EXPANSIONS OF O-MINIMAL STRUCTURES ON THE REAL FIELD BY TRAJECTORIES OF LINEAR VECTOR FIELDS

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Abstract. Let $\mathfrak{R}$ be an o-minimal expansion of the field of real numbers that defines nontrivial arcs of both the sine and exponential functions. Let $\mathcal{G}$ be a collection of images of solutions on intervals to differential equations $y' = F(y)$, where F ranges over all $\mathbb{R}$-linear transformations $\mathbb{R}^n \to \mathbb{R}^n$. Then either the expansion of $\mathfrak{R}$ by the elements of $\mathcal{G}$ is as well behaved relative to $\mathfrak{R}$ as one could reasonably hope for, or it defines the set of all integers $\mathbb{Z}$, and thus is as complicated as possible. In particular, if $\mathfrak{R}$ defines any irrational power functions, then the expansion of $\mathfrak{R}$ by the elements of $\mathcal{G}$ either is o-minimal or defines $\mathbb{Z}$.

1. Introduction

We wish to understand expansions of o-minimal structures on the real field $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot\,)$ by images of solutions of definable vector fields. In this paper, we study the fundamental case of linear vector fields.

The reader is assumed to be familiar with the basics of first-order definability theory over $\overline{\mathbb{R}}$, including o-minimality (*e.g.*, van den Dries and Miller [9]). We fix some terminology. Throughout, "definable" (in some structure) means "definable using arbitrary real constants" (in the structure). We use "expansion" and "reduct" in the sense of definability, and identify structures up to interdefinability. A **vector field** on a subset A of $\mathbb{R}^n$ is a map $F\colon A \to \mathbb{R}^n$. A **solution** of F is a differentiable map $\gamma\colon I \to \mathbb{R}^n$ defined on a nontrivial interval $I \subseteq \mathbb{R}$ of some sort such that $\gamma(I) \subseteq A$ and $\gamma'(t) = F(\gamma(t))$ for all $t \in I$. A **trajectory** of F is the image of a solution. We do not regard trajectories as carrying directions or being parameterized in any particular way. A **linear** vector field is an $\mathbb{R}$-linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ for some $n \geq 1$. See, for example, Perko [21, Chapter 1] or Arnold [1, Chapter 3] for basics of linear vector field theory.

Recall that a subset of a topological space is **locally closed** if it is relatively open in its closure, equivalently, if it is the intersection of an open set and a closed set. It is known from ODE theory that every trajectory of a linear vector field is either locally closed (even semianalytic) or dense and codense in a product of circles centered at the origin; the latter can happen only if all eigenvalues of the vector field are purely imaginary, at least two of which are rationally independent, and the trajectory is the image of a solution on an unbounded interval.

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Let $\mathbb{R}^*$ denote the set of nonzero real numbers. For $\omega \in \mathbb{R}^*$, let S_ω denote the logarithmic spiral $\{ (e^t \cos \omega t, e^t \sin \omega t) : t \in \mathbb{R} \}$. Note that S_ω is a trajectory of (the vector field arising from) the matrix $\left(\begin{smallmatrix} 1 & -\omega \\ \omega & 1 \end{smallmatrix}\right)$.

1.1. **Theorem.** *Let $\mathcal{G}$ be a collection of locally closed trajectories of linear vector fields such that each $\Gamma \in \mathcal{G}$ is the image of a solution on an unbounded interval. Then $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma \in \mathcal{G}})$ is equal to exactly one of the following:*

— $(\overline{\mathbb{R}}, (x^r)_{r \in K})$ *for some subfield K of $\mathbb{R}$*
— $(\overline{\mathbb{R}}, e^x)$
— $(\overline{\mathbb{R}}, S_\omega)$ *for some $\omega \in \mathbb{R}^*$*
— $(\overline{\mathbb{R}}, \mathbb{Z})$.

The structures $(\overline{\mathbb{R}}, (x^r)_{r \in K})$ and $(\overline{\mathbb{R}}, e^x)$ are canonical objects in o-minimality and the model theory of expansions of $\overline{\mathbb{R}}$; see [17, 18] and Wilkie [24] for details. The structures $(\overline{\mathbb{R}}, S_\omega)$ are emerging as canonical objects in non-o-minimal tameness in the sense that they are as well behaved as we could reasonably hope for given that they define infinitely spiralling subsets of the plane; see [19, 3.4] for details or 4.4 below for a brief summary. The exclusivity in 1.1 follows from some of these results, as we now show. By [18], $(\overline{\mathbb{R}}, (x^r)_{r \in K})$ defines the power function x^s if and only if $s \in K$, and $(\overline{\mathbb{R}}, (x^r)_{r \in \mathbb{R}})$ is a proper reduct of $(\overline{\mathbb{R}}, e^x)$, which is o-minimal by [24]. Evidently, neither $(\overline{\mathbb{R}}, \mathbb{Z})$ nor any $(\overline{\mathbb{R}}, S_\omega)$ are o-minimal. By [19, Corollary to Theorem 3.4.2] (or 4.4 below), every subset of $\mathbb{R}$ definable in $(\overline{\mathbb{R}}, S_\omega)$ either has interior or is nowhere dense. Thus, $(\overline{\mathbb{R}}, S_\omega)$ does not define $\mathbb{Q}$, hence also not $\mathbb{Z}$. Moreover, the proof of 1.2 below shows that ω is unique up to nonzero rational multiples.

The structure $(\overline{\mathbb{R}}, \mathbb{Z})$ defines every trajectory of any vector field, indeed, it defines every real projective set (of any arity); see Kechris [14, (37.6)]. Thus, $(\overline{\mathbb{R}}, \mathbb{Z})$ is also canonical in a sense, but not a good one as far as model theory is concerned.

All outcomes in 1.1 occur, as will become apparent during the course of the proof. Indeed, each can be realized with $\mathcal{G}$ a singleton provided that K is finitely generated. Thus, the following result of independent interest is logically necessary for 1.1.

