Supersimple Moufang polygons

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Abstract

This paper continues the work started in [4], where we showed that each class of finite Moufang polygons forms an asymptotic class, in the sense of [9] and [6]. Here, we show that all (infinite) Moufang polygons whose first order theory is *supersimple* of *finite rank* are characterized as those inherited from the finite, i.e. if Γ is a supersimple finite rank Moufang polygon, then Γ belongs to one of the families of Moufang polygons which also has finite members. The proof rests on the classification of Moufang polygons due to Tits and Weiss, [15].

1 Introduction

Among the families of Moufang polygons, which have been classified in [15], there are, up to duality, six which include arbitrary large finite ones; namely, families whose members are either Desarguesian projective planes, symplectic quadrangles, Hermitian quadrangles in projective space of dimension 3 or 4, split Cayley hexagons, twisted triality hexagons or Ree-Tits octagons, with the latter arising over *difference fields* (see Example 3.7). We call the members of these families (whether finite or infinite) *good* Moufang polygons. In this paper, we show that these are the only families whose members can be supersimple of finite rank (a model-theoretic hypothesis). We state our main result as follows, where by $\Gamma(K)$ we mean a Moufang polygon whose associated algebraic structure (in terms of Section 3) arises over some field K.

Theorem 1.1 Let $\Gamma = \Gamma(K)$ be a supersimple finite rank Moufang polygon. Then:

(i) the (difference) field K is definable in Γ;

(ii) Γ is good.

A proof of this theorem, which also appears as Corollary 9.3, is given throughout Sections 6, 7, 8 and 9. Notice that if Γ is a good Moufang polygon over a supersimple (difference) field of finite rank, then Γ is supersimple finite rank; this was not explicitly proved in [4], but it follows from the main results of [13] on classes of finite Chevalley groups, or finite twisted groups of fixed Lie type and Lie rank, generalised to the infinite case (by checking [2], they go through for infinite fields) and Theorem 8.2(ii) of [4].

The work in this paper, as well that in [4], is extracted from [5]. It originated from [8]; in fact, our results are a generalization of those in [8] from the *superstable* context (under the extra assumption of finite Morley rank, a model-theoretic notion of dimension) to the *supersimple* context. As for [8], the motivation is group-theoretic: there is a well-known project, the 'Algebraicity Conjecture', see [1], to give a model-theoretic classification of simple [1] algebraic groups by showing that they are the simple groups of finite Morley rank; a very important tool in [1] is the classification of

[1]Notice the unfortunate clash of terminology between (super)simplicity in the model-theoretic sense, and simplicity intended in the usual group-theoretic sense.

simple groups of finite Morley rank with a spherical Moufang BN-pair of Tits rank at least 2, see [8], which relies on the classification of Moufang generalized polygons of finite Morley rank, also achieved in [8]. The finite Morley rank condition is extremely strong, and eliminates many interesting Moufang generalized polygons; for example, those associated with twisted simple groups. Some of the latter, instead, do enter into the picture under the supersimplicity assumption (for instance, Moufang octagons).

Our work makes use of the classification of Moufang polygons, [15], and the key point is to interpret the (skew) field associated with the underlying algebraic structure of each Moufang polygon. In many of the cases this is done by using techniques from [8] - for example, in Section 5, when we recover the additive structure of the (skew) field.

We assume that the reader is familiar with the basic model-theoretic notions treated in this paper, i.e. supersimple structures, finite rank assumption, first-order interpretability, and so on; all the relevant information can be found in Section 5 of [4]. In particular, in Section 4 we recall the definition of S_1-rank, and we also provide a list of some of the main properties which are satisfied by a supersimple finite rank field; for more on supersimplicity see, for instance, Chapter 5 of [17]. Good Moufang polygons have already been described in some detail in Section 3 of [4]. Here, in Section 2, we will introduce the remaining Moufang polygons, namely those which are *not* good; these will also be called *bad* Moufang polygons. Also, Section 3 gives some background on the algebraic structures associated with the Moufang polygons. Section 4 then gives some model-theoretic facts related to these algebraic structures which will be used in order to prove the main theorem above. Section 5 deals with the key points regarding the interpretation of the underlying field K of Γ in Theorem 1.1(i); this is done almost exactly as in Section 1 of [8].

Finally, Sections 6, 7, 8 and 9 deal, respectively, with the families of projective planes, quadrangles, hexagons and octagons, which are not good; we will show that if $\Gamma = \Gamma(K)$ is a bad Moufang polygon, then Γ cannot arise over a supersimple finite rank (difference) field. Basically, once we define the underlying field K inside the polygon Γ, we use some model-theoretic facts in order to prove that the configuration cannot be supersimple of finite rank; for instance, in most of the cases we can define a field extension of K having a non-surjective norm map, which by the main result of [12] contradicts supersimplicity.

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2 Bad Moufang polygons

We view generalized polygons as first-order structures as follows. By $L_{\text{inc}} = (P, L, I)$ we mean a language with two disjoint unary relations P and L and a binary relation I, where $I \subseteq P \times L \cup L \times P$ is symmetric and stands for *incidence*; an L_{inc}-structure is called an *incidence structure* and, usually, the elements a satisfying P are called *points*, those satisfying L are called *lines*, and pairs (a, l), or (l, a), satisfying I are called *flags*. Also, by a k-*chain* we mean a sequence $(x_0, x_1, ..., x_k)$ of elements $x_i \in P \cup L$ such that x_i is incident with x_{i-1} for $i = 1, 2, .., k$ and $x_i \neq x_{i-2}$ for $i = 2, 3, ..., k$, and by *distance* d between any two elements $x, y \in P \cup L$, denoted by $d(x, y)$, we mean the least k such that there is a k-chain joining them.

Definition 2.1 A *generalized n-polygon*, or *generalized n-gon*, is an incidence structure $\Gamma = (P, L, I)$ satisfying the following three axioms:

(i) every element $x \in P \cup L$ is incident with at least three other elements;

(ii) for all elements $x, y \in P \cup L$ we have $d(x, y) \leq n$;

(iii) if $d(x, y) = k < n$, there is a unique k-chain $(x_0, x_1, ..., x_k)$ with $x_0 = x$ and $x_k = y$.

A *subpolygon* Γ' of Γ is an incidence substructure $\Gamma' = (P', L', I') \subseteq \Gamma$, i.e. $P' \subseteq P$, $L' \subseteq L$ and $I' = I \cap (P' \times L')$, satisfying the axioms (i)-(iii) above.

Sometimes, the definition of a generalized polygon is allowed to include, as well as (i), the condition $(i)'$: every element $x \in P \cup L$ is incident with exactly two other elements; if so, we then distinguish between thick and thin n-gons, namely, we say that an n-gon Γ is *thick* if it is as in Definition 2.1 above, while we call it *thin* if it is as in Definition 2.1 with axiom (i) replaced by $(i)'$. Also, if Γ' is a subpolygon of Γ which satisfies $(i)'$, then we also call Γ' an *ordinary subpolygon* of Γ. Moreover, we recall that a *duality* of an n-gon $\Gamma_1 = (P_1, L_1, I_1)$ onto an n-gon $\Gamma_2 = (P_2, L_2, I_2)$ is an isomorphism (a map which sends points to points, lines to lines, and preserves incidence and non-incidence) of Γ_1 onto $\Gamma_2^{\text{dual}} := (L_2, P_2, I_2)$, i.e. the polygon obtained by interchanging points and lines of Γ_2.

We recall the definition of perspectivity maps. Let $\Gamma = (P, L, I)$ be a generalized n-gon, and for some fixed $k \leq n$ and for every $x \in P \cup L$ let $B_k(x) := \{y : d(x, y) = k\}$. Consider now two elements $x, z \in P \cup L$ such that $d(x, z) = k$ for some $k < n$. Then it follows from Definition 2.1(iii) that there exists a unique element $y \in B_{k-1}(x) \cap B_1(z)$, which will be called the *projection of x over z* and denoted by $\mathrm{proj}_z x$ or $\mathrm{proj}_k(x,z)$, when we want to specify the distance between x and z. In particular, if $d(x,z)=n$ then there exists a bijection $[z,x] := B_1(x) \longrightarrow B_1(z)$ such that $[z,x](y) =: \mathrm{proj}_{n-1}(y,z)$ for every element $y \in B_1(x)$. We will call it the *perspectivity map from x to z.* We also recall the definition of a root group. Given a generalized n-gon Γ, by a *root* α we mean an n-chain $(x_0, x_1, ..., x_n)$, and by the *interior of* α we mean the set $\bar{\alpha} := \cup_{i=1}^{n-1} B_1(x_i)$. For each root α, we define the *root group* U_α to be the subgroup consisting of elements of $\mathrm{Aut}(\Gamma)$ which fix $\bar{\alpha}$ pointwise. By the *little projective group of* Γ we mean the group $\Sigma := \langle U_\alpha : \alpha \text{ root } \rangle$.

As done in Section 4 of [4], given a generalized n-gon Γ, we fix an ordinary subpolygon $A = (x_0, x_1, ..., x_{2n-1}) \subset \Gamma$, a root $\alpha = (x_0, x_1, ..., x_n) \subset A$, and the (*positive*) root groups U_i associated to the roots $\alpha_i = (x_i, x_{i+1}, ..., x_{i+n})$ for $i = 1, ..., n$, as well the (*negative* or *opposite*) root groups U_j associated to the roots $\alpha_j = (x_j, x_{j+1}, ..., x_{j+n})$ for $j = n+1, n+2, ..., 2n$ (here, as in [15], on the indices we use a modulo $2n$ sum and thus, for example, U_0 is the same thing of U_{2n}), so that the coordinatization procedure of Definition 2.8 of [4] makes sense; as a remark, this means that, model-theoretically, Γ is in the definable closure of the *hat-rack* (see Proposition 2.3 of [8]). We say that the root α is called *Moufang* if the group U_0 acts transitively on $B_1(x_0)$; in particular, Γ is said to be Moufang [2] if all its roots are Moufang.

[2]The classification of generalized polygons is not currently possible, and therefore one needs a stronger condition arising from the group action, called the Moufang condition, in order to classify them (see [15], which gives the complete list of Moufang generalized polygons); it is a strong homogeneity condition for buildings. In fact, the Moufang condition was first introduced for irreducible, spherical buildings of rank ≥ 3; it was shown that if such buildings are *thick*, then they are automatically Moufang. Generalized polygons are nothing but rank 2 residues of irreducible, spherical buildings of rank ≥ 3; in particular, if the latter are Moufang, so are their rank 2 residues. As a consequence, every thick irreducible spherical building of rank at least 3 is an amalgamation, in a certain precise sense, of Moufang generalized polygons.

