DEFINABLE VERSION OF WEDDERBURN-ARTIN THEOREM IN O-MINIMAL STRUCTURES

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Abstract. Here we work in an arbitrary (but fixed) o-minimal expansion of a dense linearly ordered set. We say that a definable ring is *definably semiprime* if squares of nontrivial two-sided ideals definable in the expansion are non-trivial. We prove a definable version of Wedderburn-Artin Theorem and give a characterization of definably semiprime rings.

Througout this article, d, k, m, n will range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers and let $S = (S, 0_S, 1_S, +_S, \cdot_S)$ denote an associative unitary ring with $1_S \neq 0_S$ (not necessarily commutative). For $a, b \in S$, we simply write $a + b$ instead of $a +_S b$; and ab instead of $a \cdot_S b$ if it is clear from the context. For subsets X and Y of S, we define the product of X and Y by

$$XY = \{\sum_{i=1}^{n} x_i y_i : n \in \mathbb{N}, x_i \in X, y_i \in Y\}$$

(where $\sum_{i=1}^{n} x_i y_i = 0_S$ when $n = 0$). Recall that a left (right) S-module is *simple* if it has no nontrivial left (right) submodules. For each ring T, let $M_n(T)$ denote the ring of $n \times n$ matrices over T.

We say that that S is *simple* if every two-sided ideal of S is either the zero ideal or S; *semisimple* if S considered as a left module over itself is a direct sum of simple left S-modules; and *semiprime* if for every nontrivial two-sided ideal I of S, I^2 is not the zero ideal. Note that S is semisimple if and only if S is semiprime and satisfies the descending chain condition on principal left ideals.

The study of characterization of classes of algebraic objects has been of interests of mathematicians. Wedderburn-Artin Theorem, which is a fundamental theorem in representation theory and noncommutative ring theory, provides a characterization of semisimple rings. By Maschke's Theorem (see e.g. [5]), the Wedderburn-Artin Theorem also gives a structure of group rings SG when G is a finite group and S is a field of characteristic zero. Historically, this theorem was first proved for finitely generated algebras over a field by J.H.M. Wedderburn in 1908 [13]; and later generalized by E. Artin [1] in 1927. Many variations of proofs of Wedderburn-Artin Theorem have been invented since then (see e.g. [3, 4, 6, 7]).

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Next, we will discuss more model theoretic perspectives of the article. Let $\mathfrak{R}$ be an expansion of a densely linearly ordered set without endpoints $(R,<)$. Here, the word "definable" always means "definable with parameters". We say that $\mathfrak{R}$ is *o-minimal* if every subset of R definable in $\mathfrak{R}$ is a finite union of open intervals and points. For more information on o-minimality, we refer to [2]. The ring S is said to be a *definable ring* in $\mathfrak{R}$ if the set S, the addition $+_S$, and the multiplication $\cdot_S$ are definable in $\mathfrak{R}$. In 1988, A. Pillay proved that every infinite definable field is either real closed or algebraically closed (see [11]). A generalization of the above result proved by Y. Peterzil and C. Steinhorn in [10] provides a characterization of infinite definable rings without zero divisors, that is, if S is an infinite ring without zero divisors definable in an o-minimal structure $\mathfrak{R}$, then S is a division ring and there is a definable ring K in $\mathfrak{R}$ such that K is real closed and S is isomorphic to either K, $K(\sqrt{-1})$, or $\mathbb{H}(K)$ (the quaternion ring over K). Note that the above theorem cannot be applied to all semisimple rings. Therefore, the main goal of this article is to study characterization of semisimple definable rings in an o-minimal stucture.

Assume that $\mathfrak{R}$ is an o-minimal structure. Suppose S is a definable ring in $\mathfrak{R}$. The ring S is said to be *definably simple* if every two-sided ideal of S definable in $\mathfrak{R}$ is either the zero ideal or S; and *definably semiprime* if for every nontrivial two-sided ideal I definable in $\mathfrak{R}$, I^2 is not the zero ideal. Obviously, if S is simple (semiprime), then S is definably simple (respectively, definably semiprime). Here we obtain:

Theorem A (Definable Wedderburn-Artin Theorem). *Suppose S is a definable ring in $\mathfrak{R}$.*

(1) If S is definably simple, then S is definably isomorphic to $M_k(D)$ where $k \in \mathbb{N}$ and D is either a finite field, K, $K(\sqrt{-1})$, or $\mathbb{H}(K)$ for some definable real closed field K.