1.2. *The collection of structures*

— $(\overline{\mathbb{R}}, (x^r)_{r \in K})$, *$K$ a subfield of $\mathbb{R}$*
— $(\overline{\mathbb{R}}, e^x)$
— $(\overline{\mathbb{R}}, S_\omega)$, $\omega \in \mathbb{R}^*$
— $(\overline{\mathbb{R}}, \mathbb{Z})$

is closed under amalgamation.[1]

The proof uses a recent result of Hieronymi; for convenience, we recall the statement. Given $\alpha \in \mathbb{R}^{>0}$, put $\alpha^{\mathbb{Z}} = \{ \alpha^k : k \in \mathbb{Z} \}$.

1.3 ([13]). *If $\alpha, \beta \in \mathbb{R}^{>0}$ are such that $\log \alpha$ and $\log \beta$ are $\mathbb{Q}$-linearly independent, then $(\overline{\mathbb{R}}, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$ defines $\mathbb{Z}$.*

[1]That is, if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ belong to the collection, then so does the expansion of $\mathfrak{R}_1$ by all sets definable in $\mathfrak{R}_2$.

(This has resolved a number of formerly-open issues, including what should be the conclusion of 1.1.)

Proof of 1.2. As $(\overline{\mathbb{R}},\mathbb{Z})$ defines all real projective sets, all of the structures are reducts of $(\overline{\mathbb{R}},\mathbb{Z})$. Evidently, $(\overline{\mathbb{R}},(x^r)_{r\in\mathbb{R}})$ is a reduct of $(\overline{\mathbb{R}},e^x)$. For every $A\subseteq\mathbb{R}$ we have $(\overline{\mathbb{R}},(x^r)_{r\in A})=(\overline{\mathbb{R}},(x^r)_{r\in\mathbb{Q}(A)})$, where $\mathbb{Q}(A)$ is the subfield of $\mathbb{R}$ generated by A.

Let $\omega\in\mathbb{R}^*$. The projection on the first coordinate of the intersection of S_ω with the positive x-axis is equal to $\alpha^{\mathbb{Z}}$ with $\alpha=e^{2\pi/\omega}$, so $(\overline{\mathbb{R}},x^r,S_\omega)$ defines both $\alpha^{\mathbb{Z}}$ and $\alpha^{r\mathbb{Z}}$. If r is irrational, then $(\overline{\mathbb{R}},x^r,S_\omega)=(\overline{\mathbb{R}},\mathbb{Z})$ by 1.3. Similarly, if $\omega,\tau\in\mathbb{R}^*$ and $\tau\notin\omega\mathbb{Q}$, then $(\overline{\mathbb{R}},S_\omega,S_\tau)=(\overline{\mathbb{R}},\mathbb{Z})$. On the other hand, if $\tau=\omega q$ for some $q\in\mathbb{Q}$ (necessarily nonzero), then $(\overline{\mathbb{R}},S_\omega,S_\tau)=(\overline{\mathbb{R}},S_\omega)$. To see this, observe that for each $t>0$ there is a unique point in S_ω of modulus t, namely, $(t\cos(\omega\log t),t\sin(\omega\log t))$. Thus, $(\overline{\mathbb{R}},S_\omega)$ defines the map

$$(\cos(\omega\log x),\sin(\omega\log x))\colon\mathbb{R}^{>0}\to\mathbb{R}^2,$$

hence also the map $(x^{1/q}\cos(\omega\log x),x^{1/q}\sin(\omega\log x))\colon\mathbb{R}^{>0}\to\mathbb{R}^2$, the image of which is equal to S_τ. (Reparameterize by e^{qx}.) Thus, $(\overline{\mathbb{R}},S_\omega)$ defines S_τ. ☐

We postpone the rest of the proof of 1.1 to Section 2.

The assumption in 1.1 that every $\Gamma\in\mathcal{G}$ be the image of a solution on an unbounded interval is necessary. To illustrate, let $r\in\mathbb{R}$. The graph of the restriction $x^r\restriction[1,2]$ is a trajectory of $\left(\begin{smallmatrix}1&0\\0&r\end{smallmatrix}\right)$. By van den Dries [4], $(\overline{\mathbb{R}},x^r\restriction[1,2])$ is o-minimal and defines no irrational power functions. By quantifier elimination and analytic continuation, $x^r\restriction[1,2]$ is definable in $\overline{\mathbb{R}}$ if and only if $r\in\mathbb{Q}$. Hence, the conclusion of 1.1 fails for $(\overline{\mathbb{R}},x^r\restriction[1,2])$ if r is irrational. It is conjectured that the topological assumption ("locally closed") is necessary, but this is open as yet; we discuss this in more detail in Section 3 below. In any case, we remove both of the extra assumptions by working over a richer ground structure, as we show next.

Let $\mathbb{R}^{\mathrm{RE}}$ denote the expansion of $\overline{\mathbb{R}}$ by the restrictions to $[0,1]$ of e^x and $\sin x$. ("RE" is short for "restricted elementary"; see [5].) It is an easy exercise to see that $\sin\restriction[0,1]$ is interdefinable with arctan over $\overline{\mathbb{R}}$; we use either as convenient.

1.4. **Theorem.** *Let $\mathcal{G}$ be a collection of trajectories of linear vector fields. Then $(\mathbb{R}^{\mathrm{RE}},(\Gamma)_{\Gamma\in\mathcal{G}})$ is equal to exactly one of the following*:

— $(\mathbb{R}^{\mathrm{RE}},(x^r)_{r\in K})$ *for some subfield K of $\mathbb{R}$*
— $(\overline{\mathbb{R}},\arctan,e^x)$
— $(\mathbb{R}^{\mathrm{RE}},S_\omega)$ *for some $\omega\in\mathbb{R}^*$*
— $(\overline{\mathbb{R}},\mathbb{Z})$.

