POINCARÉ - VERDIER DUALITY IN O-MINIMAL STRUCTURES

MÁRIO J. EDMUNDO AND LUCA PRELLI

Abstract. Here we prove a Poincaré - Verdier duality theorem for the o-minimal sheaf cohomology with definably compact supports of definably normal, definably locally compact spaces in an arbitrary o-minimal structure.

Résumé. On démontre une dualité de Poincaré - Verdier dans le cadre de la cohomologie o-minimale des faisceaux avec support compact et définissable sur un espace définissablement normaux, définissablement localement compact dans une structure o-minimale arbitraire.

Date: February 12, 2010.

2000 *Mathematics Subject Classification.* 03C64; 55N30.

The first author was supported by Fundação para a Ciência e a Tecnologia, Financiamento Base 2008 - ISFL/1/209.

Keywords and phrases: O-minimal structures, sheaf cohomology.

1. Introduction

We fix an arbitrary o-minimal structure $\mathcal{M} = (M, <, \ldots)$ and work in the category of definable spaces, X, in $\mathcal{M}$ with the o-minimal site on X, with morphisms being definable continuous maps. The o-minimal site on X is the site whose underlying category is the set of all relatively open definable subsets of X (open in the strong, o-minimal topology) with morphisms the inclusions and admissible coverings being covers by open definable sets with finite subcoverings.

The o-minimal setting generalizes the semi-algebraic and globally sub-analytic contexts ([11]), and so our first main theorem (on Subsection 2.2) generalizes the existence of sheaf cohomology with supports in semi-algebraic geometry, as described in the book [8]. This o-minimal sheaf cohomology with supports satisfies the Eilenberg-Steenrod axioms adapted to the o-minimal site - for the homotopy axiom we need to assume that $\mathcal{M}$ has definable Skolem functions and the definable space X involved is definably normal such that for every closed interval $[a, b] \subseteq M$ the projection $X \times [a, b] \longrightarrow X$ maps closed definable subsets into closed definable subsets. Other cohomology theories have been constructed for o-minimal structures of special types in the past. Simplicial and singular cohomologies were constructed in o-minimal expansions of fields by A.Woerheide in his doctoral thesis, a report of which can be found in [15]. A sheaf cohomology without supports has been constructed in [13] for o-minimal structures (with the extra technical assumptions for the homotopy axiom given above), which generalized the sheaf cohomology without supports for real algebraic geometry of Delfs, for which he proved the homotopy axiom in [7]. The theory presented here generalizes all of these and is an extension of the corresponding theory in topological spaces ([4], [17], [18] and [19]).

Following the classical proof of the Poincaré - Verdier duality for topological spaces we prove here a version Verdier duality theorem for the o-minimal sheaf cohomology with definably compact supports of definably normal, definably locally compact spaces in an arbitrary o-minimal structure (Theorem 4.5). This result is new even in the semi-algebraic context. We do not develop yet the full theory of proper direct image and its dual in the o-minimal context but nevertheless we prove our version of Verdier duality by considering inclusions of definably locally closed definable subsets. The theory of proper direct image is partially developed in the semi-algebraic case in the book by Delf's ([8]). In the sub-analytic context there are several approaches to this theory by Kashiwara and Schapira ([20]) and also L. Prelli ([23]).

From Verdier duality we derive the Poincaré and Alexander duality theorems (Theorems 4.11 and 4.14). The later results are based on a general and new orientation theory for definable manifolds which we show to be the same as the orientation theory in o-minimal expansions of fields defined in [1] and [2] using o-minimal singular homology. (See subsection 4.4).

Our Poincaré - Verdier duality theory relays heavily on the theory of normal and constructible supports and o-minimal cohomological Φ-dimension. This rather technical theory in presented in Section 3 and is the o-minimal version of the topological theory of paracompactifying families of supports and cohomological Φ-dimension and generalizes the corresponding theory in the semi-algebraic context ([8]).

The motivation for developing this general o-minimal Poincaré - Verdier duality is to be able to apply it to compute the o-minimal cohomology of definably compact definable groups defined in arbitrary o-minimal structures generalizing in this way the computation of the o-minimal singular cohomology of definable groups in o-minimal expansions of fields already presented in [14]. We hope do this in a different paper.

Acknowledgement: The contents of Subsection 2.2 and most of Subsection 3.1 were previously worked out jointly with Gareth O. Jones and Nicholas J. Peatfield. The first author would like to thank them for allowing him to include this material here.

2. Notations and review

In this section we recall some preliminaries notions about sheaves on topological spaces and the previous results about sheaves on the o-minimal spectrum of a definable space. For further details about classical sheaf theory, see for example [4], [17], [18], [19] and [21]. Good references on o-minimality are, for example, the book [11] by van den Dries and the notes [5] by Michel Coste. For semi-algebraic geometry relevant to this paper the reader should consult the work by Delfs ([7] and [8]), Delfs and Knebusch ([9]) and the book [3] by Bochnak, Coste and M-F. Roy.

2.1. **Sheaves on topological spaces.** Let X be a topological space and let k be a field. As usual, we will set $\mathrm{Mod}(k_X)$ the category of sheaves of k-modules on X. This is a Grothendieck category, hence it admits enough injectives and a family of generators (the sheaves k_U defined below). Moreover filtrant inductive limits are exact.

Let $f : X \to Y$ be a morphism of topological spaces. As usual we denote by f_* and f^{-1} the functors of direct and inverse image. In particular, when Y is a subset of X we will denote by $i_Y : Y \hookrightarrow X$ the inclusion.

When S is closed and $\mathcal{F} \in \mathrm{Mod}(k_X)$ one sets $\mathcal{F}_S = i_{S*}i_S^{-1}\mathcal{F}$, and when U is open $\mathcal{F}_U = \ker(\mathcal{F} \to \mathcal{F}_{X\setminus U})$ (or equivalently $\mathcal{F}_U$ is the sheaf associated to the presheaf $V \mapsto \Gamma(V;\mathcal{F}_U)$ which is $\Gamma(V;\mathcal{F})$ if $V \subseteq U$ and 0 otherwise). When $Z = U \cap S$ set $\mathcal{F}_Z = (\mathcal{F}_U)_S$. The functor $(\bullet)_Z$ is exact and $\mathcal{F}_Z$ is characterized by $\mathcal{F}_{Z|Z} = \mathcal{F}_{|Z}$ and $\mathcal{F}_{Z|X\setminus Z} = 0$. If Z' is another locally closed subset of X, then $(\mathcal{F}_Z)_{Z'} = \mathcal{F}_{Z\cap Z'}$. When $\mathcal{F} = k_X$ is the constant sheaf on X we just set k_Z instead of $(k_X)_Z$. If Z_1, Z_2 are locally closed and Z_1 is closed in Z_2 we have an exact sequence

$$0 \to \mathcal{F}_{Z_2\setminus Z_1} \to \mathcal{F}_{Z_2} \to \mathcal{F}_{Z_2\cap Z_1} \to 0.$$

When U is open one sets $\Gamma_U\mathcal{F} = i_{U*}i_U^{-1}\mathcal{F}$. Then we have $\Gamma(V;\Gamma_U\mathcal{F}) = \Gamma(U \cap V;\mathcal{F})$. When S is closed $\Gamma_S\mathcal{F} = \ker(\mathcal{F} \to \Gamma_{X\setminus S}\mathcal{F})$ (sections with support in S). When $Z = U \cap S$ we set $\Gamma_Z = \Gamma_U \circ \Gamma_S$. The functor $\Gamma_Z(\bullet)$ is left exact and if Z' is another locally closed subset, then $\Gamma_{Z'}(\Gamma_Z\mathcal{F}) = \Gamma_{Z\cap Z'}\mathcal{F}$. If Z_1, Z_2 are locally closed and Z_1 is closed in Z_2 we have an exact sequence

$$0 \to \Gamma_{Z_2\cap Z_1}\mathcal{F} \to \Gamma_{Z_2}\mathcal{F} \to \Gamma_{Z_2\setminus Z_1}\mathcal{F}.$$

Let Z be a locally closed subset of X. We are going to define the functor $i_{Z!}$ such that for $\mathcal{F} \in \mathrm{Mod}(k_Z)$, $i_{Z!}\mathcal{F}$ is the unique k-sheaf in $\mathrm{Mod}(k_X)$ inducing $\mathcal{F}$ on Z and zero on $X \setminus Z$. First let U be an open subset of X and let $\mathcal{F} \in \mathrm{Mod}(k_U)$. Then $i_{U!}\mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto \Gamma(V;i_{U!}\mathcal{F})$ which is $\Gamma(V;\mathcal{F})$ if $V \subseteq U$ and 0 otherwise. If S is a closed subset of X and $\mathcal{F} \in \mathrm{Mod}(k_S)$, then $i_{S!}\mathcal{F} = i_{S*}\mathcal{F}$. Now let $Z = U \cap S$ be a locally closed subset of X, then one defines

$i_{Z!} = i_{U!} \circ i_{S!} \simeq i_{S!} \circ i_{U!}$. The functor $i_{Z!}$ is exact and has a right adjoint, denoted by $i^!_Z$, when Z is open we have $i^!_Z \simeq i^{-1}_Z$, when Z is closed $i^!_Z \simeq i^{-1}_Z \Gamma_Z$. With these definitions one has

$$\mathcal{F}_Z \simeq \mathcal{F} \otimes k_Z \simeq i_{Z!} i^{-1}_Z \mathcal{F} \text{ and } \Gamma_Z \mathcal{F} \simeq \mathcal{H}om(k_Z, \mathcal{F}) \simeq i_{Z*} i^!_Z \mathcal{F}.$$

Let X be a topological space and Φ a family of supports on X (i.e. a collection of closed subsets of X such that: (i) Φ is closed under finite unions and (ii) every closed subset of a member of Φ is in Φ). Recall that for $\mathcal{G} \in \mathrm{Mod}(k_X)$, an element $s \in \Gamma(X; \mathcal{G})$ is in $\Gamma_\Phi(X; \mathcal{G})$ if and only if its *support*,

$$\operatorname{supp} s = X \setminus \cup\{U \subseteq X : U \text{ is open in } X \text{ and } s_{|U} = 0\},$$

is in Φ, i.e.

$$\Gamma_\Phi(X; \mathcal{G}) = \varinjlim_{S \in \Phi} \Gamma_S(X; \mathcal{G}).$$

The following fact (see [4], Chaper I, Proposition 6.6) will also be useful later:

Proposition 2.1. *Let X be a topological spaces, Φ a family of supports on X, Z a locally closed subset of X and let $i_Z : Z \to X$ be the inclusion. Let $\mathcal{F}$ be a sheaf in* $\mathrm{Mod}(k_Z)$. *Then*

$$\Gamma_\Phi(X; i_{Z!}\mathcal{F}) \simeq \Gamma_{\Phi|Z}(Z; \mathcal{F}).$$

2.2. **Sheaves on o-minimal spectral spaces.** Let $\mathcal{M} = (M, <, \ldots)$ be our fixed arbitrary o-minimal structure. First observe that in M we have the order topology generated by open definable intervals and in M^k we have the product topology generated by the open boxes. Thus every definable set $X \subseteq M^k$ has the induced topology and we say that a definable subset $Z \subseteq X$ is open (resp. closed) if it is open (resp. closed) with the induced topology. Similarly, we can talk about continuous definable maps $f : X \longrightarrow Y$ between definable sets. This topology has however a problem: in non-standard o-minimal structures definable sets are usually totally disconnected and never connected or locally compact or compact. So we have to introduce definable analogues of these and other topological notions.