Moreover, we call $(U_{[1,n]}, U_1, U_2, ..., U_n)$ a *root group sequence* of Γ, where $U_{[1,n]} := \langle U_i : 1 \leq n \rangle$.

We will need the following result throughout Sections 6-9.

Lemma 2.2 (Proposition 5.6 of [15]) Let Γ be a Moufang n-polygon. Then there exists a bijection φ from the set theoretic product $U_1 \times U_2 \times ... \times U_n$ to $U_{[1,n]}$ given by $\varphi(u_1, u_2, ..., u_n) = u_1 u_2 ... u_n$; that is, every element u of $U_{[1,n]}$ is uniquely expressible as $u = u_1 u_2 ... u_n$, for $u_i \in U_i$.

In the following table we list the good Moufang polygons up to duality. By a quadric of type D_4 over a field K in a 7-dimensional projective space $\mathrm{PG}(7, K)$ we mean the quadric containing projective 3-spaces whose standard equation is $X_0X_1 + X_2X_3 + X_4X_5 + X_6X_7 = 0$ (see Section 2.4.2 of [16]). Also, by a *metasymplectic space* we mean that the octagon comes from a spherical building of type F_4; see Theorem 2.5.2 of [16].

n	notation from [4]	ambient info	little proj. group	duality
3	$\mathrm{PG}(2, K)$	projective plane	$\mathrm{PSL}_3(K)$	self-dual
4	$W(K)$	4-dim projective space	$\mathrm{PS}_{p_4}(K)$	$Q(5, K)$
4	$HQ(4, K)$	5-dim projective space	$\mathrm{PSU}_4(K)$	$Q(6, K)$
4	$HQ(5, K)$	6-dim projective space	$\mathrm{PSU}_5(K)$	$Q(8, K)$
6	$H(K)$	Quadric of type D_4	$G_2(K)$	self-dual
6	$H(K^3, K)$	Quadric of type D_4	${}^3D_4(K)$	self-dual
8	$O(K, \sigma)$	metasymplectic space	${}^2F_4$	self-dual

Table 2.1

Remark 2.3 Here we give a sort of geography of the content of [15], and also build a bridge between this current section and the following one. All the results on Moufang polygons that we mention from [15] are intended up to duality. In [15], Moufang polygons are uniquely determined by their root group sequences, and they are also uniquely identified with some associated algebraic structures arising from the classification. In this section we list Moufang polygons from their root group sequences point of view, by specifying their root groups together with the associated commutator relations, while in the next section we list the algebraic structures associated to such Moufang polygons.

As it is shown by [15, Uniqueness Lemma 7.5], the class of Moufang polygons isomorphic to Γ is uniquely determined by a fixed root group sequence of Γ. In fact, suppose that an $(n+1)$-tuple $(U_{[1,n]}, U_1, U_2, ..., U_n)$ from an abstract group is given (therefore not necessarily from a little projective group of a Moufang polygon), and that it satisfies the conditions $\mathcal{M}_1 - \mathcal{M}_4$ as in [15, Definition 8.7] which are also satisfied by any root group sequence; then, from such an $(n+1)$-tuple, in [15, Chapters 7 and 8] is shown how to construct a graph, or more precisely a point-line incidence structure, which turns out to be isomorphic to some Moufang polygon; see [15, Existence Lemma 8.11].

In [15, Chapters 19-31], all Moufang polygons are classified. Basically, given a Moufang polygon Γ, its root group sequence is described in terms of some commutator relations defining $U_{[1,n]}$ inside the little projective group Σ, and these data are described in terms of some underlying algebraic structures; the latter are all described in [15, Chapters 9-15] (see also the following Section 3). Therefore, a classification of such algebraic structures provides a classification of the associated Moufang polygons.

Following the remark above, this and the next section are organized as follows: in this section we give examples of bad Moufang polygons in terms of their root group sequences and commutator relations, by only mentioning the associated algebraic strucuters; then in the following section we introduce the relevant algebraic structures in some detail (see also Remark 3.1). However, in the following examples, and elsewhere, we do not specify how the polygon is determined by its root group sequence. Below, throughout Examples 2.4 - 2.9, we will denote the elements of the root groups U_i by $x_i(s)$ for $s \in S_i$, where the S_i are the arising algebraic structures and are given case by case.

Example 2.4 Moufang projective planes ($n = 3$): We recall that generalized triangles are nothing but projective planes. As shown in Chapter 19 of [15], the underlying algebraic structure arising from a Moufang projective plane is an alternative division ring, i.e. a structure A satisfying all the axioms of division rings except associativity of multiplication, which is replaced by: $x(xy) = (xx)y$ and $(yx)x = y(xx)$ for all $x, y \in A$. Also, every alternative division ring yields a Moufang projective plane. An alternative division ring is associative if and only if it is a field or a *skew field*, by which we mean a non-commutative division ring.

By the *Bruck-Kleinfeld Theorem*, a proof of which can be found in Chapter 20 of [15], the only non-associative alternative division rings are the *Cayley-Dickson* algebras; see Example 3.2.

In terms of its root group sequence and underlying algebraic structure, a Moufang projective plane $\mathrm{PG}(2, A)$ has the following commutator relation (see 16.1 of [15]), where A is the underlying alternative division ring and $x_i \in U_i$ are the elements of the (positive) root groups U_1, U_2 and U_3, which are all isomorphic to the additive group of A:

$[x_1(t), x_3(u)] = x_2(tu)$, for all $t, u \in A$.

With this data, a *bad Moufang projective plane* is a projective plane $\mathrm{PG}(2, A)$ whose associated alternative division ring A is isomorphic to a Cayley-

Dickson algebra.

Example 2.5 Moufang orthogonal and Hermitian quadrangles ($n = 4$): A description of Moufang orthogonal/Hermitian quadrangles has already been given in Example 3.2 of [4]; according to Definition 3.6 of [4], the Moufang orthogonal quadrangles $Q(5, K)$ and $Q(6, K)$, and the Moufang Hermitian quadrangles $HQ(4, K)$ and $HQ(5, K)$, are good Moufang polygons. Thus, throughout the rest of this paper, by a *bad orthogonal* or *Hermitian quadrangle* we mean, respectively, $Q(l, K)$ for $l \geq 7$, and $HQ(l, K)$ for $l \geq 6$.

From the classification results given in [15] (see Chapters 23, 25 and 26), orthogonal and Hermitian Moufang quadrangles depend on some algebraic structures given by Example 3.3 below; thus, we will denote a Moufang orthogonal quadrangle $Q(l, K)$ by $Q(K, L_0, q)$, for some quadratic space (K, L_0, q), and a Moufang Hermitian quadrangle $HQ(l, K)$ by $Q(K, K_0, \sigma, L_0, q)$, for some pseudo-quadratic space (K, K_0, σ, L_0, q). Then $Q(K, L_0, q)$ has a root group sequence $(U_{[1,4]}, U_1, U_2, U_3, U_4)$ whose root groups U_1 and U_3 are isomorphic to the additive group of K, the root group U_2 and U_4 are isomorphic to the additive group of L_0, and the following commutator relations hold (see 16.3 of [15]):

$$[x_2(a), x_4(b)^{-1}] = x_3(f(a, b))$$
$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(tq(a))$$

for all $t \in K$ and $a, b \in L_0$, where f is the bilinear form associated to the quadratic form q.

Likewise, the root group sequence $(U_{[1,4]}, U_1, U_2, U_3, U_4)$ of $Q(K, K_0, \sigma, L_0, q)$ satisfies the following commutator relations (see 16.5 of [15]), where the group T is defined to be $T = (T, \cdot) := \{(a, t) \in L_0 \times K : q(a) - t \in K_0\}$ and $(a, t) \cdot (b, u) = (a + b, t + u + f(b, a))$ for all $(a, t), (b, u) \in T$:

$$[x_1(a,t), x_3(b,u)^{-1}] = x_2(f(a,b))$$
$$[x_2(v), x_4(w)^{-1}] = x_3(0, v^\sigma w + w^\sigma v)$$
$$[x_1(a,t), x_4(v)^{-1}] = x_2(tv)x_3(av, v^\sigma tv)$$

with the root groups U_1 and U_3 being isomorphic to T, and the root groups U_2 and U_4 to the additive group of K. In the particular case in which $L_0 = 0$ we obtain $HQ(4, K)$.

Example 2.6 Moufang mixed quadrangles ($n = 4$) These quadrangles are associated with indifferent sets, with the latter defined in Example 3.4. Thus, given a Moufang mixed quadrangle Γ, we will denote it by $Q(L, L_0, K, K_0)$. Then, according to Chapter 24 of [15], $Q(L, L_0, K, K_0)$ has a root group sequence $(U_{[1,4]}, U_1, U_2, U_3, U_4)$ whose root groups U_1 and U_3 are isomorphic to the additive group of K_0, the root groups U_2 and U_4 are isomorphic to the additive group of L_0, and the following commutator relation holds (see 16.4 of [15]):

$$[x_1(t), x_4(a)] = x_2(t^2a)x_3(ta)$$

for all $t \in K_0$ and $a \in L_0$.

These quadrangles are subquadrangles of symplectic quadrangles $W(K')$ (i.e. quadrangles whose points and lines are, respectively, 1 and 2-dimensional subspaces of a 4-dimensional vector space V over a field K' of characteristic 2, with V equipped with a symplectic form), and contain orthogonal quadrangles as subpolygons; the latter are associated with a subfield K'' of K', e.g. $K' = K$ and $K'' = K_0$. We say that $Q(L, L_0, K, K_0)$ is a *bad mixed quadrangle* unless $L = K_0 = K$, in which case we obtain nothing but the associated orthogonal subquadrangle.

Example 2.7 Moufang exceptional quadrangles ($n = 4$): There are four types of exceptional quadrangles, and they are *all* bad. Following the notation from [15], we will denote by $Q_i(K, L_0, q)$ the Moufang exceptional quadrangle of type E_i, for $i = 6, 7$ and 8, and by $Q_{F_4}(L, K, L', K')$ the Moufang exceptional quadrangle of type F_4; or, when it is clear from the context, we will sometimes denote them by, respectively, Q_i and Q_{F_4}, for ease.

Since for our purposes we will only make use of dimensionality data of the algebraic structures associated to such quadrangles (see Example 3.5), we refer to Chapters 12, 13 and 14 of [15] for their description, and to Examples 16.6 and 16.7 of [15] for the associated commutator relations.