Proof. By [4], $\mathbb{R}^{\mathrm{RE}}$ is o-minimal and defines no irrational power functions, so the proof of exclusivity is the same as for 1.1. All restrictions of e^x, $\sin x$ and $\cos x$ to bounded intervals are definable in $\mathbb{R}^{\mathrm{RE}}$, hence so are all *solutions* (not just their images) on bounded intervals to linear vector fields (see [21, p. 42 Corollary]). It follows easily from recent work of Tychonievich [23] that $\mathbb{Z}$ is definable if some $\Gamma\in\mathcal{G}$ is not locally closed; see 2.4 below. The rest is immediate from 1.1. ☐

When combined with growth dichotomy [17] and results from [16, Chapter IV] (see 4.1 below for the statement), we immediately obtain:

1.5. **Theorem.** *Let $\mathcal{G}$ be a collection of trajectories of linear vector fields and $\mathfrak{R}$ be an o-minimal expansion of $\mathbb{R}^{\mathrm{RE}}$. Then exactly one of the following holds for $\mathfrak{R}' := (\mathfrak{R}, (\Gamma)_{\Gamma\in\mathcal{G}})$.*

— *$\mathfrak{R}$ does not define e^x and there is a subfield K of $\mathbb{R}$ such that $\mathfrak{R}' = (\mathfrak{R}, (x^r)_{r\in K})$.*
— *$\mathfrak{R}' = (\mathfrak{R}, e^x)$.*
— *$\mathfrak{R}$ defines no irrational powers and there exists $\omega \in \mathbb{R}^*$ such that $\mathfrak{R}' = (\mathfrak{R}, S_\omega)$.*
— *$\mathfrak{R}' = (\overline{\mathbb{R}}, \mathbb{Z})$.*

In each of the first three outcomes, the sets definable in $\mathfrak{R}'$ are as well behaved relative to those definable in $\mathfrak{R}$ as one could reasonably hope for, in particular, $\mathfrak{R}'$ is o-minimal in the first two. It would take us too far afield to give details here, but I provide a summary of the key results in an appendix (Section 4). Hence, loosely speaking, either $\mathfrak{R}'$ is as wild as possible or it is as tame over $\mathfrak{R}$ as reasonably possible. In particular,

1.6. *If $\mathfrak{R}$ defines any irrational power functions, then $\mathfrak{R}'$ either is o-minimal or defines $\mathbb{Z}$.*

Remarks. (i) $\mathbb{R}^{\mathrm{RE}}$ is interesting in its own right; see [5]. (ii) By Bianconi [2], $(\overline{\mathbb{R}}, \arctan)$ does not define $e^x\restriction[0,1]$, nor does $(\overline{\mathbb{R}}, e^x)$ define arctan. (iii) By Pfaffian closure (see Speissegger [22]), if $\mathfrak{M}$ is an o-minimal expansion of $\overline{\mathbb{R}}$, then so is $(\mathfrak{M}, e^x, \arctan)$, but it is not yet known if the field of exponents of $\mathfrak{M}$ is necessarily preserved under expanding by $e^x\restriction[0,1]$ or arctan, nor even if polynomial boundedness is necessarily preserved. Thus, although we can apply 1.5 with $\mathfrak{R} = (\mathfrak{M}, e^x\restriction[0,1], \arctan)$, we do not yet know what to conclude over $\mathfrak{M}$. (iv) Currently, we know of no examples of expansions of o-minimal structures $\mathfrak{R}$ on $(\overline{\mathbb{R}}, e^x)$ by collections $\mathcal{G}$ of trajectories of C^1 vector fields definable in $\mathfrak{R}$ such that $(\mathfrak{R}, (\Gamma)_{\Gamma\in\mathcal{G}})$ neither is o-minimal nor defines $\mathbb{Z}$.

Theorem 1.5 characterizes expansions of o-minimal structures on $\mathbb{R}^{\mathrm{RE}}$ by collections of trajectories of linear vector fields. Another paper is currently in preparation, joint with Patrick Speissegger, on the next natural level of complexity: expansions by collections of trajectories near isolated singularities of definable planar vector fields, with emphasis on the analytic category. We already have a result similar to 1.5, but more care is needed for its statement. Applications of 1.5 to the analysis of trajectories of definable vector fields of arbitrary arity are hoped for, but the situation is unclear as yet even for $n = 3$.

Here is an outline of the rest of this paper. We prove 1.1 in Section 2. We discuss issues of optimality in Section 3. In Section 4, an appendix, we recall for the reader's convenience some results from other sources.

2. Proof of 1.1

First, we declare an important convention and note some associated facts. Throughout, we tend to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and use complex notation, though differentiability is always taken in the real sense. *In particular:* "coordinate" means "complex coordinate" until further notice. For $a + ib \in \mathbb{C}$ and $t > 0$, put $t^{a+ib} = t^a(\cos(b\log t) + i\sin(b\log t))$. Observe that the function $x^{a+ib}\colon \mathbb{R}^{>0} \to \mathbb{C}$ is the restriction to $\mathbb{R}^{>0}$ of an appropriate branch of the complex power function z^{a+ib}, so we may call x^{a+ib} a power function. We identify x^{a+i0} with the real power function x^a whenever appropriate. If $c \in \mathbb{R}$, then $x^{a+ib} \circ x^c = x^{ac+ibc}$, but $x^{a+ib} \circ x^{c+id}$ does not even make sense unless $d = 0$. Note the following facts; they will be used often (indeed, some we have used already).

2.1. *Let* $a, b \in \mathbb{R}^*$.
— $\{\, t^{a+ib} : t \in \mathbb{R}^{>0} \,\} = S_{b/a}$
— $(\overline{\mathbb{R}}, x^{a+ib}) = (\overline{\mathbb{R}}, x^a, x^{ib})$
— $(\overline{\mathbb{R}}, x^{ib}) = (\overline{\mathbb{R}}, x^{ib}{\restriction}[1,2], e^{2\pi\mathbb{Z}/b})$
— *If* $a \notin \mathbb{Q}$, *then* $(\overline{\mathbb{R}}, x^{a+ib}) = (\overline{\mathbb{R}}, \mathbb{Z})$.
— *For* $w \in \mathbb{C}$, $(\overline{\mathbb{R}}, S_b)$ *defines* x^w *if and only if* $w \in \mathbb{Q} + ib\mathbb{Q}$.