Since we do not want to restrict our work to the affine definable setting, we introduce the notion of definable spaces. A *definable space* is a triple $(X, (X_i, \phi_i)_{i=1}^k)$ where:

(i) $X = \cup\{X_i : i = 1, \ldots, k\}$;
(ii) each $\phi_i : X_i \longrightarrow M^{l_i}$ is a bijection such that $\phi_i(X_i)$ is a definable subset of M^{l_i};
(iii) for all j, $\phi_i(X_i \cap X_j)$ is open in $\phi_i(X_i)$ and the transition maps $\phi_{ij} : \phi_i(X_i \cap X_j) \longrightarrow \phi_j(X_i \cap X_j) : x \mapsto \phi_j(\phi_i^{-1}(x))$ are definable homeomorphisms.

The *dimension* of a definable space X is defined as

$$\dim X = \max\{\dim \phi_i(X_i) : i = 1, \ldots, k\}.$$

A definable space has a topology such that each X_i is open and the ϕ_i's are homeomorphisms: a subset U of X is an open in the basis for this topology if and only if for each i, $\phi_i(U \cap X_i)$ is an open definable subset of $\phi_i(X_i)$. We also say that a subset A of X is definable if and only if for each i, $\phi_i(A \cap X_i)$ is a definable subset of $\phi_i(X_i)$. A map between definable spaces is definable if when it is read through the charts it is definable. Thus we have the category of definable spaces with definable continuous maps.

We say that a definable space X is:

- *definably connected* if it is not the disjoint union of two open and closed definable subsets;
- *definably compact* if for every continuous definable map $\sigma : (a,b) \subseteq M \cup \{-\infty, +\infty\} \longrightarrow X$, the limits $\lim_{t\to a^+}\sigma(t)$ and $\lim_{t\to b^-}\sigma(t)$ exist and belong to X.
- *definably locally compact* if for every definably compact subset Z with open definable neighborhood U in X, there is a definably compact neighborhood of Z in U.
- *definably normal* if for every disjoint closed definable subsets Z_1 and Z_2 of X there are disjoint open definable subsets U_1 and U_2 of X such that $Z_i \subseteq U_i$ for $i = 1, 2$.
- *definable manifold of dimension n* if $\phi_i(X_i)$ is an open definable subset of M^n for every $i = 1, \dots, k$.

The *o-minimal site* on a definable space X is the category whose objects are open definable subsets of X, the morphisms are the inclusions and the admissible covers are covers by open definable subsets with finite subcoverings.

The following results are an easy adaptation of Propositions 6.4.1 and 6.3.3 of [20], replacing $\mathcal{T}_c$ with open definable (indeed we just need the site generated by a family of open subsets closed under finite intersections and whose coverings admit a finite subcover). The first result gives an easy way to construct o-minimal k-sheaves:

Proposition 2.2. *Suppose that X be a definable space. Let $\mathcal{F}$ be a k-presheaf on X relative to the o-minimal site on X and assume that:*

(1) $\mathcal{F}(\emptyset) = 0$;
(2) *for any U and V open definable subsets of X the canonical sequence*

$$0 \longrightarrow \mathcal{F}(U \cup V) \longrightarrow \mathcal{F}(U) \oplus \mathcal{F}(V) \longrightarrow \mathcal{F}(U \cap V)$$

is exact.

Then $\mathcal{F}$ is a k-sheaf on X relative to the o-minimal site on X.

The second result shows that in this setting the global sections functor commutes with filtrant inductive limits:

Proposition 2.3. *Let U be a open definable subset of X and let $(\mathcal{F}_i)_{i\in I}$ be a filtrant inductive family of sheaves on the o-minimal site associated to X. Then*

$$\Gamma(U; \varinjlim_i \mathcal{F}_i) \simeq \varinjlim_i \Gamma(U; \mathcal{F}_i).$$

We define the *o-minimal spectrum* $\widetilde{X}$ of a definable space X as in [5], [6] and [22]: it is the set of ultrafilters of definable subsets of X. The o-minimal spectrum $\widetilde{X}$ of a definable space X is T_0, quasi-compact and a spectral topological space when equipped with the topology generated by the open subsets of the form $\widetilde{U}$, where U is an open definable subset of X. That is: (i) it has a basis of quasi-compact open subsets, closed under taking finite intersections; and (ii) each irreducible closed subset is the closure of a unique point.

The *dimension* of the o-minimal spectrum $\widetilde{X}$ of a definable space X is defined as

$$\dim \widetilde{X} = \dim X.$$

By a *constructible subset of* $\widetilde{X}$ we mean a subset of the form $\widetilde{A}$ where A is a definable subset of X.

We also have the *o-minimal spectrum* $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ *of a continuous definable map* $f : X \longrightarrow Y$ between definable spaces: given an ultrafilter $\alpha \in \widetilde{X}$, $\widetilde{f}(\alpha)$ is the ultrafilter in $\widetilde{Y}$ determined by the collection $\{A : f^{-1}(A) \in \alpha\}$.

We now recall some results from [13] about this tilde functor. Note that these results were stated in [13] in the category of definable sets but are true in the category of definable spaces with exactly the same proofs.

As we saw in [13] we have:

Remark 2.4. The tilde functor is an isomorphism between the boolean algebra of definable subsets of a definable space X and the boolean algebra of constructible subsets of its o-minimal spectrum $\widetilde{X}$ and it commutes with image and inverse image under definable maps.

Another useful property is the following result:

Theorem 2.5 ([13])**.** *Let X be a definable space. Then the following hold:*

(i) *X is definably connected if and only if its o-minimal spectrum $\widetilde{X}$ is connected.*
(ii) *X is definably normal if and only if its o-minimal spectrum $\widetilde{X}$ is normal.*

Also we have the following shrinking lemma:

Proposition 2.6 ([13], The shrinking lemma)**.** *Suppose that X is a definably normal definable space (resp. a normal o-minimal spectrum of a definable space). If $\{U_i : i = 1, \dots, n\}$ is a covering of X by open definable subsets (resp. open subsets) of X, then there are definable (resp. constructible) open subsets V_i and definable (resp. constructible) closed subsets K_i of X $(1 \leq i \leq n)$ with $V_i \subseteq K_i \subseteq U_i$ and $X = \cup\{V_i : i = 1, \dots, n\}$.*

Since the o-minimal spectrum of a definable space is quasi-compact, as in the proof of Propositions 2.2 and 2.3, we have:

Remark 2.7. Suppose that X is an object in the category of o-minimal spectra of definable spaces. Let $\mathcal{F}$ be a k-presheaf on X and assume that:

(1) $\mathcal{F}(\emptyset) = 0$;
(2) for any U and V open constructible subsets of X the canonical sequence
$$0 \longrightarrow \mathcal{F}(U \cup V) \longrightarrow \mathcal{F}(U) \oplus \mathcal{F}(V) \longrightarrow \mathcal{F}(U \cap V)$$
is exact.

Then $\mathcal{F}$ is a k-sheaf on X. Moreover sections on open constructible subsets commute with filtrant $\varinjlim$.

We have a morphism of sites naturally induced by the above tilde functor from the category of definable spaces with continuous definable maps into the category of o-minimal spectral spaces with the o-minimal spectra of continuous definable maps. This morphism of sites induces the following isomorphism:

Theorem 2.8 ([13])**.** *Let X be a definable space. Then there is an isomorphism between the category of k-sheaves on X relative to o-minimal site on X and the category of k-sheaves on the o-minimal spectrum $\widetilde{X}$ of X relative to the spectral topology on $\widetilde{X}$.*

The isomorphism of Theorem 2.8 allowed the development of o-minimal sheaf cohomology without supports in [13] by defining concepts and also sometimes proving results via this tilde isomorphism. In this paper we will continue to use this technique but allowing now the presence of supports.

We will now define the notion of family of supports on a definable set. Our treatment of this will follow the corresponding theory in semi-algebraic geometry in [8] (Chapter II, Sections 1 - 5) and in topological spaces in [4] (Chapter I, Section 6 and Chapter II, Sections 9 - 13). Note also that since, as we saw in [13], the role of paracompactness in sheaf theory on topological spaces has to be replaced by normality in sheaf theory on o-minimal spectral spaces, we will continue to do this here.

Definition 2.9. Let X be a definable space. A *family of definable supports* is a family of closed definable subsets of X such that:

(1) every closed definable subset of a member of Φ is in Φ;
(2) Φ is closed under finite unions.

Φ is said to be a *family of definably normal supports* if in addition:

(3) each element of Φ is definably normal;
(4) for each element S of Φ, if U is an open definable neighborhood of S in X, then there exists a (closed) definable neighborhood of S in U which is in Φ.

Example 2.10. Let X be a definable space and let c be the collection of all definably compact definable subsets of X. Then c is a family of definable supports on X. Moreover, if X is definably normal and definably locally compact, then c will be a family of definably normal supports on X.

If Y is a definable subset of the definable space X and Φ a family of definable supports on X, then we have families of definable supports

$$\Phi \cap Y = \{A \cap Y : A \in \Phi\}$$

and

$$\Phi_{|Y} = \{A \in \Phi : A \subseteq Y\}$$

on Y.

If $f : X \longrightarrow Z$ is a continuous definable map between definable spaces and Φ is a family of definable supports on Z, then we have a family of definable supports

$$f^{-1}\Phi = \{A \subseteq X : A \text{ is closed, definable and } \exists B \in \Phi\ (A \subseteq f^{-1}(B)\}$$

on X.