Example 2.8 Moufang hexagons ($n = 6$): As shown in Chapters 15 and 30 of [15], Moufang hexagons are classified by studying their underlying algebraic structures called hexagonal systems $(J, F, N, \sharp, T, \times, 1)$ (with the notation of [15]). For the split Cayley hexagon and the twisted triality hexagon, we already defined them in Example 3.4 of [4], since they are good hexagons; thus, by a *bad Moufang hexagon* we will mean an hexagon whose underlying hexagonal system is one of those summarized and listed in Example 3.6.

According to 16.8 of [15], a Moufang hexagon has a root group sequence $(U_{[1,6]}, U_1, U_2, ..., U_6)$ whose groups U_1, U_3, U_5 are isomorphic to the additive group of J, the root groups U_2, U_4, U_6 are isomorphic to the additive group of the field F, and the following commutator relations hold:

$$[x_1(a), x_3(b)] = x_2(T(a, b)),$$
$$[x_3(a), x_5(b)] = x_4(T(a, b)),$$
$$[x_1(a), x_5(b)] = x_2(-T(a^\sharp, b))x_3(a \times b)x_4(T(a, b^\sharp)),$$
$$[x_2(t), x_6(u)] = x_4(tu) \text{ and}$$
$$[x_1(a), x_6(t)] = x_2(-tN(a))x_3(ta^\sharp)x_4(t^2N(a))x_5(-ta),$$

for all $a, b \in J$ and $t, u \in F$.

Example 2.9 Moufang octagons ($n = 8$): They are classified in Section 31 of [15]. The classification depends on some mixed quadrangle arising from the octagon, and it also depends on a polarity (i.e. automorphism of order two) of this quadrangle. This polarity gives rise to the Tits endomorphism σ (see Example 3.7) associated with the field K coordinatizing the octagon, as well as the associated mixed quadrangle. Thus, with the notation of [15], we will denote a Moufang octagon by $O(K, \sigma)$.

By 16.9 and 17.7 of [15], the root groups U_1, U_3, U_5 and U_7 are isomorphic to the additive group of K, and the root groups U_2, U_4, U_6 and U_8 are isomorphic to the group $K_\sigma^{(2)} := (K \times K, \cdot)$, where $(t, u) \cdot (s, v) = (t+s+u^\sigma v, u+v)$ for all $(t, u), (s, v) \in K_\sigma^{(2)}$; for the commutator relations, see 16.9 of [15].

By a *bad Moufang octagon* $O(K, \sigma)$ we mean that either K is not perfect, σ is not bijective, or possibly both.

3 Underlying algebraic structures of Moufang polygons

In Section 2 of [4] we explained the coordinatization procedure in order to give coordinates (with respect to a fixed ordinary subpolygon) to all point/line elements of a Moufang polygon; this is the viewpoint of [8] and [16]. This is a generalization of the usual coordinatization procedure of (Moufang) projective planes to n-gons for $n \geq 4$; see Chapter 3 of [16]. However, the classification results obtained in [15] are purely group-theoretic, and also the coordinatization procedure is given group-theoretically.

Throughout the remaining part of this paper, we will sometimes use the following informal meaning of coordinatization: given a generalized polygon Γ, we say that Γ and S are *associated* if S is one of the algebraic structures listed in Chapters 9-15 of [15] and arising from Γ as proved in Chapters 19-31 of [15]; see also the remark below. This is not used in a precise model-theoretic sense. In the examples below, we only give the main algebraic structures which will be relevant to our proofs later on.

Remark 3.1 Typically, there is an algebraic structure S (e.g. an alternative division ring, a vector space over a field, a Jordan division algebra, and so on), two subsets S_1 and S_2 of S, and functions from $S_1 \times S_1$, $S_1 \times S_2$ and/or $S_2 \times S_1$ to S_1 and/or S_2 (e.g. a bilinear form, a quadratic form, a norm map, and so on) which 'determine', up to duality, the associated generalized polygon, and vice versa. For instance, sometimes S_1 has the structure of a field, S_2 that of a vector space over S_1, and the map $S_2 \longrightarrow S_1$ is given by a quadratic form (this is the case of an orthogonal quadrangle - see Section 2.3 of [16]).

As done in [15], we will identify the domains of the S_i (as in the remark above) with the corresponding root groups U_i, via maps $x_i : S_i \longrightarrow U_i$, say. Then, model-theoretically, as done in [4, Section 4], the S_i play the role of *parameter sets* in the coordinatization procedure as in Definition 5.4 and, in the polygon language L_{inc}, we definably identify them with the associated root groups U_i, which are definable (together with their action) in the polygon language L_{inc}. However, sometimes the algebraic structures S, and therefore the S_i, carry more structure than the U_i, and we thus need to define this extra-structure; for instance, in the case of Moufang projective planes, the structure $S = S_i$, for all i, is an alternative division ring $A = (A, +, \cdot)$, and $(A, +)$ is definably isomorphic to U_i together with its root group action, but we also need to define the ring multiplication $\cdot$ in the language L_{inc}; see Section 6.

Example 3.2 Cayley Dickson algebra ($n = 3$): Let E be a field, let σ be an automorphism of E of order 2 and let $K = \text{Fix}_E(\sigma)$ be the subfield of E

fixed by σ. Then E/K is a separable quadratic extension, and we denote by N and T, respectively, the norm and trace maps of this field extension. We then consider a *quaternion algebra* $Q = (E/K, \beta) \subseteq M(2, E)$, as defined by [15, Definition 9.3], with $\dim_K(Q) = 2$ and $M(2, E)$ being the set of 2×2 matrices with entries from E, and $\beta \in K^\star$; then Q is associative but not commutative. There is a unique way of extending σ to an anti-automorphism of Q whose order is two, and as shown by 9.2 of [15] in terms of σ we can extend N and T to Q so that these maps turn out to be the restriction to Q of, respectively, the norm and trace maps of $M(2, E)$. Then, by 9.3 and 9.4 of [15], Q is a division algebra if and only if $\beta \notin N(E)$.

In a similar fashion, for some $\gamma \in K^\star$, we can construct an algebra $A \subseteq M(4, Q)$ of dimension 4 over Q, and therefore of dimension 8 over K, so that we still have a unique way of extending N to A; see 9.8 of [15]. We denote this algebra by $A = (Q, \gamma)$, and call it *Cayley-Dickson* algebra. This algebra A is not necessarily associative. The key point is that A is an alternative division ring if and only if Q is a division algebra and $\gamma \notin N(Q)$; see 9.9(v) of [15].

Example 3.3 Quadratic and pseudo-quadratic spaces ($n = 4$): We recall that the notation used in Example 2.5 is that from [16]; in particular, there is an ambient right vector space V over some (skew) field K and a field anti-automorphism σ of order at most 2, so that V is equipped with a σ-*quadratic form* $q : V \longrightarrow K/K_\sigma$, with $K_\sigma := \{t - t^\sigma : t \in K\}$ and q defined ad follows: $q(a + b) \equiv q(a) + q(b) + g(a, b) (\bmod K_\sigma)$, $q(at) \equiv t^\sigma q(a)t \pmod{K_\sigma}$, for all $a, b \in V$ and all $t \in K$, and $q(a) \equiv 0 \pmod{K_\sigma}$ if and only if $a = 0$, for all $a \in V$, where g is either the bilinear or Hermitian form associated to q according to whether, respectively, σ is the identity map or not. Below by the vector space L_0 of a (pseudo)quadratic space associated to an orthogonal or Hermitian quadrangle, we mean the vector space V_0 of V satisfying the following: by Proposition 2.3.4 of [16], if q is such a non-degenerate σ-quadratic form (which is also assumed to have *Witt index* 2), then there exist four vectors e_i, $i \in \{-2,-1,1,2\}$, a subvectorspace V_0 of V, a direct sum decomposition $V = e_{-2}K \oplus e_{-1}K \oplus V_0 \oplus e_1K \oplus e_2K$ and a non-degenerate anisotropic σ-quadratic form $q_0 : V_0 \longrightarrow K/K_\sigma$ such that for all $v = e_{-2}x_{-2} + e_{-1}x_{-1} + v_0 + e_1x_1 + e_2x_2$, with $x_i \in K$, $i \in \{-2,-1,1,2\}$ and $v_0 \in V_0$, we have $q(v) = x_{-2}^\sigma x_2 + x_{-1}^\sigma x_1 + q_0(v_0)$.

A *quadratic space* is a triple (K, L_0, q) where K is a field, L_0 a vector space over K and q a quadratic form on L_0. Also, it is called *anisotropic* if $q(a) = 0$ if and only if $a = 0$.

In the following we consider the σ-quadratic form $q : L_0 \longrightarrow K/K_0$ rather than just K/K_σ, for some additive subgroup K_0 of K such that $K_\sigma \subseteq K_0 \subseteq \mathrm{Fix}_K(\sigma) := \{a \in K : a^\sigma = a\}$. Let L_0 be a right vector space over K equipped with an anisotropic σ-quadratic form q on L_0 whose values are in the quotient K/K_0. An *anisotropic σ-quadratic space*, or *pseudo-quadratic space*, is a quintuple (K, K_0, σ, L_0, q) such that $K_\sigma \subseteq K_0 \subseteq Fix_K(\sigma)$, $a^\sigma K_0 a \subseteq K_0$, for all $a \in K$, and $1 \in K_0$.

Example 3.4 Indifferent sets ($n = 4$): Let K be a field of characteristic 2, and K_0 be a subfield of K such that K_0 contains $K^2 := \{k^2 : k \in K\}$. Then we say that a quadruple (L, L_0, K, K_0) is an *indifferent set* if L and L_0 are vector subspaces of, respectively, K and K_0, and viewed as vector spaces over, respectively, K_0 and K^2; moreover, we also require that $1 \in L \cap L_0$, and that L and L_0 generate, respectively, K and K_0 as rings.

Example 3.5 Exceptional quadratic spaces ($n = 4$): Exceptional quadrangles of type E_i, for $i = \{6, 7, 8\}$, are associated to certain quadratic spaces (K, L_0, q) which are a generalization of those defined above, in terms of a so-called *norm splitting* (basically, q is uniquely defined in terms of a fixed chosen basis of L_0 over E, with E/K being a separable quadratic extension); see Definition 12.9 of [15]. These quadratic spaces are very much related to

Clifford algebras $C(q) := K \oplus L_0 \oplus (L_0 \oplus_K L_0) \oplus (L_0 \oplus_K L_0 \oplus_K L_0) \oplus .../(u \oplus u - q(u) \cdot 1)$; see Definition 12.21 of [15]. The key fact, to us, is that for E_6, E_7 and E_8, the dimension of L_0 over K is, respectively, 10, 12 and 15.

