.l.o.g. Y is μ -good, where $\mu := \Pi_m^l$. Let $g := \mu \circ f : \mathbb{K}^n \rightarrow \mathbb{K}^m$. By induction on l , there exists $b \in \mathbb{K}^m$ such that $\dim(\tilde{g}^{-1}(b)) \geq d$. Moreover, since Y is μ -good, $\dim(Y_b) = 0$. Let $\tilde{X} := \upharpoonright g^{-1}(b)$ and $\tilde{f} = f \upharpoonright g^{-1}(b) : \tilde{X} \rightarrow Y_{[b]}$. By the case $m = 0$, there exists $a \in Y_{[b]}$, such that $\dim(\tilde{f}^{-1}(a)) \geq d$. Hence, w.l.o.g. $l = m$ and $Y = \mathbb{K}^m$.
- Hence, we are reduced to the situation $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$, with $n = m + d$. By Lemma 8.9, w.l.o.g. f is \mathcal{C}^1 . Let $Y' := f(\mathbb{K}^n)$ and $\Sigma_f \subseteq Y$ be the set of singular values for f . If, for contradiction, $\dim(f^{-1}(a)) < d$ for every $a \in \mathbb{K}^m$, then $Y' = \Sigma_f$. Thus by Sard's Lemma, $\dim Y' < m$. Therefore, by induction on m , there exists $a \in Y'$ such that $\dim(f^{-1}(a)) \geq d$, a contradiction. \square

Lemma 9.5. *Let \mathbb{K} be i -minimal with DSF, $X \subseteq \mathbb{K}^n$, $Y \subseteq \mathbb{K}^m$, and $f : X \rightarrow Y$ be definable. Then:*

1. *if f is surjective, then $\dim Y \leq \dim X$;*
2. *if f is injective, then $\dim Y \geq \dim X$;*
3. *if f is bijective, then $\dim Y = \dim X$.*

Proof. 1) Assume, for contradiction, that $k := \dim X < d := \dim Y$. By DSF, there exists a definable function $g : Y \rightarrow X$, such that $f \circ g = \text{id}_Y$. By Lemma 9.4, there exists $a \in X$ such that $\dim(g^{-1}(a)) \geq d - k \geq 1$, contradicting the fact that g is injective.

2) follows from 1) and definable choice, and 3) follows from 1) and 2). \square

10 D-minimal structures

Definition 10.1. \mathbb{K} is d-minimal if for every $\mathbb{K}' \equiv \mathbb{K}$, every definable subset of \mathbb{K} is the union of an open set and finitely many discrete sets.

Remark 10.2. Let $A \subseteq \mathbb{K}^n$ be definable. If A is a union of N discrete sets, then A is a union of N definable and discrete sets.

Remark 10.3. Every d-minimal structure is constructible.

Definition 10.4. Let $d \leq n \in \mathbb{N}$, $\Pi(n, d)$ be the set of projections from \mathbb{K}^n onto d -dimensional coordinate spaces, and $\mu \in \Pi(n, d)$. For every $p \in \mathbb{N}$, let $\text{reg}_\mu^p(A)$ and $\text{reg}^p(A)$ be defined as in [Mil05, Def. 8.4 and §3.4]. As in the case when \mathbb{K} is an expansion of \mathbb{R} , $\text{reg}_\mu^p(A)$ is definable, open in A , and a \mathcal{C}^p -submanifold of \mathbb{K}^n of dimension d .

For every $A \subseteq \mathbb{K}^n$, let $\text{isol}(A)$ be the set of isolated points of A . Notice that $\text{isol}(A)$ is discrete.

Lemma 10.5. *Suppose that \mathbb{K} is i -minimal, and let $A \subseteq \mathbb{K}^{m+n}$ be definable, such that $B := \{x \in \mathbb{K}^m : \text{isol}(A_x) \neq \emptyset\}$ has interior. Then, for every $p \in \mathbb{N}$, $\text{reg}_\pi^p(A) \neq \emptyset$, where $\pi := \Pi_m^{n+m}$.*

Proof. Fix $p \in \mathbb{N}$; let $V \subseteq B$ be a non-empty open box, and $C := \bigsqcup_{v \in V} (\{v\} \times \text{isol}(A_v))$. Notice that $V \subseteq \pi(C)$. By Definable Choice, there exists a definable function $f : V \rightarrow \mathbb{K}^n$ such that $\langle x, f(x) \rangle \in C$ for every $x \in V$. For every $x \in V$, define

$$\begin{aligned} f^+(x) &:= \min(f(x) + 1, \inf\{y \in A_x : y > f(x)\}), \\ f^-(x) &:= \max(f(x) - 1, \sup\{y \in A_x : y < f(x)\}). \end{aligned}$$

Notice that $f^- < f < f^+$ on all V . By i -minimality, after shrinking V , we can assume that f, f^+ and f^- are \mathcal{C}^p on V . It is easy to see that $\Gamma(f) \subseteq \text{reg}_\pi^p(A)$. \square

Lemma 10.6. *Suppose that every 0-dimensional definable subset of \mathbb{K} has an isolated point, and let $p \in \mathbb{N}$. Then:*

1. \mathbb{K} is i -minimal.
2. Let \mathcal{A} be a finite collection of definable subsets of \mathbb{K}^n . Then, there is a Π -good partition \mathcal{P} of \mathcal{A} , compatible with \mathcal{A} , such that $P \setminus \text{reg}_\mu^0(P)$ is nowhere dense in P for every projection μ and every $P \in \mathcal{P}$ such that P is μ -good.
3. $A \setminus \text{reg}^p(A)$ is nowhere dense in A , for every definable set A .

Proof. (1). Let $X \subseteq \mathbb{K}$ be definable and with empty interior. Suppose, for contradiction, that \overline{X} contains a non-empty open interval I , and let $Y := X \cap I$. Notice that $\dim Y = 0$ and Y is dense in I , and therefore it has no isolated points.

(2) and (3) have the same proof as [Mil05, Prop. 8.4] (using Thm. 8.4 and Lemma 10.5). \square

Lemma 10.7. *The following are equivalent:*

1. \mathbb{K} is d -minimal;
2. for every $\mathbb{K}' \equiv \mathbb{K}$, every subset of \mathbb{K} is the union of a definable open set and finitely many definable discrete sets;
3. for every $m \in \mathbb{N}$ and every definable $A \subseteq \mathbb{K}^{m+1}$ there exists $N \in \mathbb{N}$ such that, for all $x \in \mathbb{K}^m$, either A_x has interior or is a union of N definable discrete sets;
4. for every $m, n \in \mathbb{N}$ and definable $A \subseteq \mathbb{K}^{n+m}$ there exists $N \in \mathbb{N}$ such that for every $x \in \mathbb{K}^m$, either $\dim A_x > 0$, or A_x is a union of N definable discrete sets.

Proof. (1 \Leftrightarrow 2) follows from Remark 10.2.

(2 \Leftrightarrow 3) is a routine compactness argument.

(3) is the case $n = 1$ of (4).

(3 \Rightarrow 4). Induction on n . The case $n = 1$ is the hypothesis. Let $n > 1$, and assume we have already proved (3) for each $n' < n$. Let $C := \{x \in \mathbb{K}^m : \dim(A_x) = 0\}$. Let $x \in C$. For each $y \in K$, we have $\dim(A_{(x,y)}) = 0$, and therefore $A_{(x,y)}$ is a union of N discrete definable sets (for some N

independent from x and y). Moreover, for each $x \in C$, the set $D(x) := \{y \in \mathbb{K} : A_{(x,y)} \neq \emptyset\}$ has empty interior, because \mathbb{K} is i -minimal. Thus, for every $x \in C$, $D(x)$ is a union of M definable discrete sets (for some M independent from x). Hence, for every $x \in C$, A_x is a union of NM discrete sets, which can be taken definable. \square

Definition 10.8 (Cantor-Bendixson Rank). Let T be a Hausdorff topological space. For every $X \subseteq T$, and every ordinal α , let $X^{(0)} := X$, $X^{(\alpha+1)}$ be the set of non-isolated points of $X^{(\alpha)}$, and $X^{(\alpha)} := \bigcap_{\beta < \alpha} X^{(\beta)}$ if α is a limit ordinal. Let $\text{rk}^{CB}(X)$, the Cantor-Bendixson rank of X , be the smallest ordinal α such that $X^{(\alpha)} = \emptyset$ (or $\text{rk}^{CB}(X) = +\infty$ if such α does not exist). For every $a \in X$, let $\text{rk}_X^{CB}(a)$ be the supremum of ordinals α such that $a \in X^{(\alpha)}$.

Notice that each $X^{(\alpha)} \setminus X^{(\alpha+1)}$ is discrete, and that X is a finite union of discrete sets iff $\text{rk}^{CB}(X) < \omega$. Moreover, $X^{(\text{rk}^{CB}(X))} = \emptyset$ (if $\text{rk}^{CB}(X) < +\infty$). Besides, $\text{rk}^{CB}(X) = 0$ iff X is empty, $\text{rk}^{CB}(X) = 1$ iff X is discrete (and non-empty).

\mathbb{K} is a -minimal iff every definable subset of \mathbb{K} of dimension 0 has rank at most 1 (to be proved).

Remark 10.9. If X and X' are subsets of a Hausdorff topological space T , then $\text{rk}^{CB}(X \cup X') \leq \text{rk}^{CB}(X) \oplus \text{rk}^{CB}(X')$, where \oplus is the Cantor sum of ordinals. Hence, a set X is a union of n discrete sets iff $\text{rk}^{CB}(X) \leq n$.

