A PROPERTY OF SMALL GROUPS

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Abstract. A group is small if it has countably many pure n-types for each integer n. It is shown that in a small group, subgroups which are definable with parameters in a finitely generated algebraic closure satisfy local descending chain conditions. An infinite small group has an infinite abelian subgroup, which may not be definable. A nilpotent small group is the central product of a definable divisible group with one of bounded exponent. In a group with simple theory, any set of pairwise commuting elements is contained in a definable finite-by-abelian subgroup. As a corollary, a group with small and simple theory has an infinite definable finite-by-abelian subgoup.

A connected group of Morley rank one is abelian [20, Reineke]. Better, a connected omega-stable group of minimal Morley rank is abelian, from which it follows that every infinite omega-stable group has a definable infinite abelian subgroup [6, Cherlin]. Berline and Lascar generalised this to superstable groups in [4]. More recently, Poizat introduced d-minimal structures which englobe minimal ones, and proved a d-minimal group to be abelian-by-finite [17]. He went further showing that an infinite group of finite Cantor rank has a definable abelian infinite subgroup [18]. More generally, we show that an infinite weakly small group has an infinite abelian subgroup, which may however not be definable.

We then turn to groups definable in a small and simple theory. Recall that an omega-categorical superstable group is abelian-by-finite [3, Baur, Cherlin and Macintyre]. In [23], Wagner showed any small stable infinite group to have an infinite abelian subgroup. Later on, Evans and Wagner proved that an omega-categorical supersimple group is finite-by-abelian-by-finite and has finite SU-rank [7]. We show that an infinite group the theory of which is small and simple has an infinite definable finite-by-abelian subgroup. However it is still unknown whether a stable group must have an infinite abelian subgroup.

Definition 1. A theory is *small* if it has countably many n-types without parameters for every integer n. A structure is small if its theory is so.

Note that smallness is preserved by interpretation, and by adding finitely many parameters to the language. Small theories arise when one wishes to count the number of pairwise non-isomorphic countable models of a complete first order theory in a countable language. Such a theory having fewer than the maximal number of pairwise non-isomorphic models is small indeed.

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Definition 2. *(Belegradek)* A structure is *weakly small* if it has countably many 1-types over a for any finite tuple a coming from the structure.

Weakly small structures were introduced by Belegradek to give a common generalisation of small and minimal structures.

Definition 3. *(Poizat [17])* An infinite structure is *d-minimal* if any of its partitions has no more than d infinite definable subsets.

Every d-minimal structure in a countable language is weakly small, as there are at most d non algebraic types over every finite parameter set. Note that weak smallness neither is a property of the theory, nor allows the use of compactness, nor guarantees that the set of 2-types be countable. It allows arguments using formulae in one free variable only. Those formulae, the parameters of which lie in a fixed finite set, are ranked by Cantor rank and degree.

Examples. Let G be the sum over all prime numbers p of cyclic groups of order p. For every set of primes P, the type saying that x is p-divisible if and only if $p \in P$ is finitely consistent. So G is not weakly small.

Let p be a prime, and H the sum over all natural numbers n of the cyclic groups of order p^n. The theory of H eliminates quantifiers up to pp-formulae, so every definable subset of H is a boolean combination of cosets of subgroups of the form $p^n H$, or $p^n x = 0$. This allows only countably many 1-types over every finite subset, so H is weakly small. But Fact 4.1 and Remark 4.3 show that it is not small.

1. Cantor rank

Given a structure M, a set A of parameters lying inside M, and an A-definable subset X of M, we define the *Cantor rank of X over A* by the following induction :

> $CB_A(X) \geq 0$ if X is not empty,
> $CB_A(X) \geq \alpha + 1$ if there are infinitely many disjoint A-definable subsets of X having Cantor rank over A at least α.
> $CB_A(X) \geq \lambda$ for a limit ordinal λ, if $CB_A(X)$ is at least α for every α less than λ.