Proof. Observe that: (i) $x^{1+ib/a}$ is the unique point of modulus x in $S_{b/a}$; (ii) $\left|x^{a+ib}\right| = x^a$ and $x^{ib} = x^{a+ib}/x^a$; (iii) every $x > 0$ can be written uniquely as cy for some $c \in [1, e^{2\pi/|b|})$ and $y \in e^{2\pi\mathbb{Z}/|b|}$; (iv) all restrictions of x^{ib} to compact subintervals of $\mathbb{R}^{>0}$ are definable in $(\overline{\mathbb{R}}, x^{ib}{\restriction}[1,2])$. The rest follows easily from 1.3 and the proof of 1.2. □

We now begin the proof proper. Let $\mathcal{G}$ be a collection of locally closed trajectories of linear vector fields such that each $\Gamma \in \mathcal{G}$ is the image of a solution on an unbounded interval. We must show that $(\overline{\mathbb{R}}, (\Gamma)_{\Gamma\in\mathcal{G}})$ is one of the following structures:
— $(\overline{\mathbb{R}}, (x^r)_{r\in K})$ for some subfield K of $\mathbb{R}$
— $(\overline{\mathbb{R}}, e^x)$
— $(\overline{\mathbb{R}}, S_\omega)$ for some $\omega \in \mathbb{R}^*$
— $(\overline{\mathbb{R}}, \mathbb{Z})$.

It suffices to do the case that $\mathcal{G}$ consists of a single trajectory Γ (by 1.2), and to show that $(\overline{\mathbb{R}}, \Gamma)$ is one of the following structures:
— $(\overline{\mathbb{R}}, (x^r)_{r\in R})$ for some finite $R \subseteq \mathbb{R}$
— $(\overline{\mathbb{R}}, e^x)$
— $(\overline{\mathbb{R}}, x^{i\omega})$ for some $\omega \in \mathbb{R}^*$
— $(\overline{\mathbb{R}}, \mathbb{Z})$.

There is a real $n \times n$ matrix M, a column vector $c \in \mathbb{R}^n$, and an unbounded interval $I \subseteq \mathbb{R}$ such that $\Gamma = \{\, e^{Mt}c : t \in I \,\}$. By replacing M with $M + iM$, c with $c + ic$, and Γ with $\Gamma + i\Gamma$, we reduce to the case that M is an $n \times n$ complex matrix for some $n \geq 1$, and we work over $\mathbb{C}$. Thus, for computation, we regard $\Gamma \subseteq \mathbb{C}^n$ as a (real-time) trajectory of the vector field $\mathbb{C}^n \to \mathbb{C}^n$ arising from M, but we also regard Γ as a subset of $\mathbb{R}^{2n}$ in the usual way. (For information on complexification and decomplexification of linear systems, see [1, §20].) By replacing Γ with its image under some invertible linear transformation, we reduce to the case that M is in Jordan normal form (over $\mathbb{C}$). Thus, there exist
— a nonnegative integer J
— nonnegative integers $n_0, \dots, n_J$ such that $n_0 + \cdots + n_J = n$ and $n_1, \dots, n_J \geq 2$
— an $n_0 \times n_0$ diagonal matrix M_0 (allowing $n_0 = 0$)
— $n_j \times n_j$ Jordan blocks M_j for $j = 1, \dots, J$ (allowing $J = 0$)
— column vectors $c_j \in \mathbb{C}^{n_j}$ for $j = 0, \dots, J$

such that $(\overline{\mathbb{R}}, \Gamma)$ is equal to the expansion of $\overline{\mathbb{R}}$ by the image of I under the map
$$\gamma := \left(e^{M_0x}c_0, \dots, e^{M_Jx}c_J\right) : \mathbb{R} \to \mathbb{C}^{n_0} \times \cdots \times \mathbb{C}^{n_J} \cong \mathbb{C}^n.$$
(In this setting, a Jordan block is a complex square matrix such that all diagonal entries are equal, all superdiagonal entries are equal to 1, and all other entries are 0.

2.2. By taking coordinate projections of Γ, we may eliminate any constant component functions of γ as we see fit (provided that Γ remains a trajectory of a linear vector field). Thus, we may reduce to the case that all diagonal entries of M_0 are nonzero, all coordinates of c_0 are nonzero, and for each $j = 1, \dots, J$, the last coordinate of c_j is nonzero. (If the last coordinate of c_j is 0, then $e^{M_j x} c_j = (e^{Bx} b, 0)$, where B is the result of deleting the last row and column from M_j and b is the result of deleting the last coordinate of c_j. Observe that B is again a Jordan block.)

2.3. By 2.2, we may replace Γ with its image under some invertible diagonal transformation to reduce to the case that:

— Every coordinate function of $e^{M_0 x} c_0$ is of the form $e^{\lambda x}$ where λ is a simple (that is, of multiplicity 1) nonzero eigenvalue of M.

— For each $j = 1, \dots, J$, the projection of $e^{M_j x} c_j$ on its last two coordinates is of the form $((\xi + x)e^{\lambda x}, e^{\lambda x})$, where λ is a nonsimple eigenvalue of M and $\xi \in \mathbb{C}$. (Divide $e^{M_j x} c_j$ by the last coordinate of c_j.)

The proof now proceeds by a case distinction on the nature of the set of eigenvalues of M; we outline the steps in order to motivate the details. If all eigenvalues are simple, then M is diagonal, which then involves a few subcases. If M has a nonsimple nonreal eigenvalue, then, after permuting coordinates, the projection of Γ on the last two coordinates defines $\mathbb{Z}$ over $\overline{\mathbb{R}}$. Thus, we reduce to the case that all nonreal eigenvalues of M are diagonal entries of M_0. If some M_j arises from a nonsimple nonzero real eigenvalue, then again after permuting coordinates, the projection of Γ on the last two coordinates defines e^x over $\overline{\mathbb{R}}$, allowing an easy finish by consideration of the possible outcomes of the diagonal case. We are then left only with the case that 0 is a nonsimple eigenvalue of M and all other eigenvalues of M are diagonal entries of M_0. Hence, the map $\left(e^{M_1 x} c_1, \dots, e^{M_J x} c_J\right)$ is nontrivial and polynomial (by nilpotence), and the projection of Γ on the last two coordinates is equal to $\{\, (\xi + t, 1) : t \in I \,\}$ for some $\xi \in \mathbb{C}$. Thus, if M_0 is trivial (that is, 0×0), then $(\overline{\mathbb{R}}, \Gamma) = \overline{\mathbb{R}}$. Otherwise, $(\overline{\mathbb{R}}, \Gamma)$ defines the map $e^{M_0 x} c_0 \restriction I$ (not just its image), allowing us to finish by an easy case distinction on the eigenvalues of M_0. We now proceed to details via a sequence of lemmas.