Remark 2.11. Note that a family of definable supports Φ on a definable space X determines a family of supports

$$\widetilde{\Phi} = \{A \subseteq \widetilde{X} : A \text{ is closed and } \exists B \in \Phi\ (A \subseteq \widetilde{B})\}$$

on the topological space $\widetilde{X}$. By Remark 2.4 it follows that

$$\widetilde{\Phi \cap Y} = \widetilde{\Phi} \cap \widetilde{Y},\ \widetilde{\Phi_{|Y}} = \widetilde{\Phi}_{|\widetilde{Y}} \text{ and } \widetilde{f^{-1}\Phi} = \widetilde{f}^{-1}\widetilde{\Phi}.$$

We will say that the family of supports on $\widetilde{X}$ is *constructible* if it is obtained by applying tilde to some family of definable supports on X.

By theorem 2.5 it follows that Φ is definably normal if and only if $\widetilde{\Phi}$ is normal. Here, we say that Ψ is a *family of normal supports* on the spectral topological space $\widetilde{X}$ if is a family of supports and:

(1) each element of Ψ is normal;
(2) for each element S of Φ, if U is an open neighborhood of S in $\widetilde{X}$, then there exists a (closed) constructible neighborhood of S in U which is in Φ.

Definition 2.12. Let X be a definable space, Φ a family of definable supports in X and $\mathcal{F}$ a k-sheaf on X relative to the o-minimal site on X. We define the *o-minimal sheaf cohomology groups with definable supports in* Φ via the tilde isomorphism of Theorem 2.8 by

$$H^*_\Phi(X;\mathcal{F}) = H^*_{\widetilde{\Phi}}(\widetilde{X};\widetilde{\mathcal{F}}),$$

where $\widetilde{\mathcal{F}}$ is the image of $\mathcal{F}$ via the isomorphism between the category of k-sheaves on X relative to o-minimal site on X and the category of k-sheaves on the o-minimal spectrum $\widetilde{X}$ of X.

If $f : X \longrightarrow Y$ is a continuous definable map, we define the induced homomorphism

$$f^* : H^*_\Phi(Y;\mathcal{F}) \longrightarrow H^*_{f^{-1}\Phi}(X; f^{-1}\mathcal{F})$$

in cohomology to be the same as the induced homomorphism

$$\widetilde{f}^* : H^*_{\widetilde{\Phi}}(\widetilde{Y};\widetilde{\mathcal{F}}) \longrightarrow H^*_{\widetilde{f}^{-1}\widetilde{\Phi}}(\widetilde{X}; \widetilde{f}^{-1}\widetilde{\mathcal{F}})$$

in cohomology of the continuous map $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ of topological spaces.

The proof of the o-minimal Vietoris-Begle theorem with supports below is similar to its analogue without supports ([13] Theorem 4.3) using classical arguments:

Theorem 2.13 (Vietoris-Begle theorem)**.** *Let $f : X \longrightarrow Y$ be a surjective morphism in the category of o-minimal spectra of definable spaces that maps constructible closed subsets of X onto closed subsets of Y. Let $\mathcal{F} \in \mathrm{Mod}(k_Y)$, Φ a constructible family of supports on Y and suppose that Y is a subspace of a normal space in the category of o-minimal spectra of definable spaces. Assume that $f^{-1}(\beta)$ is connected and $H^q(f^{-1}(\beta); f^{-1}\mathcal{F}_{|f^{-1}(\beta)}) = 0$ for $q > 0$ and all $\beta \in Y$. Then the induced map*

$$f^* : H^*_\Phi(Y;\mathcal{F}) \longrightarrow H^*_{f^{-1}\Phi}(X; f^{-1}\mathcal{F})$$

is an isomorphism.

We have in this context the Eilenberg-Steenrod axioms with definable supports adapted to the o-minimal site. Indeed, once we pass to the category of o-minimal spectra of definable spaces the proofs of the exactness and excision axioms are purely algebraic. See [4]. The dimension axiom is also immeadiate. On the other hand, from the Vietoris-Begle theorem (Theorem 2.13) we obtain:

Theorem 2.14 (Homotopy axiom)**.** *Suppose that X is a definable space and $\mathcal{F}$ is a k-sheaf on X relative to the o-minimal site on X. Let $[a,b] \subseteq M$ be a closed interval. Assume that $\mathcal{M}$ has definable Skolem functions, X is definably normal and the projection $\pi : X \times [a,b] \longrightarrow X$ maps closed definable subsets of $X \times [a,b]$ onto closed definable subsets of X. If for $d \in [a,b]$,*

$$i_d : X \longrightarrow X \times [a,b]$$

is the continuous definable map given by $i_d(x) = (x, d)$ *for all* $x \in X$*, then*

$$i_a^* = i_b^* : H^n_{\Phi\times[a,b]}(X \times [a, b]; \pi^{-1}\mathcal{F}) \longrightarrow H^n_\Phi(X; \mathcal{F})$$

for all $n \in \mathbb{N}$.

Proof. The homotopy axiom will follow once we show that the projection map $\pi : X \times [a, b] \longrightarrow X$ induces an isomorphism

$$\pi^* : H^n_\Phi(X; \mathcal{F}) \longrightarrow H^n_{\Phi\times[a,b]}(X \times [a, b]; \pi^{-1}\mathcal{F})$$

since by functoriality we obtain

$$i_a^* = i_b^* = (\pi^*)^{-1} : H^n_{\Phi\times[a,b]}(X \times [a, b]; \pi^{-1}\mathcal{F}) \longrightarrow H^n_\Phi(X; \mathcal{F})$$

for all $n \in \mathbb{N}$. Equivalently we need to show that

$$\widetilde{\pi}^* : H^n_{\widetilde{\Phi}}(\widetilde{X}; \widetilde{\mathcal{F}}) \longrightarrow H^n_{\widetilde{\Phi\times[a,b]}}(\widetilde{X \times [a, b]}; \widetilde{\pi}^{-1}\widetilde{\mathcal{F}})$$

is an isomorphism. For this we need to verify the hypothesis of the Vietoris-Begle theorem (Theorem 2.13), but this was done in the proof of the homotopy axiom for o-minimal sheaf cohomology without supports ([13] Theorem 4.4). □

Remark 2.15. In this context we also have the exactness for triples of closed definable subsets and the Mayer-Vietoris theorem for Φ-excisive pairs of definable sets. See [4].

3. Φ-soft sheaves

The results we present below are in the category of o-minimal spectra of definable spaces but by the isomorphism of Theorem 2.8 they have a suitable, but more restrictive, analogue in the category of definable spaces. In fact these results are the analogue of classical results on paracompactifying families of supports on topological spaces ([4]) adapted to normal and constructible families of supports on spectral spaces.

3.1. **Normal and constructible supports.**

We start the subsection with the following useful result:

Proposition 3.1. *Assume that* X *is an object in the category of o-minimal spectra of definable spaces and let* $(\mathcal{F}_i)_{i\in I}$ *be a filtrant inductive family of sheaves in* $\mathrm{Mod}(k_X)$ *and* Φ *a constructible family of supports on* X*. Then*

$$\Gamma_\Phi(X; \varinjlim_{i\in I} \mathcal{F}_i) = \varinjlim_{i\in I} \Gamma_\Phi(X; \mathcal{F}_i).$$

Proof. First observe that by definition for any $\mathcal{G} \in \mathrm{Mod}(k_X)$ we have

$$\Gamma_\Phi(X; \mathcal{G}) = \varinjlim_{S\in\Phi} \Gamma_S(X; \mathcal{G}).$$

Thus it is enough to show that for each $S \in \Phi$ constructible we have

$$\Gamma_S(X; \varinjlim_{i\in I} \mathcal{F}_i) = \varinjlim_{i\in I} \Gamma_S(X; \mathcal{F}_i).$$

For this consider the following commutative diagram where the vertical arrows are the canonical maps

$$\begin{array}{ccccccc}
0 \longrightarrow & \varinjlim_{i\in I}\Gamma_S(X;\mathcal{F}_i) & \longrightarrow & \varinjlim_{i\in I}\Gamma(X;\mathcal{F}_i) & \longrightarrow & \varinjlim_{i\in I}\Gamma(X\setminus S;\mathcal{F}_i) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \Gamma_S(X;\varinjlim_{i\in I}\mathcal{F}_i) & \longrightarrow & \Gamma(X;\varinjlim_{i\in I}\mathcal{F}_i) & \longrightarrow & \Gamma(X\setminus S;\varinjlim_{i\in I}\mathcal{F}_i).
\end{array}$$

The rows are exact by definition of Γ_S (i.e. $\Gamma_S(X;\mathcal{G}) = \ker(\Gamma(X;\mathcal{G}) \to \Gamma(X\setminus S;\mathcal{G})$ for any $\mathcal{G} \in \mathrm{Mod}(k_X)$) and by the exactness of filtrant $\varinjlim$. Since X and $X \setminus S$ are open constructible subsets of X and sections on open constructible subsets commute with filtrant $\varinjlim$ (Remark 2.7), it follows that the two vertical arrows on the right are isomorphisms. Hence the first vertical arrow is also an isomorphism as required. □

The following lemma is fundamental in this Subsection:

Lemma 3.2. *Assume that Z is a subspace of a normal space X in the category of o-minimal spectra of definable spaces, $\mathcal{G}$ is a sheaf in $\mathrm{Mod}(k_Z)$ and Y is a quasi-compact subset of Z. Then the canonical morphism*

$$\varinjlim_{Y\subseteq U}\Gamma(U\cap Z;\mathcal{G}) \longrightarrow \Gamma(Y;\mathcal{G}_{|Y})$$

where U ranges through the family of open constructible subsets of X, is an isomorphism.

Proof. Since Y is quasi-compact, the family of open neighborhoods of Y in Z of the form $V\cap Z$ where V is an open constructible subset of X is a fundamental system of neighborhoods of Y in Z. Hence, the morphism of the lemma is certainly injective.