If the structure is weakly small, this process eventually stops, and X has ordinal Cantor rank over A. The *Cantor degree of X over A* is the greatest integer d such that there is a partition of X into d A-definable sets having maximal Cantor rank over A. We shall write $dCB_A(X)$ for this degree. It is also the number of complete types over A having maximal Cantor rank over A.

Definable bounded-to-one maps preserve Cantor rank, and the degree variations can be bounded by the maximal size of the finite fibres :

Lemma 1.1. *Let X and Y be A-definable sets, and f an A-definable map from X to Y, then*

1. *If f is onto, $CB_A(X) \geq CB_A(Y)$.*
2. *If f is n-to-one, $CB_A(Y) \geq CB_A(X)$.*
3. *If f is n-to-one and onto, then X and Y have the same Cantor rank over A, and*
$$dCB_A(Y) \leq dCB_A(X) \leq n \cdot dCB_A(Y)$$

Remark 1.2. The first two points appear for one-to-one maps together with the introduction of Morley's rank [12, Theorem 2.3]. Poizat extends them for n-to-one maps to groups with finite Cantor rank [18, Lemme 1]. To the author's knowledge, the result concerning the degree is new.

Proof. We may add A to the language. For point one, we show inductively that $CB(X)$ is at least $CB(Y)$. If $CB(Y) \geq \alpha + 1$, there are infinitely many disjoint definable sets Y_i in Y of rank at least α. Their pre-images are disjoint and have rank at least α by induction, so $CB(X) \geq \alpha + 1$.

For point two, we show inductively that $CB(Y)$ is at least $CB(X)$. Suppose $CB(X) \geq \alpha + 1$. In X, there are infinitely many disjoint definable sets X_i of rank at least α. As f is n-to-one, $f(X_1)$ intersects only finitely many of the sets $f(X_i)$. One may remove those sets and assume $f(X_1)$ intersects none. Iterating, we may assume that all the sets $f(X_i)$ are disjoint. By induction hypothesis, their rank is at least α, so $CB(Y) \geq \alpha + 1$.

For the third point, if Y has degree d, then there is a partition of Y in definable sets $Y_1, \ldots, Y_d$ with maximal rank. The pre-images of the sets Y_i have maximal rank according to the first point and form a partition of X, so the degree $dCB(X)$ is at least $dCB(Y)$.

For the converse inequality, let Y have degree d, and let Y_1 be a subset of Y of degree one. It is enough to show that $f^{-1}(Y_1)$ has degree at most n. Suppose there are $n+1$ disjoint definable subsets $X_0, \ldots, X_n$ of $f^{-1}(Y_1)$ with maximal rank. As the map f is n-to-one, the intersection $\bigcap_{i=0}^{n} f(X_i)$ is empty, so there is a least intersection $\bigcap_{i\in I} f(X_i)$ having same rank as Y. Thus, the intersection of $\bigcap_{i\in I} f(X_i)$ and $f(X_i)$ has small rank for every i out of I, and $dCB(Y_1)$ is at least two, a contradiction. □

Remark 1.3. In point (3), to deduce that X and Y have the same Cantor rank, the fibres of f must be bounded, and not only finite. Consider for instance Y to be the set of all natural numbers together with the ordering, and X to be the set of pairs of integers (x, y) so that $y \leq x$. When projecting on the second coordinate, every fibre is infinite, so $CB_{\mathbf{N}}(X) = 2$; when projecting on the first coordinate, the fibres are finite, but still $CB_{\mathbf{N}}(Y) = 1$.