Proof. By induction on $\alpha := \text{rk}^{CB}(X)$ and $\beta := \text{rk}^{CB}(Y)$. The basic case when X is a singleton is obvious. For contradiction, let $a \in (X \cup Y)^{(\alpha+\beta)}$. W.l.o.g., $a \in X$; let $\gamma := \text{rk}_X^{CB}(a) < \alpha$.

Let V be an open neighbourhood of a such that $X^{(\gamma)} \cap V = \{a\}$, and $X' := X \cap V \setminus \{a\}$. Notice that $\alpha' := \text{rk}^{CB}(X') \leq \gamma < \alpha$. Hence, by inductive hypothesis, $\text{rk}^{CB}(X' \cup Y) \leq \alpha' \oplus \beta < \alpha \oplus \beta$. Thus, $\text{rk}^{CB}(X' \cup Y \cup \{a\}) \leq \alpha' \oplus \beta \oplus 1 \leq \alpha \oplus \beta$. Thus, $\text{rk}_{X \cup Y}^{CB}(a) < \alpha \oplus \beta$ for every $a \in X \cup Y$, and we are done. \square

Corollary 10.10. *If \mathbb{K} d -minimal iff for every $n \in \mathbb{N}$ and every definable $X \subseteq \mathbb{K}^{n+1}$, there exists $d \in \mathbb{N}$ such that, for all $a \in \mathbb{K}^n$, $\text{rk}^{CB}(X_a \setminus \text{int}(X_a)) \leq d$.*

Definition 10.11. A definable d -dimensional \mathcal{C}^p -submanifold M is weakly μ -special if for each $y \in \mu M$ and $x \in M_y$ there exist $U \subseteq \mathbb{K}^d$ open box around y and $W \subseteq \mathbb{K}^{n-d}$ open box around x , such that $A \cap (U \times W) = \Gamma(f)$ for some (definable) \mathcal{C}^p -map $f : U \rightarrow W$. M is μ -special if the box U in the above definition does *not* depend on x (but only on y). M is special if it is μ -special for some $\mu \in \Pi(n, d)$.

Lemma 10.12. *Assume \mathbb{K} is d -minimal, let $p \in \mathbb{N}$, and \mathcal{A} be a finite collection of definable subsets of \mathbb{K}^n . Then, there exists a finite partition of \mathbb{K}^n into weakly special definable \mathcal{C}^p -submanifolds compatible with \mathcal{A} .*

Proof. As in the proof of [Mil05, Thm. 3.4.1], we are reduced to show that if $A \in \mathcal{A}$, with $0 < d := \dim A < n$; then A can be partitioned into special \mathcal{C}^p -manifolds. Moreover, we can further assume that A is a π -good, where $\pi := \Pi_d^n$, that $A \setminus \text{reg}_\pi^p(A)$ is nowhere dense in A , and each A_x is discrete, for every $x \in \mathbb{K}^d$.

Let $M := \text{reg}_\pi^p(A)$. Notice that M is a weakly π -special \mathcal{C}^p -manifold. It suffices to prove that $\pi(A \setminus M)$ is nowhere dense (in \mathbb{K}^d) to conclude the proof

(since then $\text{fdim}(A \setminus M) < \text{fdim}(A)$). Assume, for contradiction, that $B \subset \pi(A \setminus M)$ is a non-empty open box, and let $N := \text{reg}_\pi^p(A \setminus M)$. By shrinking B , we might assume that $\text{cl}(M_x) = (\text{cl } M)_x$ for every $x \in B$. By Lemma 10.6, M is dense in A , and therefore, for every $x \in B$, $\text{cl}(A_x) = (\text{cl } A)_x = (\text{cl } M)_x = \text{cl}(M_x)$, that is M_x is dense in A_x . However, A_x is discrete, and thus $A_x = M_x$. \square

Conjecture 10.13. *Assume \mathbb{K} is d -minimal, let $p \in \mathbb{N}$, and \mathcal{A} be a finite collection of definable subsets of \mathbb{K}^n . Then, there exists a finite partition of \mathbb{K}^n into special definable C^p -submanifolds compatible with \mathcal{A} .*

11 Dense pairs

11.1 Cauchy completion

For particular cases of some of the results in this section, see also [LS95].

Definition 11.1. If $\overline{\mathbb{K}}$ is an ordered field, we denote by $\overline{\mathbb{K}}^C$ the **Cauchy completion** of $\overline{\mathbb{K}}$, that is, the maximal linearly ordered group such that $\overline{\mathbb{K}}$ is dense in $\overline{\mathbb{K}}^C$; notice that $\overline{\mathbb{K}}^C$ is, in a canonical way, a real closed field [Sco69].

A **cut** $\Lambda := (\Lambda^L, \Lambda^R)$ of \mathbb{K} is a partition of \mathbb{K} into two disjoint subsets Λ^L, Λ^R , such that $y < x$ for every $y \in \Lambda^L$ and $x \in \Lambda^R$. A cut (Λ^L, Λ^R) is a **gap** if Λ^L is non-empty and has no maximum, and Λ^R is non-empty and has no minimum. A cut Λ is **regular** if $\hat{\Lambda} = 0$, where $\hat{\Lambda} := \inf\{x - y : x, y \in \mathbb{K} \text{ \& } y < \Lambda < x\}$.

The Cauchy completion of \mathbb{K} is the disjoint union of \mathbb{K} and the set of regular gaps of \mathbb{K} , with suitably defined order $<$ and operations $+$ and \cdot .

Lemma 11.2. *Let \mathbb{K} be definably complete, $\mathbb{K}^* \succ \mathbb{K}$, $b \in \mathbb{K}^* \setminus \mathbb{K}$. For every $X \subseteq \mathbb{K}^m$ definable, let $X^* \subseteq \mathbb{K}^{*m}$ be the interpretation of X in \mathbb{K}^* .*

Let $E \subset \mathbb{K}$ be closed, \mathbb{K} -definable, and with $n := \text{rk}^{CB}(E) < \omega$, let Λ be the cut of \mathbb{K} determined by b . If Λ is regular, then $b \notin E^$.*

If moreover \mathbb{K} is d -minimal, and $D \subset \mathbb{K}$ is \mathbb{K} -definable and of dimension 0, then $b \notin D$.

Proof. We will prove the lemma by induction on $n := \text{rk}^{CB}(E)$. W.l.o.g., E is bounded (because b is \mathbb{K} -bounded). If $n = 0$, then $E = \emptyset$. If $n = 1$, then E is discrete; thus, E is pseudo-finite; let $\delta := \delta(E)$; notice that $0 < \delta \in \mathbb{K}$. Since Λ is regular, there exist $a', a'' \in \mathbb{K}$ such that $a' < b < a''$ and $a'' - a' < \delta$. Thus, $(a', a'')^* \cap E^* = \{b\}$, and therefore b is \mathbb{K} -definable, absurd.

If $n > 1$, let $G := E^{(n-1)}$; notice that G is closed, discrete and non-empty; thus, $\text{rk}^{CB}(G) = 1$, and G is pseudo-finite. By the case $n = 1$, $b \notin G^*$; let $a', a'' \in \mathbb{K}$ such that $a' < b < a''$ and $G \cap [a', a''] = \emptyset$ (a' and a'' exist by the proof of the case $n = 1$). Let $F := E \cap [a', a'']$. Notice that F is d -compact. Moreover, $\text{rk}^{CB}(F) < n$. Therefore, by inductive hypothesis, $b \notin F^*$. Hence, $b \notin E^*$.

If \mathbb{K} is d -minimal, let E be the closure of D in \mathbb{K} . By d -minimality, $\text{rk}^{CB}(D) < \omega$; thus, $b \notin D^*$. \square

From the proof of the above lemma, we can deduce the following.

Lemma 11.3. *Let \mathbb{K} be d -minimal, and $\mathbb{K} \prec \mathbb{K}^*$. Let $c \in \mathbb{K}^* \setminus \mathbb{K}$, Λ be the cut determined by c over \mathbb{K} , and $X^* \subseteq \mathbb{K}^*$ be \mathbb{K} -definable. If Λ is a regular gap and $c \in X^*$, then there exists an open interval I with end-points in \mathbb{K} such that $c \in I \subseteq X^*$.*

Lemma 11.4. *Let \mathbb{K} be definably complete, $\mathbb{K}^* \succ \mathbb{K}$, $b \in \mathbb{K}^* \setminus \mathbb{K}$, Λ be the cut of \mathbb{K} determined by b , $f : \mathbb{K} \rightarrow \mathbb{K}$ be definable, $c := f^*(b) \in \mathbb{K}^*$, and Γ be the cut determined by c over \mathbb{K} . Assume that f is continuous and strictly monotone. Then, Γ is a regular gap, and $\mathbb{K}\langle c \rangle = \mathbb{K}\langle b \rangle$, where $\mathbb{K}\langle b \rangle$ is the definable closure of $\mathbb{K} \cup \{b\}$.*

Proof. Let $0 < \varepsilon \in \mathbb{K}$; by uniform continuity, there exists $0 < \delta \in \mathbb{K}$, such that, if $x, y \in \mathbb{K}$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let $y', y'' \in \mathbb{K}$ such that $y' < b < y''$ and $y'' - y' < \delta$ (they exist because Λ is a regular cut). Thus, $f(y') < c < f(y'')$ and $f(y'') - f(y') < \varepsilon$, and therefore Γ is regular. Moreover, f is invertible, and therefore $b = f^{-1}(c) \in \mathbb{K}\langle c \rangle$. \square