Exceptional quadrangles of type F_4 are related to mixed quadrangles as explained by the following. Let K be a field of characteristic 2, L be a separable quadratic extension of K, and σ be a non-trivial ('involutary') field automorphism of L fixing K pointwise; let also K' be a subfield of K containing K^2, and L' be the subfield of L generated by L^2 and K', so that L'/K' is still a separable quadratic extension (this comes free, since σ restricts to an automorphism of L', and its fixed subfield is K'). If Q_{F_4} is an exceptional quadrangle of type F_4 associated with a *quadratic space of type* F_4, which we denote by (L, K, L', K'), then the root groups U_1 and U_3 are isomorphic to the direct product $L' \times L' \times K$, and the root groups U_2 and U_4 isomorphic to the direct product $L \times L \times K'$ (both additively); by restricting the coordinates to $\{0\} \times \{0\} \times K$ and $\{0\} \times \{0\} \times K'$, we obtain the mixed quadrangle $Q(K, K', K, K')$, which is thus an orthogonal quadrangle if and only if $K = K'$.

Example 3.6 Hexagonal systems ($n = 6$): As proved in Chapters 29 and 30 of [15], hexagonal systems are the underlying algebraic structures arising from Moufang hexagons; a complete list of hexagonal systems is given in Chapter 15 of [15]. They are denoted by $(J, F, N, \sharp, T, \times, 1)$, where F is a commutative field, J a vector space over F, N is a function from J to F called the *norm of* J/F, $\sharp$ is a function from J to itself called *adjoint*, T is a symmetric bilinear form from $J \times J$ to J, and 1 is a distinguished element of $J^\star$ called the *identity*; we are not going to give the precise definition (see 15.15 of [15]), but we just quote 15.16 of [15], which says that in an hexagonal system $(J, F, N, \sharp, T, \times, 1)$ the functions N, T and $\times$, as well the identity 1, are all uniquely determined by the function $\sharp$. It follows that we can restrict our attention to triples $(J, K, \sharp)$.

Apart from the split Cayley hexagon and the twisted triality hexagon (see Section 2.4 of [16]), already treated in Example 3.4 of [4] since they are good hexagons (see also Section 2.4 of [16]), there are four other examples of Moufang hexagons; throughout this paper we will refer to them with the same notation from [15]. However, below we also include the other two cases: Type 1/F corresponds to split Cayley hexagons when $E = F$, and Type 3/F corresponds exactly to twisted triality hexagons.

<u>TYPE 1/F</u>: Let E/F be a field extension such that $E^3 := \{x^3 : x \in E\} \subset F$. We have two possibilities: $F = E$ or $\mathrm{char}(F) = 3$ and the extension E/F is purely inseparable. If $\sharp$ is defined so that $a^\sharp = a^2$ for all $a \in E$, then $(E, F, \sharp)$ is an hexagonal system with $N(a) = a^3$, $a \times b = 2ab$ and $T(a, b) = 3ab$ for all $a, b \in E$.

<u>TYPE 3/F</u>: Let E/F be a separable field extension of degree three. We denote by L/F the normal closure of E/F, and by σ the element of order three in $\mathrm{Gal}(L/F)$. We define the function $\sharp$ as follows: $a^\sharp = a^\sigma a^{\sigma^2}$ for all $a \in E$. Then $(E, F, \sharp)$ is an hexagonal system with N and T, respectively, the norm and trace maps of the extension E/F.

<u>TYPE 9/F</u>: Let E/F be a separable cubic field extension. Suppose that the extension is normal. Let also σ be an element of the $\mathrm{Gal}(E/F)$. Choose $\gamma \in F^\star$. We then consider the *cyclic algebra of degree three* $D \subseteq M(3, E)$ determined by E, σ and γ, with $M(3, E)$ being the set of $3{\times}3$ matrices on E; see Example 15.5 of [15]. It is an algebra of dimension 9 over its centre $F = Z(D)$. The key point is that D is a division algebra if and only if $\gamma \notin N(E)$, where N ($\sharp$) coincides with the restriction to D of the norm map (adjoint map) of $M(3, E)$; see 15.7, 15.8 and 15.28 of [15]. Here, by $\gamma \in D$ we mean its image under the map $E \longrightarrow D$ which sends a to the diagonal

matrix $\mathrm{diag}(a, a^\sigma, a^{\sigma^2})$, and in particular γ to $\mathrm{diag}(\gamma, \gamma, \gamma)$.

If D is a cyclic division algebra of degree three with centre F, norm map N and adjoint map $\sharp$ as above, then D is an hexagonal system; see 15.27 of [15].

TYPE 27/F: Let D be a cyclic division algebra of degree three with centre F and norm map N, as above. Suppose that there exists an element $\gamma \in F \backslash N(D)$. Then, as shown by 15.23 of [15], we can extend the maps $\sharp, N$ and T to $J = D \oplus D \oplus D$, so that J is still an hexagonal system, and it has dimension 27 over F.

TYPE 9K/F: Let D be a cyclic division algebra of degree three over a field K, as above (which was defined of type 9/F), and let τ be an *involution* of D *of the second kind*, i.e. an anti-automorphism of order 2 which operates non-trivially on $Z(D) = K$. Let N, T and $\sharp$ denote, respectively, the norm, trace and adjoint maps of D. Let σ denote the restriction of τ to K and let $F = \mathrm{Fix}_K(\sigma)$ be the fixed field of σ in K. Then τ is a σ-*involution* of D, i.e. an automorphism of D as a vector space over F such that $\tau^2 = 1$ and $(ad)^\tau = a^\sigma d^\tau$ for all $a \in K$ and $d \in D$. Finally, let $J = \mathrm{Fix}_D(\tau)$. Then, by 15.30 of [15], J is closed under $\sharp$ and is an hexagonal system with $\dim_F(J) = \dim_K(D) = 9$.

TYPE 27K/F: Let J_0 be an hexagonal system of type 27/K (rather than F as in the notation above). It is then possible to construct from J_0 a new hexagonal system in the same fashion we constructed the haxagonal system of type 9K/F from that of type 9/F: let σ be an automorphism of K of order 2 and let $F = \mathrm{Fix}_K(\sigma)$; then suppose that τ is a σ-involution of J_0 which commutes with $\sharp$, and define $J = \mathrm{Fix}_{J_0}(\tau)$. By 15.30 of [15], it follows that J is an hexagonal system with $\dim_F J = 27$.

Example 3.7 Octagonal sets ($n = 8$): According to Definition 10.12 of [15], an *octogonal set* is a pair (K, σ), where K is a field of characteristic 2 and σ is an endomorphism of K such that σ^2 is the Frobenius endomorphism which sends x to x^2. In the particular case in which σ is an automorphism, we also call the pair (K, σ) a *difference field*. For example, if K is a finite field $F_{2^{2k+1}}$, then the Tits endomorphism is always the automorphism $x \longrightarrow x^{2^k}$, and the pair $(F_{2^{2k+1}}, x \longrightarrow x^{2^k})$ is a finite difference field; see Lemma 7.6.1 of [16].

4 Some model-theoretic facts for supersimplicity related to the algebraic structures of Section 3

The crucial model-theoretic concept in this paper is that of supersimplicity. Supersimple theories represent a subclass of simple theories equipped with a rank on types. A nice account of supersimple theories can be found in Chapter 5 of [17]. As examples of supersimple structures, we mention pseudofinite fields and also smoothly approximable structures; for the latter, see [3].

The appropriate rank for supersimple theories is the so-called SU-rank (see, for instance, Definition 5.1.1 of [17]), which also makes sense for arbitrary first-order theories; also, supersimple theories allow a further notion of rank called D-rank (see, for instance, Definition 5.1.13 of [17]). However, we define the S_1-rank which seems to be more suitable in a finite rank situation; besides, in any supersimple theory T and for any formula $\varphi(\bar{x})$, $\mathrm{SU}(\varphi(\bar{x})) = \mathrm{D}(\varphi(\bar{x})) = \mathrm{S}_1(\varphi(\bar{x}))$ whenever one of the three is finite (Lemma 6.13 and Proposition 6.14 of [7]). Therefore, since all these notions of ranks agree if and only if they are finite, throughout the rest of this paper we will just talk about supersimple structures of *finite rank*.

Definition 4.1 Given a formula $\varphi(\bar{x})$ in a language L, with parameters contained in a set A, we define the S_1-*rank* of $\varphi(\bar{x})$ as follows:

(*i*) $\mathrm{S}_1(\varphi(\bar{x})) = -1$ if $\varphi(\bar{x})$ is inconsistent; otherwise $\mathrm{S}_1(\varphi(\bar{x})) \geq 0$;

(*ii*) for $n \geq 0$, $\mathrm{S}_1(\varphi(\bar{x})) \geq n+1$ if there is a formula $\psi(\bar{x}, \bar{y}) \in L$ and an A-indiscernible sequence $(\bar{c_i} : i < \omega)$ such that $\models \psi(\bar{x}, \bar{c_i}) \longrightarrow \varphi(\bar{x})$ for some (any) i, and such that if $i \neq j$ then $\mathrm{S}_1(\psi(\bar{x}, \bar{c_i})) \geq n$ and $\mathrm{S}_1(\psi(\bar{x}, \bar{c_i}) \wedge \psi(\bar{x}, \bar{c_j})) < n$.

Moreover, we say that a first order theory is an S_1-*theory* if every formula has finite S_1-rank, and for every formula $\psi(\bar{x}, \bar{y})$ and every $m \geq 0$ the set $\{\bar{b} : \mathrm{S}_1(\psi(\bar{x}, \bar{b})) = m\}$ is definable.

We now list some model-theoretic facts which will be used later on in order to prove our main Theorem 1.1.

Proposition 4.2 (4.7 of [12]) Let F be a supersimple field, and D a finite dimensional division algebra over F. Then D is a field.

The following is the main result from [12].

Proposition 4.3 Let L be a finite Galois extension of a supersimple field K. Then, the norm map $N_{L/K} : L^\star \longrightarrow K^\star$ is surjective.

In Section 7 we will need the following proposition, which is adapted from the proofs of 34.2 and 34.3 of [15] with almost no change; basically, we just need to replace the finite field assumption with a supersimple field, and make sure that all the methods used for finite fields in the proof of [15] are also applicable for supersimple fields of finite rank.

Proposition 4.4 Let (K, L_0, q) be an anisotropic quadratic space, where K is a supersimple field of finite rank. Then $\dim_K L_0 \leq 2$.