(2) If S is definably semiprime, then S is definably isomorphic to a finite direct sum of $M_{k_1}(D_1), \dots, M_{k_l}(D_l)$ where $k_1, \dots, k_n \in \mathbb{N}$ and each D_i is either a finite field, K, $K(\sqrt{-1})$, or $\mathbb{H}(K)$ for some definable real closed field K.

As a consequence, we obtain that if S is definable in $\mathfrak{R}$ and definably semiprime, then every ideal of S is definable in $\mathfrak{R}$. In [11], A. Pillay proved the descending chain condition on definable subgroups of a definable group in $\mathfrak{R}$. We have the descending chain condition on definable left (right) ideals of S. Since every principal left ideal is definable, by Theorem A, we immediately obtain:

Theorem B. *Suppose S is a definable ring. Then the following are equivalent:*

(1) T is semisimple;

(2) T is semiprime;

(3) T is definably semiprime.

Conventions and notations

Throughout the rest of this article, let $\mathfrak{R}$ be an arbitrary (but fixed) o-minimal structure and assume that S is a definable ring in $\mathfrak{R}$. For each ideal I of S and $a \in I$, we denote by Ia the left ideal $\{sa : s \in I\}$ of S.

1. Ideals definable in $\mathfrak{R}$

Observe that from the definition of the product of two ideals, there is no guarantee whether the product of two ideals definable in $\mathfrak{R}$ is also definable. We can see that the products of ideals is contained in the definition of definably semiprime ring. To understand definably semiprime rings, we first study properties of ideals of S definable in $\mathfrak{R}$.

Let I be a left ideal of S. (Note that the discussion below will work when I is a right ideal of S.) We say that I is *finitely generated* if there exist $a_1, \ldots, a_n \in I$ such that

$$I = \{\sum_{i=1}^{n} s_i a_i : s_1, \ldots, s_n \in S\} = \sum_{i=1}^{n} S a_i.$$

First, we will show that every left ideal definable in $\mathfrak{R}$ is finitely generated. This is a special case of Theorem B in [12]. We include the proof of this special case for the sake of readers.

A group $G = (G, e_G, *_G)$ is a *definable group* in $\mathfrak{R}$ if the set G and the group operation $*_G$ are definable in $\mathfrak{R}$. In [9], Y. Peterzil and S. Starchenko showed that:

1.1. **Lemma.** *If $\mathfrak{R}$ is $\aleph_0$-saturated and G is a group definable in $\mathfrak{R}$, then there exists a finite collection $a_1, \ldots, a_n \in G$, where $n \le \dim G + [G : G^0]$, such that the only definable subgroup of G containing these element must be G.*

Obviously, every left ideal definable in $\mathfrak{R}$ is a definable group.

1.2. **Lemma.** *Suppose I is a left ideal of S definable in $\mathfrak{R}$ and $T \subseteq S$ is a definable ring with respect to the induced ring operations on S.*

(1) Then there exist $a_1, \ldots, a_n \in I$ such that

$$I = \{\sum_{i=1}^{n} t_i a_i : t_1, \ldots, t_n \in T\} = \sum_{i=1}^{n} T a_i.$$

(2) If $It \subseteq I$ for every $t \in T$, then there exist $b_1, \ldots, b_k \in I$ such that

$$I = \{\sum_{i=1}^{k} b_i t_i : t_1, \ldots, t_k \in T\} = \sum_{i=1}^{k} b_i T.$$

In particular, I is finitely generated.