Lemma 11.5. *Let \mathbb{K} be a d -minimal structure, and $\mathbb{K}^* \succ \mathbb{K}$, such that \mathbb{K} is dense in \mathbb{K}^* . Then, \mathbb{K}^* has the exchange property relative to \mathbb{K} ; that is, if $A \subset \mathbb{K}^*$, and $b, c \in \mathbb{K}^*$ satisfy $c \in \mathbb{K}\langle A, b \rangle \setminus \mathbb{K}\langle A \rangle$, then $b \in \mathbb{K}\langle A, c \rangle$.*

Proof. Let b and c be as in the hypothesis. Let $\mathbb{K}' := \mathbb{K}\langle A \rangle$; w.l.o.g., $\mathbb{K} = \mathbb{K}'$. Then, since \mathbb{K} has definable Skolem function, $b = f^*(c)$ for some K -definable $f : \mathbb{K} \rightarrow \mathbb{K}$. By Theorem 8.4-III there exists $D \subseteq \mathbb{K}$ nowhere dense and K -definable, such that f is continuous and either constant or strictly monotone on each sub-interval of $\mathbb{K} \setminus D$. By Lemma 11.2, $b \notin D^*$. Therefore, by Lemma 11.4, either $c \in \mathbb{K}$, or $b \in \mathbb{K}\langle c \rangle$. \square

Notice that the hypothesis that \mathbb{K} is dense in \mathbb{K}^* in the above lemma is necessary: [DMS10, 1.17] shows that if \mathbb{K}^* does not satisfy UF and it is sufficiently saturated, then \mathbb{K}^* does not satisfy the exchange property.

Proposition 11.6 (Cauchy completion). *Let \mathbb{K} be a d -minimal structure, expanding the ordered field $\overline{\mathbb{K}}$, $L = (0, 1, +, \cdot, <, \dots)$ be the language of \mathbb{K} . There exists a unique expansion of the Cauchy completion $\overline{\mathbb{K}}^C$ to an L -structure \mathbb{K}^C , such that \mathbb{K} is an elementary substructure of \mathbb{K}^C .*

Proof. Let \mathcal{S} be set of elementary extensions \mathbb{K}' of \mathbb{K} , such that \mathbb{K} is a dense in \mathbb{K}' ; order \mathcal{S} by elementary inclusion. Let \mathbb{K}' be a maximal element of \mathcal{S} (\mathbb{K}' exists by Zorn's lemma). We claim that $\mathbb{K}' = \mathbb{K}^C$ (and, thus, \mathbb{K}^C can be expanded to an L -structure).

Suppose not. W.l.o.g., $\mathbb{K}' = \mathbb{K}$. Let Λ be a regular gap of \mathbb{K} , \mathbb{K}^* be an elementary extension of \mathbb{K} , and $b \in \mathbb{K}^*$ filling the gap Λ . Let $\mathbb{K}\langle b \rangle$ be the definable closure of $\mathbb{K} \cup \{b\}$ in \mathbb{K}^* ; since \mathbb{K} has definable Skolem functions (by definable choice), $\mathbb{K} \prec \mathbb{K}\langle b \rangle \preceq \mathbb{K}^*$.

We claim that \mathbb{K} is dense in $\mathbb{K}\langle b \rangle$, contradicting the maximality of \mathbb{K} . In fact, let $c \in \mathbb{K}\langle b \rangle$; thus, $c = f(b)$ for some \mathbb{K} -definable $f : \mathbb{K}^* \rightarrow \mathbb{K}^*$. We have to prove that either $c \in \mathbb{K}$, or that Γ is a regular gap, where Γ is the cut determined by c over \mathbb{K} . By Theorem 8.4-III there exists $D \subseteq \mathbb{K}$ nowhere dense and definable, such that f is continuous and either constant or strictly monotone on each sub-interval of $\mathbb{K} \setminus D$. By Lemma 11.2, $b \notin D^*$. Therefore, by Lemma 11.4, either $c \in \mathbb{K}$, or Γ is a regular gap.

It remains to prove that the L -structure on $\overline{\mathbb{K}}^C$ is unique. Let \mathbb{K}_1^C and \mathbb{K}_2^C be two expansion of $\overline{\mathbb{K}}^C$ to elementary extensions of \mathbb{K} . Let \mathbb{K}' be a maximal common elementary substructure of \mathbb{K}_1^C and \mathbb{K}_2^C extending \mathbb{K} . Assume, for contradiction, that $\mathbb{K}' \neq \mathbb{K}^C$; w.l.o.g., we can assume that $\mathbb{K} = \mathbb{K}'$. Let $b \in \mathbb{K}^C \setminus \mathbb{K}$, and let $\phi(x)$ be an L -formula with parameters in \mathbb{K} . In order to reach a contradiction, we must prove that $\mathbb{K}_1^C \models \phi(b)$ iff $\mathbb{K}_2^C \models \phi(b)$. W.l.o.g., $\mathbb{K}_1^C \models \phi(b)$; let $X := \phi(\mathbb{K})$. Moreover, for every $Y \subseteq \mathbb{K}^n$ definable, let Y_i^C be the interpretation of Y in \mathbb{K}_i^C , for $i = 1, 2$. Notice that $X = U \cap D$, where $U := \overset{\circ}{X}$ is open and definable, and $D := X \setminus \overset{\circ}{X}$ is definable, with $\dim D = 0$. Since \mathbb{K} is d -minimal, $\text{rk}^{CB}(D) < \omega$. Thus, by Lemma 11.2, $b \notin D_1^C$. Hence, $b \in U_1^C$; since \mathbb{K} is dense in \mathbb{K}_1^C , and U_1^C is open, there exist $y' < y'' \in \mathbb{K}$ such that $b \in (y', y'')_1^C \subseteq U_1^C$. Since $\mathbb{K} \preceq \mathbb{K}_1^C$, $(y', y'') \subseteq U$, and since $\mathbb{K} \preceq \mathbb{K}_2^C$, $b \in (y', y'')_2^C \subseteq U_2^C \subseteq X_2^C$. \square

Corollary 11.7 (of the proof). *Let \mathbb{K} be a d -minimal structure, $\mathbb{K}^* \succ \mathbb{K}$, $b \in \mathbb{K}^* \setminus \mathbb{K}$, and Λ be the cut of \mathbb{K} determined by b . If Λ is a regular gap, then it uniquely determines the type of b over \mathbb{K} ; moreover, \mathbb{K} is dense in $\mathbb{K}\langle b \rangle$; besides, for every $c \in \mathbb{K}\langle b \rangle \setminus \mathbb{K}$, we have $\mathbb{K}\langle c \rangle = \mathbb{K}\langle b \rangle$.*

Corollary 11.8. *An Archimedean locally o -minimal structure is o -minimal*

Proof. If \mathbb{K} is Archimedean, then $\overline{\mathbb{K}}^C = \overline{\mathbb{R}}$; thus, \mathbb{K} has an elementary extension $\tilde{\mathbb{R}}$ that is an expansion of \mathbb{R} . Therefore, $\tilde{\mathbb{R}}$ is o -minimal, and thus \mathbb{K} is o -minimal. \square

11.1.1 Polish structures and theories

Lemma 11.9. *Let \mathbb{F} be an ordered field. Assume that \mathbb{F} contains a countable dense subset (not necessarily definable) and that \mathbb{F} is Cauchy complete. Then:*

1. \mathbb{F} has cofinality ω .
2. \mathbb{F} is a Polish space (i.e., a Cauchy complete separable metric space).
3. $|\mathbb{F}| = 2^{\aleph_0}$.
4. If $X \subset \mathbb{F}^n$ is perfect, non-empty, and a \mathcal{G}_δ (in the topological sense), then $|X| = 2^{\aleph_0}$.
5. If $X \subset \mathbb{F}^n$ is closed and $|X| < 2^{\aleph_0}$, then $\text{isol}(X)$, the set of isolated points of X , is dense in X .
6. If $X \subseteq \mathbb{F}^n$ is a non-empty \mathcal{G}_δ (in the topological sense), then it is a Baire space (again, in the topological sense), and it is even strong Choquet [Kec95].

Proof. (1) is obvious.

(2) requires us to define a metric. If \mathbb{F} is Archimedean, then \mathbb{F} is homeomorphic to the reals, and we are done. Otherwise, let v be the natural valuation on \mathbb{F} induced by the ordering, and G be the value group of \mathbb{F} .

Claim 14. The topology induced by v on \mathbb{F} is the same as the order topology.

Notice that the claim is false if \mathbb{F} is Archimedean.

Claim 15. G is countable.

Thus, there exists a coinital order-reversing embedding ι of $(G, >)$ in $(\mathbb{R}_+, <)$ as ordered sets (notice that ι ignores the group structure). For every $x, y \in \mathbb{F}$, define

$$d(x, y) := \begin{cases} \iota(v(x - y)) & \text{if } x \neq y; \\ 0 & \text{otherwise.} \end{cases}$$

Claim 16. (\mathbb{F}, d) is a metric space.

Actually, (\mathbb{F}, d) satisfies the ultra-metric inequality.