To prove that it is surjective, consider a section $s \in \Gamma(Y;\mathcal{G}_{|Y})$. There is a covering $\{U_j : j \in J\}$ of Y by open constructible subsets of X and sections $s_j \in \Gamma(U_j \cap Z;\mathcal{G}_{|U_j\cap Y})$, $j \in J$, such that $s_{j|U_j\cap Y} = s_{|U_j\cap Y}$. Since Y is quasi-compact, we can assume that J is finite, and so $\cup\{U_j : j\in J\}$ is an open constructible subset of X. Since X is normal, by the shrinking lemma (Proposition 2.6), there are open constructible subsets $\{V_j : j \in J\}$ of this union such that $\overline{V}_j \subseteq U_j$ for every $j \in J$ and $Y \subseteq \cup\{V_j : j \in J\}$. For $x \in Z$ set $J(x) = \{j \in J : x \in \overline{V}_j\}$. Each x has a constructible neighborhood W_x with $J(y) \subseteq J(x)$ for each $y \in W_x$. This is defined by

$$W_x = (\bigcap_{x\in V_l} V_l \cap \bigcap_{j\in J(x)} U_j) \setminus \bigcup_{k\notin J(x)} \overline{V}_k.$$

Observe that for all $i, j \in J(x)$ we have that W_x is an open subset of both U_i and U_j. Hence, for every $i, j \in J(x)$ we have $s_{i|W_x\cap Y} = s_{|W_x\cap Y} = s_{j|W_x\cap Y}$. So, for $y \in W_x \cap Y$, we have $(s_i)_y = (s_j)_y$ for any $i, j \in J(x)$. This implies that the set

$$W = \{x \in (\bigcup_{j\in J} V_j) \cap Z : (s_i)_x = (s_j)_x \text{ for any } i, j \in J(x)\}$$

contains Y (clearly $Y \subseteq \bigcup_{x\in Z} W_x \cap Y \subseteq (\bigcup_{j\in J} V_j) \cap Z$). On the other hand, the condition $(s_i)_z = (s_j)_z$ for any $i, j \in J(x)$ and the fact that $J(x)$ is finite implies that z has an open neighborhood in Z on which $s_i = s_j$ for any $i, j \in J(x)$. Thus W is an open neighborhood of Y in Z. Since Y is quasi-compact we may assume that W is of the form $U \cap Z$ for some open constructible subset U of X. Since $s_{i|W\cap V_i\cap V_j} = s_{j|W\cap V_i\cap V_j}$ there exists $t \in \Gamma(W;\mathcal{G})$ such that $t_{|W\cap V_j} = s_{j|W\cap V_j}$. This proves that the morphism is surjective. $\square$

A general form of Lemma 3.2 is:

Lemma 3.3. *Assume that X is an object in the category of o-minimal spectra of definable spaces, Z is a subspace of X, $\mathcal{G}$ is a sheaf in $\mathrm{Mod}(k_Z)$, Φ is a normal and constructible family of supports on X and Y is a subset of Z such that $D \cap Y$ is a quasi-compact subset for every $D \in \Phi$. Then the canonical morphism*

$$\varinjlim_{Y\subseteq U} \Gamma_{\Phi\cap U\cap Z}(U\cap Z;\mathcal{G}) \longrightarrow \Gamma_{\Phi\cap Y}(Y;\mathcal{G}_{|Y})$$

where U ranges through the family of open constructible subsets of X, is an isomorphism.

Proof. Let us prove injectivity. Let $s \in \Gamma_{D\cap U\cap Z}(U\cap Z;\mathcal{G})$, with $D \in \Phi$ and $U \supset Y$ open constructible subset of X and such that $s_{|D\cap Y} = 0$. Since Φ is a normal and constructible family of supports on X, there is a constructible and normal $E \in \Phi$ which is a closed neighborhood of D in X. Thus $D\cap Z$ is a subspace of a normal space E in the category of o-minimal spectra of definable spaces and $D\cap Y$ is a quasi-compact subset of $D\cap Z$. By Lemma 3.2 applied to E, $D\cap Z$ and $D\cap Y$, there exists an open (in E) constructible neighborhood V' of $D\cap Y$ such that $s_{|V'\cap D\cap Z} = 0$. Of course we may assume that $V' = V\cap E$ for some open constructible subset V of X. So there exists an open (in X) constructible neighborhood V of $D\cap Y$ such that $s_{|V\cap D\cap Z} = 0$. Also, by replacing V with its intersection with U if necessary we may assume that $V \subseteq U$. Set $W = V \cup (U \setminus D)$. Then W is open constructible in X, $Y \subseteq W \subseteq U$ and $s_{|W\cap Z} = 0$.

Let us prove that the morphism is surjective. Let $s \in \Gamma_{\Phi\cap Y}(Y;\mathcal{G}_{|Y})$ and consider normal constructible sets C, D and E in Φ such that D is a closed neighborhood of C in X, E is a closed neighborhood of D in X and the support of s is contained in $C\cap Y$. We shall find $\widetilde{t} \in \Gamma_D(U\cap Z;\mathcal{G})$ such that $\widetilde{t}_{|Y} = s$. After applying Lemma 3.2 above to E, $D\cap Z$ and $D\cap Y$ we see that there exists an open in $E\setminus\partial E$ (and hence in X) constructible neighborhood V of $D\cap Y$ and a section $t \in \Gamma(V\cap D\cap Z;\mathcal{G})$ such that $t_{|D\cap Y} = s_{|D\cap Y}$. Since $t_{|\partial D\cap Y} = 0$, then each point x of $\partial D\cap Y$ has an open constructible neighborhood $W_x \subset V$ such that $t_{|W_x\cap D\cap Z} = 0$. Using quasi-compactness of $\partial D\cap Y$ (it is closed on the quasi-compact set $D\cap Y$), there exists a finite number of points $x_1,\dots,x_n$ such that $\partial D\cap Y \subset \bigcup_{i=1}^n W_{x_i} := W$. We have $t_{|W\cap D\cap Z} = 0$ and W is open constructible. Let $U_1 = (V\cap(D\setminus\partial D))\cup W$. Then U_1 is open constructible and $D\cap Y \subseteq U_1 \subseteq V$. Define $t' \in \Gamma(U_1\cap Z;\mathcal{G})$ by: $t'_{|V\cap(D\setminus\partial D)\cap Z} = t_{|V\cap(D\setminus\partial D)\cap Z}$ and $t'_{|W\cap Z} = 0$. This is well defined since $t_{|W\cap D\cap Z} = 0$ and $(V\cap(D\setminus\partial D)\cap Z)\cap(W\cap Z) \subseteq W\cap D\cap Z$. Observe also that $t'_{|U_1\cap D\cap Z} = t$. Let $U_2 = X\setminus D$. Then $U = U_1\cup U_2$ is open constructible, $Y \subseteq U$, $U_1\cap U_2 \subseteq W$ and we can define $\widetilde{t} \in \Gamma(U\cap Z;\mathcal{G})$ in the following way: $\widetilde{t}_{|U_1\cap Z} = t'_{|U_1\cap Z}$, $\widetilde{t}_{|U_2\cap Z} = 0$. It is well defined since $t'_{|W\cap Z} = 0$ and $U_1\cap U_2 \subset W$. Moreover $\operatorname{supp}\widetilde{t} \subseteq D$ and $\widetilde{t}_{|Y} = s$ as required. $\square$

Recall that a sheaf $\mathcal{F}$ on a topological space X with a family of supports Φ is Φ-*soft* if and only if the restriction $\Gamma(X;\mathcal{F}) \longrightarrow \Gamma(S;\mathcal{F}_{|S})$ is surjective for every $S \in \Phi$. If Φ consists of all closed subsets of X, then $\mathcal{F}$ is simply called *soft*.

Proposition 3.4. *Let X be a topological space and $\mathcal{F}$ is a sheaf in* $\mathrm{Mod}(k_X)$. *If Φ is a family of supports on X such that every $C \in \Phi$ has a neighborhood D in X with $D \in \Phi$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is Φ-soft;*
(2) *$\mathcal{F}_{|S}$ is soft for every $S \in \Phi$;*
(3) $\Gamma_\Phi(X;\mathcal{F}) \longrightarrow \Gamma_{\Phi_{|S}}(S;\mathcal{F}_{|S})$ *is surjective for every closed subset S of X;*

If in addition X is an object in the category of o-minimal spectra of definable spaces and Φ is a constructible family of supports on X, then the above are also equivalent to:

(4) *$\mathcal{F}_{|Z}$ is soft for every constructible subset Z of X which is in Φ;*

If moreover Φ is a normal and constructible family of supports on X, then the above are also equivalent to:

(5) $\Gamma(X;\mathcal{F}) \longrightarrow \Gamma(Z;\mathcal{F}_{|Z})$ *is surjective for every constructible subset Z of X which is in Φ;*

Proof. The equivalence of (1), (2) and (3) is shown in [4] Chapter II, 9.3. (Our hypothesis is sufficient in the proof given there). The equivalence of (2) and (4) is obvious since every $S \in \Phi$ is contained in some constructible subset of X which is in Φ.

Clearly (1) implies (5). Assume (5) and let $S \in \Phi$ and $s \in \Gamma(S;\mathcal{F}_{|S})$. Since Φ is normal and constructible, there is a normal closed and constructible neighborhood D of S which is in Φ. By Lemma 3.2, s can be extended to a section $t \in \Gamma(W;\mathcal{F})$ of $\mathcal{F}$ over a neighborhood W of S in D. Applying the shrinking lemma we find a closed constructible neighborhood Z of S in W. Since $D \in \Phi$ we have $Z \in \Phi$. So $t_{|Z} \in \Gamma(Z;\mathcal{F}_{|Z})$, $(t_{|Z})_{|S} = s$ and $t_{|Z}$ can be extended to X by (5). Hence, (5) implies (1).

□

Corollary 3.5. *Assume that X is an object in the category of o-minimal spectra of definable spaces and Φ is a normal and constructible family of supports on X. Then filtrant inductive limits of Φ-soft sheaves in* $\mathrm{Mod}(k_X)$ *are Φ-soft.*

Proof. It follows combining Propositions 3.1 and 3.4 (5) and the exactness of filtrant inductive limits. □

The following topological result will also be useful below:

Proposition 3.6. *Let X be a topological space and Φ is a family of supports on X such that every $C \in \Phi$ has a neighborhood D in X with $D \in \Phi$. Let W be a locally closed subset of X. The following hold:*

(i) *if $\mathcal{F} \in \mathrm{Mod}(k_X)$ is Φ-soft, then $\mathcal{F}_{|W}$ is $\Phi_{|W}$-soft.*
(ii) *$\mathcal{G}$ in $\mathrm{Mod}(k_W)$ is $\Phi_{|W}$-soft if and only if $i_{W!}\mathcal{G}$ is Φ-soft.*
(iii) *if $\mathcal{F} \in \mathrm{Mod}(k_X)$ is Φ-soft, then $\mathcal{F}_W$ is Φ-soft.*

Proof. (i) If W is open it is obvious. If W is closed it follows from Proposition 3.4 (3). Combining these two cases (i) follows.

(ii) The "if " part follows from Proposition 3.4 (2). For the "only if" part note that by Proposition 2.1 (applied to X and S respectively and using the fact that $(\Phi_{|S})_{|W} = \Phi_{|W\cap S})$ we have $\Gamma_\Phi(X; i_{W!}\mathcal{G}) \simeq \Gamma_{\Phi|W}(W;\mathcal{G})$ and $\Gamma_{\Phi|S}(S; i_{W!}\mathcal{G}) = \Gamma_{\Phi|W\cap S}(S\cap W;\mathcal{G})$ for any closed subset S of W. Then apply Proposition 3.4 (3).