Put $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

2.4. *Let* $X = \{\, (\xi_1 e^{ib_1 t}, \dots, \xi_m e^{ib_m t}) : t \in I \,\}$, *where* $b_1, \dots, b_m \in \mathbb{R}$ *and* $\xi_1, \dots, \xi_m \in \mathbb{C}^*$. *The following are equivalent.*

(1) *There exist* $j, k \in \{1, \dots, m\}$ *such that* $b_k \notin b_j \mathbb{Q}$.
(2) X *is not definable in* $\overline{\mathbb{R}}$.
(3) X *is not locally closed.*
(4) X *is dense and co-dense in* $|\zeta_1|\, S^1 \times \cdots \times |\zeta_m|\, S^1$.
(5) $(\overline{\mathbb{R}}, X, \arctan) = (\overline{\mathbb{R}}, \mathbb{Z})$.

Proof. As $\xi_1, \dots, \xi_m \neq 0$, it suffices to consider the case that $\xi_1 = \cdots = \xi_m = 1$.

The equivalence of the first four items is essentially just a rephrasing of known ODE facts (see [1, §24]). The o-minimality of $(\overline{\mathbb{R}}, \arctan)$ yields (5)⇒(2), so we also have (5)⇒(1). It suffices now to show that (1)⇒(5). Assume that, say, $b_1 \neq 0$ and b_2/b_1 is irrational. Let

G be the subgroup of S^1 generated by $e^{i2\pi b_2/b_1}$. Let πX denote the projection of X on the first two coordinates. Since I is unbounded, the set

$$\{\, z \in \mathbb{C} : (1,z) \in \pi X \,\} \cup \{\, \overline{z} \in \mathbb{C} : (1,z) \in \pi X \,\}$$

is a cofinite subset of G, so $(\overline{\mathbb{R}}, X)$ defines G. By [23], $(\overline{\mathbb{R}}, G, \arctan)$ defines $\mathbb{Z}$, so the same is true of $(\overline{\mathbb{R}}, X, \arctan)$. $\square$

In addition to 2.1 and 2.3, the following easy observations should now be kept in mind.

2.5. *Let $a > 0$.*

— $(\overline{\mathbb{R}}, x^w) = (\overline{\mathbb{R}}, x^w\restriction(0,a)) = (\overline{\mathbb{R}}, x^w\restriction(a,\infty))$ *for each* $w \in \mathbb{C}$.
— $(\overline{\mathbb{R}}, e^x) = (\overline{\mathbb{R}}, e^x\restriction(-\infty,a)) = (\overline{\mathbb{R}}, e^x\restriction(a,\infty))$.
— $(\overline{\mathbb{R}}, \alpha^{\mathbb{Z}}) = (\overline{\mathbb{R}}, \alpha^{\mathbb{Z}} \cap (0,a)) = (\overline{\mathbb{R}}, \alpha^{\mathbb{Z}} \cap (a,\infty))$ *for every* $\alpha > 0$.
— $(\overline{\mathbb{R}}, S_\omega) = (\overline{\mathbb{R}}, S_\omega \cap \{\, z \in \mathbb{C} : |z| < a \,\}) = (\overline{\mathbb{R}}, S_\omega \cap \{\, z \in \mathbb{C} : |z| > a \,\})$ *for every* $\omega \in \mathbb{R}^*$.
— $(\overline{\mathbb{R}}, \mathbb{Z}) = (\overline{\mathbb{R}}, \mathbb{Z} \cap (-\infty,a)) = (\overline{\mathbb{R}}, \mathbb{Z} \cap (-\infty,a))$.

2.6. *If M is diagonal, then $(\overline{\mathbb{R}}, \Gamma)$ is equal to one of the following*:

— $(\overline{\mathbb{R}}, (x^r)_{r\in R})$ *for some finite* $R \subseteq \mathbb{R}$
— $(\overline{\mathbb{R}}, x^{i\omega})$ *for some* $\omega \in \mathbb{R}^*$
— $(\overline{\mathbb{R}}, \mathbb{Z})$.

Proof. We have $\Gamma = \{\, (e^{\lambda_1 x}, \dots, e^{\lambda_n x}) : x \in I \,\}$ where $\lambda_1, \dots, \lambda_n \in \mathbb{C}^*$. If $(\overline{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$, then we are done, so assume otherwise. Recall 2.1.

If all λ_k are purely imaginary, then $(\overline{\mathbb{R}}, \Gamma) = \overline{\mathbb{R}}$ by 2.4 (Γ is locally closed), so assume that at least one λ_j has nonzero real part.

Suppose that some λ_j is real. By permuting coordinates and reparameterizing, we may take $\lambda_1 = 1$. Then $\Gamma = \{\, (x, x^{\lambda_2}, \dots, x^{\lambda_n}) : x \in e^I \,\}$, so

$$(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, x^{\operatorname{Re}\lambda_2}, \dots, x^{\operatorname{Re}\lambda_n}, x^{i\operatorname{Im}\lambda_2}, \dots, x^{i\operatorname{Im}\lambda_n}).$$

If $\lambda_2, \dots, \lambda_n \in \mathbb{R}$, then we are done, so assume that $\lambda_2 \notin \mathbb{R}$. As we have assumed that $\mathbb{Z}$ is not definable, we have $\lambda_2, \dots, \lambda_n \in \mathbb{Q} + i(\operatorname{Im}\lambda_2)\mathbb{Q}$, so $(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, x^{i\operatorname{Im}\lambda_2})$.