Claim 17. (\mathbb{F}, d) is homeomorphic to (\mathbb{F}, v) (and hence to $(\mathbb{F}, <)$).

Claim 18. If $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in (\mathbb{F}, d) , then $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in (\mathbb{F}, v) .

Hence, assume that $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequences in (\mathbb{F}, d) . Since (\mathbb{F}, v) is Cauchy complete (by assumption), $a_n \rightarrow a$ for some $a \in \mathbb{F}$, according to the topology induced by v . However, v and d induce the same topology, and therefore $a_n \rightarrow a$ according also to d ; thus, (\mathbb{F}, d) is a complete metric space. Finally, (\mathbb{F}, d) is separable by assumption.

$|\mathbb{F}| \leq 2^{\aleph_0}$ is easy. The opposite inequality follows from (3).

Notice that a \mathcal{G}_δ non-empty subset of \mathbb{F}^n is a Polish space [Kec95, 3.11].

(3) X itself is a non-empty perfect polish space. The conclusion follows from [Kec95, 6.2].

(4) Assume, for contradiction, that $\text{isol}(X)$ is not dense in X ; let $B \subseteq \mathbb{K}^n$ be a closed box, such that $\overset{\circ}{B} \cap X \neq \emptyset$, and $Y := X \cap B$ contains no isolated points. Hence, Y satisfies the hypothesis of (3), absurd.

(5) Every Polish space is Baire and strong Choquet [Kec95, 8.17]. \square

Definition 11.10. Let T be a complete theory expanding the theory of ordered fields, in a language \mathcal{L} , expanding the language \mathcal{L}_{OF} of ordered fields. We say that T is a **Polish theory** if, for every finite language \mathcal{L}' , such that $\mathcal{L}_{OF} \subseteq \mathcal{L}' \subseteq \mathcal{L}$, the restriction of T to \mathcal{L}' has a model which is separable and Cauchy complete. If T is not complete, we say that T is a Polish theory if every completion of T is Polish.

Lemma 11.11. *Let T be a definably complete theory (expanding RCF). If T is Polish, $\mathbb{K} \models T$ and $X \subseteq \mathbb{K}^n$ is definable and \mathcal{G}_δ (in the definable sense), then X is definably Baire.*

Proof. W.l.o.g., \mathbb{K} is Cauchy complete and separable. Hence, by Lemma 11.9, X is topologically Baire, and *a fortiori* definably Baire. \square

Lemma 11.12. *A d -minimal theory T is Polish. In particular, if \mathbb{K} is d -minimal and $X \subseteq \mathbb{K}^n$ is definable, then X is Baire.*

Proof. W.l.o.g., the language of T is countable. Let \mathbb{F}' be a countable model of T , and \mathbb{F} be the Cauchy completion of \mathbb{F}' . \mathbb{F} is a model of T . \square

11.2 The Z-closure

Definition 11.13. Let $A \subseteq \mathbb{K}$, and $c \in \mathbb{K}$. We define the Z-closure of A inside \mathbb{K}

$$\text{Zcl}^{\mathbb{K}}(A) := \bigcup \{C \subseteq \mathbb{K} : C \text{ nowhere dense and definable with parameters from } A\}.$$

If \mathbb{K} is clear from the context, we drop the subscript \mathbb{K} . $A \subseteq \mathbb{K}$ is Z-closed in \mathbb{K} if $\text{Zcl}^{\mathbb{K}}(A) = A$.

The notion above is most interesting when A is an elementary substructure of \mathbb{K} .

Remark 11.14. If \mathbb{K} is o-minimal, then $\text{Zcl} = \text{dcl}$.

Remark 11.15. If \mathbb{K} is o-minimal and $A \subseteq \mathbb{K}$, then A is Z-closed in \mathbb{K} if and only if A is an elementary substructure of \mathbb{K} .

Remark 11.16. For every $A \subseteq \mathbb{K}$, $\text{dcl}(A) \subseteq \text{Zcl}(A)$, and $\text{Zcl}(\text{dcl}(A)) = \text{Zcl}(A) = \text{Zcl}(\text{dcl}(A))$.

Remark 11.17. If A has DSF, then $\text{Zcl}^{\mathbb{K}}(A) \preceq \mathbb{K}$.

Lemma 11.18. If $A \subseteq \mathbb{K}$ has non-empty interior, then $\text{dcl}(A) = \mathbb{K}$, and therefore $\text{Zcl}(A) = \mathbb{K}$.

Proof. Since $A \subseteq \text{dcl} A = \text{dcl}(\text{dcl}(A))$, w.l.o.g. $A = \text{dcl} A$. Let $\varepsilon > 0$ and $a \in \mathbb{K}$ such that $B(a; \varepsilon) \subseteq A$. Hence, $(-\varepsilon, \varepsilon) \subseteq A$. Thus, $(1\varepsilon, +\infty) \subseteq A$. Let $b \in \mathbb{K}$; we want to prove that $b \in A$; w.l.o.g., $b > 0$. Let $a := (1/\varepsilon)$ and $a' := b + 1/\varepsilon$; notice that a and a' are in A , and therefore $b = a' - a \in A$. \square

Remark 11.19. Let $\mathbb{K} \prec \mathbb{F}$ be a dense substructure. If \mathbb{K} is d-minimal, then \mathbb{K} is Z-closed in \mathbb{F} .

Proof. By Lemma 11.2. \square

Remark 11.20. Given $A \subseteq \mathbb{K}$, $\text{dcl}(A)$ does not depend on \mathbb{K} : that is, if $\mathbb{K} \preceq \mathbb{K}'$, then the definable closure of A inside \mathbb{K} and the definable closure of A inside \mathbb{K}' are the same set. Instead, $\text{Zcl}^{\mathbb{K}}(A)$ may depend on \mathbb{K} : for instance, an infinite nowhere dense subset of \mathbb{K} is A -definable and if \mathbb{K}' is a κ -saturated elementary extension of \mathbb{K} , then $|\text{Zcl}^{\mathbb{K}'}(A)| \geq \kappa$. If \mathbb{K} is i -minimal but not o-minimal, then dcl depends on \mathbb{K} : for instance there exists some $\mathbb{K}' \succ \mathbb{K}$ such that $\text{Zcl}^{\mathbb{K}'}(\emptyset) \neq \text{Zcl}^{\mathbb{K}}(\emptyset)$.

Remark 11.21. If $A \subseteq \mathbb{K} \preceq \mathbb{K}'$, then $\text{Zcl}^{\mathbb{K}}(A) = \text{Zcl}^{\mathbb{K}'}(A) \cap \mathbb{K}$.

Definition 11.22. Let $f : X \rightsquigarrow Y$ be a definable application (i.e., a multi-valued partial function), with graph F . Assume that \mathbb{K} is i -minimal. For every $x \in X$, let $f(x) := \{y \in Y : \langle x, y \rangle \in F\} \subseteq Y$. Such an application f is a **Z-application** if, for every $x \in X$, $\dim(f(x)) = 0$ (thus, the domain of f is all X); it is a partial Z-application if for every $x \in X$, $\dim(f(x)) \leq 0$.

Remark 11.23. Let $A \subseteq \mathbb{K}$, and $b \in \mathbb{K}$. Then, $b \in \text{Zcl} A$ iff there exists an \emptyset -definable Z-application $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ and $\bar{a} \in A$, such that $b \in f(\bar{a})$. Moreover, if $\bar{c} \in \mathbb{K}^n$, then $b \in \text{Zcl}(A\bar{c})$ iff there exists an A -definable Z-application $f : \mathbb{K}^n \rightarrow \mathbb{K}$, such that $b \in f(\bar{c})$.

Proof. The “if” direction is clear: $f(\bar{a})$ is nowhere-dense. For the converse, let $Z \subset \mathbb{K}$ be nowhere-dense and A -definable, such that $b \in Z$. Let $\phi(x, \bar{a})$ be the formula defining Z . Let $\psi(x, \bar{y})$ be the formula “($\psi(\bar{x}, \bar{y})$ and $\psi(\mathbb{K}, \bar{y})$ is nowhere-dense) or ($x = 0$ and $\phi(\mathbb{K}, \bar{y})$ is somewhere-dense)”. Then, ψ defines a Z -application $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$, and $b \in f(\bar{a})$.

The “moreover” part is clear. \square

Proviso. For the rest of this subsection, \mathbb{K} is i -minimal with DSF.

We will show that, under the above condition, $\text{Zcl}^{\mathbb{K}}$ is a matroid (a.k.a. combinatorial pregeometry); we will write Zcl for $\text{Zcl}^{\mathbb{K}}$.

Definition 11.24. A formula $\phi(x, \bar{y})$ is x -narrow if, for every $\bar{c} \in \mathbb{K}^n$, $\phi(\mathbb{K}, \bar{c})$ is nowhere-dense.

Remark 11.25. For every $A \subseteq \mathbb{K}$,

$$\begin{aligned} \text{Zcl}(A) &= \bigcup \{C \subset \mathbb{K} : C \text{ is } 0\text{-dimensional and } A\text{-definable}\} = \\ &= \bigcup \{\phi(\mathbb{K}, \bar{a}) : \phi(x, \bar{y}) \text{ is } x\text{-narrow and } \bar{a} \subseteq A\}. \end{aligned}$$

If \mathbb{K} is d -minimal, then

$$\text{Zcl}(A) = \bigcup \{C \subset \mathbb{K} : C \text{ is discrete and } A\text{-definable}\}.$$

Remark 11.26. Let $A \subseteq \mathbb{K}$, and $D \subseteq \mathbb{K}^n$ be A -definable and 0-dimensional. If $\bar{b} \in D$, then each coordinate of \bar{b} is in $\text{Zcl } A$.