If A is a subset of B and X an A-definable set, then $CB_A(X)$ is less or equal to $CB_B(X)$. Note that the Cantor rank (respectively degree) of X over A or over the definable closure of A are the same. The Cantor rank over A also does not change when adding algebraic parameters to A, and the degree variation can be bounded :

Lemma 1.4. *Let X be a definable set without parameters, and a an algebraic tuple of degree n over the empty set, then*

(1) $CB_a(X) = CB_\emptyset(X)$
(2) $dCB_\emptyset(X) \leq dCB_a(X) \leq n \cdot dCB_\emptyset(X)$

Proof. For the first point, let us show inductively that $CB_\emptyset(X)$ is at least $CB_a(X)$. Suppose $CB_a(X)$ is at least $\alpha+1$. There are infinitely many disjoint a-definable X_i of rank at least α. By induction hypothesis, for every i, the set X_i and a conjugate of X_1 have same rank (computed over the set $\bar{a}$ of all conjugates of a). So a conjugate of X_1 intersects only finitely many X_i in a set of maximal rank over $\bar{a}$. One can take off these X_i, cut off a small ranked subset of the remaining X_i and assume that the conjugates of X_1 do not intersect any X_i. Iterating, one may assume that no conjugate of X_i intersects X_j when i differs from j. Let $\overline{X_i}$ denote the finite union of the conjugates of X_i under the action of the automorphisms group of the structure. $CB_a(\overline{X_i})$ equals $CB_a(X_i)$, so by induction hypothesis, $CB_\emptyset(\overline{X_i})$ is at least α. As the sets $\overline{X_i}$ are disjoint, $CB_\emptyset(X) \geq \alpha + 1$. Conversely, the Cantor rank of a set increases when one allows new calculation parameters, so the first point is proved.

For the second point, we may assume that X has degree one over the empty set. Suppose that X has degree at least $n+1$ over a. Let X_1 be a subset of X definable over a with maximal rank over a and degree one. The union $\overline{X_1}$ of its conjugates has degree at most n over a, so $\overline{X_1}$ and its complement in X both have maximal rank over a, hence over the empty set, a contradiction. □

Remark 1.5. As the degree may increase when adding algebraic parameters, X may not have ordinal Cantor rank over the algebraic closure of a. We shall call *local Cantor rank of X over $acl(a)$* its Cantor rank over any parameter b defining X and having the same algebraic closure as a.

Remark 1.6. Lemmas 1.1 and 1.4 are both corollaries of the following topological result, the proof of which follows exactly the one of Lemma 1.1 : *if X and Y are two compact Hausdorff topological spaces and f is an open continuous n-to-one surjection from X onto Y, then X and Y have the same Cantor-Bendixson rank and the inequalities $dCB(Y) \leq dCB(X) \leq n \cdot dCB(Y)$ hold.* Let R be a continuous equivalence relation on a topological space X, that is a relation such that the canonical map $X \to X/R$ be open. If every equivalence class of R has size at most n, as the map $X \to X/R$ is also continuous by definition, it follows that X has the same Cantor-Bendixson rank as the quotient space X/R and the inequalities $dCB(X) \leq dCB(X/R) \leq n \cdot dCB(X)$ hold. Applied to the space of types over the algebraic parameter a of a given structure M, modulo the equivalence relation "to be conjugated under the action of the automorphisms group of M fixing a", the latter yields Lemma 1.4.

2. General facts about weakly small groups

As an immediate corollary of Lemma 1.1 we obtain a result of Wagner :

Corollary 2.1. (Wagner [23]) *If f is a definable group homomorphism of a weakly small group G, the kernel of which has at most n elements, then $f(G)$ has index at most n in G.*

Proof. Otherwise, one can find a finite tuple a over which at least $n+1$ cosets of $f(G)$ are definable, so G has degree over a at least $(n+1) \cdot dCB_a(f(G))$, a contradiction with Lemma 1.1. □

Corollary 2.2. *In a weakly small group, there are at most n conjugacy classes of elements the centraliser of which has order at most n.*

Proof. Otherwise, let us pick $n+1$ conjugacy classes $C_1, \ldots, C_{n+1}$ of elements the centraliser of which has order at most n, and choose a finite tuple a over which these classes are definable. According to Lemma 1.1, each class C_i has maximal Cantor rank over a and degree at least $dCB_a(G)/n$, a contradiction. □

For any set X definable in an omega-stable group, one can define the stabiliser of X up to some small Morley ranked set. In a weakly small group, we can define a local stabiliser, where local means "in a finitely generated algebraic closure".