(iii) The result follows from (i) and (ii), since $\mathcal{F}_W = i_{W!}\mathcal{F}_{|W}$. □

A special and useful case of Proposition 3.6 is when X is an object in the category of o-minimal spectra of definable spaces and Φ is a normal and constructible family of supports on X.

Proposition 3.7. *Assume that X is an object in the category of o-minimal spectra of definable spaces, Φ is a normal and constructible family of supports on X and Y is a subspace of X such that $D\cap Y$ is a quasi-compact subset for every $D\in\Phi$. Then the full additive subcategory of* $\mathrm{Mod}(k_Y)$ *of $\Phi\cap Y$-soft k-sheaves is $\Gamma_{\Phi\cap Y}(Y;\bullet)$-injective, i.e.:*

1. *For every $\mathcal{F}\in \mathrm{Mod}(k_Y)$ there exists a $\Phi\cap Y$-soft $\mathcal{F}'\in\mathrm{Mod}(k_Y)$ and an exact sequence $0\to\mathcal{F}\to\mathcal{F}'$.*
2. *If $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ is an exact sequence in $\mathrm{Mod}(k_Y)$ and $\mathcal{F}'$ is $\Phi\cap Y$-soft, then $0\longrightarrow\Gamma_{\Phi\cap Y}(Y;\mathcal{F}')\longrightarrow\Gamma_{\Phi\cap Y}(Y;\mathcal{F})\longrightarrow\Gamma_{\Phi\cap Y}(Y;\mathcal{F}'')\longrightarrow 0$ is an exact sequence.*
3. *If $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ is an exact sequence in $\mathrm{Mod}(k_Y)$ and $\mathcal{F}'$ and $\mathcal{F}$ are $\Phi\cap Y$-soft, then $\mathcal{F}''$ is $\Phi\cap Y$-soft.*

Proof. The result for the full additive subcategory of $\mathrm{Mod}(k_Y)$ of injective (and flabby) k-sheaves is classical for topological spaces (see for example [19], Proposition 2.4.3). Thus (1) holds for the $\Phi\cap Y$-soft case since injective k-sheaves are $\Phi\cap Y$-soft.

We now prove (2). Let $s''\in\Gamma_{\Phi\cap Y}(Y,\mathcal{F}'')$. Then since Φ is normal and constructible, $\operatorname{supp} s''\subset V$, with V open constructible in X and $\overline{V}\in\Phi$. Now, let us consider the exact sequence

$$0\to\mathcal{F}'_{V\cap Y}\to\mathcal{F}_{V\cap Y}\to\mathcal{F}''_{V\cap Y}\to 0.$$

By Proposition 3.6 (iii) we have that $\mathcal{F}'_{Y\cap V}$ is still $\Phi\cap Y$-soft. Replacing $\mathcal{F}',\mathcal{F},\mathcal{F}''$ with $\mathcal{F}'_{V\cap Y},\mathcal{F}_{V\cap Y},\mathcal{F}''_{V\cap Y}$ we are reduced to prove that the sequence

$$0\to\Gamma(Y;\mathcal{F}')\to\Gamma(Y;\mathcal{F})\to\Gamma(Y;\mathcal{F}'')\to 0$$

is exact when $Y = Y\cap\overline{V}$. Let $s''\in\Gamma(Y;\mathcal{F}'')$, and let $\{D_i\}_{i=1}^n$, $D_i\in\Phi\cap Y$ be a finite covering of Y such that there exists $s_i\in\Gamma(D_i;\mathcal{F})$ whose image is $s''|_{D_i}$. There exists such a covering since Φ is normal and $Y\cap\overline{V}$ is quasi-compact. For $n\geq 2$ on $D_1\cap D_2$ s_1-s_2 defines a section of $\Gamma(D_1\cap D_2;\mathcal{F}')$ which extends to $s'\in\Gamma(Y;\mathcal{F}')$ since $\mathcal{F}'$ is $\Phi\cap Y$-soft. Replace s_1 with s_1-s'. We may suppose that $s_1=s_2$ on $D_1\cap D_2$. Then there exists $\widetilde{s}\in\Gamma(D_1\cup D_2;\mathcal{F})$ such that $\widetilde{s}|_{D_i}=s_i$, $i=1,2$. Thus the induction proceeds.

Finally, (3) follows at once from (2) by a simple diagram chase using Proposition 3.4 (3): let Z be a set in $\Phi\cap Y$ and consider the following commutative diagram

$$\begin{tikzcd}
\Gamma_{\Phi\cap Y}(Y;\mathcal{F}) \arrow[r, "\delta"] \arrow[d, "\alpha"] & \Gamma_{\Phi\cap Y}(Y;\mathcal{F}'') \arrow[d, "\gamma"] \\
\Gamma_{\Phi\cap Y\cap Z}(Z;\mathcal{F}_{|Z}) \arrow[r, "\beta"] & \Gamma_{\Phi\cap Y\cap Z}(Z;\mathcal{F}''_{|Z}).
\end{tikzcd}$$

By hypothesis on $\mathcal{F}$, α is surjective. By (2) β is surjective. Therefore, γ is surjective as required. $\square$

Hence, if X is an object in the category of o-minimal spectra of definable spaces, Φ is a normal and constructible family of supports on X and Y is a subspace of X such that $D \cap Y$ is a quasi-compact subset for every $D \in \Phi$. Then one can take a $\Phi \cap Y$-soft resolution of $\mathcal{F}$ to compute $H^*_{\Phi\cap Y}(Y;\mathcal{F})$.

Example 3.8. Some particular cases of Proposition 3.7 are:

- if $Y = U$ is open constructible such that $\overline{U} \in \Phi$, then the family of $\Phi \cap U$-soft sheaves in $\mathrm{Mod}(k_U)$ is $\Gamma(U; \bullet)$-injective.
- If $Y = D \in \Phi$, then the family of $\Phi_{|D}$-soft sheaves in $\mathrm{Mod}(k_D)$ is $\Gamma(D; \bullet)$-injective.

Corollary 3.9. *Assume that X is an object in the category of o-minimal spectra of definable spaces. Suppose either that Φ is a normal and constructible family of supports on X and W is a (constructible) locally closed subset of X or that Φ is any family of supports on X and W is a closed subset of X. If $\mathcal{F} \in \mathrm{Mod}(k_W)$, then*

$$H^*_\Phi(X; i_{W!}\mathcal{F}) = H^*_{\Phi|W}(W;\mathcal{F}).$$

Proof. The second case is covered by [4] Chapter II, 10.1. If W is closed in an open subset U of X, then $\Phi_{|U}$ is a normal and constructible family of supports on U and $\Phi_{|W} = \Phi_{|U} \cap W$. And the result follows from Propositions 3.7, 3.6 (ii) and 2.1. $\square$

The following will be useful in the next subsection:

Proposition 3.10. *Assume that X is an object in the category of o-minimal spectra of definable spaces, $\mathcal{F}$ is a sheaf in $\mathrm{Mod}(k_X)$ and Φ is a normal and constructible family of supports on X. The following are equivalent:*

1. *$\mathcal{F}$ is Φ-soft;*
2. *$\mathcal{F}_U$ is Γ_Φ-acyclic for all open and constructible $U \subseteq X$;*
3. *$H^1_\Phi(X;\mathcal{F}_U) = 0$ for all open and constructible $U \subseteq X$;*

Proof. (1) $\Rightarrow$ (2) follows from Propositions 3.7 and 3.6 (iii). (2) $\Rightarrow$ (3) is trivial. To show that (3) implies (1), consider a constructible closed set C in Φ and the exact sequence $0 \longrightarrow \mathcal{F}_{X\setminus C} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_C \longrightarrow 0$. The associated long exact cohomology sequence

$$\cdots \to \Gamma_\Phi(X;\mathcal{F}) \to \Gamma_\Phi(X;\mathcal{F}_C) \to H^1_\Phi(X;\mathcal{F}_{X\setminus C}) \to \dots$$

shows that $\Gamma_\Phi(X;\mathcal{F}) \longrightarrow \Gamma_\Phi(X;\mathcal{F}_C)$ is surjective. Hence $\mathcal{F}$ is Φ-soft by Proposition 3.4 (5). $\square$

3.2. **Cohomological Φ-dimension.** Recall that for a topological space X and Φ a family of supports on X, the *cohomological Φ-dimension of X* is the smallest n such that $H^q_\Phi(X;\mathcal{F}) = 0$ for all $q > n$ and all sheaves $\mathcal{F}$ in $\mathrm{Mod}(k_X)$.

The following holds:

Proposition 3.11. *Assume that X is an object in the category of o-minimal spectra of definable spaces and Φ is a normal and constructible family of supports on X. Let $\mathcal{F}$ be a sheaf in $\mathrm{Mod}(k_X)$. Then the following are equivalent:*

(1) *If* $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots \longrightarrow \mathcal{I}^n \longrightarrow 0$ *is an exact sequence of sheaves in* $\mathrm{Mod}(k_X)$ *such that* $\mathcal{I}^k$ *is* Φ*-soft for* $0 \leq k \leq n-1$*. Then* $\mathcal{I}^n$ *is* Φ*-soft.*
(2) $\mathcal{F}$ *has a* Φ*-soft resolution of length* n*;*
(3) $H^k_\Phi(X;\mathcal{F}_U) = H^k_{\Phi|U}(U;\mathcal{F}_{|U}) = 0$ *for all open and constructible* $U \subseteq X$ *and all* $k > n$*.*

Proof. The result follows from Proposition 3.10 (2) and is a particular case of a general result of homological algebra ([19], Exercise I.19): let F be a left exact functor and let $\mathcal{J}$ be the family of F-acyclic objects. Suppose that $\mathcal{J}$ is cogenerating. Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) with $\mathcal{J}$ instead of Φ-soft and F instead of $\Gamma_\Phi(X;(\bullet)_U)$. $\square$

Theorem 3.12. *Let* X *be an object in the category of o-minimal spectra of definable spaces and let* Φ *be a normal and constructible family of supports on* X*. Then the cohomological* Φ*-dimension of* X *is bounded by* $\dim X$*.*

Proof. To prove our theorem we will use (1) of Proposition 3.11. Let $n = \dim X$. Then, in this situation it suffices to prove that $\mathcal{I}^n_{|Z}$ is soft for every constructible subset Z of X which is in Φ (Proposition 3.4 (4)). Since Φ is normal, there is a constructible neighborhood Y of Z in X which is in Φ. If we show that $\mathcal{I}^n_{|Y}$ is soft, then it will follow that $\mathcal{I}^n_{|Z}$ is soft (Proposition 3.4 (2)).