Proof. It is enough to show that $b_1 \in \text{Zcl } A$. Let $D_1 := \Pi_1^n D$. Then, D_1 is 0-dimensional, and $b_1 \in D_1$. \square

Lemma 11.27. Zcl is transitive. That is, $\text{Zcl}(\text{Zcl } A) = \text{Zcl } A$.

Proof. Let $b \in \text{Zcl}(\text{Zcl } A)$. Then, there exists $\bar{c} \in (\text{Zcl } A)^n$ and an x -narrow formula $\phi(x, \bar{y})$, such that $\mathbb{K} \models \phi(b, \bar{c})$. For every $i \leq n$, let Y_i be an A -definable nowhere-dense set containing c_i . Let $\psi(x, \bar{y})$ be the $L(A)$ -formula

$$\phi(x, \bar{y}) \wedge \bigwedge_{i \leq n} y_i \in Y_i$$

and $Z := \psi(\mathbb{K}^{n+1})$.

Claim 19. $\dim Z = 0$.

By Lemma 9.4.

Since Z is A -definable, and $\langle b, \bar{c} \rangle \in Z$, Remark 11.26 implies that $b \in \text{Zcl } A$. \square

Lemma 11.28. Zcl has the exchange property. That is, if $a \in \text{Zcl}(Bc) \setminus \text{Zcl}(B)$, then $c \in \text{Zcl}(Ba)$.

Proof. Assume $a \in \text{Zcl}(Bc)$ and $c \notin \text{Zcl}(Ba)$; we want to conclude that $a \in \text{Zcl}(B)$. Let $\phi(x, y)$ be an x -narrow $L(B)$ -formula, such that $\mathbb{K} \models \phi(a, c)$. Define $\psi(x, y)$ as

$$\exists r > 0 \forall y' (|y - y'| < r \rightarrow \phi(x, y'));$$

that is, $\mathbb{K} \models \phi(x, y)$ iff $y \in \text{int}(\phi(x, \mathbb{K}))$. Notice that $\mathbb{K} \models \psi(a, c)$, because $\phi(a, \mathbb{K}) \setminus \psi(a, \mathbb{K})$ has dimension 0. Moreover, for every $a' \in \mathbb{K}$, $\psi(a', \mathbb{K})$ is open: let $\theta(x, y)$ be the formula

$$\psi(x, y) \ \& \ y \text{ is the centre of a definably connected component of } \psi(x, \mathbb{K}).$$

Notice that, for every $a' \in \mathbb{K}$, $\theta(a', \mathbb{K})$ is discrete. Let $d \in D$ be the centre of the connected component of $\psi(a, \mathbb{K})$ containing c , and $Z := \theta(\mathbb{K}^2)$.

Claim 20. $\dim Z = 0$.

Let $\pi := \Pi_1^2$, and $\mu : \mathbb{K}^2 \rightarrow \mathbb{K}$ be the projection onto the second coordinate. If both $\pi(Z)$ and $\mu(Z)$ have dimension 0, then the claim is true. If $\dim(\pi(Z)) = 0$, then, since $\dim(Z_x) \leq 0$ for every x , Lemma 9.4 implies that $\dim Z = 0$. If, for contradiction, $\dim(\pi(Z)) > 0$, then, since $Z \subseteq \psi(\mathbb{K}^2)$, and $\psi(x, \mathbb{K})$ is open for every x , DSF and Theorem 8.4(III) imply that there exist $g < h : I \rightarrow \mathbb{K}$ B -definable and continuous, such that, for every $x \in I$, $(g(x), h(x)) \subseteq \psi(x, \mathbb{K})$. Hence, $\psi(\mathbb{K}^2)$ contains an open set, and is contained in $\phi(\mathbb{K}^2)$, contradicting the fact that $\phi(x, y)$ is x -narrow.

Since $\langle a, d \rangle \in Z$ and Z is B -definable, the claim and Remark 11.26 imply that $a \in \text{Zcl}(B)$. \square

The above two lemmata imply that Zcl is a matroid. Hence, we can speak about Z -generating and Z -independent sets, Z -basis and Z -dimension: E is Z -generating set of B/A if $\text{Zcl}(AE) = \text{Zcl}(B)$; E is Z -independent over A if, for every $e_1, \dots, e_{n+1} \in E$, $e_{n+1} \notin \text{Zcl}(Ae_1 \dots e_n)$; E is a Z -basis if it is both Z -generating and Z -independent. The Z -dimension of B/A is $\text{Zdim}(B/A)$, the cardinality of some Z -basis of B/A . It is important to notice that the above notions do not depend on the ambient space \mathbb{K} : by Remark 11.21 if $A \subseteq B \subseteq \mathbb{K} \preceq \mathbb{K}'$, and $E \subseteq \mathbb{K}$, then E is a Z -generating set of B/A (resp. Z -independent over A , resp. a Z -basis of B/A) in \mathbb{K} iff it is a Z -generating set of B/A (resp. Z -independent over A , resp. a Z -basis of B/A) in \mathbb{K}' . Hence, $\text{Zdim}(B/A)$ is also independent from \mathbb{K} .

Let $\mathbb{M} \succ \mathbb{K}$ be “the” monster model. Let κ be an infinite cardinal, such that $\kappa > |T|$, where T is the theory of \mathbb{K} ; we will assume that $|\mathbb{K}| < \kappa$ and \mathbb{M} is κ -saturated and strongly κ -homogeneous. We will say that A is a proper subset of \mathbb{M} if $A \subset \mathbb{M}$ and $|A| < \kappa$.

Lemma 11.29. *Let A be a proper subset of \mathbb{M} , and $c \in \mathbb{M}$. Let $\Xi(c/A)$ be the set of conjugates of c over A . T.f.a.e.:*

1. $c \in \text{Zcl}(A)$;
2. $\Xi(c/A)$ has empty interior;
3. $\Xi(c/A)$ is nowhere dense.

If moreover \mathbb{M} is d -minimal, then $c \in \text{Zcl}(A)$ iff $\Xi(c/A)$ is discrete.

Proof. $(1 \Rightarrow 3 \Rightarrow 2)$ is clear.

$(2 \Rightarrow 1)$. Let $\phi(x)$ be any $\mathcal{L}(A)$ -formula, such that $\mathbb{M} \models \phi(c)$. Since $c \notin \text{Zcl}(A)$, $c \in \text{int}(\phi(\mathbb{M}))$, and therefore there exist $d, d' \in \mathbb{M}$, such that $d < c < d'$ and $(d, d') \subseteq \phi(\mathbb{M})$. Let $\Gamma(v, v')$ be the set of $\mathcal{L}(Ac)$ -formulae

$$v < c < v' \ \& \ (v, v') \subseteq \phi(\mathbb{M}),$$

where $\phi(x)$ varies in $\text{tp}(c/A)$. By what we said above, Γ is consistent: hence, by saturation, there exist $d, d' \in \mathbb{M}$ satisfying Γ . We claim that $(d, d') \subseteq \Xi(c/A)$. In fact, if $c' \in (d, d')$, then by definition, c' satisfies all the $\mathcal{L}(A)$ -formula satisfied by c : therefore, $\text{tp}(c'/A) = \text{tp}(c/A)$, and, by homogeneity, $c' \in \Xi(c/A)$.

Assume now that \mathbb{M} is d-minimal. If $c \in \text{Zcl}(A)$, then there exists a discrete set X , definable with parameters from A , such that $c \in X$; therefore, $\Xi(c/A) \subseteq X$, and thus $\Xi(c/A)$ is discrete. The converse is clear. \square

Corollary 11.30. *$\text{Zcl}^{\mathbb{M}}$ is an existential matroid. I.e., let $\bar{a} \in \mathbb{M}^n$, $\bar{b} \in \mathbb{M}^m$, $c \in \mathbb{M}$, and $\phi(x, \bar{y}, \bar{z})$ be x -narrow. Assume that, for every conjugate c' of c over \bar{a} , $\mathbb{M} \models \phi(c', \bar{a}, \bar{b})$. Then, c (and all its conjugates over \bar{a}) is in $\text{Zcl}^{\mathbb{M}}(\bar{a})$.*

Proof. It is [For10, Theorem 9.8]. The corollary is also a direct consequence of [For10, Lemma 3.22]. \square

Definition 11.31. Given A, B, C proper subsets of \mathbb{M} , we say that A and C are Z -free over B , written $A \downarrow_B^Z C$, if some (every) Z -basis of A over B remains Z -independent over BC .

The above three lemmata imply the following result.

Theorem 11.32. *Zcl is an existential matroid. Hence, \downarrow^Z is an independence relation in the sense of [Adl05] (and in particular it is symmetric), and satisfies $a \downarrow_B^Z a$ iff $a \in \text{Zcl}^{\mathbb{K}}(B)$, for every $a \in \mathbb{K}$, $B \subseteq \mathbb{K}$.*

Notice that if \mathbb{K} is o-minimal, then $\text{Zcl} = \text{dcl}$, and therefore $\downarrow^Z = \downarrow^{\mathbb{P}}$. The converse is also true (remember the assumption that \mathbb{K} is i -minimal with DSF).