Definition 2.3. Let X be a definable set without parameters in a weakly small group G, and let Γ stand for the algebraic closure of a finite tuple g in G. One defines the *local almost stabiliser* of X in Γ to be

$$Stab_\Gamma(X) = \{x \in \Gamma : CB_{x,g}(xX\Delta X) < CB_g(X)\}$$

Corollary 2.4. *$Stab_\Gamma(X)$ is a subgroup of Γ. If X is invariant under conjugation by Γ, then $Stab_\Gamma(X)$ is normal in Γ.*

Proof. Let a and b be in $Stab_\Gamma(X)$. The sets X, aX and bX have the same types of maximal rank computed over g, a, b, so $CB_{g,a,b}(aX\Delta bX)$ is smaller than $CB_g(X)$. As the rank is preserved under definable bijections, and when adding algebraic parameters,

$$CB_{g,a,b}(aX\Delta bX) = CB_{g,a,b}(b^{-1}aX\Delta X) = CB_{g,b^{-1}a}(b^{-1}aX\Delta X)$$

so $b^{-1}a$ belongs to $Stab_\Gamma(X)$. □

Recall that for a definable generic set X of an omega-stable group G, the stabiliser of X has finite index in G. For a weakly small group, we have a local version of this fact :

Proposition 2.5. *Let G be a weakly small group, g a finite tuple of G, and X a g-definable subset of X. If δ is a subgroup of $dcl(g)$ and if X has maximal Cantor rank over g, then $Stab_\delta(X)$ has finite index in δ.*

Proof. Let m and l be the degree of G, and X. There are m types of maximal rank in G, which we call its *generic* types. Thus, for a translate of X by an element of δ, there are at most C_m^l choices for its generic types. If one chooses $C_m^l + 1$ cosets of X, at least two of them have the same generic types. □

Weakly small groups definable over a finitely generated algebraic closure satisfy a local descending chain condition :

Lemma 2.6. *Let G be a weakly small group, and $H_2 \leq H_1$ two subgroups of G definable without parameters. If $H_2 \cap acl(\emptyset)$ is properly contained in $H_1 \cap acl(\emptyset)$, then either $CB(H_2) < CB(H_1)$, or $dCB(H_2) < dCB(H_1)$. If H_1 and H_2 have same Cantor rank, then $H_2 \cap acl(\emptyset)$ has finite index in $H_1 \cap acl(\emptyset)$.*

Proof. If b is an element of $acl(\emptyset)$ in $H_1 \setminus H_2$, the finite union $\overline{b.H_2}$ of conjugates of $b.H_2$ under the action of automorphisms group of G is definable without parameters, and is disjoint from H_2. If H_1 and H_2 have the same Cantor rank, $\overline{b.H_2}$ has maximal Cantor rank in H_1, so there must be only finitely many choices for $\overline{b.H_2}$, and thus for $b.H_2$. □

Theorem 2.7. *In a weakly small group, the trace over $acl(\emptyset)$ of a descending chain of $acl(\emptyset)$-definable subgroups becomes stationary after finitely many steps.*

Proof. Let $G_1 \geq G_2 \geq \dots$ be a descending chain of $acl(\emptyset)$-definable subgroups. After finitely many steps, the local Cantor rank becomes constant according to Lemma 1.4, so we may assume it is constant. Then $G_i \cap acl(\emptyset)$ has finite index in $G_1 \cap acl(\emptyset)$ for every i after Lemma 2.6. We may add finitely many algebraic parameters to the language and assume that G_1 be definable over the empty set. We write $\mathring{G}_i \cap acl(\emptyset)$ for the intersection of conjugates of $G_i \cap acl(\emptyset)$ under the action of the automorphisms group of the structure. The intersection of the $\mathring{G}_i \cap acl(\emptyset)$ is the intersection of finitely many of them by Lemma 2.6 : it is a subgroup of $G_0 \cap acl(\emptyset)$ the index of which is finite, and which belongs to every G_i. The sequence of indexes $[G_0 \cap acl(\emptyset) : G_i \cap acl(\emptyset)]$ is thus bounded, and bounds the length of the chain $G_1 \cap acl(\emptyset) \geq G_2 \cap acl(\emptyset) \geq \dots$. □