Suppose that no λ_j is real. After permuting coordinates and reparameterizing, we write $\lambda_1 = 1 + i\omega$ with $\omega \in \mathbb{R}^*$. We show that $(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, x^{i\omega})$. By 1.2 and an easy induction, it is enough to do the case $n = 2$, that is, $\Gamma = \{\, (e^{(1+i\omega)t}, e^{(a+ib)t}) : t \in I \,\}$ for some $a, b \in \mathbb{R}$ with $b \neq 0$. It suffices now to show that $a \in \mathbb{Q}$ and $b \in \omega\mathbb{Q}$. The projection of Γ on its first coordinate defines S_ω over $\overline{\mathbb{R}}$, so $(\overline{\mathbb{R}}, \Gamma)$ defines $e^{2\pi\mathbb{Z}/\omega}$. If $a = 0$, then $\{\, |z| : (z,1) \in \Gamma \,\} = \{\, e^{2\pi k/b} : k \in \mathbb{Z} \cap I \,\}$. Since $(\overline{\mathbb{R}}, \Gamma)$ does not define $\mathbb{Z}$, we have $b \in \omega\mathbb{Q}$. Suppose that $a \neq 0$. The projection of Γ on its last coordinate then defines $S_{b/a}$, so $b/a \in \omega\mathbb{Q}$ and $\Gamma = \{\, (e^{(1+i\omega)t}, e^{(1+i\omega)at}) : t \in I \,\}$. Now, for every $x \in e^I$, there are unique $z, w \in \mathbb{C}$ such that $(z,w) \in \Gamma$ and $|z| = x$; for this w, we have $|w| = x^a$. Thus, $(\overline{\mathbb{R}}, \Gamma)$ defines the function x^a, so $a \in \mathbb{Q}$, which in turn yields $b \in \omega\mathbb{Q}$ (since $b/a \in \omega\mathbb{Q}$). $\square$

2.7. *Suppose that* $M = \left(\begin{smallmatrix} a+ib & 1 \\ 0 & a+ib \end{smallmatrix}\right)$ *with* $a, b \in \mathbb{R}$.

(1) *If $b \neq 0$, then* $(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, \mathbb{Z})$.

(2) *If $a \neq 0$ and $b = 0$, then $(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, e^x)$.*

Proof. We have $\Gamma = \{\, ((\xi + t)e^{(a+ib)t}, e^{(a+ib)t}) : t \in I \,\}$ for some $\xi \in \mathbb{C}$.

(1). Suppose that $b \neq 0$. If $a = 0$, then

$$\{\, z \in \mathbb{C} : (\xi + z, 1) \in \Gamma \,\} = \{\, 2\pi k/b : k \in \mathbb{Z} \cap I \,\},$$

so $(\overline{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$. Suppose that $a \neq 0$. By projecting on the second coordinate, $(\overline{\mathbb{R}}, \Gamma)$ defines $S_{b/a}$. For every $x \in e^I$, $\log x$ is the unique $y \in \mathbb{R}$ for which there exists $z \in \mathbb{C}$ such that $|z| = x$ and $((\xi + y/a)z, z) \in \Gamma$. Thus, $(\overline{\mathbb{R}}, \Gamma)$ defines both $S_{b/a}$ and log, hence also $\mathbb{Z}$.

(2). Suppose that $a \neq 0$ and $b = 0$. Reparameterizing and interchanging the coordinates of Γ yields the graph of the function $\xi x + (1/a)x \log x \restriction e^I$, which is interdefinable over $\overline{\mathbb{R}}$ with e^x. Hence, $(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, e^x)$. $\square$

Remark. Of course, $(\overline{\mathbb{R}}, \Gamma) = \overline{\mathbb{R}}$ if $a = b = 0$.

2.8. *Suppose that $M = \begin{pmatrix} a+ib & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ with $a, b \in \mathbb{R}$.*

— *If $b \neq 0$, then $(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, \mathbb{Z})$.*
— *If $b = 0$, then $(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, e^x)$.*

Proof. We have $\Gamma = \{\, (e^{(a+ib)t}, \xi + t, 1) : t \in I \,\}$ for some $\xi \in \mathbb{C}$, so $(\overline{\mathbb{R}}, \Gamma)$ defines the function $e^{(a+ib)x} \restriction I$. Recall that $a + ib \neq 0$ (2.2)k. The rest is immediate. $\square$

We are now ready to finish the

Proof of Theorem 1.1. If M is diagonal, then we are done by 2.6. Suppose that M is not diagonal. Then $J \geq 1$ and $M_1, \ldots, M_J$ arise from nonsimple eigenvalues of M.

Suppose that M has a nonsimple, nonreal eigenvalue $a+ib$. After permuting coordinates, we may assume that $a + ib$ is an eigenvalue of M_J. Project Γ on its last two coordinates and apply 2.7 to obtain that $(\overline{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$.

By the preceding paragraph, we may reduce to the case that all nonreal eigenvalues of M are simple. Note that by basic ODE theory, the map

$$x \mapsto \big(e^{M_1 x} c_1, \ldots, e^{M_J x} c_J\big) : \mathbb{R} \to \mathbb{C}^{n-n_0}$$

is then definable in $(\overline{\mathbb{R}}, e^x)$, hence so is the projection of Γ on the last $n - n_0$ coordinates.

Suppose that M has a nonsimple, nonzero real eigenvalue a. By permuting coordinates, we may assume that a is an eigenvalue of M_J. Project Γ on its last two coordinates and apply 2.7 to obtain that $(\overline{\mathbb{R}}, \Gamma)$ defines e^x. By 2.6, $(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, \mathbb{Z})$ if M_0 has a nonreal eigenvalue, and $(\overline{\mathbb{R}}, \Gamma) = (\overline{\mathbb{R}}, e^x)$ otherwise.

Finally, suppose that 0 is the only nonsimple eigenvalue of M. Then the map

$$\big(e^{M_1 x} c_1, \ldots, e^{M_J x} c_J\big)$$

is polynomial, so the projection of Γ on the last $n - n_0$ coordinates is definable in $\overline{\mathbb{R}}$. Hence, $(\overline{\mathbb{R}}, \Gamma) = \overline{\mathbb{R}}$ if M_0 is trivial. Suppose that M_0 is nontrivial. By 2.8 and 2.6, $(\overline{\mathbb{R}}, \Gamma)$ is equal to $(\overline{\mathbb{R}}, \mathbb{Z})$ or $(\overline{\mathbb{R}}, e^x)$, depending on whether or not M_0 has a nonreal eigenvalue. $\square$

3. Optimality

It is clear from its proof that, with enough patience, a version of 1.1 can be stated that describes precisely the outcomes in terms of the collection of eigenvalues of the matrices that give rise to the trajectories in $\mathcal{G}$, provided that the trajectories and matrices are suitably normalized (recall 2.2). Similarly, "definable" can be refined to "$\emptyset$-definable" uniformly with respect to $\mathcal{G}$ by tracking parameters and adding constants as needed to the structures in the conclusion of 1.1. We leave formulations of these variants to the interested reader.