Proof: Suppose that $\dim_K L_0 = l \geq 2$ and let f denote the bilinear form associated with q. Let ε be an arbitrary element of L_0^*. Replacing q by $q/q(\varepsilon)$, we can assume that $q(\varepsilon) = 1$. Choose an arbitrary 2-dimensional subspace M_0 of L_0 containing ε and let $M_0^{\perp} = \{a \in L_0 : f(a, M_0) = 0\}$, where $f(a, M_0) = \{f(a, m) : m \in M_0\}$. It is clear that, by fixing a basis of M_0 over K, and by restricting q to M_0, the whole structure $(K, M_0, q_{|M_0})$, where $q_{|M_0}$ denotes the restriction of q to M_0, is interpretable in K, and therefore supersimple of finite rank. We now need the following; to ease the notation, we will still denote $q_{|M_0}$ by q.

Claim: M_0 can be identified with a field E containing K in such a way that q corresponds to the norm of the extension E/K.

For let δ be a vector such that ε and δ span M_0 over K and let $p(x) = x^2 - f(\varepsilon, \delta)x + q(\delta) \in K[x]$. Also, let E be a splitting field of p over K, let N denote the norm function of the extension E/K and let η be a root of p in E. Since q is anisotropic and $t\varepsilon - \delta \neq 0$ for all $t \in K$, it follows that $0 \neq q(t\varepsilon - \delta) = q(t\varepsilon) + q(-\delta) + f(t\varepsilon, \delta) = t^2 + q(\delta) - tf(\varepsilon, \delta)$ for all $t \in K$. Thus, the polynomial p has no root in K. Then since E is the splitting field of p over K, for the non-trivial element $\sigma \in \mathrm{Gal}(E/K)$ we have $N(s+t\eta) = (s+t\eta)(s+t\eta)^{\sigma} = (s+t\eta)(s+t\eta^{\sigma}) = (s+t\eta)(s+t(f(\varepsilon, \delta)-\eta)) = s^2 + stf(\varepsilon, \delta) + t^2 q(\delta) = q(s\varepsilon + t\delta)$, for all $s, t \in K$; notice that the second equality follows from the fact that σ fixes K, while the third equality follows from the fact that since η is a root of p, which implies that $q(\delta) = \eta\eta^{\sigma}$ and $f(\varepsilon, \delta) = \eta^{\sigma} - \eta$, we have $N(\eta) = \eta\eta^{\sigma} = q(\delta) = \eta f(\varepsilon, \delta) - \eta^2 = \eta(f(\varepsilon, \delta) - \eta)$, thus that $\eta^{\delta} = f(\varepsilon, \delta) - \eta$. Therefore, q and N correspond under the K vector space isomorphism from M_0 to E sending ε to 1 and δ to η. This ends the *claim*.

Therefore, by the claim, there is a field E containing K and an isomorphism from M_0 to E such that q restricted to M_0 corresponds to the norm of the extension E/K under the isomorphism. Thus, by the choice of p and by the construction of E as the splitting field of p, the restriction of f to M_0 corresponds to the trace of E/K (i.e. the vectors ε and δ generate M_0, and the trace of the minimal polynomial of E/K is $f(\varepsilon, \delta)$, since the minimal polynomial of the extension E/F is p). Since K is supersimple, it is perfect (see Fact 4.1(ii) of [12]); hence, every finite degree extension of K is separable. Thus, the extension E/K is separable and its trace is non-degenerate (if not, by the claim, E could not be the splitting field of p). This implies that $M_0 \cap M_0^{\perp} = 0$. By Proposition 4.3, $q(M_0) = K$. Let $b \in M_0^{\perp}$. Then $q(a+b) = q(a) + q(b)$ for all $a \in M_0$. Since $q(M_0) = K$, we can choose $a \in M_0$ such that $q(a+b) = 0$. Since q is anisotropic, it follows that $a + b = 0$, so $b \in M_0 \cap M_0^{\perp} = 0$. Thus, $M_0^{\perp} = 0$. Hence, $L_0 = M_0$. This proves that $\dim_K L_0 \leq 2$. (Q.E.D.)

Applying similar methods from the study of finite Moufang polygons as we did for the above proposition, we can prove the following; again, this is a result which derives from [15] (see pages 377 and 378) with almost no change. We omit the proof.

Proposition 4.5 Let (K, K_0, σ, L_0, q) be an anisotropic σ-quadratic space, where (K, σ) is a supersimple difference field of finite rank. Then $\dim_K(L_0) \leq 1$.

Remark 4.6 Since the main objective of the remaining sections will be to prove Theorem 1.1(i), we mention some properties which are satisfied by fields whose first-order theory is supersimple of finite rank. At the moment it is not yet clear how to characterize a supersimple finite rank field, but we do have examples: for instance, *measurable* fields (in the sense of Definition

5.1 of [9]) which are indeed supersimple of finite rank. Measurable fields are conjectured to be pseudofinite, that is to be perfect, to have a unique Galois extension of each finite degree, and to be PAC (pseudo-algebraically closed), i.e. every absolutely irreducible variety has a rational point; the first two properties are satisfied by measurable fields, while we do not know about the PAC condition. More generally, perfect PAC fields with a *small* Galois group are also known to be supersimple of finite rank (where by 'small' we mean that for each natural number n there is a bounded number of non-isomorphic extensions of degree n). Among other results on supersimple fields, also oriented towards the above conjecture, we mention [10] and [11].

If the measurable field conjecture were true, by the main result from [4] and Theorem 1.1, it would follow that measurable Moufang polygons are just those infinite Moufang polygons which arise over pseudofinite fields.

5 Definability of the field from the polygon and restriction of coordinates

In this section we focus on Moufang polygons whose associated algebraic structures arise over some field K (for instance, here we do not consider projective planes over Cayley Dickson algebras). When this happens, we aim to define the field K. For given $x \in P \cup L$, it is sometimes possible to define a multiplication $\cdot$, say, on the right loop structure $(B, +) = (\{y \in P \cup L : d(x, y) = 1\}, +)$ defined as in Section 1 of [8], so that $(B, +, \cdot)$ becomes a field. We have a very useful result which allows us to define a multiplication on $(B, +)$. The idea comes from [8] and makes use of some results stated in Section 8.4 of [16]. First, we need to collect a series of facts.

Proposition 5.1 (Lemma 4.9 of [8]) Let K be a field and $G \subseteq \mathrm{PGL}(2, K)$ be a 2-transitive subgroup with respect to the usual action on the projective line $\mathrm{P}^1_K = K \cup \{\infty\}$. Then $\mathrm{PSL}_2(K) \subseteq G$; in particular, $\mathrm{PSL}_2(K)$ has no proper 2-transitive subgroups.

Proposition 5.2 (Lemma 3.3 of [8]) Let Γ be a Moufang polygon. Suppose that one of the groups of projectivities $\Pi(x)$ is (as a permutation group) isomorphic to a subgroup of $\mathrm{PGL}(2, K)$, for some field K. Then $\mathrm{PSL}_2(K)$ is a definable subgroup of $\Pi(x)$.

Proposition 5.3 (Lemma 3.1 of [8]) Let K be a field. Then in the group $\mathrm{PSL}_2(K)$ a copy K' of the field K is definable.

Recall from Section 8.4 of [16] the notion of projectivity groups. For many Moufang polygons $\Gamma = (P, L, I)$ the group of projectivities $\Pi(x)$, for some $x \in P \cup L$, satisfies the assumption on G in Proposition 5.2. Therefore, in such cases, the field is definable by the propositions above; to see this, we quote the Table 8.2 on page 378 of [16], which gives the representations of the projectivity groups of some Moufang polygons as groups acting on the set $B(x)$ for some $x \in P \cup L$. For 'representation' we mean the equivalence class of the permutation representation $(\Pi(x), B_1(x))$, which depends only on the sort of element x, i.e. the permutation representation $(\Pi(x), B_1(x))$ may differ from $(\Pi(x'), B_1(x'))$ only if x is a point (line) and x' is a line (point); see Lemma 1.5.1 of [16]. Notice that, in particular, all the good Moufang polygons are covered in the table below, though some of them only over finite fields.

The representation of the projectivity group associated to any line l is denoted by $(\Pi(\Gamma), X(\Gamma))$, while that associated to any point x is denoted by $(\Pi^*(\Gamma), X^*(\Gamma))$. Notice that the labelling used to denote the polygons differs from that of Table 8.2 of [16], since we want consistency with our notation. Apart from the classical notation (i.e. $PG(1, K)$, $PGL_2(K)$, $PSL_2(K)$, etc.), for the 'mysterious' groups $PSL_2^{K'}(K)$, $PGL_2^q(q^2)$, and so on, we refer to Section 8.4 of [16]; likewise, see Section 7.5.3 of [16] for the Hermitian unital $U_H(q)$, and Section 7.6 of [16] for the Suzuki quadrangle $Sz(K,\sigma)$ and its Suzuki-Tits ovoid $\Theta_{ST}(K,\sigma)$. Also, $H(K,K',K,K')$ denotes the *mixed hexagon* as defined in 3.5.3 of [16], which is, in our notation, the hexagon of Type 1/K over a field K of characteristic 3.

n	the polygon Γ	$(\Pi(\Gamma), X(\Gamma))$	$(\Pi^*(\Gamma), X^*(\Gamma))$
3	PG$(2,K)$	$(PGL_2(K), PG(1,K))$	$(PGL_2(K), PG(1,K))$
4	$Q(K,K',K,K')$	$(PSL_2^{K'}(K), PG(1,K))$	$(PSL_2^{K^2}(K'), PG(1,K'))$
4	$W(K)$	$(PGL_2(K), PG(1,K))$	$(PSL_2(K), PG(1,K))$
4	$HQ(4,q^2)$	$(PSL_2^q(q^2), PG(1,q^2))$	$(PGL_2(q), PG(1,q))$
4	$HQ(5,q^2)$	$(PGL_2^q(q^2), PG(1,q^2))$	$(PGU_3(q), U_H(q))$
6	$H(K,K',K,K')$	$(PGL_2(K), PG(1,K))$	$(PGL_2(K'), PG(1,K'))$
6	$H(K)$	$(PGL_2(K), PG(1,K))$	$(PGL_2(K), PG(1,K))$
6	$T(q^3,q)$	$(PGL_2(q^3), PG(1,q^3))$	$(PGL_2(q), PG(1,q))$
8	$O(K,\sigma)$	$(PGL_2(K), PG(1,K))$	$(Sz(K,\sigma), \theta_{ST}(K,\sigma))$

Table 5.1

We close this section by giving some further information on the coordinatization procedure of Moufang polygons; this will be relevant is some steps through Sections 6-9. First, we recall Definition 2.8 of [4], but we rephrase it in a more self-contained way, and refer the reader to Section 2 of [4] for more details. In particular, below, by a *Schubert cell* we mean the set $B_k(x,y) := B_k(y)\backslash B_{k-1}(x)$ for some $x,y \in \Gamma$ and sone fixed $k < n$. For ease of notation, from now on we will denote by U_i the root group U_{α_i} associated to the root $\alpha_i = (x_i, x_{i+1}, ..., x_{i+n})$.