Remark 2.8. We shall call this result *the weakly small chain condition.* Note that Theorem 2.7 is trivial for an $\aleph_0$-categorical group, as well as if one replaces the algebraic closure by the definable closure. Also, the same proof shows that in a weakly small structure, the trace over $acl(\emptyset)$ of a descending chain of $acl(\emptyset)$-definable equivalence relations becomes constant, provided that all two classes of each relation be in definable bijection.

3. A property of small groups

Proposition 3.1. *An infinite group whose centre has infinite index, and with only one non-central conjugacy class, is not weakly small.*

Remark 3.2. This is the analogue of the stable case [16, Théorème 3.10] stating that *an infinite group with only one non-trivial conjugacy class is unstable*, which itself comes from the minimal case [20, Reineke].

Proof. Note that the group has no second centre. Moding out the centre, we may suppose that the centre is trivial. If there is a non-trivial involution, every element is an involution and the group is abelian, a contradiction. Any non-trivial element g is conjugated to g^{-1} by some element, say h. So h is non-trivial and conjugated to h^2, which equals h^k for some k. Write δ for the definable closure of h and k. The chain

$$C(C(h)) \cap \delta > C(C(h^k)) \cap \delta > C(C(h^{k^2})) \cap \delta > \cdots$$

is infinite, contradicting the weakly small chain condition 2.7. □

Proposition 3.3. *An infinite non-abelian weakly small group has arbitrary large proper centralisers.*

Proof. For a contradiction, suppose all the centralisers be finite of bounded size. Note that the group has finite exponent, for otherwise it would have arbitrary large proper centralisers.

(1) The group G has finitely many conjugacy classes. As the centralisers have bounded size, we apply Corollary 2.2. We may add a member a_i of each class to the language.

(2) We may assume every proper normal subgroup of G to be trivial. A normal subgroup must be central or have finite index in G. One may replace G by $H \cup \{1\}$, where H is a minimal union of conjugacy classes stable under multiplication.

(3) G is not locally finite. A group of bounded exponent cannot have finitely generated subgroups of arbitrary large finite size, as the size of a Sylow of every finite subgroup is bounded (a Sylow subgroup has a non-trivial centre, the centraliser of any element of which contains the whole Sylow). Moreover, as the group has finite exponent, the number of Sylow subgroups of a finite group is bounded. So, let Γ be a finitely generated infinite algebraic closure.

(4) Γ has finitely many conjugacy classes. Every x in Γ can be written a_i^y, with y algebraic over a_i and x as the centralisers are finite.

(5) One may assume the proper normal subgroups of Γ to be trivial. No proper sub-union of conjugacy classes of G is stable by multiplication. We may add finitely many parameters witnessing this fact to the language.

(6) For every conjugacy class a^G, the group $Stab_\Gamma(a^G)$ equals Γ. The local stabiliser of a^G in Γ is a normal subgroup of Γ, non-trivial by Proposition 2.5.

(7) G has only one non-central conjugacy class. We use an argument of Poizat in [17], which we shall call *Poizat's symmetry argument*. Let a and b be non-central elements. For every conjugate xbx^{-1} of b except a set of small Cantor rank over a and b, the elements $axbx^{-1}$ and b are conjugates. As a surjection with bounded fibres preserves the rank, for all x except a set of small rank, $axbx^{-1}$ and b are conjugates. Symmetrically, for all x except a set of small rank, $x^{-1}axb$ and a are conjugates : one can find some x such that $axbx^{-1}$ and $x^{-1}axb$ are conjugated respectively to b and a. Thus, b and a lie in the same conjugacy class.