Proof. Let $\mathfrak{R}'$ be an elementary extension of $\mathfrak{R}$ that is $\aleph_0$-saturated. Suppose $S \subseteq R^m$. Let $\bar{x} = (x_1, \ldots, x_m)$ be an m-tuple of pairwise distinct variables. By definability, there exist $\bar{b} \in R^k$ and $\varphi_S(\bar{x}, \bar{y})$, $\varphi_T(\bar{x}, \bar{y})$, $\psi(\bar{y}, \bar{z})$ be formulas such that (1) $\bar{y} = (y_1, \ldots, y_k)$ is a k-tuple of pairwise distinct fresh variables, (2) $\varphi_S(\bar{x}, \bar{b})$ defines S, (3) $\varphi_T(\bar{x}, \bar{b})$ defines T, and (4) $\psi(\bar{x}, \bar{b})$ defines I in $\mathfrak{R}$. Since $\mathfrak{R}'$ is an elementary extension of $\mathfrak{R}$, we have that (1) $\varphi_S(\bar{x}, \bar{b})$ defines a ring S' in $\mathfrak{R}'$, (2) $\varphi_T(\bar{x}, \bar{b})$ defines a ring T' in $\mathfrak{R}'$ and $T' \subseteq S'$, (3) $\psi(\bar{x}, \bar{b})$ defines a left ideal I' of S' in $\mathfrak{R}'$. By Lemma 1.1, there exist $a_1', \ldots, a_n' \in I'$ such that the only definable subgroup of the additive group of I' containing $a_1', \ldots, a_n'$ is I'. Observe that $\sum_{i=1}^{n} T' a_i'$ is a definable subgroup of I' containing $a_1', \ldots, a_n'$. Therefore, $I' = \sum_{i=1}^{n} T' a_i'$.

Let $\chi(\bar{x}, \bar{x}_1, \ldots, \bar{x}_n, \bar{b})$ be the formula representing

$$\psi(\bar{x}, \bar{b}) \leftrightarrow \exists \bar{z}_1 \ldots \exists \bar{z}_n, \bigwedge_{i=1}^{n} \varphi_T(\bar{z}_i, \bar{b}) \wedge \bar{x} = \sum_{i=1}^{n} \bar{z}_i \bar{x}_i.$$

Hence

$$\mathfrak{R}' \models \exists \bar{x}_1 \ldots \exists \bar{x}_n (\bigwedge_{i=1}^{n} \psi(\bar{x}_i, \bar{b}) \wedge \forall \bar{x}, \chi(\bar{x}, \bar{x}_1, \ldots, \bar{x}_n, \bar{b})).$$

Since $\mathfrak{R}$ is an elementary substructure of $\mathfrak{R}'$ and $\bar{b} \in M^k$,

$$\mathfrak{R} \models \exists \bar{x}_1 \ldots \exists \bar{x}_n (\bigwedge_{i=1}^{n} \psi(\bar{x}_i, \bar{b}) \wedge \forall \bar{x}, \chi(\bar{x}, \bar{x}_1, \ldots, \bar{x}_n, \bar{b})).$$

Therefore, $I = \sum_{i=1}^{n} T a_i$.

By a similar argument, we obtain that if $It \subseteq I$ for every $t \in T$, then there exist $b_1, \ldots, b_k \in I$ such that $I = \sum_{i=1}^{k} b_i T$. □

Next, we will show that if $\mathfrak{R}$ is sufficiently saturated, then the product of a definable left ideal and the ring itself is definable in $\mathfrak{R}$.

1.3. **Lemma.** *Suppose $\mathfrak{R}$ is $\aleph_0$-saturated. Let I be a left ideal of S definable in $\mathfrak{R}$. Then IS is a two-sided ideal of S definable in $\mathfrak{R}$.*

Proof. Obviously, IS is a two-sided ideal of S. It suffices to prove that IS is definable. By Lemma 1.1, there exist $a_1, \ldots, a_n \in I$ such that the only definable subgroup of the additive group of I containing $a_1, \ldots, a_n$ is I. Observe that $\sum_{i=1}^{n} a_i S = \{\sum_{i=1}^{n} a_i s_i : s_i \in S\}$ is a definable group in $\mathfrak{R}$. Hence it is enough to prove that $IS = \sum_{i=1}^{n} a_i S$. It is easy to see that $a_1, \ldots, a_n \in \sum_{i=1}^{n} a_i S \subseteq IS$. Since $a_1, \ldots, a_n \in \sum_{i=1}^{n} a_i S$, we have $I \subseteq \sum_{i=1}^{n} a_i S$. Therefore,

$$IS \subseteq (\sum_{i=1}^{n} a_i S) S \subseteq \sum_{i=1}^{n} a_i S.$$

Hence $IS = \sum_{i=1}^{n} a_i S$. □

We end this section by the following definable analogue of Brauer's Lemma.