Definition 5.4 Let Γ be a Moufang n-polygon. We fix an ordinary subpolygon $A = (x_0, x_1, ..., x_{2n-1}) \subset \Gamma$ and a root $\alpha_0 = (x_0, x_1, ..., x_n) \subset A$, and label the corrisponding root groups $U_1, U_2, ..., U_n$, as well the opposite root groups $U_{n+1}, U_{n+2}, ..., U_{2n}$. We then consider an element $x \in B_k(x_{2n-1}, x_0)$, for some $k < n$, and let $(x_{2n-1}, x_0, x'_1, x'_2, ..., x'_k = x)$ denote the corresponding $(k+1)$-chain. Note that $d(x'_i, x_{n+i}) = n$ for $i = 1, 2, ..., k$, so we may put $t_i(x) = \mathrm{proj}_{n-1}(x'_i, x_{n+i-1}) \in T_i$, where $T_i = B_1(x_{n+i-1}) \backslash \{x_{n+1}\}$ are the *parameter sets* of the coordinatization of Γ. We have therefore attached coordinates $t_1(x), t_2(x), ..., t_k(x) \in T_1 \times T_2 \times ... \times T_k$ to the element x.

Let Γ be a Moufang n-polygon, and let us fix A, α and U_i as in the definition above. In the language L_{inc} of Γ, we can define a (right) left loop $(B_0, +_0)$ on the set $B_1(x_0) = \{y : d(x, y) = 1\}$, as done in Section 1 of [8]; see also Section 4 of [4], where we definably identify (with parameter from A) the root groups U_{2i} with $(B_0, +_0)$. Also, by Lemma 4.1 of [4] we can definably extend the group action of U_{2i} to the whole of Γ; this expands the proof of [8, Lemma 3.2] by making use of the Beth's Definability Theorem. Similarly, we can define another (right) left loop $(B_1, +_1)$ on the set $B_1(x_1) = \{y : d(x_1, y) = 1\}$, and then definably identify it with the root groups U_{2i+1}. Moreover, by Lemma 2.2, every element u of $U_{[1,n]}$ is uniquely expressible as $u = u_1 u_2 ... u_n$, for $u_i \in U_i$. Then, we can see $U_{[1,n]}$ as living in the little projective group Σ associated to Γ. Since we have defined each U_i, for $i = 1, 2, ..., n$, and because of the uniqueness assumption of Lemma 2.2, it follows that $U_{[1,n]}$ is definable too by just specifying for every element $u = u_1 u_2 ... u_n \in U_{[1,n]}$ the action of each u_i on the whole of Γ. Finally, as already discussed in Section 3, from the root sequence $(U_{[1,n]}, U_1, U_2, ..., U_n)$ and the commutator relations which characterize the polygon in terms of Chapter 16 of [15], some algebraic structures S_i arise, and their underlying sets will form the parameter sets T_i of the coordinatization of Γ: there will be definable maps u_i which associate to each $t_i \in T_i$ the (unique) corrisponding element $u_i = u_i(t_i)$, for each i.

Discussion 5.5 We now discuss how to construct, in a definable way, a certain (sub)polygon Γ' associated with some Moufang polygon Γ; below, we list all the possible cases where such a situation arises.

(1) All the Moufang Hermitian quadrangles $\Gamma = HQ(l, K)$ for $l \geq 5$ are extensions of the Moufang Hermitian quadrangle $\Gamma' = HQ(4, K)$. This is Theorem 21.11 of [15], which is proved in Chapter 25 of [15]. See also Remark 21.16 of [15]. An important point is that given a Moufang Hermitian quadrangle $HQ(l, K)$ for $l \geq 4$, with associated σ-quadratic space (K, K_0, σ, L_0, q) as defined in the last paragraph of Example 3.3, by excluding either $L_0 = 0$ or $\sigma = 1$ we can have any Hermitian quadrangle Γ but $\Gamma' = HQ(4, K)$. We are in the situation in which we can label the root groups of Γ' exactly as those of Γ; namely, we fix an apartment $A' = (x'_0, x'_1, ..., x'_7) \subset \Gamma'$ and a root $\alpha'_0 = (x'_0, x'_1, ..., x'_4) \subset A'$, and label the corresponding root groups by U'_i such that if U_i is the root group of Γ associated with the root $\alpha_i = (x_i, x_{i+1}, ..., x_{i+4})$, then U'_i is the root group of Γ' associated with the root $\alpha'_i = (x'_i, x'_{i+1}, ..., x'_{i+4})$. Then we know that the root groups U'_{2i} are all equal to the roots U_{2i} and isomorphic to the additive group of K, while the root groups U'_{2i+1} are subgroups of the U_{2i+1} and isomorphic to a certain group $(T, \cdot)$ as given in Example 2.5. In particular, as shown in the proof of [15, Theorem 21.11], the root groups U'_{2i+1} are identified with $Y_i := U^{[2]}_{[1,4]}$, i.e. the pointwise stabilizer in $U_{[1,4]}$ of $\{x \in \Gamma : d(x, x_0) \leq 2\}$, which is a definable set in Γ. We can now consider the new root group sequence $(U_{[1,4]}, U'_1, U_2, U'_3, U_4)$, which is thus definable in Γ. From the latter, we can then reconstruct the Moufang Hermitian quadrangle Γ' via coordinatization as in Definition 5.4, where the coordinates of every element will be a string of alternating coordinates from the underlying sets of K and K_0; the latter, as well the anti-automorphism σ associated to K, is also definable as $\mathrm{Fix}_K(\sigma)$ (see Proposition 7.2 for more details).

(2) We have a similar result to the above also for Moufang mixed quadrangles and exceptional quadrangles of type E_i, for $i = 6, 7$ and 8. In both cases the arising subpolygon is a Moufang orthogonal quadrangle. With similar notation, we recover the root groups U_i' as $U_i' = [U_{i-1}, U_{i+1}]$.

(3) Slightly different is the case of a Moufang octagon Γ. We need to reconstruct a mixed Moufang quadrangle Γ' which is associated to Γ as explained in Section 9. We recall from Example 3.7 that Γ is uniquely associated to an octagonal set (K, σ), and from Example 2.9 that its root groups U_{2i+1} are isomorphic to the additive group of K while the root groups U_{2i} are isomorphic to a certain group $K_\sigma^{(2)}$. To define Γ', we first restrict the coordinates $(t_1, t_2, ..., t_8)$ of the elements of Γ to (t_1, t_3, t_5, t_7), via the coordinates restriction procedure from $U_1 \times U_2 \times ... \times U_8$ to $U_1 \times \{1_{U_2}\} \times U_3 \times \{1_{U_4}\} ... U_7 \times \{1_{U_7}\}$, and then reconstruct Γ' from the new root group sequence $(U'_{[1,4]}, U'_1, U'_2, U'_3, U'_4)$ as in Chapter 24 of [15]. In general, from the latter, it does not follow that $U'_{[1,4]} = \langle U_1, U_3, U_5, U_7 \rangle$ and $U'_i = U_i$, but this is indeed the case for supersimple Moufang octagons, since the arising Moufang mixed quadrangle Γ' will be an orthogonal quadrangle; see Proposition 7.6.

6 Moufang projective planes

Let $\Gamma = \mathrm{PG}(2, A)$ be a Moufang projective plane associated with an alternative division ring $A = (A, +, \cdot)$, as in Example 2.4. Fix an ordinary subpolygon $\Gamma_0 = (x_0, x_1, ..., x_6) \subset \Gamma$, a root $\alpha = (x_0, x_1, x_2, x_3) \subset \Gamma_0$, and label the corresponding positive root groups by U_1, U_2 and U_3, and the opposite root groups by U_4, U_5 and U_6.

Lemma 6.1 Let $\Gamma = \mathrm{PG}(2, A)$ be a Moufang projective plane over an alternative division ring A. Then A is definable in $\mathrm{PG}(2, A)$.

Proof: We fix the above setting: Γ_0, α, U_i, and so on. It follows from Section 1 of [8] that all the root groups, together with their action on the whole of Γ, are parameter definable (with parameters from Γ_0) and isomorphic to $(A, +)$, and that there exists a parameter definable identification between any two of them; see also Section 4 of [4].

We now aim to define the multiplication $\cdot$ of A. By Lemma 2.2, with $n = 3$, there exists a bijection φ from the set theoretic product $U_1 \times U_2 \times U_3$ to $U_{[1,3]} := \langle U_1, U_2, U_3 \rangle$; that is, every element u of $U_{[1,3]}$ is uniquely expressible as $u = u_1 u_2 u_3$, for $u_i \in U_i$. Then, we can see $U_{[1,3]}$ as living in the little projective group Σ, say, associated to Γ. Since we have defined each U_i, for $i = 1, 2, 3$, then $U_{[1,3]}$ is definable. Let now $[\ ,\ ]$ denote the group theoretic commutator, interpreted in the little projective group Σ. Then, again by Lemma 2.2, for every $a, b \in A$ there exists a unique $c \in A$ such that $[u_1(a), u_3(b)] = u_2(c)$ and $c = ab$; thus, the multiplication $\cdot$ is interpreted in $U_{[1,3]}$ by the following first-order definable operation $\times$:

$$u_1(a) \times u_3(b) := [u_1(a), u_3(b)] = u_2(c)$$

where $u_1(a)$, $u_2(c)$ and $u_3(b)$ are the unique elements which identify, respectively, a, c and b. (Q.E.D)

Corollary 6.2 Any supersimple Moufang projective plane of finite rank is necessarily associated with a supersimple field of finite rank.

Proof: Let $\Gamma = \mathrm{PG}(2, A)$ be a supersimple Moufang projective plane of finite rank over an alternative division ring A. Then, by Lemma 6.1, A is definable in Γ. Hence, A is a supersimple alternative division ring of finite rank. We have two possibilities: if A is associative, then A is a either a field or a skew-field, but in this case A is necessarily a field (i.e. A is commutative) because every supersimple division ring is commutative (from [12]); if A is non-associative, then A has to be isomorphic to a Cayley Dickson algebra $Q = (Q, \sigma) = ((E/K), \beta)$ (see Example 3.2), with K a subring of Q, and therefore of A, as well the centre of Q. Hence, K is definable in A, and so in Γ, and therefore supersimple of finite rank. Moreover, since the quaternion division algebra Q is an algebra of degree 4 over K, we can also define Q (making use of a basis over K).