(8) Final contradiction. G is a group with bounded exponent and only one non-trivial conjugacy class. Such a group does not exist [20, 17, Reineke]. For instance, as a group of exponent 2 is abelian, the group should have exponent a prime $p \neq 2$. If $x \neq 1$, the elements x and x^{-1} would be conjugated by some element y of order 2 modulo the centraliser of x, which prevents the group from having exponent p. □

Theorem 3.4. *A small infinite omega-saturated group has an infinite abelian subgroup.*

Proof. If it is not abelian, by compactness and saturation, it has an infinite proper centraliser. Iterating, one either ends on an infinite abelian centraliser, or builds an infinite chain of pairwise commuting elements. □

Appealing to Hall-Kulatilaka-Kargopolov, who use Feit-Thomson's Theorem, one can say much more. Recall that Hall-Kulatilaka-Kargopolov state that *an infinite locally finite group has an infinite abelian subgroup* [10].

Theorem 3.5. *An infinite weakly small group has an infinite abelian subgroup.*

Proof. Let G be a counter-example. We may assume that its centre is trivial. According to Hall-Kulatilaka-Kargopolov, the group G is not locally finite. An infinite finitely generated subgroup γ splits into finitely many conjugacy classes (in the sense of G). The almost stabiliser of every class is a normal subgroup of finite index in γ. After Poizat's symmetry argument, it is the union of the neutral element and only one class C, which we shall call the *privileged class*, as it is the same for every finitely generated group γ.

Let g be an element in the algebraic closure Γ of γ. The almost stabiliser of g^G in Γ is a normal subgroup of Γ containing one element of the privileged class C ; as two elements in $C \cap \Gamma$ are conjugated in Γ, this almost stabiliser contains $C \cap \Gamma$. It follows that the almost stabiliser of every conjugacy class of elements in Γ is $C \cap \Gamma$ plus the neutral. As this holds for every finitely generated Γ, the set $C \cup \{1\}$ is a subgroup of G. Replacing G by the later, we are back to the case where all centralisers are bounded, a contradiction with Proposition 3.3. □

Remark 3.6. The initial proof of Theorem 3.4 used Hall-Kulatilaka-Kargopolov. The author is grateful to Poizat who adapted the proof to a weakly small group and made clarifying remarks.

Remark 3.7. One cannot expect the infinite abelian group to be definable, as Plotkin found infinite $\aleph_0$-categorical groups without infinite definable abelian subgroups [15].

4. Small nilpotent groups

We now switch to small nilpotent groups. Let us first recall that the structure of small abelian pure groups is already known :

Fact 4.1. (Wagner [24]) *A small abelian group is the direct sum of a definable divisible group with one of bounded exponent.*

Remark 4.2. This does not hold for a weakly small abelian group : consider the sum over n of cyclic groups of order p^n. One just has that for every n, any element is the sum of an n-divisible element with one of finite order.

Remark 4.3. Since Prüfer and Baer, one knows that a divisible abelian group is isomorphic to direct sums of copies of $\mathbf{Q}$ and Prüfer groups, whereas an abelian group of bounded exponent is isomorphic to a direct sum of cyclic groups [9]. It follows that the theory of a small pure group has countably many denumerable pairwise non-isomorphic models ; thus, Vaught's conjecture holds for the theory of a pure abelian group. More generally, Vaught's conjecture holds for every complete first order theory of module over a countable Dedekind ring (and thus for a module over $\mathbf{Z}$), as well as for several classes of modules over countable rings [19, Puninskaya].

In an abelian group, every divisible group is a direct summand [2, Theorem 1]. This may not be true for a central divisible subgroup of an arbitrary group, even if the ambient group is nilpotent. For instance, consider the subgroup of $GL_3(\mathbf{C})$ the elements of which are upper triangular matrices with one entries on the main diagonal ; it is a nilpotent group whose center Z is divisible, isomorphic to $\mathbf{C}^{\times}$, but Z is no direct summand. However, we claim the following :