1.4. **Lemma.** *Let S be a definably semiprime ring and I be a nonzero left ideal definable in $\mathfrak{R}$. Then there exists a nonzero idempotent e (that is, $e^2 = e$) in I such that eSe is a definable division ring.*

Proof. By a similar argument as in the proof of Lemma 1.2, we may assume that $\mathfrak{R}$ is $\aleph_0$-saturated. Suppose S is definably semiprime and I is a nonzero left ideal definable in $\mathfrak{R}$. By the descending chain condition on definable ideals of S, we may assume further that I is minimal among left ideals of S that are definable in $\mathfrak{R}$. By Lemma 1.3, we know that IS is a two-sided ideal of S that contains I and is definable in $\mathfrak{R}$. Since S is definably semiprime and $SI = I$,

$$\{0\} \neq (IS)^2 = (I(SI))S \subseteq I^2 S.$$

Therefore, $I^2 \neq \{0_S\}$. Then there is $a \in I$ such that Ia is a nonzero left ideal of S that is definable in $\mathfrak{R}$. By the minimality of I, we have $Ia = I$. Let $e \in I$ such that $ea = a$. Hence, $e^2a = ea$, that is, $(e^2 - e)a = 0_S$. Let $J = \{s \in I : sa = 0_S\}$. Then $e^2 - e \in J$ and J is a left ideal of S that is definable in $\mathfrak{R}$. Since $ea = a \neq 0_S$ and $e \in I$, we have J is a proper subset of I. Therefore, $J = \{0_S\}$ and so $e^2 = e$. Thus, e is an idempotent. Next, we prove that $Se = I$. Since $e^2 = e$ and $e \in I$, we have $Se \subseteq I$ and Se is a nonzero left ideal of S that is definable in $\mathfrak{R}$. It follows from the minimality of I that $Se = I$.

To prove that eSe is a definable division ring, it is enough to show that eSe has a left multiplicative identity and every nonzero element has a left multiplicative inverse. Obviously, $e(ese) = e^2se = ese$ for every $s \in S$. Then e is a left multiplicative identity. Let $b \in eSe$ be nonzero. Then there exists $t \in S$ such that $b = ete$. Note that $\{0_S\} \neq Sb \subseteq Se = I$. By the minimality of I, we have $Sb = Se$. Let $s \in S$ such that $e = sb$. Then

$$(ese)b = (ese)(ete) = ese^2te = es(ete) = e(sb) = e^2 = e$$

Therefore, $ese \in eSe$ is a left inverse of b. □

2. Proof of Theorem A

In this section, we will give the proof of Theorem A. Note that, by an argument similar to the proof of Lemma 1.2, it suffices to proof Theorem A when $\mathfrak{R}$ is $\aleph_0$-saturated. First, let us consider when S is definably simple.

2.1. **Theorem.** *Suppose $\mathfrak{R}$ is $\aleph_0$-saturated and S is definably simple. Then S is definably isomorphic to $M_k(D)$ where $k \in \mathbb{N}$ and D is either a finite field, K, $K(\sqrt{-1})$, or $\mathbb{H}(K)$ for some definable real closed field K.*

Proof. By Lemma 1.4, let $e \in S$ be a nonzero idempotent such that $D := eSe$ is a definable division ring. Let $I = Se$. Since e is an idempotent and $D = eSe$, we have that $ID = (Se)(eSe) \subseteq I$. By Lemma 1.2 and the fact that D is a division ring, there exist $b_1, \dots, b_k \in I$ such that $I = \sum_{i=1}^k b_i D$ and for each j if $b_j = \sum_{i \neq j} b_i d_i$, then $d_i = 0_S$ for all $i \neq j$. In other word, we consider I as a right D-module with base $\{b_1, \dots, b_k\}$.