Let U be an open and constructible subset of Y. By hypothesis and Proposition 3.10 each $(\mathcal{I}^k_{|Y})_U$ is acyclic for $0 \leq k \leq n-1$. Let $\mathcal{Z}^k = \ker((\mathcal{I}^k_{|Y})_U \longrightarrow (\mathcal{I}^{k+1}_{|Y})_U)$. Then the long exact cohomology sequences of the short exact sequences $0 \longrightarrow \mathcal{Z}^k \longrightarrow (\mathcal{I}^k_{|Y})_U \longrightarrow \mathcal{Z}^{k+1} \longrightarrow 0$ show that

$$H^q(Y;(\mathcal{I}^n_{|Y})_U) = H^q(Y;\mathcal{Z}^n) = H^{q+1}(Y;\mathcal{Z}^{n-1}) = \cdots = H^{q+n}(Y;\mathcal{Z}^0) = H^{q+n}(Y;(\mathcal{F}_{|Y})_U).$$

Since Y is normal, constructible and $\dim Y = n$ we have $H^q(Y;\mathcal{G}) = 0$ for $q > n$ and every sheaf $\mathcal{G}$ on Y ([13] Proposition 4.2). Thus $H^1(Y;(\mathcal{I}^n_{|Y})_U) = 0$. Since U was an arbitrary open and constructible subset of Y, it follows from Proposition 3.10 that $\mathcal{I}^n_{|Y}$ is soft as required. $\square$

Proposition 3.13. *Assume that* X *is an object in the category of o-minimal spectra of definable spaces and* Φ *is a normal and constructible family of supports on* X*. If* $\mathcal{G} \in \mathrm{Mod}(k_X)$ *is* Φ*-soft, then for every* $\mathcal{F} \in \mathrm{Mod}(k_X)$ *we have that* $\mathcal{G}\otimes\mathcal{F} \in \mathrm{Mod}(k_X)$ *is* Φ*-soft.*

Proof. By Theorem 3.12, X has finite cohomological Φ-dimension. Suppose that the cohomological Φ-dimension of X is n. Since the family of the constant sheaves $\{k_U\}$, U constructible open subset of X is generating, there is a resolution of $\mathcal{F}$

$$\mathcal{P}_{n-1} \overset{\partial_{n-1}}{\to} \mathcal{P}_{n-2} \cdots \mathcal{P}_1 \overset{\partial_1}{\to} \mathcal{P}_0 \to \mathcal{F} \to 0$$

where the $\mathcal{P}_i$'s are direct sums of sheaves of the form k_U, U constructible (see [19], Proposition 2.4.12). From Proposition 3.6 (iii) it follows that $\mathcal{G}_U \simeq \mathcal{G}\otimes k_U$ is Φ-soft. Since the direct sum of Φ-soft sheaves in $\mathrm{Mod}(k_X)$ is Φ-soft (Corollary 3.5) each $\mathcal{G}\otimes\mathcal{P}_i$ is Φ-soft.

From the resolution above we obtain an exact sequence of sheaves

$$\mathcal{G}\otimes\mathcal{P}_{n-1} \overset{\partial_{n-1}}{\to} \mathcal{G}\otimes\mathcal{P}_{n-2} \cdots \to \mathcal{G}\otimes\mathcal{P}_1 \overset{\partial_1}{\to} \mathcal{G}\otimes\mathcal{P}_0 \to \mathcal{G}\otimes\mathcal{F} \to 0.$$

By Proposition 3.11, since $\mathcal{G} \otimes \mathcal{P}_i$ is Φ-soft for $i = 0, \dots, n-2$, we conclude that $\mathcal{G} \otimes \mathcal{F}$ is Φ-soft. □

4. Duality with coefficient in a field

In this section we will work in the category of definable spaces with continuous definable maps and k-sheaves on such spaces will be considered always relative to the o-minimal site. In our results we will have a definably normal, definably locally compact definable space X and the family of definable supports c on X of definably compact definable subsets of X. By Example 2.10 and Remark 2.11 the corresponding constructible family of supports on the o-minimal spectra of X will be a normal and constructible family of supports. Hence, by the tilde isomorphism in the category of k-sheaves given by Theorem 2.8 and our Definition 2.12, in our proofs we will apply the results of Section 3 since they transfer to this definable setting.

Remark 4.1. We observe that since all the results of this section depend only on Section 3, they hold on an arbitrary definable space X replacing c by a definably normal family of definable supports Φ on X. In particular, these results hold on any definable space X on which c is a definably normal family of definable supports.

4.1. Sheaves of linear forms.

Here we shall work with a fixed field k. For a k-vector space N we let $N^\vee$ denote the dual k-vector space, i.e. $N^\vee = \mathrm{Hom}_k(N, k)$.

Let X be a definably normal, definably locally compact definable space and $\mathcal{F}$ a k-sheaf on X. From now on, given a locally closed subset Z of X, we will write $\Gamma_c(Z; \mathcal{F})$ instead of $\Gamma_{c|Z}(Z; \mathcal{F})$ for short. The inclusion $V \longrightarrow U$ of two open definable subsets of X will induce a map

$$\begin{array}{ccc} \Gamma_c(X; \mathcal{F}_V) & \longrightarrow & \Gamma_c(X; \mathcal{F}_U) \\ \downarrow \wr & & \downarrow \wr \\ \Gamma_c(V; \mathcal{F}) & \longrightarrow & \Gamma_c(U; \mathcal{F}) \end{array}$$

"extension by zero". (Where the vertical isomorphisms are a consequence of Proposition 2.1 with $\Phi = c$). The k-linear dual of this

$$\Gamma_c(U; \mathcal{F})^\vee \longrightarrow \Gamma_c(V; \mathcal{F})^\vee$$

gives rise to restriction maps in a presheaf $\mathcal{F}^\vee$ defined by

$$\Gamma(U; \mathcal{F}^\vee) = \Gamma_c(U; \mathcal{F})^\vee.$$

Proposition 4.2. *Let X be a definably normal, definably locally compact definable space. For every c-soft k-sheaf $\mathcal{F}$ on X, the presheaf $\mathcal{F}^\vee$ is a sheaf.*

Proof. By Proposition 2.2, it is enough to show that for any two open definable subsets W and V of X the sequence

$$0 \longrightarrow \Gamma(V \cup W; \mathcal{F}^\vee) \longrightarrow \Gamma(V; \mathcal{F}^\vee) \oplus \Gamma(W; \mathcal{F}^\vee) \longrightarrow \Gamma(V \cap W; \mathcal{F}^\vee)$$

formed by the sum and difference between two restriction maps is exact.

Consider the Mayer-Vietoris sequence

$$0 \longrightarrow \Gamma_c(V \cap W; \mathcal{F}) \longrightarrow \Gamma_c(V; \mathcal{F}) \oplus \Gamma_c(W; \mathcal{F}) \longrightarrow \Gamma_c(V \cup W; \mathcal{F}) \longrightarrow H^1_c(V \cap W; \mathcal{F})$$

and notice that $H^1_c(V \cap W; \mathcal{F}) = 0$ since the restriction of $\mathcal{F}$ to $V \cap W$ is c-soft by Proposition 3.6 (i). The result now follows by taking the k-linear dual of the Mayer-Vietoris sequence. $\square$

Proposition 4.3. *Let X be a definably normal, definably locally compact definable space. Let $\mathcal{G}$ be a c-soft k-sheaf on X. There is a natural isomorphism*

$$\Gamma_c(X; \mathcal{F} \otimes \mathcal{G})^\vee \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{G}^\vee)$$

as $\mathcal{F}$ varies through the category of k-sheaves on X.

Proof. Let U be an open definable subset. Consider the natural maps

$$\Gamma(U; \mathcal{F}) \otimes \Gamma_c(U; \mathcal{G}) \longrightarrow \Gamma_c(U; \mathcal{F} \otimes \mathcal{G}) \to \Gamma_c(X; \mathcal{F} \otimes \mathcal{G})$$

The dual of the composite can be written

$$\Gamma_c(X; \mathcal{F} \otimes \mathcal{G})^\vee \longrightarrow \mathrm{Hom}(\Gamma(U; \mathcal{F}), \Gamma_c(U; \mathcal{G})^\vee)$$

By variation of U this defines a map

$$\Gamma_c(X; \mathcal{F} \otimes \mathcal{G})^\vee \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G}^\vee) \tag{1}$$

which we must show that it is an isomorphism.

(i) First we consider the case where $\mathcal{F} = k_U$ where U is an open definable subset. We have

$$\Gamma_c(X; \mathcal{G}_U)^\vee = \Gamma_c(U; \mathcal{G})^\vee = \Gamma(U; \mathcal{G}^\vee) = \mathrm{Hom}(k_U, \mathcal{G}^\vee).$$

These identifications transform the map (1) into the identity.

(ii) For the general case, consider a presentation of $\mathcal{F}$ of the form

$$\mathcal{P} \longrightarrow \mathcal{Q} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{P}$ and $\mathcal{Q}$ are direct sums of sheaves of the form k_U as above (see [19], Proposition 2.4.12). Let us consider the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_c(X; \mathcal{F} \otimes \mathcal{G})^\vee \arrow[r] \arrow[d] & \Gamma_c(X; \mathcal{Q} \otimes \mathcal{G})^\vee \arrow[r] \arrow[d] & \Gamma_c(X; \mathcal{P} \otimes \mathcal{G})^\vee \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{F}, \mathcal{G}^\vee) \arrow[r] & \mathrm{Hom}(\mathcal{Q}, \mathcal{G}^\vee) \arrow[r] & \mathrm{Hom}(\mathcal{P}, \mathcal{G}^\vee).
\end{tikzcd}$$

The two functors of (1) transform direct sums into direct products. It follows that the two vertical maps to the right are isomorphisms. Then it follows from the five lemma that the first vertical arrow is an isomorphism. $\square$

Corollary 4.4. *Let X be a definably normal, definably locally compact definable space. Let $\mathcal{G}$ be a c-soft k-sheaf on X. Then the sheaf $\mathcal{G}^\vee$ is injective in the category of k-sheaves on X.*

Proof. By Proposition 4.3, we must show that

$$\mathcal{F} \mapsto \Gamma_c(X; \mathcal{F} \otimes \mathcal{G})^\vee$$

is an exact functor. But this follows from Propositions 3.13 and 3.7 and the exactness of $\vee$ in $\mathrm{Mod}(k)$. $\square$

4.2. **Verdier duality.** If X is a definably normal, definably locally compact definable space we will let $\mathrm{D}^+(k_X)$ denote the derived category of bounded below complexes of k-sheaves on X. We are now ready to prove our main result:

Theorem 4.5 (Verdier duality)**.** *Let X denote a definably normal, definably locally compact definable space. Then there exists an object $\mathcal{D}^*$ in $\mathrm{D}^+(k_X)$ and a natural isomorphism*

$$\mathrm{RHom}(\mathcal{F}^*, \mathcal{D}^*) \simeq \mathrm{RHom}(R\Gamma_c(X; \mathcal{F}^*), k)$$

as $\mathcal{F}^*$ varies through $\mathrm{D}^+(k_X)$.