We may thus assume that A is a Cayley Dickson algebra, otherwise the assertion follows. Then, A, K and Q are definable. In particular, see Example 3.2, Q is a supersimple division algebra over the field K. With this proviso, it follows from Proposition 4.2 that Q is a field, therefore commutative. This contradicts the construction of Q (and therefore of A), which is non-commutative. Hence, A cannot be a Cayley Dickson algebra, and Γ can only be associated with fields (which are therefore supersimple because definable). (Q.E.D)

7 Moufang quadrangles

We start with orthogonal and Hermitian quadrangles. In the following, and later on, we should not confuse the vector space L_0 with the ambient vector space V as given by Example 3.2 of [4] (see also Example 3.3); in particular, care must be taken about the respective dimensions, over the base field, of L_0 and V.

Proposition 7.1 Let $Q(l, K)$ be a supersimple Moufang orthogonal quadrangle of finite rank over a field K in an l-dimensional vector space V, with $l \geq 5$. Then:

(i) K is definable, and therefore supersimple of finite rank;

(*ii*) $Q(l, K)$ is, up to duality, isomorphic to either $Q(5, K)$ or $Q(6, K)$.

Proof: Let $\Gamma = Q(K, L_0, q)$ be a Moufang orthogonal quadrangle associated with an anisotropic quadratic space (K, L_0, q) as defined in Example 2.5; then L_0 is a vector subspace of V, and it is, up to isomorphism, V_0 in the decomposition $V = e_{-2}K \oplus e_{-1}K \oplus V_0 \oplus e_1K \oplus e_2K$, as explained in the last sentence of the first paragraph of Example 3.3. It follows that $V_0 \neq 0$ (otherwise, $\dim_K(V) \leq 4$ and so Γ would not be thick, contradicting Corollary 2.3.6 of [16]). Since the elements of the root groups of Γ are induced by linear maps of the ambient vector space V (see Chapter 8 of [16], and also Proposition 3.5 of [8]), the little projective group induces a subgroup of $\mathrm{PGL}_2(K)$ on any line pencil/point row of Γ; thus, the induced group of projectivities satisfies the assumption of Propositions 5.1 and 5.2 (see Section 8.5 of [15] for more details about projective embeddings of generalized quadrangles). Then, by Proposition 5.3, we can define the field K, which is therefore supersimple of finite rank; hence, we can define the vector space L_0 over K (basically making use of a chosen basis over K). It follows, by Proposition 4.4, that $\dim_K(L_0) \leq 2$.

If $\dim_K(L_0) = 1$, then $L_0 = K$ and the quadrangle Γ is, up to duality (see Proposition 3.4.13 of [16]), the orthogonal quadrangle $Q(5, K)$ in some 5-dimensional vector space V over the field K. If $\dim_K(L_0) = 2$, then there exists a 2-dimensional vector space E over K (and the associated σ, q and N) as in the proof of Proposition 4.4. In this case Γ is, up to duality (see Proposition 3.4.9 of [16]), the orthogonal quadrangle $Q(6, K)$. (Q.E.D)

Next, we give an analogue of Proposition 7.1 for Moufang Hermitian quadrangles; however, attention must be paid to the case in which we have a skew field K. We need to make sure that no Moufang Hermitian quadrangle $H(4, K)$ over a skew field K can be supersimple of finite rank. Hence, we just need to show that K is definable in the polygon. For the latter would imply that K is commutative, contradicting the fact that a skew field is non-commutative. Recall that, see Section 2, by L_{inc} we denote the language of polygons.

Proposition 7.2 Let K be a (skew) field with an antiautomorphism σ, and consider a Moufang Hermitian quadrangle $H(4, K)$ over K. Then we have the following:

(i) K and σ are definable in the language L_{inc} of $H(4, K)$;

(ii) if K is a skew field, then $H(4, K)$ is not supersimple of finite rank.

Proof: A proof of (i) is given by [8, Proposition 3.7], which explicitly handles only the case of a skew field (in [8] the authors refer to skew fields as 'proper' skew fields); however, the proof is also valid for fields. Notice that in [8] the statement of the proposition refers to the finite Morley rank case; however, the proof makes no use of finite Morley rank. Thus, we use their proof to define the (skew) field also in the supersimple case. Then (ii) follows from (i). (Q.E.D)

Remark 7.3 This can be used as an alternative to the proof above. Since we are classifying Moufang quadrangles (whose first order theory is supersimple of finite rank) up to duality, we refer to Propositions 3.4.9, 3.4.11 and 3.4.13 of [15], for the only three existing cases of an isomorphism between an orthogonal quadrangle and the dual of an Hermitian one, and vice versa, and between a symplectic quadrangle and the dual of an orthogonal quadrangle, and vice versa; in particular, Proposition 3.4.11 tells us that the Hermitian quadrangle $H(4, L)$, for some skew field L over its centre K (as a quaternion algebra), is isomorphic to the dual of an orthogonal quadrangle

$Q(8, K)$ associated with an anisotropic quadratic space (K, L_0, q) where the dimension of L_0 over K is 4. Hence, by Proposition 4.4 such an orthogonal quadrangle cannot be supersimple of finite rank.

As we said in Discussion 5.5, all the Moufang Hermitian quadrangles $HQ(l, K)$ for $l \geq 5$ are extensions of the Moufang Hermitian quadrangle $HQ(4, K)$. We next prove that the latter together with $HQ(5, K)$ are the only supersimple Moufang Hermitian quadragles of finite rank; first, we need the following lemma.

Lemma 7.4 Let $\Gamma = HQ(l, K)$, for $l \geq 5$, be a supersimple Moufang Hermitian quadrangle of finite rank which extends $\Gamma' = HQ(4, K)$. Then Γ' is interpretable in Γ.

Proof: This is just Discussion 5.5(1). (Q.E.D)

Corollary 7.5 Let $HQ(l, K)$ be a supersimple Moufang Hermitian quadrangle of finite rank over a (skew) difference field (K, σ) in an l-dimensional vector space V, $l \geq 4$. Then:

(i) K and σ are definable, and therefore (K, σ) is a supersimple difference field of finite rank;

(ii) $HQ(l, K)$ is isomorphic, up to duality, either to $HQ(4, K)$ or $HQ(5, K)$.

Proof: Let $\Gamma = HQ(l, K)$, $l \geq 4$, be a supersimple Moufang Hermitian quadrangle associated with an anisotropic σ-quadratic space (K, K_0, σ, L_0, q) as in Example 2.5. Suppose first that $\Gamma = HQ(4, K)$. Then, by Proposition 7.2, (K, σ) is definable and therefore supersimple of finite rank. Suppose now that $\Gamma = HQ(l, K)$, for $l \geq 5$. Then, it follows from Lemma 7.4 that $\Gamma' = HQ(4, K)$ is parameter definable in Γ, hence supersimple of finite rank. Therefore, as before, (K, σ) is definable in Γ. This proves (i).

Thus, (K, σ) is a supersimple difference field of finite rank, and we can apply Proposition 4.5; it follows that $\dim_K(L_0) \leq 1$. Recall now that we have the decomposition of the ambient vector space V in $V = e_{-2}K \oplus e_{-1}K \oplus V_0 \oplus e_1 K \oplus e_2 K$, as in Example 3.3, where V_0 is L_0. Hence, we have only two cases: if $\dim(L_0) = 0$ then Γ is $HQ(4, K)$, and if $\dim(L_0) = 1$ then Γ is $HQ(5, K)$; the latter has actually two subcases, according to the characteristic of K, but we refer to Section 5.5.5, Step II, of [16], and Remark 21.16 of [15], for the details. (Q.E.D)

We now turn to Moufang mixed quadrangles, see Example 2.6. The following is similar to [8, Proposition 3.6], which states that a Moufang mixed quadrangle Γ has finite Morley rank if and only if K is algebraically closed, in which case Γ is exactly an orthogonal quadrangle; and, more precisely, it is in fact $Q(5, K)$. As usual, the task is to define K in Γ.

Since the definability of K in Proposition 3.6 of [8] makes no use of finite Morley rank (although the latter assumption implies that K is algebraically closed), we apply it below with our supersimple finite rank assumption; since the proof is practically identical to that of [8], we do not give all the details, but we do indicate the key point.

Proposition 7.6 Let $Q(L, L_0, K, K_0)$ be an infinite supersimple Moufang mixed quadrangle of finite rank. Then:

(i) K coincides with K_0 and is definable;

(ii) $Q(L, L_0, K, K_0)$ is, up to duality, definably isomorphic to either $Q(5, K)$ or $Q(6, K)$.

Sketch of the proof: Let Γ be a Moufang mixed quadrangle $Q(L, L_0, K, K_0)$ as in Example 2.6, and suppose that its first-order theory is supersimple of finite rank. There is an orthogonal quadrangle Γ' associated to Γ, and the argument of restricting coordinates and definably reconstructing it is identical to that of the Hermitian case in Lemma 7.4. We omit the details. Since Γ' is definable in Γ, and therefore supersimple of finite rank, it follows by Proposition 7.1 that Γ' is, up to duality, either $Q(5, K')$ or $Q(6, K')$, where K' is the definable underlying field of Γ', therefore supersimple of finite rank. Since, by [12], every supersimple field is perfect, it follows that $(K')^2 = K'$. Hence, $K^2 \subseteq K' = (K')^2 \subseteq K^2$, namely $K' = K^2$; thus, since $K^2 = K' = (K')^2 = (K^2)^2$, it follows that $K = K^2$, and therefore K and K' coincide, and so K is definable. So Γ and $Q(5, K')$ (or $Q(6, K')$) coincide, since they arise from the same algebraic structure and have the same parameter sets. Thus, we conclude that Γ is an orthogonal quadrangle over a supersimple field K of finite rank. (Q.E.D)

Finally, we are left with Moufang exceptional quadrangles of type E_i, with $i \in \{6, 7, 8\}$, and F_4. They can be handled all together in the next proposition.

Proposition 7.7 Let Γ be an exceptional Moufang quadrangle either of type E_i, for $i = 6, 7, 8$, or F_4. Then Γ cannot be supersimple of finite rank.