We will show that S is definably isomorphic to $M_k(D)$. Note that we may identify the matrix $M_k(D)$ by $D^{k \times k}$. Define a homomorphism $h\colon S \to D^{k\times k}$ by

$h(a)$ = the matrix representation of the homomorphism $x \mapsto ax\colon I \to I$ with respect to the base $\{b_1, \dots, b_k\}$.

Observe that h is definable.

To prove that h is injective, let $a \in S$ such that $h(a) = 0$. Then $aI = \{0_S\}$; and so $aIS = \{0_S\}$. By Lemma 1.3, IS is a two-sided ideal of S that is definable in $\mathfrak{R}$. Since S is definably semisimple and $e \in (Se)S = IS$, we have $IS = S$. Hence $aS = \{0\}$. Therefore, $a = 0_S$.

To prove the surjectivity, let $\Lambda \in M_k(D)$. For notational simplicity, we also denote the corresponding homomorphism of Λ by Λ itself. Since $1_S \in S = IS = SeS$, we write $1_S = \sum_{i=1}^k s_i e t_i$ where $s_i, t_i \in S$. Let $a = \sum_{i=1}^k \Lambda(s_i e) e t_i$. It remains to show that $h(a) = \Lambda$.

Let $x \in I$. Since $I = Se$ and $D = eSe$, we have $et_i x \in D$ for every $i \in \{1, \dots, k\}$. By the D-linearity of Λ, we have

$$(h(a))(x) = ax = \sum_{i=1}^{k} \Lambda(s_i e) et_i x = \Lambda((\sum_{i=1}^{k}(s_i et_i)x) = \Lambda(x).$$

Hence, $h(a) = \Lambda$.

Therefore, h is a definable isomorphism from S to $M_k(D)$. We can see that if S is finite, then $D = eSe$ is a finite division ring; hence, a field. Suppose S is infinite. Then D is an infinite definable division ring. Therefore, there is a definable real closed field K such that D is definably isomorphic to K, $K(\sqrt{-1})$ or $\mathbb{H}(K)$. This completes the proof. □

Let $X \subseteq S$. We say that X is *orthogonal* if $ab = 0_S$ for all distinct $a, b \in X$. Suppose X is a set of idempotents in S. We now define a partially order relation $\preccurlyeq$ on X by

$$a \preccurlyeq b \text{ if and only if } ab = a = ba.$$

We say that X has the *maximum condition* if for every ascending $\preccurlyeq$-chain $(a_n)_{n\in\mathbb{N}}$ in X, there exists $n \in \mathbb{N}$ such that $a_n = a_k$ for all $k \geq n$.

2.2. **Lemma.** *Let X be a set of idempotents in S. Then*

(1) if X is orthogonal, then X is finite; and

(2) X has the maximum condition.

Proof. To prove (1), suppose to the contrary that X is infinite. Without loss of generality, assume $X = \{a_n : n \in \mathbb{N}\}$ and $0_S \notin X$. For each $n \in \mathbb{N}$, let $I_n = S(1_S - \sum_{i=1}^{n} a_i)$ be the left ideal of S generated by $1_S - \sum_{i=1}^{n} a_i$. By the descending chain condition on definable left ideals, it suffices to show that I_{n+1} is a proper subset of I_n for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Since X is an orthogonal set of idempotents, we have

$$\begin{aligned}(1_S - \sum_{i=1}^{n+1} a_i)(1_S - \sum_{j=1}^{n} a_j) &= 1_S - \sum_{i=1}^{n+1} a_i - \sum_{j=1}^{n} a_j + \sum_{i=1}^{n+1} a_i \sum_{j=1}^{n} a_j \\ &= 1_S - \sum_{i=1}^{n+1} a_i - \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} a_j \\ &= 1_S - \sum_{i=1}^{n+1} a_i.\end{aligned}$$

Therefore, $I_{n+1} \subseteq I_n$. Next, suppose that $1_S - \sum_{j=1}^{n} a_j \in I_{n+1}$. Then there exists $t \in S$ such that $1_S - \sum_{j=1}^{n} a_j = t(1 - \sum_{i=1}^{n+1} a_i)$. Hence, $a_{n+1} = (1_S - \sum_{j=1}^{n} a_j)a_{n+1} = t(1_S - \sum_{i=1}^{n+1} a_i)a_{n+1} = ta_{n+1} - ta_{n+1} = 0_S$ which is absurd. Thus, I_{n+1} is a proper subgroup of I_n.