Lemma 11.33. *T.f.a.e.:*

1. $\text{Zcl}^{\mathbb{M}} = \text{dcl}$;
2. $\downarrow^Z = \downarrow^{\mathbb{P}}$ (in the monster model);
3. \mathbb{K} is o-minimal.

Proof. (3 \Rightarrow 1) is clear. If (1) holds, then $\downarrow^Z = \downarrow^{\mathbb{M}}$; moreover, since \downarrow^Z is symmetric, $\downarrow^{\mathbb{M}}$ is also symmetric, and therefore $\downarrow^{\mathbb{M}} = \downarrow^{\mathbb{P}}$ [Adl05].

Assume that (2) holds. Let $a \in \text{Zcl}(B)$. Then, $a \downarrow_B^Z a$, therefore $a \downarrow_B^{\mathbb{P}} a$, and thus $a \in \text{dcl } B$: hence, (1) also holds. We have to prove that \mathbb{M} is o-minimal. Assume, for contradiction, that $A \subset \mathbb{M}$ is definable with parameters \bar{b} , infinite and with empty interior. Then, $A \subseteq \text{Zcl}^{\mathbb{M}}(\bar{b}) = \text{dcl}(\bar{b})$. However, since A is infinite, $|A| \geq \kappa$, and therefore $|\text{dcl}(\bar{b})| \geq \kappa > |T|$, which is impossible. \square

Lemma 11.34. *The dimension induced by \downarrow^Z and the geometric notion of dimension coincide. That is, if $X \subseteq \mathbb{M}^n$ is definable, then $\dim X = \max\{\text{rk}^Z(\bar{x}) : \bar{x} \in X\}$.*

Contrast the situation of \downarrow^Z to the notion of M -dividing independence (defined in [Adl05]), where, $A \downarrow_B^M C$ iff, for every $\bar{c} \subset \text{dcl}(BC)$,

$$\text{dcl}(A\bar{c}) \cap \text{dcl}(BC) = \text{dcl}(B\bar{c}).$$

Lemma 11.35. *Assume that T is d -minimal, but not o -minimal. Then, $\downarrow^{\mathbb{M}}$ is not symmetric (and therefore dcl does not have the Exchange Property). However, $\downarrow^{\mathbb{M}}$ does satisfy the existence and extension axioms, and therefore coincides with $\downarrow^{\mathbb{P}}$, the b -forking relation. Hence, T is not rosy, and in particular $\downarrow^{\mathbb{P}}$ is not symmetric.*

Proof. The fact that $\downarrow^{\mathbb{M}}$ satisfy existence and extension and coincides with $\downarrow^{\mathbb{P}}$ is immediate from [For10, Lemma 3.22].

Let $\mathbb{K} \prec \mathbb{M}$ such that \mathbb{K} is not Cauchy complete, and $|\mathbb{K}| < \kappa$. By expanding the language with less than κ new constants, w.l.o.g. we can assume that \mathbb{K} is the prime model of T . Let π be a regular gap of \mathbb{K} ; choose c_0 and c_1 such that $c_1 > \mathbb{K}$ and $c_0 \models \pi$.

Claim 21. There exist a_0 and b , and a finite set C such that:

1. $c_0, c_1 \in C$;
2. $b \in \text{Zcl}(C)$;
3. $b \in \text{dcl}(Ca_0) \setminus \text{dcl}(C)$;
4. $a_0 \notin \text{Zcl}(Cb)$;
5. $a_0 > 0$ and a_0 is infinitesimal w.r.t. \mathbb{K} .

Let X be a definable (with parameters) subset of \mathbb{M} which is discrete and infinite (such a set exists by Corollary 8.7, because T is i -minimal but not o -minimal), and let C be any finite set containing c_0 , c_1 , and the parameters of X . By saturation, there exists $b \in X \setminus \text{dcl}(C)$. Let I be an open interval containing b , such that $I \cap X = \{b\}$. By Lemma 11.18, since $\text{Zrk}(\mathbb{M}/\mathbb{K}) \geq \kappa$, there exists $a'_0 \in I \setminus \text{Zcl}(Cb)$. W.l.o.g., $a'_0 > 0$. If a'_0 is infinitesimal (w.r.t. \mathbb{K}), let $a_0 := a'_0$; if a'_0 is finite but not infinitesimal, let $a_0 := a'_0/c_1$; if a'_0 is infinite, let $a_0 := 1/a'_0$. Notice that $\text{Zcl}(Cb) = \text{Zcl}(C)$.

Claim 22. With b and C as in the above Claim, there exists $a \in \mathbb{M}$ such that:

1. $a \models \pi$ (and therefore a is in a Cauchy completion of \mathbb{K});
2. $a \notin \text{Zcl}(Cb)$;
3. $b \in \text{dcl}(Ca) \setminus \text{dcl}(C)$.

Let $a := c_0 + a_0$. Since a_0 is infinitesimal, $a \models \pi$. Since $a_0 \notin \text{Zcl}(C)$ and $c_0 \in C$, the second point follows. Since $\text{dcl}(Ca) = \text{dcl}(Ca_0) \ni b$, the third point follows.

Claim 23. $a \not\downarrow^{\mathbb{M}} Cb$.

In fact, let $C' := C$; then, $b \in \text{dcl}(C'a) \cap \text{dcl}(Cb)$, but $b \notin \text{Zcl}(C')$.

Claim 24. $Cb \not\downarrow^{\mathbb{M}} a$.

In fact, let $A := \text{dcl}(a) = \mathbb{K}\langle a \rangle$, and $A' \subseteq A$. Define $Y := \text{dcl}(CA'b) \cap A$; we have to prove that $Y = A'$. Since a satisfies a regular gap over \mathbb{K} , $\text{dcl}(\emptyset) = \mathbb{K}$ is dense in A ; therefore, dcl^A satisfies EP. Hence, either $A' = \mathbb{K}$, or $A' = A$. If $A' = A$, the conclusion is obvious. If $A' = \mathbb{K}$, then $a \notin Y$, because $Y \subset \text{Zcl}(Cb)$, and $a \notin \text{Zcl}(Cb)$; therefore, since dcl^A satisfies EP, $Y = \mathbb{K}$. \square

We do not know if the above lemma extends to i -minimal theories with DSF, or to constructible theories.

Lemma 11.36. *Let $f : \mathbb{K} \rightsquigarrow \mathbb{K}$ be a Z -application, definable with parameters \bar{c} . Let $b \in \mathbb{K}$ and $U \subseteq \mathbb{K}$ be non-empty and open, such that, for every $a \in U$, $b \in g(a)$. Then, $b \in \text{Zcl}(\bar{c})$.*

11.3 Dense pairs of d-minimal structures

Dense pairs of o-minimal structures were studied in [vdD98a].

Proviso. \mathbb{K} is an constructible structure with DSF, and $T := \text{Th}(\mathbb{K})$.

We have seen that the Z -closure is an existential matroid on \mathbb{K} . Moreover, $A \subseteq \mathbb{K}$ is topologically dense iff it is dense w.r.t. to the matroid Zcl , that is iff X intersects every definable subset of \mathbb{K} of dimension 1.

We can apply the results in [For10] to T , and obtain the following results.

Theorem 11.37. *Let T^d be the theory of pairs $\mathbb{A} \prec \mathbb{B} \models T$, such that \mathbb{A} is dense in \mathbb{B} and $\text{Zcl}(\mathbb{A}) = \mathbb{A}$. Then, T is consistent and complete. Besides, \mathbb{B} is the open core of $\langle \mathbb{B}, \mathbb{A} \rangle$. If moreover T is d-minimal, then T^d is the theory of pairs $\mathbb{A} \prec \mathbb{B} \models T$, such that \mathbb{A} is dense in \mathbb{B} (the fact that \mathbb{A} is Zcl -closed in \mathbb{B} follows).*

Similar results can be shown for dense tuples of models of T [For10, §13].

More results can be proved for T^d , e.g. a form of elimination of quantifiers [For10]. If moreover T is d-minimal, then also the results in [For10, §9] apply to T .

We will give some additional results and conjectures that are more specific to our situation.

Theorem 11.38 ([vdD98a, Theorem 2]). *Let $\langle \mathbb{B}, \mathbb{A} \rangle \models T^d$. Given a set $Y \subset \mathbb{A}^n$, t.f.a.e.:*

1. Y is definable in $\langle \mathbb{B}, \mathbb{A} \rangle$;
2. $Y = Z \cap \mathbb{A}^n$ for some set $Z \subseteq \mathbb{B}^n$ that is definable in \mathbb{B} .

If moreover T is constructible, then the above two conditions are equivalent to:

3. Y is definable in the structure $\langle \mathbb{A}, (\mathbb{A} \cap (0, b))_{b \in \mathbb{B}} \rangle$.

Remember that constructible structures are i -minimal with DSF.

Proof. (1 \Rightarrow 2) and, under the extra condition on T , (2 \Rightarrow 3) are as in [vdD98a, Theorem 2]. (3 \Rightarrow 1) and (2 \Rightarrow 1) are obvious (and true without the extra condition on T). \square

We do not know whether we do really need the extra condition that T is constructible to prove that (2) implies (3) in the above Theorem.

Question 11.39. Let T' be a complete a-minimal theory. Is there an existential matroid on T' ? [DMS10, 6.2] and [For10] prove that if T is equal to either T^d or T^g (see [DMS10] for the definition of T^g) for some o-minimal theory T , then T' admits such a matroid (in the case of T^g , the matroid is acl).