Proof. For a complex L^* of k-vector spaces we put $L^{*\vee} = \mathrm{Hom}^*(L^*, k)$ with the notation of [18] I.4.3. Notice also that $L^{*\vee}$ is a complex of k-vector spaces whose p'th differential is given by

$$(-1)^{p+1}(\partial^{-p-1})^\vee : (L^{-p})^\vee \longrightarrow (L^{-p-1})^\vee.$$

This formula will also be used to extend the functor $\mathcal{G} \mapsto \mathcal{G}^\vee$ on the category of k-sheaves on X given by Proposition 4.2 to complexes of k-sheaves.

By Theorem 3.12 X has finite cohomological c-dimension, hence by Proposition 3.11 (1) the constant sheaf k_X admits a bounded c-soft resolution $\mathcal{G}^*$. By Corollary 4.4, $\mathcal{G}^{*\vee}$ is a bounded complex of injective k-sheaves. For an injective complex $\mathcal{I}^*$ quasi-isomorphic to $\mathcal{F}^*$ in the derived category of bounded below complexes of k-sheaves on X and integers p and q we have, by Proposition 4.3, a canonical isomorphism

$$\Gamma_c(X; \mathcal{I}^p \otimes \mathcal{G}^q)^\vee = \mathrm{Hom}(\mathcal{I}^p, \mathcal{G}^{q\vee}).$$

giving an isomorphism of complexes

$$\Gamma_c(X; \mathcal{I}^* \otimes \mathcal{G}^*)^\vee = \mathrm{Hom}^*(\mathcal{I}^*, \mathcal{G}^{*\vee}). \tag{2}$$

From the quasi-isomorphism $k_X \longrightarrow \mathcal{G}^*$ we deduce a quasi-isomorpism

$$\Gamma_c(X; \mathcal{I}^* \otimes \mathcal{G}^*)^\vee \longrightarrow \Gamma_c(X; \mathcal{I}^*)^\vee$$

which yields a final quasi-isomorphism

$$\mathrm{Hom}^*(\mathcal{I}^*, \mathcal{G}^{*\vee}) \longrightarrow \Gamma_c(X; \mathcal{I}^*)^\vee. \tag{3}$$

Finally put $\mathcal{D}^* = \mathcal{G}^{*\vee}$. □

The complex $\mathcal{D}^*$ above is called the *dualizing complex*. It is a bounded below complex of injective k-sheaves uniquely determined up to homotopy and so the cohomology k-sheaves $\mathcal{H}^p\mathcal{D}^*$, $p \in \mathbb{Z}$, are uniquely determined up to isomorphism.

The proof above shows the following:

Remark 4.6. The dualizing complex for a definably normal, definably locally compact definable space of cohomological c-dimension n can be represented by a complex $\mathcal{D}^*$ of injective k-sheaves where

$$\mathcal{D}^i = 0 \quad \text{for} \quad i \notin [-n, 0].$$

Recall that the inclusion $V \longrightarrow U$ of open definable subsets of X give rise to the extension by zero map

$$H^p_c(V; k_X) \longrightarrow H^p_c(U; k_X)$$

whose k-linear dual

$$H^p_c(U; k_X)^\vee \longrightarrow H^p_c(V; k_X)^\vee$$

gives rise to a presheaf $U \mapsto H^p_c(U; k_X)^\vee$.

Proposition 4.7. *Let $\mathcal{D}^*$ denote the dualizing complex for the definably normal, definably locally compact definable space X. For any integer p, the cohomology k-sheaf $\mathcal{H}^{-p}\mathcal{D}^*$ is the sheaf associated to the k-presheaf*

$$U \mapsto H^p_c(U;k_X)^\vee.$$

Proof. Recall the isomorphism $H^p_c(U;k_X) \simeq H^p_c(X;k_U)$. Passing to the dual and using Theorem 4.5 we have the chain of isomorphisms

$$H^p_c(U;k_X)^\vee \simeq H^p_c(X;k_U)^\vee \simeq H^{-p}\mathrm{Hom}(k_U,\mathcal{D}^*) \simeq H^{-p}(U;\mathcal{D}^*)$$

and the result follows since $\mathcal{H}^{-p}\mathcal{D}^*$ is the k-sheaf associated to the k-presheaf $U \mapsto H^{-p}(U;\mathcal{D}^*)$. $\square$

Corollary 4.8. *On a definably normal, definably locally compact definable space X of cohomological c-dimension n, the k-presheaf*

$$U \mapsto H^n_c(U;k_X)^\vee$$

is a k-sheaf.

Proof. By Remark 4.6 we have an exact sequence

$$0 \longrightarrow \Gamma(U;\mathcal{H}^{-n}\mathcal{D}^*) \longrightarrow \Gamma(U;\mathcal{D}^{-n}) \longrightarrow \Gamma(U;\mathcal{D}^{-n+1}).$$

On the other hand $H^{-n}(U;\mathcal{D}^*) = \ker(\Gamma(U;\mathcal{D}^{-n}) \to \Gamma(U;\mathcal{D}^{-n+1})$. Moreover, as we saw above

$$H^{-n}(U;\mathcal{D}^*) \simeq H^n_c(U;k_X)^\vee.$$

Then $\Gamma(U;\mathcal{H}^{-n}\mathcal{D}^*) \simeq H^n_c(U;k_X)^\vee$ and the result follows. $\square$

4.3. **Poincaré and Alexander duality.** Here we derive Poincaré and Alexander duality from the Verdier duality.

Definition 4.9. Let X be a definably normal, definably locally compact definable manifold of dimension n. We say that X *has an orientation k-sheaf* if for every open definable subset U of X there exists a finite cover of U by open definable subsets $U_1,\dots,U_\ell$ of U such that for each i we have

$$H^p_c(U_i;k_X) = \begin{cases} k & \text{if} \quad p=n \\ \\ 0 & \text{if} \quad p\neq n. \end{cases}$$

If X has an orientation sheaf, we call the k-sheaf $\mathcal{O}r_X$ on X with sections

$$\Gamma(U;\mathcal{O}r_X) = H^n_c(U;k_X)^\vee$$

the *orientation k-sheaf on X*. By Theorem 3.12, the cohomological c-dimension of X is n and $H^n_c(U_i;k_X) = H^n_c(X;k_{U_i}) \neq 0$ for $i=1,\dots,\ell$, hence X must have cohomological c-dimension n. So $\mathcal{O}r_X$ is indeed a k-sheaf on X by Corollary 4.8).

Note also that, since the o-minimal spectra $\widetilde{X}$ of X is a quasi-compact (spectral) topological space, X has an orientation k-sheaf if and only if for every $\beta \in \widetilde{X}$ and every open definable subset V of X such that $\beta \in \widetilde{V}$, there is an open definable subset U of V such that $\beta \in \widetilde{U}$ and

$$H^p_c(U; k_X) = \begin{cases} k & \text{if} \quad p = n \\ \\ 0 & \text{if} \quad p \neq n. \end{cases}$$

Example 4.10. Suppose that $\mathcal{M}$ is an o-minimal expansion of an ordered field. Let X be a Hausdorff definable manifold of dimension n. Since then X is affine and every definable set is definably normal, X is definably normal ([11] Chapter 6, Lemma 3.5). Since also X and any open definable subset of X can be covered by finitely many definable sub-balls ([12] Theorem 1.2), X is definably locally compact and, computing the o-minimal cohomology with definably compact supports of definable sub-balls, it follows that X has an orientation k-sheaf. Observe that the result on coverings by definable sub-balls is related to [1] Theorem 4.3 (and can be read off from the proofs of Lemmas 4.1 and 4.2 there) and also to Wilkie's result ([25] Theorem 1.3) which says that every bounded open definable set can be covered by finitely open cells.

Let X be a definably normal, definably locally compact definable manifold of dimension n with an orientation k-sheaf $\mathcal{O}r_X$. Then the k-sheaf $\mathcal{O}r_X$ is locally isomorphic to k_X.

Theorem 4.11 (Poincaré duality). *Let X be a definably normal, definably locally compact definable manifold of dimension n with an orientation k-sheaf $\mathcal{O}r_X$. There exists an isomorphism*

$$H^p(X; \mathcal{O}r_X) \longrightarrow H^{n-p}_c(X; k_X)^\vee.$$

Proof. Proposition 4.7 and the fact that X has an orientation k-sheaf, imply that

$$\mathcal{H}^{-p}\mathcal{D}^* = 0 \qquad ; p \neq n.$$

On the other hand, by Corollary 4.8 we have $\mathcal{H}^{-n}\mathcal{D}^* = \mathcal{O}r_X$. Thus we have a quasi-isomorphism

$$\mathcal{O}r_X[n] \simeq \mathcal{D}^* \tag{4}$$

Therefore we have

$$H^p(X; \mathcal{O}r_X) \simeq H^{p-n}(X; \mathcal{D}^*) \simeq H^{p-n}\mathrm{Hom}(k_X, \mathcal{D}^*).$$

By Verdier duality (Theorem 4.5) with $\mathcal{F}^* = k_X$ the later is also isomorphic to $H^{n-p}_c(X; k_X)^\vee$.

□

Definition 4.12. Let X be a definably normal, definably locally compact definable manifold of dimension n with an orientation k-sheaf $\mathcal{O}r_X$. By a *k-orientation* we understand an isomorphism

$$k_X \simeq \mathcal{O}r_X$$

of k-sheaves. We shall say that X is *k-orientable* if a k-orientation exists and *k-unorientable* in the opposite case.

Proposition 4.13. *Let X be a definably connected, definably normal, definably locally compact definable manifold of dimension n with an orientation k-sheaf $\mathcal{O}r_X$. Then*

(1) $H^n_c(X; k_X) \simeq k$ *if X is k-orientable.*
(2) $H^n_c(X; k_X) \simeq 0$ *if X is k-unorientable.*

Proof. Since X is definably normal and definably connected, Proposition 4.1 in [13] implies that $H^0(X; k_X) = k$ and so (1) follows at once from the Poincaré duality (Theorem 4.11).