We now begin the proof of (2). Observe that if $0_S \in X$, then 0_S is the $\preccurlyeq$-least element in X. Therefore, we may assume that $0_S \notin X$. Suppose $(a_n)_{n\in\mathbb{N}}$ is an infinite ascending $\preccurlyeq$-chain in X where a_n's are pairwise distinct. For each $n \in \mathbb{N}$, let $b_n = a_n - a_{n-1}$ where $a_{-1} = 0_S$. Let $Y = \{b_n : n \in \mathbb{N}\}$. By (1), it is enough to prove that Y is an infinite orthogonal set of idempotents in S. Obviously, $Y \subseteq S$ and each b_n is nonzero. For each $n \in \mathbb{N}$, we have

$$b_n^2 = (a_n - a_{n-1})^2 = a_n^2 - a_n a_{n-1} - a_{n-1}a_n + a_{n-1}^2 = a_n - a_{n-1} - a_{n-1} + a_{n-1} = b_n,$$

that is, b_n is an idempotent in S. If $n < m$, then

$$\begin{aligned} b_n b_m &= (a_n - a_{n-1})(a_m - a_{m-1}) \\ &= a_n a_m - a_{n-1}a_m - a_n a_{m-1} + a_{n-1}a_{m-1} \\ &= a_n - a_{n-1} - a_n + a_{n-1} \\ &= 0_S, \end{aligned}$$

and, similarly, $b_m b_n = 0_S$. Since all b_n's are nonzero idempotents in S and $b_n b_m = 0_S$ for distinct n, m, we have $b_n \neq b_m$ for all distinct n, m. Therefore, Y is an infinite orthogonal set of idempotents in S, which is absurd. □

As a consequence of the above lemma, we know that every set of idempotents has a $\preccurlyeq$-maximal element. We are now ready to complete the proof of Theorem A.

2.3. **Theorem.** *Suppose $\mathfrak{R}$ is $\aleph_0$-saturated and S is definably semiprime. Then S is definably isomorphic to a finite direct sum of $M_{k_1}(D_1), \dots, M_{k_l}(D_l)$ where $k_1, \dots, k_n \in \mathbb{N}$ and each D_i is either a finite field, K, $K(\sqrt{-1})$, or $\mathbb{H}(K)$ for some definable real closed field K.*

Proof. By Theorem 2.1, it is enough to prove that S is a finite direct sum of definably simple rings. In addition, by the descending chain condition and mathematical induction, it suffices to prove that S is a direct sum of a definably simple ring T and a definably semiprime ring J. Let I be minimal among nonzero left ideals of S that are definable in $\mathfrak{R}$. Set $T = IS$ and $J = \{a \in S : Sa = \{0_S\}\}$. Observe that J is a two-sided ideal of S that is definable in $\mathfrak{R}$ and $TJ = \{0_S\}$. By Lemma 1.3, T is minimal among two-sided ideals of S that contain I and are definable in $\mathfrak{R}$. Since $(T \cap J)^2 \subseteq TJ = \{0_S\}$, we have $T \cap J = \{0_S\}$. To prove that S is a direct sum of T and J, it now remains to prove that $S = T + J$. Since S is definably semiprime and I is a nonzero left ideal definable in $\mathfrak{R}$, by Lemma 1.4, I contains nonzero idempotents. Let X be the set of all nonzero idempotents in T. Since $I \subseteq T$, $X \neq \emptyset$. By Lemma 2.2, let $e \in X$ be a $\preccurlyeq$-maximal idempotent in X. Note that if $1_S - e \in J$, then $1_S \in T + J$ and so $S = T + J$. Suppose to the contrary that $1_S - e \notin J$. Then $T(1_S - e)$ is a nonzero left ideal of S that is definable in $\mathfrak{R}$ and $T(1_S - e) = (IS)(1_S - e) \subseteq T$. By Lemma 1.4, let g be a nonzero idempotent in $T(1_S - e)$. Let $h = e + g - eg \in T$. Since $(1_S - e)e = 0_S$ and $g \in T(1_S - e)$, we have $ge = 0_S$. Then