It is suspected that "locally closed" is necessary in 1.1, but this is open as yet. The proof of 1.1 shows that, essentially, the problem is to understand what happens if some $\Gamma \in \mathcal{G}$ are complex multiplicative groups $\{ (e^{it}, e^{ib_2 t}, \dots, e^{ib_n t}) : t \in \mathbb{R} \}$ where $n \geq 2$ and b_2 is irrational. By the proof of 2.4, $(\overline{\mathbb{R}}, \Gamma)$ then defines an infinite cyclic subgroup G of S^1. Hence, it also defines a dense-and-codense subset of $[-1, 1]$, so the first three outcomes of 1.1 fail for $(\overline{\mathbb{R}}, \Gamma)$. It is suspected that $(\overline{\mathbb{R}}, \Gamma)$ does not define $\mathbb{Z}$—at least, for generic enough choices of $b_2, \dots, b_n$—so that the conclusion of 1.1 fails. For example, suppose that $n = 2$ and b_2 is not $\emptyset$-definable in $(\overline{\mathbb{R}}, e^x)$. The current conjecture is that $(\overline{\mathbb{R}}, \Gamma)$ defines no new (relative to $\overline{\mathbb{R}}$) open sets; work on this is currently underway by other researchers (Caycedo, Günaydın and Hieronymi). If so, then it is fair to say that $(\overline{\mathbb{R}}, \Gamma)$ is as topologically tame as possible; see Dolich *et al.* [3], Miller and Speissegger [20], and [19, §5] for more information. It is worth noting that although $(\overline{\mathbb{R}}, \arctan, G)$ defines $\mathbb{Z}$ by [23], $(\overline{\mathbb{R}}, G)$ defines no new open sets; see Günaydın and Hieronymi [12] for details. Of course, we could also hope to show that the conclusion of 1.1 fails for expansions of $(\overline{\mathbb{R}}, e^x)$ or some $(\overline{\mathbb{R}}, S_\omega)$ by collections of such Γ, but so far, there is no evidence to suggest that any such collections define any new open sets over either $(\overline{\mathbb{R}}, e^x)$ or $(\overline{\mathbb{R}}, S_\omega)$.

It is not yet known what is the optimal version of 1.1 if we drop only the assumption that all trajectories be images of solutions on unbounded intervals. Work on this is currently underway by Tychonievich.

Much of the proof of 1.1 goes through in greater generality with a little extra work. For example, instead of having all $\Gamma \in \mathcal{G}$ be trajectories of linear vector fields, we could consider images of intervals under arbitrary $\mathbb{C}$-linear combinations of functions of the form $x^m e^{\lambda x}$ where $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. However, troubles arising from purely imaginary λ become even more pronounced—*e.g.*, consider functions $x(e^{ib_1 x} + \cdots + e^{ib_n x})$ for rationally independent reals $b_1, \dots, b_n$—thus making for more technical statements. It is unclear to me as yet whether the extra generality is worth formulating as a stand-alone result.

Finally, to what extent might the results of this paper hold over o-minimal expansions of arbitrary real closed fields? I regard this question as unproductive for now, if for no other reason than that the works of Hieronymi [13] and Tychonievich [23] appear to rely heavily on working over $\mathbb{R}$.

4. Appendix

For the reader's convenience, I provide here a brief summary of some results from elsewhere. *The original sources should be consulted for complete information, including history and attributions.*

Let $\mathfrak{R}$ be a polynomially bounded o-minimal expansion of $\overline{\mathbb{R}}$ with field of exponents K (that is, the set of all $r \in \mathbb{R}$ such that $\mathfrak{R}$ defines the power function x^r). By Pfaffian closure [22], $(\mathfrak{R}, \arctan, e^x)$ is o-minimal, hence so are all reducts of $(\mathfrak{R}, \arctan, e^x{\restriction}[0,1])$ over $\mathfrak{R}$. But in this generality, it is not yet known if these reducts also have field of exponents K, nor even if they are polynomially bounded. This lack of knowledge accounts for some awkwardness in the statements below.

For technical reasons, put $x^r = 0$ for $x \leq 0$.

4.1 ([16]). *Let $S \subseteq \mathbb{R}$ be such that $\mathfrak{R}$ defines $x^s{\restriction}[1,2]$ for each $s \in S$.*

(1) *Relative to $\mathfrak{R}$, the expansion $(\mathfrak{R}, (x^r)_{r \in K(S)})$ admits QE (quantifier elimination) and is explicitly universally axiomatizable.*
(2) *Every function definable in $(\mathfrak{R}, (x^s)_{s \in S})$ is given piecewise by compositions of powers x^r with $r \in K(S)$ and functions definable in $\mathfrak{R}$.*
(3) *$(\mathfrak{R}, (x^s)_{s \in S})$ is polynomially bounded with field of exponents $K(S)$.*

(This has not yet been published elsewhere in this form.)

Sketch of proof. (1). By definability of Skolem functions [7, p. 94], we may assume that $\mathfrak{R}$ admits quantifier elimination and is universally axiomatizable. The rest is a routine modification of the proof of [18, 2.5], using [6, Theorem C] instead of [18, 1.2]. (In the proof of [18, 2.4], use K instead of $\mathbb{Q}$, and the reduct of A to the language of $\mathfrak{R}$ instead of A_{an}.)

(2) is a standard model-theoretic consequence of (1); see *e.g.* van den Dries *et al.* [8].