For (2), suppose that $H^n_c(X; k_X) \neq 0$. Then by Theorem 4.11 there is a non trivial section s of $\mathcal{O}r_X$ over X. By our Definition 2.12, the support of s is a closed subset of the o-minimal spectrum of X. Since $\mathcal{O}r_X$ is locally isomorphic to k_X it follows that the support of s is also an open subset of the o-minimal spectrum of X. But since the o-minimal spectrum of X is connected (Theorem 2.5) it follows that the support of s is the o-minimal spectrum of X. Thus $\mathcal{O}r_X \simeq k_X$. □

Theorem 4.14 (Alexander duality)**.** *Let X be a definably normal, definably locally compact, k-orientable definable manifold of dimension n. For Z a closed definable subset of X there exists an isomorphism*

$$H^p_Z(X; k_X) \longrightarrow H^{n-p}_c(Z; k_X)^\vee.$$

Proof. By (4) we have $H^p_Z(X; k_X) \simeq H^{p-n}_Z(X; \mathcal{D}^*) \simeq H^{p-n}\mathrm{Hom}(k_Z, \mathcal{D}^*)$. By Verdier duality (Theorem 4.5) with $\mathcal{F}^* = k_Z$ the later is also isomorphic to $H^{n-p}_c(X; k_Z)^\vee \simeq H^{n-p}_c(Z; k_X)^\vee$. □

4.4. **Duality in o-minimal expansions of fields.** In this subsection we assume that the o-minimal structure $\mathcal{M}$ is an expansion of an ordered field.

Let X be a Hausdorff definable manifold of dimension n. Then has we saw in Example 4.10 X is affine, definably normal with an orientation k-sheaf.

In o-minimal expansions of fields we have o-minimal singular homology and cohomology theories satisfying the Eilenberg-Steenrod axioms adapted to the o-minimal site ([15], [24]). By [15] the o-minimal singular cohomology theory with coefficients in a field k is isomorphic to the o-minimal sheaf cohomology theory with coefficients in the constant sheaf k_X. Because of this isomorphism, below we will use the standard notation from o-minimal singular cohomology and write k for k_A and

$$H^*(A, B; k) \quad \text{for} \quad H^*_{A \setminus B}(A; k)$$

where $B \subseteq A \subseteq X$ are definable subsets of X.

O-minimal singular homology theory can be used to obtain an orientation theory for definable manifolds ([1], [2]). (In the papers [1] and [2], orientation is defined by taking homology with coefficients in $\mathbb{Z}$ but replacing $\mathbb{Z}$ by k and considering homology groups as k-vector spaces one gets the theory of k-orientations.) Our goal here is to show an Alexander duality for homology and to conclude that the two orientation theories agree.

First observe that if $B \subseteq A$ are definably locally closed definable subsets of X, then

$$H^*_c(A \setminus B; k) = \varinjlim_{B \subseteq C \subseteq A,\ C \text{ closed},\ \overline{A \setminus C} \in c} H^*(A, C; k). \tag{5}$$

Let Λ be the directed system of definably locally closed subsets D of A such that $B \subseteq D \subseteq A$ and $\overline{A \setminus D} \in c$, directed by reverse inclusion. Since the map that sends $D \in \Lambda$ into $\overline{D}$ is cofinal (even surjective) in the directed system of definable closed subsets C of A such that $B \subseteq C \subseteq A$ and $\overline{A \setminus C} \in c$, directed by reverse inclusion, it follows that to prove (5) it is enough to show that

$$H_c^*(A \setminus B; k) = \varinjlim_{D \in \Lambda} H^*(A, D; k),$$

i.e., we have to show that the natural homomorphism

$$\varinjlim_{B \subseteq U \subseteq A,\ A \setminus U \in c} H^*(A, U; k) \longrightarrow \varinjlim_{D \in \Lambda} H^*(A, D; k)$$

is an isomorphism. But this is a consequence of the following. If $D \in \Lambda$, then there exists an open definable subset O of A such that D is closed in O. So, by [11] Chapter VIII, 3.3 and 3.4, there is an open definable neighborhood U of D in O such that D is a definable deformation retract of U. Therefore, the inclusion $D \longrightarrow U$ induces an isomorphism $H^*(A, U; k) \longrightarrow H^*(A, D; k)$.

We are now ready to show the Alexander duality for o-minimal homology. This is the o-minimal version of [10] Chapter VIII, Theorem 7.14 and the generalization of Theorem 3.5 in [16].

Theorem 4.15. *Let X be a definable manifold of dimension n which is k-orientable with respect to homology. Let $L \subseteq K \subseteq X$ be closed definable sets with $K - L$ closed in $X - L$. Then there is an isomorphism*

$$H_c^q(K \setminus L; k) \longrightarrow H_{n-q}(X \setminus L, X \setminus K; k)$$

for all $q \in \mathbb{Z}$ which is natural with respect to inclusions.

Proof. Let $K' = K \setminus L$, $X' = X \setminus L$, A a definable closed subset of K' such that $\overline{K' \setminus A} \in c$ and $C = \overline{K' \setminus A}$. Then we have the following commutative diagram

$$\begin{tikzcd} H^q(K', A; k) \arrow[r] \arrow[d] & H_{n-q}(X' \setminus A, X' \setminus K'; k) \arrow[d] \\ H^q(K' \cap C, A \cap C; k) \arrow[r, "\cap \zeta'_{K' \cap C}"] & H_{n-q}(X' \setminus A \cap C, X' \setminus K' \cap C; k). \end{tikzcd}$$

where the vertical arrows are the inclusion homomorphisms which, by the excision axiom, are isomorphisms. The bottom arrow is the isomorphism of Theorem 3.5 in [16]. This diagram goes to the limit to give the isomorphism of the theorem by (5) and

$$H_*(X', X' - K'; k) = \varinjlim_{A \subseteq K',\ A \text{ closed},\ \overline{K' \setminus A} \in c} H_*(X' - A, X' - K'; k)$$

(as $X' = \cup\{X' - A : A \subseteq K',\ A \text{ closed},\ \overline{K' \setminus A} \in c\}$). $\square$

Combining Alexander duality for homology (Theorem 4.15) and for cohomology (Theorem 4.14) we show:

Corollary 4.16. *Let X be a Hausdorff definable manifold. Then X is k-orientable with respect to homology if and only if X is k-orientable with respect to cohomology.*

Proof. Indeed, let X be a Hausdorff definable manifold of dimension n. If X is k-orientable with respect to homology, then Theorem 4.15 implies that for every definably connected, definably compact definable subset K of X we have an isomorphism $H_n(X, X\setminus K; k) \simeq k$ which is compatible with inclusions. Applying the dual universal coefficients theorem and going to the limit we obtain $H^n_c(X; k) \simeq k$ showing that X is k-orientable (Proposition 4.13). If X is k-orientable with respect to cohomology, then Theorem 4.14 applied to K and X implies that for every definably connected, definably compact definable subset K of X we have an isomorphism $H^n(X, X \setminus K; k) \simeq k$ which is compatible with inclusions. Applying the dual universal coefficients theorem we get an isomorphism $H_n(X, X \setminus K; k) \simeq k$ compatible with inclusions which allows us to define a k-orientation for X relative to homology. □

References

[1] A. Berarducci and M.Otero *Transfer methods for o-minimal topology* J. Symb. Logic **68** (3) (2003) 785–794.

[2] A. Berarducci, M. Edmundo and M. Otero *Corrigendum to "Transfer methods for o-minimal topology"* J. Symb. Logic **72** (3) (2007) 1079–1080.

[3] J. Bochnak, M. Coste and M-F. Roy *Real Algebraic Geometry* Springer-Verlag 1998.

[4] G. Bredon *Sheaf theory* Second Edition Springer-Verlag 1997.

[5] M. Coste *An introduction to o-minimal geometry* Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000). Available in RAAG preprint server (`http://ihp-raag.org/`).

[6] M. Coste and M.-F. Roy *La topologie du spectre réel* in Ordered fields and real algebraic geometry, Contemporary Mathematics **8** (1982) 27–59.

[7] H. Delfs *The homotopy axiom in semi-algebraic sheaf cohomology* J. reine angew. Maths. **355** (1985) 108–128.

[8] H. Delfs *Homology of locally semialgebraic spaces* LNM 1484 Springer-Verlag 1991.

[9] H. Delfs and M. Knebusch *On the homology of algebraic varieties over real closed fields* J. reine u.angew. Math. **335** (1982) 122–163.

[10] A. Dold *Lectures on algebraic topology* Springer Verlag 1995.

[11] L. van den Dries *Tame topology and o-minimal structures* Cambridge University Press 1998.

[12] M. Edmundo *Covering definable manifolds by open definable subsets* In: Logic Colloquium '05, Lecture Notes in Logic **28** (2008) (ed., C. Dimitracopoulos et al.) Cambridge University Press.

[13] M. Edmundo, G. Jones and N. Peatfield *Sheaf cohomology in o-minimal structures* J. Math. Logic **6** (2) (2006) 163–179.

[14] M. Edmundo and M. Otero *Definably compact abelian groups.* J. Math. Log. **4** (2) (2004) 163–180.

[15] M. Edmundo and A. Woerheide *Comparation theorems for o-minimal singular (co)homology* Trans. Amer. Math. Soc. **360** (9) (2008) 4889–4912.

[16] M. Edmundo and A. Woerheide *The Lefschetz coincidence theorem in o-minimal expansions of fields* Topology Appl. **156** (15) (2009) 2470–2484.

[17] R. Godement *Théorie des faisceaux* Hermann 1958.

[18] B. Iversen *Cohomology of sheaves* Springer Verlag 1986.

[19] M. Kashiwara and P. Schapira *Sheaves on manifolds* Springer Verlag 1990.

[20] M. Kashiwara and P. Schapira *Ind-sheaves* Astérisque **271** (2001) 136 pp.

[21] M. Kashiwara and P. Schapira *Categories and sheaves* Springer Verlag 2005.

[22] A. Pillay *Sheaves of continuous definable functions* J. Symb. Logic **53** (4) (1988) 1165–1169.

[23] L. Prelli *Sheaves on subanalytic sites* Rend. Sem. Mat. Univ. Padova **120** (2008) 167–216.

[24] A. Woerheide *O-minimal homology* PhD. Thesis (1996), University of Illinois at Urbana-Champaign.

[25] A. Wilkie *Covering definable open subsets by open cells* In: O-minimal Structures, Proceedings of the RAAG Summer School Lisbon 2003, Lecture Notes in Real Algebraic and Analytic Geometry (M. Edmundo, D. Richardson and A. Wilkie eds.,) Cuvillier Verlag 2005.

Universidade Aberta & CMAF Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
E-mail address: `edmundo@cii.fc.ul.pt`

Università di Padova, Dipartimento di Matematica Pura ed Applicata, Via Trieste 63, 35121 Padova, Italy
E-mail address: `lprelli@math.unipd.it`