$$\begin{aligned} h^2 &= (e + g - eg)^2 \\ &= e^2 + g^2 + egeg + eg + ge - e^2 g - ege - geg - eg^2 \\ &= e + g + 0 + eg + 0 - eg - 0 - 0 - eg \\ &= h \end{aligned}$$

and

$$he = e^2 + ge - ege = e = e + eg - eg = e^2 + eg - e^2 g = eh.$$

Hence, $e \preccurlyeq h$. By the maximality of e in X, we have $e = h = e + g - eg$. Hence $g = eg$. Therefore, $g = g^2 = g(eg) = 0$ which is absurd. Therefore, S is the direct sum of T and J.

To complete the proof it remains to show that T is a definably simple ring and J is a definably semiprime ring. Since T, J are two-sided ideals of S and $T \cap J = \{0_S\}$, for all $x \in T$,

$$x(1_S - e) = (1_S - e)x = 0_S; \text{ and so}$$
$$xe = xe + x(1_S - e) = x(e + 1_S - e) = x = (e + 1_S - e)x = ex + (1_S - e)x = ex.$$

In addition, for all $y \in J$, we have

$$\begin{aligned} ye = ey &= 0_S; \text{ and} \\ y(1_S - e) = y(1_S - e) + ye &= y(1_S - e + e) \\ &= y \\ &= (1_S - e + e)y = (1_S - e)y + ey = (1_S - e)y. \end{aligned}$$

Hence, e is the multiplicative identity of T and $(1_S - e)$ is the multiplicative identity of J. With respect to the ring operations on S, we have that T is a definable ring with unit e and J is a definable rings with unit $(1_S - e)$.

To show that J is definably semiprime, let I' be a nonzero two-sided ideal of J that is definable in $\mathfrak{R}$. Since S is the direct sum of T and J, I' is also a nonzero two-sided ideal of S. Since S is definably semiprime, we have $(I')^2 \neq \{0_S\}$. Hence J is definably semiprime. Therefore, it remains to show that T is definably simple. Observe that every two-sided ideal of T that is definable in $\mathfrak{R}$ is also a two-sided ideal of S. Hence, it suffices to prove that T is minimal among two-sided ideals of S that is definable in $\mathfrak{R}$.

Let I'' be a nonzero two-sided ideal of S such that I'' is definable in $\mathfrak{R}$ and $I'' \subseteq T$. Since S is definably semiprime, we have

$$\{0_S\} \neq (I'')^2 \subseteq I''(IS) \subseteq (I'' \cap I)S.$$

Therefore, $I'' \cap I$ is a nonzero left ideal of S that is definable in $\mathfrak{R}$. By the minimality of I, we have $I = I'' \cap I \subseteq I''$. Hence $T = IS \subseteq I''S = I''$.

This completes the proof. □

3. Remarks

Suppose $\mathfrak{R}$ is an o-minimal expansion of a real closed field $(R, <, 0, 1, +, \cdot)$. In [8], M. Otero, Y. Peterzil and A. Pillay proved that every infinite definable ring without zero divisors is definably isomorphic to either R, $R(\sqrt{-1})$, or $\mathbb{H}(R)$. Therefore, we obtain the following corollary to Theorem A.

3.1. **Corollary.** *Suppose $\mathfrak{R}$ is an o-minimal expansion of a real closed field $(R, <, 0, 1, +, \cdot)$ and S is a definable ring in $\mathfrak{R}$.*

1. *If S is definably simple, then S is definably isomorphic to $M_k(D)$ where $k \in \mathbb{N}$ and D is either a finite field, R, $R(\sqrt{-1})$, or $\mathbb{H}(R)$.*
2. *If S is definably semiprime, then S is definably isomorphic to a finite direct sum of $M_{k_1}(D_1), \ldots, M_{k_l}(D_l)$ where $k_1, \ldots, k_n \in \mathbb{N}$ and each D_i is either a finite field, R, $R(\sqrt{-1})$, or $\mathbb{H}(R)$.*

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