Example 11.40. We show that Zcl does not satisfy Existence on $\mathbb{M} := \langle \mathbb{B}, \mathbb{A} \rangle$, a monster model of T^d . Write $x' \equiv_C^2 x$ if the $\mathcal{L}(U)$ -type of x and x' over C are the same, and let $\Xi^2(x/C) := \{x' \in \mathbb{M} : x' \equiv_C^2 x\}$.

Choose $a_1, a_2 \in \mathbb{A}$ and $b \in \mathbb{B} \setminus \mathbb{A}$, such that a_1 and a_2 are \mathbb{Z} -independent over b . Let $c := a_1b + a_2$, and $f : \mathbb{A}^2 \rightarrow \mathbb{B}$ be the definable function $f(x_1, x_2) := x_1 \cdot c + x_2$. By hypothesis, $\text{Zrk}(a_1a_2/b) = 2$, and therefore $\text{Zrk}(a_1, a_2/bc) \geq 1$. Thus, either $a_1 \notin \text{Zcl}(bc)$, or $a_2 \notin \text{Zcl}(bc)$; w.l.o.g., $a_1 \notin \text{Zcl}(bc)$. However, f is injective, and therefore a_1 and a_2 are $\mathcal{L}(U)$ -definable over bc , hence, $\Xi^2(a_1/bc) = \{a_1\} \subseteq \text{Zcl}(bca_1)$. If Zcl did satisfy existence, then $a_1 \in \text{Zcl}(bc)$, absurd.

11.3.1 The open core

Proviso. For this subsection, we assume that T is d -minimal. Let $\mathbb{M} := \langle \mathbb{B}, \mathbb{A} \rangle \models T^d$.

We have seen that \mathbb{B} is the open core of \mathbb{M} . Hence, since every \mathcal{F}_σ subset of \mathbb{M}^n is definable in the open core of \mathbb{M} , every such set is constructible. We will prove some additional results about this topic. Scl is the **small closure** on \mathbb{M} and Sdim is the corresponding dimension function, as defined in [For10].

Lemma 11.41. Let $(X_t)_{t \in \mathbb{M}}$ be a definable increasing family of subsets of \mathbb{M}^n , and $X := \bigcup_t X_t$. Let $d \leq n$ and assume that, for every $t \in \mathbb{M}$, $\text{Sdim}(X_t) \leq d$. Then, $\text{Sdim}(X) \leq d$.

Proof. [For10, Lemma 3.71], applied to Scl . □

Corollary 11.42. \mathbb{M} is Baire.

Proof. By [For10], if $X \subseteq S$ is definable in \mathbb{M} and nowhere-dense, then $\text{Sdim } X = 0$.

The conclusion then follows from Lemma 11.41. □

Lemma 11.43. T^d is a Polish theory.

Proof. W.l.o.g., the language of T is countable. Let \mathbb{A}' be a countable model of T and \mathbb{B}' be its Cauchy completion. Notice that $\mathbb{B}' \neq \mathbb{A}'$ and therefore $\langle \mathbb{B}', \mathbb{A}' \rangle$ is a Cauchy complete and separable model of T^d . □

12 Types in locally o-minimal structures

As usual, \mathbb{K} is a definably complete structure. Let $+\infty$ be the partial 1-type over \mathbb{K} , given by $\{x > a : a \in \mathbb{K}\}$.

Remark 12.1. T.f.a.e.:

- \mathbb{K} is locally o-minimal;
- $+\infty$ is a complete type;
- for every cut Λ of \mathbb{K} , if $\hat{\Lambda} = 0^+$, then Λ is a complete type over \mathbb{K} .

Proof. See Lemma 6.1 and Corollary 11.7. □

Lemma 12.2. Let \mathbb{K} be locally o-minimal, and $p \in S_1(\mathbb{K})$. p is a definable type iff either $p = \pm\infty$, or $p = a^\pm$, or $p = a$, for some $a \in \mathbb{K}$.

Proof. By the above remark, the listed types are complete types. The “only if” direction is clear. For the other direction, let $p = +\infty$; we have to prove that p is definable (the other cases are similar). Fix c a realization of p in some $\mathbb{K}' \succ \mathbb{K}$. Let $\phi(x, \bar{y})$ be a formula, and $D := \{\bar{y} \in \mathbb{K}^n : \mathbb{K} \models \phi(c, \bar{y})\}$. We have to prove that D is a definable subset of \mathbb{K} . Let $F := \phi(\mathbb{K}')$ and $F_c := \phi(c, \mathbb{K}')$; notice that $D = F_c \cap \mathbb{K}^n$. Define

$$G := \liminf F_t = \{\bar{a} \in \mathbb{K}^n : \exists t_0 \forall t > t_0 \bar{a} \in F_t\}.$$

Notice that G is definable without parameters; let $G' := G(\mathbb{K}) = G \cap \mathbb{K}^n$.

Claim 25. $D = G'$.

The claim implies the conclusion. Let us prove that $d \subseteq D$; let $d \in D$. Thus, $d \in F_a \cap \mathbb{K}$, and therefore “ $d \in F_x$ ” is in the type of c over \mathbb{K} . Thus, $d \in F_t$ eventually, and therefore $d \in G$. The opposite inclusion is trivial. \square

Question 12.3. Give a characterization of definable n -types in locally o-minimal structure, along the lines of Marker-Steinhorn’s theorem.

13 D-minimal open core

We have seen that if T is d-minimal, then T^d has d-minimal open core. We want to give some characterization of when a structure \mathbb{K} as d-minimal open core. Unfortunately, we were not able to prove what we wanted, so here is a conjecture.

Conjecture 13.1. *Assume that, for every $\mathbb{K}' \succ \mathbb{K}$, every \mathcal{F}_σ subset of \mathbb{K}' is the union of an open set and finitely many discrete sets. Then, for every $n \in \mathbb{N}$, every \mathcal{F}_σ subset of \mathbb{K}^n is constructible. Moreover, \mathbb{K} is Baire.*

In the above situation, we say that \mathbb{K} has d-minimal open core.

Idea of proof. If \mathbb{K} is a-minimal, then \mathbb{K} has locally o-minimal open core, and a fortiori d-minimal open core. Hence, we may assume that \mathbb{K} is not a-minimal, and therefore there exists a pseudo- \mathbb{N} subset of \mathbb{K} .

First, we prove that \mathbb{K} is Baire. If not, let $N \subseteq \mathbb{K}$ be a pseudo- \mathbb{N} subset of \mathbb{K} , and $\mathcal{C} := (C_t)_{t \in N}$ be a definable family of d-compact nowhere dense subsets of \mathbb{K} , such that $\mathbb{K} = \bigcup_{t \in N} C_t$. By assumption, there exists $M \in \mathbb{N}$ such that each C_t is a finite union of M discrete sets. Therefore, \mathcal{C} is a uniform family of (at most) pseudo-enumerable sets, and therefore \mathbb{K} is pseudo-enumerable, absurd.

Let $A \subseteq \mathbb{K}^n$ be an \mathcal{F}_σ set. We have to prove that A is constructible. W.l.o.g., \mathbb{K} is ω -saturated. We proceed by induction on n and on $(d, k) := \text{fdim}(A)$. If $n = 1$, A is constructible by hypothesis. Hence, w.l.o.g. $n > 1$. If $d = n$, let $B := A \setminus \overset{\circ}{A}$. Thus, $A = \overset{\circ}{A} \cup B$. Since $\dim(B) < d$, by induction on d , B is constructible, and therefore A is constructible. If $d = 0$, for $i = 1, \dots, n$, let A_i be the projection of A onto the i th coordinate axis. By assumption on A , A_i is an \mathcal{F}_σ with empty interior, and therefore, by hypothesis, A_i is a finite union of discrete sets. Moreover, $A \subseteq A_1 \times \dots \times A_n$, therefore A is a finite union of discrete sets, and hence A is constructible.

Hence, we may assume that $0 < d < n$. Let $\pi := \Pi_d^n$; w.l.o.g., $\pi(A)$ has non-empty interior. Let $C := \ulcorner A \urcorner = A \setminus \text{lc}(A)$.

Claim 26. W.l.o.g., A is bounded.

In fact, it suffices to prove that $\phi(A)$ is constructible, where ϕ is a homeomorphism between \mathbb{K}^n and $(0, 1)^n$.

Given $X \subseteq \mathbb{K}^d$, let $A_X := A \cap \pi^{-1}(X)$.

Claim 27. W.l.o.g., $\pi(C) = \pi(A) = B$ for some closed box B with non-empty interior.

If $\text{fdim}(C) < \text{fdim}(A)$, then, by inductive hypothesis, C is constructible, and therefore A is also constructible. Thus, $\pi(C)$ contains a closed box $B' \subseteq B$ with non-empty interior. If we prove that $A_{B'}$ is constructible for any such B' , then, since \mathbb{K} is ω -saturated, there exists $M \in \mathbb{N}$ such that $A_B^{\ulcorner M \urcorner} = \emptyset$ for every such box B' . Hence, $A^{\ulcorner M \urcorner} \cap \pi^{-1}(B') = \emptyset$ for every closed box $B' \subseteq \pi(C)$ with non-empty interior, and therefore $\text{fdim}(A^{\ulcorner M \urcorner}) < \text{fdim} A$. Thus, by inductive hypothesis, $A^{\ulcorner M \urcorner}$ is constructible, and thus A is also constructible.