Proof: Suppose first that Γ is of type E_i. As in Example 3.5, we know that the underlying algebraic structure of Γ is some generalization of a quadratic space (K, L_0, q), and that the dimension of L_0 over K is, respectively, 10, 12 and 15. However, by Theorem 21.12(ii) of [15], we also know that Γ extends an orthogonal quadrangle Γ', and that Γ' is definable in Γ by Discussion 5.5(2). In particular, we reconstruct Γ' over a quadratic space (K', L'_0, q') with $K' = K$. Then, by Proposition 7.1, Γ' is either $Q(5, K)$ or $Q(6, K)$. We can thus define the field K, which is then supersimple of finite rank. However, since the $\dim_K L_0$ is either 10, 12 or 15, and since we can define the vector space L_0 over K, it follows from Proposition 4.4 that Γ cannot be of type E_i.

Suppose now that Γ is of type F_4; then, from the description in [15], Γ has two kind of mixed subquadrangles up to isomorphism. Let Γ' denote such a subquadrangle of Γ. Then, again with the same argument of restricting coordinates, Γ' is definable in Γ. Hence, as in the previous cases, we can also define the field K, say, corresponding to Γ'. However, by Chapter 28 of [15], the Moufang quadrangle $Q_{F_4}(L, K, L', K')$ can only be associated with a field K that is not perfect, which contradicts the assumption of supersimplicity; see Fact 4.1(ii) of [12], and Remark 4.6. (Q.E.D)

Corollary 7.8 Let Γ be a supersimple Moufang quadrangle of finite rank. Then Γ is, up to duality, either $Q(5, K)$, $Q(6, K)$, $H(4, K)$ or $H(5, K)$, for some supersimple field K of finite rank.

Proof: This is an immediate consequence of Propositions 7.1, 7.2, 7.6 and 7.7, and Corollary 7.5. (Q.E.D)

8 Moufang hexagons

We do have Moufang hexagons whose first order theory is supersimple of finite rank. Indeed, as a result of the main theorem from [4], there are two examples inherited from the corresponding classes of finite Moufang hexagons as non-principal ultraproducts of these classes; namely, with the notation of Example 3.6, the split Cayley hexagon associated with an hexagonal system of type 1/F (only the case in which $E = F$) and the triality twisted hexagon associated with an hexagonal system of type 3/F. Therefore, we are left with the Moufang hexagons whose hexagonal systems are of type 9/F, 27/F, 9K/F, 27K/F and the case of type 1/F in characteristic three. We prove in the following that these remaining Moufang hexagons cannot be supersimple of finite rank.

Theorem 8.1 Let H be a supersimple Moufang hexagon of finite rank. Then H is either isomorphic to a split Cayley hexagon or a twisted triality hexagon over a supersimple field of finite rank.

Proof: Let H be a supersimple Moufang hexagon of finite rank, which we also suppose to be bad (one of those listed above). Fix an ordinary subpolygon $A = (x_0, x_1, ..., x_{11})$, a root $\alpha = (x_0, x_1, ..., x_6) \subseteq A$, and the corresponding root groups $U_1, U_2, ..., U_6$. Suppose first that H is associated with any of the hexagonal systems of type 9/F, 27/F, 9K/F or 27K/F. These four cases can be treated all together since all of them arise over an hexagonal system J which depends on the construction of a cyclic division algebra D of degree three as given in the case of type 9/F of Definition 3.8; notice that, in particular, the latter case is the only situation in which J is exactly D. Then, as explained in the beginning of Section 5, we can define (with parameters) the root groups U_i given in Example 2.8; in particular, we can define the additive group $(F, +)$ of the field F and the additive group $(J, +)$ of the vector space J. Therefore, we can definably identify the root groups U_i, for $i = 2, 4, 6$, with $(F, +)$.

We now proceed with exactly the same argument we used in Lemmas 6.1 and 7.4, by applying Lemma 2.2 (with $n = 6$): there exists a bijection φ from the set theoretic product $U_1 \times U_2 \times ... \times U_6$ to $U_{[1,6]} := \langle U_1, U_2, ..., U_6 \rangle$; that is, every element u of $U_{[1,6]}$ is uniquely expressible as $u = u_1 u_2 ... u_6$, for $U_i \in U_i$. Then we can see $U_{[1,6]}$ as living in the little projective group Σ, say, of H. Hence, by the fourth commutator relation in Example 2.8, since we can define the commutator operation in Σ, we can define the multiplicative operation $\times$, say, of the field F as follows: for all $t, u \in F$, we define $t \times u = v$, where v is the unique element of F such that $x_2(t) \times x_6(u) = x_4(v)$. Hence, the whole field structure of F is definable, and so F is a supersimple field of finite rank. From F we can therefore definably recover the structure of J as a 9-dimensional vector space over F; in particular, we can also define the corresponding norm map N. Then, by Proposition 4.3, N has to be surjective, but this is a contradiction since N is not surjective; see again Example 3.6.

Hence, we may assume that H is associated with a hexagonal system of type 1/F with $\mathrm{char}(F) = 3$. We know by definition that F is not perfect. We can proceed exactly as in the previous cases, and so define F. Since F is supersimple, the field F is perfect (by Fact 4.1(ii) of [12]), which is a contradiction. (Q.E.D)

9 Moufang Octagons

Let $\Gamma = O(K, \sigma)$ be a Moufang octagon as defined in Example 2.9. We assume that Γ is supersimple of finite rank. We fix an ordinary suboctagon $A = (x_0, x_1, ..., x_{15}) \subseteq \Gamma$ and a root $\alpha = (x_0, x_1, ..., x_8) \subseteq A$. Also, associated to A, we label the corresponding root groups by U_i, for $i = \{1, 2, ..., 8\}$. By Section 4 of [4], we can define all the root groups U_i together with their action on the whole of Γ. In particular, we can define (inside the little projective group associated to Γ) the group $U_{[1,8]} := \langle U_1, U_2, ..., U_8 \rangle$, which by Lemma 2.2 (with $n = 8$) is in bijection with the product $U_1 \times U_2 \times ... \times U_8$; also, by restricting only to the units $1_{U_i} \in U_i$ for $i = \{2, 4, 6, 8\}$, we can define $U'_{[1,4]} := \langle U_1, U_3, U_5, U_7 \rangle \leq U_{[1,8]}$ (recall that, see Example 2.9, the root groups U_1, U_3, U_5 and U_7 are chosen and labelled in a way that they all are isomorphic to the additive group of K).

In Chapters 7 and 8 of [15] is shown how to construct from the definable quintuple $(U'_{[1,4]}, U_1, U_3, U_5, U_7)$ a Moufang quadrangle, which is of mixed type by 31.8 of [15]. Since, by the data above, this construction is first order definable, the resulting mixed Moufang quadrangle is definable in Γ. Denote this quadrangle by $\Gamma' = Q(L, L_0, K, K_0)$. It follows by Proposition 7.6 that Γ' is either $Q(5, K)$ or $Q(6, K)$; in particular, the field K is definable, therefore K is supersimple of finite rank. It remains to prove the definability of the endomorphism σ of K.

We need the following proposition from 6.1 of [15], which is also valid for every Moufang n-polygon, so not just for octagons (replacing 8 by n and the labelling of the root groups accordingly); below, we label the root groups by elements of the integers modulo 8, i.e. $U_i = U_{i+8}$ for all $i \in Z$ (see Definition 4.14 of [15]).

Proposition 9.1 For each i, there exist unique functions $\kappa_i, \lambda_i : U_i \backslash \{1_{U_i}\} \longrightarrow U_{i+8} \backslash \{1_{U_{i+8}}\}$ such that $x_{i-1}^{a_i \lambda_i(a_i)} = x_{i+1}$ and $x_{i+1}^{\kappa_i(a_i) a_i} = x_{i-1}$ for all $a_i \in U_i \backslash \{1_{U_i}\}$. The product $\mu_i(a_i) := \kappa_i(a_i) a_i \lambda_i(a_i)$ fixes x_i and x_{i+8}, reflects A, and $U_j^{\mu_i(a_i)} = U_{2i+8-j}$ for each $a_i \in U_i \backslash \{1_{U_i}\}$ and each j.

Proposition 9.2 Let $\Gamma = O(K, \sigma)$ be a Moufang octagon. Then:

(i) (K, σ) is definable in Γ;

(ii) if Γ is supersimple of finite rank, then (K, σ) is definable difference field.

Proof: Let $\Gamma = O(K, \sigma)$ be a Moufang octagon, and let us fix the above setting, namely A, α and U_i. To prove (i), we need to show that (K, σ) is definable in the language L_{inc} of Γ. By the uniqueness property of Proposition 9.1, the functions κ_i and λ_i are A-definable, hence μ_i, as an element of $U_{[1,8]} := \langle U_1, U_2, ..., U_8 \rangle$, is A-definable. By 31.9(i) of [15] there exists an element $e_8 \in U_8 \backslash \{1_{U_8}\}$ such that $\mu_8(e_8)^2 = 1$. The element e_8 will play the role of a parameter. It follows from the proposition above that $U_i^{\mu_8(e_8)} = U_{8-i}$ for every $i \in \{1, 3, 5, 7\}$. Therefore $\mu_8(e_8)$ acts on the mixed quadrangle

$\Gamma' = Q(L, L_0, K, K_0)$ associated with Γ; as remarked above, we can define both Γ' and K in Γ, and that $K = K_0$. Put now $\alpha := \mu_8(e_8)$. It then follows by 24.6 of [15] that there exist an endomorphism σ of K such that (K, σ) is an octagonal set as defined in Example 3.7, and that $x_3(t)^\alpha = x_2(t^\sigma)$ for all $t \in K$. Since α is definable, from the equation $x_3(t)^\alpha = x_2(t^\sigma)$ it follows that σ is definable. This proves (i). For (ii), if Γ is a supersimple Moufang octagon of finite rank, it then follows from (i) that (K, σ) is definable. Thus, to show that (K, σ) is a difference field (see Example 3.7), we have to show that the endomorphism σ of K is actually an automorphism. By definition of a Tits endomorphism, we know that $K^{\sigma^2} = K^2$. Since K is of characteristic 2, it follows that $K^{\sigma^2} = K^2 = K$, and therefore that $K^\sigma = K$. (Q.E.D)

We summarize the results of Sections 6, 7, 8 and 9 into the following main result.

Corollary 9.3 Let $\Gamma = \Gamma(K)$ be a supersimple finite rank Moufang polygon. Then:

(i) the (difference) field K is definable in Γ;

(ii) Γ is good.

Proof. It follows immediately from Corollaries 6.2 and 7.8, and Proposition 8.1 and 9.2. (Q.E.D)

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