(3). Without loss of generality, we may assume that $\mathfrak{R}$ has no relation symbols other than $<$. By [8, 5.8], we regard the Hardy field H of $(\mathfrak{R}, (x^r)_{r \in K(S)})$ as an elementary extension of $(\mathfrak{R}, (x^r)_{r \in K(S)})$. In the second and third paragraphs of the proof of [18, 2.4], again modified as above, let x be the germ of the identity function on $\mathbb{R}$. Put $A = (\mathfrak{R}, (x^r)_{r \in K(S)})$ and $B = H$. Then the resulting structure C is a model of $\mathrm{Th}(\mathfrak{R}, (x^r)_{r \in K(S)})$ containing $\mathbb{R}(x)$ as a Hardy field. On the other hand, by [8, 5.8], H is the smallest model of $\mathrm{Th}(\mathfrak{R}, (x^r)_{r \in K(S)})$ containing $\mathbb{R}(x)$. Hence, $C = H$, $v(H) = K(S).v(x)$, and $K(S)$ is the field of exponents of $(\mathfrak{R}, (x^r)_{r \in K(S)})$. $\square$

Remark. By [8, 5.5 and 5.12], $(\mathfrak{R}, (x^r)_{r \in K(S)})$ is o-minimal without appeal to Pfaffian closure.

The next result is due to van den Dries and Speissegger. Set $\log x = 0$ for $x \leq 0$.

4.2. *Suppose that $\mathfrak{R}$ defines $e^x{\restriction}[0,1]$.*

(1) *Relative to $\mathfrak{R}$, the expansion $(\mathfrak{R}, e^x, \log)$ admits QE and is explicitly universally axiomatizable* [10, Theorem B].
(2) *Every function definable in $(\mathfrak{R}, e^x)$ is given piecewise by compositions of e^x,* log, *and functions definable in $\mathfrak{R}$.*
(3) *$(\mathfrak{R}, e^x)$ is exponentially bounded* [10, 9.6].

Again, (2) is a routine consequence of (1), and the proof of (1) yields o-minimality of $(\mathfrak{R}, e^x)$ without appeal to Pfaffian closure. On the other hand, (3) holds without assuming that $\mathfrak{R}$ defines $e^x{\restriction}[0,1]$ by Pfaffian closure and more precise results from Lion *et al.* [15].

Note that every real restricted power $x^r\restriction[1,2]$ is definable in $(\overline{\mathbb{R}}, e^x\restriction[0,1])$, so the conclusion of 4.1 holds for any $S \subseteq \mathbb{R}$.

See [11] for more detailed statements related to 4.1 and 4.2.

Given $\alpha > 0$, define the function $\lfloor\ \rfloor_\alpha \colon \mathbb{R} \to \mathbb{R}$ by $\lfloor x \rfloor_\alpha = 0$ if $x \leq 0$, and $\lfloor x \rfloor_\alpha = \max\bigl(\alpha^{\mathbb{Z}} \cap (-\infty, x]\bigr)$ for $x > 0$. Note that $(\mathbb{R}, <, \lfloor\ \rfloor_\alpha) = (\mathbb{R}, <, \alpha^{\mathbb{Z}})$.

4.3 ([19]). *Suppose that $K = \mathbb{Q}$.*

(1) *Relative to $\mathfrak{R}$, the expansion $(\mathfrak{R}, \lfloor\ \rfloor_\alpha)$ admits QE and is explicitly universally axiomatizable.*
(2) *Every function definable in $(\mathfrak{R}, \alpha^{\mathbb{Z}})$ is given piecewise by compositions of $\lfloor\ \rfloor_\alpha$ and functions definable in $\mathfrak{R}$.*
(3) *Let p, n be positive integers and $\mathcal{A}$ be a finite collection of subsets of $\mathbb{R}^n$ definable in $(\mathfrak{R}, \alpha^{\mathbb{Z}})$. Then there is a finite partition $\mathcal{P}$ of $\mathbb{R}^n$ into embedded C^p-submanifolds (not required to be connected), each of which is definable in $(\mathfrak{R}, \alpha^{\mathbb{Z}})$, and there is countable decomposition $\mathcal{C}$ of $\mathbb{R}^n$ into C^p-cells definable $\mathfrak{R}$ such that $\mathcal{C}$ is compatible with $\mathcal{P}$, which in turn is compatible with $\mathcal{A}$.*

Note. Antongiulio Fornasiero has found a gap in the proof of (3) as given in [19]; it can be repaired in this case by deriving (3) directly from (1) (and (2)), as opposed to the rather roundabout method suggested in [19].

4.4. *Suppose that $K = \mathbb{Q}$. Let $\omega \in \mathbb{R}^*$ be such that $\mathfrak{R}$ defines $x^{i\omega}\restriction[1,2]$. Let $p \in \mathbb{N}$ and $\mathcal{A}$ be a finite collection of subsets of $\mathbb{R}^n$ definable in $(\mathfrak{R}, S_\omega)$. Then there is a finite partition $\mathcal{P}$ of $\mathbb{R}^n$ into embedded C^p-submanifolds (not required to be connected), each of which is definable in $(\mathfrak{R}, S_\omega)$, and there is countable decomposition $\mathcal{C}$ of $\mathbb{R}^n$ into C^p-cells definable $\mathfrak{R}$ such that $\mathcal{C}$ is compatible with $\mathcal{P}$, which in turn is compatible with $\mathcal{A}$.*

Proof. By 2.1, we have $(\mathfrak{R}, S_\omega) = (\mathfrak{R}, \lfloor\ \rfloor_{e^{2\pi/\omega}})$. Apply 4.3. $\square$

Note that $\mathbb{R}^{\mathrm{RE}}$ defines all restricted complex powers $x^z\restriction[1,2]$. The conclusion of 4.4 should be regarded as a good candidate for the best general tameness condition that could hold for expansions of $\mathfrak{R}$ by infinitely spiralling subsets of the plane. Of course, if $K \neq \mathbb{Q}$, then the conclusion fails spectacularly, as then $(\mathfrak{R}, S_\omega)$ defines $\mathbb{Z}$ by 1.3. Thus, we obtain the following dichotomy:

4.5. *Given any o-minimal expansion $\mathfrak{M}$ of $\overline{\mathbb{R}}$ and $\omega \in \mathbb{R}^*$, either $(\mathfrak{M}, S_\omega)$ defines $\mathbb{Z}$ or the conclusion of 4.4 holds for $(\mathfrak{M}, S_\omega)$.*

Proof. If $(\mathfrak{M}, S_\omega)$ does not define $\mathbb{Z}$, then it defines no irrational powers, so the same is true of $(\mathfrak{M}, x^{i\omega}\restriction[1,2])$. Apply 4.4 with $\mathfrak{R} = (\mathfrak{M}, x^{i\omega}\restriction[1,2])$. $\square$

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