By hypothesis, there exists an increasing definable family $(A(t))_{t \in \mathbb{K}}$ of d-compact subsets of \mathbb{K}^n , such that $A = \bigcup_t A(t)$.

Claim 28. W.l.o.g., $\pi(A(t)) = B$ for every $t \in \mathbb{K}$.

In fact, since \mathbb{K} is Baire, there exists $t_0 \in \mathbb{K}$ and $B' \subseteq B$ closed box with non-empty interior, such that $B' \subseteq \pi(A(t_0))$. If we prove that $A_{B'}$ is constructible for any such box B' contained in some $A(t)$, then, since \mathbb{K} is ω -saturated, there exists $M \in \mathbb{N}$ such that $A_B^{\ulcorner M \urcorner} = \emptyset$ for every such box B' . Since every $x \in B$ is contained in some B' as above, $\text{fdim}(A^{\ulcorner M \urcorner}) < \text{fdim} A$, and therefore, by inductive hypothesis, A is constructible. Thus, w.l.o.g. $B \subseteq \pi(A(t_0))$ for some $t_0 \in \mathbb{K}$. Define $A'(t) := A(t_0)$ if $t \leq t_0$, and $A'(t) = A(t)$ if $t > t_0$. Then, $\pi(A'(t)) = B$, each $A'(t)$ is d-compact, and $A = \bigcup_t A'(t)$.

Claim 29. W.l.o.g., $\dim(A_x) = 0$ for every $x \in B$.

In fact, let $D := \{x \in \mathbb{K}^d : \dim(A_x) > 0\}$. By Lemma 4.9, D is an \mathcal{F}_σ of dimension less than d . Hence, by induction on $\text{fdim}(A)$, A_D is constructible. Thus, it suffices to prove that $A \setminus A_D$ is constructible.

By assumption, and since \mathbb{K} is ω -saturated, there exists $M \in \mathbb{N}$, such that, for every $x \in \mathbb{K}^d$, A_x is the union of at most M discrete sets.

Claim 30. W.l.o.g., for every $x \in B$, A_x is discrete.

To be done.

Claim 31. $\text{lc}(A) \neq \emptyset$.

Assume, for contradiction, that $\text{lc}(A)$ is empty. Let $\tilde{A} := A(0)$; remember that \tilde{A} is d-compact and $\pi(\tilde{A}) = B$; let $f : B \rightarrow \mathbb{K}$, $f(x) := \min(\tilde{A}_x)$. To simplify the notation, assume that $n = d + 1$. By [FS09], $\mathcal{D}(f)$ is a meager \mathcal{F}_σ subset of \mathbb{K}^d ; thus, by inductive hypothesis, $\mathcal{D}(f)$ is nowhere dense. Hence, after shrinking B , we may assume that f is continuous. We want to prove that, after possibly further shrinking B , $\Gamma(f) \subseteq \text{lc}(A)$. Let $A^+ := \{\langle x, y \rangle : (y \in A_x \ \& \ y > f(x)) \vee y = f(x) + 1\}$. Since f is continuous and A is an \mathcal{F}_σ , A^+ is an \mathcal{F}_σ sets. Define $f^+ : B \rightarrow \mathbb{K}$ as $f^+(x) := \min(A_x^+)$, and define in a symmetric fashion $f^- : B \rightarrow \mathbb{K}$. Notice that $f^- < f < f^+$ (because each A_x is discrete). We claim that f^+ is continuous outside a nowhere dense subset of B . Let $A_+ := \bigcup_{t \in N} C(t)$, where N is a pseudo- \mathbb{N} subset of \mathbb{K} , and $(C(t))_{t \in N}$ is a definable increasing family of d-compact sets, such that $B \times \{1\} \subseteq C(t)$ for every t . Let $f_t(x) := \min(C(t))$, and $D(t) := \mathcal{D}(f_t)$. Each $D(t)$ is a meager \mathcal{F}_σ

subset of \mathbb{K}^d , hence, by inductive hypothesis, each $D(t)$ is nowhere dense. Hence, $D := \bigcup_{t \in N} D(t)$ is meager, and therefore, by inductive hypothesis, nowhere dense; let $B' := B \setminus \overline{D}$, and $g := f^+ \upharpoonright B'$. Since B' is open and definable, B' is Baire; moreover, g is the pointwise limit of the continuous functions $f_t \upharpoonright B'$; thus, by Lemma 7.6 and inductive hypothesis, g is continuous outside a nowhere dense subset of B' . Reasoning in the same way for f^- , we see that there exists B'' open dense subset of B , such that both f^+ and f^- are continuous on B'' . Hence, $\Gamma(f) \cap \pi^{-1}(B'') \subseteq \text{lc}(A)$ and therefore the latter is non-empty.

Claim 32. $\text{lc}(A)$ is dense in A .

For every $U \subseteq \mathbb{K}^n$ open box, such that $A \cap U$ is non-empty, the above claim implies that $\text{lc}(A) \cap U = \text{lc}(A \cap U)$ is non-empty.

Let $A' := \text{lc}(A)$. By definition, $A' = \overline{A'} \cap U$, for some open definable set U . By Lemma 4.27, $\mathfrak{B}_d(U)$ is an \mathcal{F}_σ meager subset of \mathbb{K}^d , and therefore, by inductive hypothesis, $\mathfrak{B}_d(U)$ is constructible, and therefore nowhere dense. Hence, after shrinking B , we may assume that $\mathfrak{B}_d(U)$ is empty.

Claim 33. $\mathfrak{B}_d(A')$ is empty.

Again, by Thm. 4.33.

Since $A' \subseteq A \subseteq \overline{A'}$, Remark 4.26 implies that $\mathfrak{B}_d(A) \subseteq \mathfrak{B}_d(A') = \emptyset$.

Therefore, $A' = \text{lc}(A) \subseteq A$, $\overline{A'} = \overline{A}$, and $\text{cl}(A'_x) = (\overline{A'})_x = (\overline{A})_x = \text{cl}(A_x)$ for every $x \in B$; hence, A'_x is dense in A_x for every $x \in B$. Since, for every $x \in B$, A_x is discrete, this means that $A_x = A'_x$; thus, $A' = A$, and hence A is constructible. \square

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References

- [Adl05] Hans Adler. *Explanation of Independence*. PhD thesis, Albert-Ludwigs-Universität, Freiburg im Breisgau, June 2005.
- [All96] Jean-Paul Allouche. Note on the constructible sets of a topological space. In *Papers on general topology and applications (Gorham, ME, 1995)*, volume 806 of *Ann. New York Acad. Sci.*, pages 1–10. New York Acad. Sci., New York, 1996.
- [DM01] Randall Dougherty and Chris Miller. Definable Boolean combinations of open sets are Boolean combinations of open definable sets. *Illinois J. Math.*, 45(4):1347–1350, 2001.
- [DMS10] Alfred Dolich, Chris Miller, and Charles Steinhorn. Structures having o-minimal open core. *Trans. Amer. Math. Soc.*, 362:1371–1411, 2010.
- [For10] Antongiulio Fornasiero. Dimensions, matroids, and dense pairs of first-order structures. Submitted, 2010.

- [FS09] Antongiulio Fornasiero and Tamara Servi. Definably complete and baire structures. Submitted, 2009.
- [Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [Kur66] Kazimierz Kuratowski. *Topology. Vol. I*. New edition, revised and augmented. Translated from the French by J. Jaworowski. Academic Press, New York, 1966.
- [LS95] Michael C. Laskowski and Charles Steinhorn. On o-minimal expansions of Archimedean ordered groups. *J. Symbolic Logic*, 60(3):817–831, 1995.
- [Mil96] Chris Miller. A growth dichotomy for o-minimal expansions of ordered fields. In *Logic: from foundations to applications (Staffordshire, 1993)*, Oxford Sci. Publ., pages 385–399. Oxford Univ. Press, New York, 1996.
- [Mil01] Chris Miller. Expansions of dense linear orders with the intermediate value property. *J. Symbolic Logic*, 66(4):1783–1790, 2001.
- [Mil05] Chris Miller. Tameness in expansions of the real field. In *Logic Colloquium '01*, volume 20 of *Lect. Notes Log.*, pages 281–316. Assoc. Symbol. Logic, Urbana, IL, 2005.
- [MS99] Chris Miller and Patrick Speissegger. Expansions of the real line by open sets: o-minimality and open cores. *Fund. Math.*, 162(3):193–208, 1999.
- [Oxt71] John C. Oxtoby. *Measure and category. A survey of the analogies between topological and measure spaces*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 2.
- [Pil87] Anand Pillay. First order topological structures and theories. *J. Symbolic Logic*, 52(3):763–778, 1987.
- [Poi85] Bruno Poizat. *Cours de théorie des modèles*. Bruno Poizat, Lyon, 1985. Une introduction à la logique mathématique contemporaine.
- [Rob74] Abraham Robinson. A note on topological model theory. *Fund. Math.*, 81(2):159–171, 1973/74. Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, II.
- [Sco69] Dana Scott. On completing ordered fields. In *Applications of Model Theory to Algebra, Analysis, and Probability (Internat. Sympos., Pasadena, Calif., 1967)*, pages 274–278. Holt, Rinehart and Winston, New York, 1969.
- [vdD98a] Lou van den Dries. Dense pairs of o-minimal structures. *Fund. Math.*, 157(1):61–78, 1998.
- [vdD98b] Lou van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.