Proposition 4.4. *Let G be a group, and D a divisible subgroup of the centre. There exists a subset A of G, invariant under conjugation and containing every power of its elements, with in addition*

$$G = D \cdot A \quad \textit{and} \quad D \cap A = 1$$

Proof. If $A_1 \subset A_2 \subset \cdots$ is an increasing chain of subsets each of which contains all its powers and such that $A_i \cap D$ be trivial, then $\bigcup A_i$ still contains all its powers and $\bigcup A_i \cap D$ is trivial too. By Zorn's Lemma there is a maximal subset A with these properties. We show that $D \cdot A$ equals G. Otherwise, there exists an x not in $D \cdot A$. By maximality of A, there is an integer n greater than one, and some d in D

so that x^n equal d. We may chose n minimal with this property. Let e be an nth root of d^{-1} in D, and let y equal xe. Then y^n equals one, and y does not belong to $D \cdot A$ either. But the set of powers of y intersects D : there is some integer $m < n$ such that y^m lie in D, a contradiction. $\square$

In [13, Nesin], it is shown that an omega-stable nilpotent group is the central product of a definable group with one of bounded exponent. We show that this also holds for a small nilpotent group. Recall that a group G is *the central product* of two of its subgroups, if it is the product of these subgroups and if moreover their intersection lies in the centre of G. For a group G and a subset A of G, we shall write A^n for the set of nth-powers of A, and G' for the derived subgroup of G. The following algebraic facts about nilpotent groups can be found in [5, Chapter 1].

Fact 4.5. *In a nilpotent group, any divisible subgroup commutes with every element of finite order.*

Fact 4.6. *A nilpotent group G is divisible if and only if G/G' is so.*

Proposition 4.7. *Let G be a nilpotent small group, and D a divisible subgroup containing G^n. Then G equals the product $D \cdot F$ where F has bounded exponent.*

Proof. By induction on the class of nilpotency of G. If G is abelian, Baer Theorem [2] concludes. Suppose that the result hold for any small nilpotent group of class c, suppose that G be nilpotent of class $c+1$, and let $Z(G)$ be the center of G. The group $G/Z(G)$ is nilpotent of class c. The quotient $D \cdot Z(G)/Z(G)$ is a divisible subgroup and contains $(G/Z(G))^n$. By induction, $G/Z(G)$ equals the product $(D \cdot Z(G)/Z(G)) \cdot (C/Z(G))$ with $C/Z(G)$ of finite exponent, say m. On the other hand, the centre is the sum of a divisible subgroup D_0 with a subgroup F_0 of finite exponent,say l, so C^{lm} is included in D_0. By Proposition 4.4, D_0 is a "direct summand" of C by some set A say ; but A^{lm} is included in $D_0 \cap A$, so A has finite exponent, and

$$G = D \cdot Z(G) \cdot D_0 \cdot A = (D \cdot D_0) \cdot (F_0 \cdot A)$$

Note that D contains D_0, so G equals $D \cdot B$ where B is a set having finite exponent. As the Sylow p-subgroups are normal in G, the subgroup generated by B has bounded exponent. $\square$

Theorem 4.8. *A small nilpotent group is the central product of a definable divisible group with one of bounded exponent.*

Proof. If the group is abelian, the result follows from Fact 4.1. If G is nilpotent of class $c+1$, then $G/Z(G)$ is the central sum of some divisible $A/Z(G)$ and some $B/Z(G)$ of exponent n. Besides, $Z(G)$ equals $D_0 \oplus F_0$ where F_0 has exponent m. The product $A^m \cdot D_0$ is a divisible part that we call X. Let D be the subgroup generated by X. The abelian group D/D' generated by $X \cdot D'/D'$ is divisible, and so is D by Fact 4.6 ; so D equals X. As G^{mn} is included in D, we may apply Proposition 4.7 and conclude by Fact 4.5. $\square$

5. Groups in a small and simple theory

We shall not define what a simple structure is, but refer the interested reader to [25, Wagner]. We just recall the uniform descending chain condition up to finite index in a group with simple theory :

Fact 5.1. (Wagner [25, Theorem 4.2.12]) *In a group with simple theory, let $H_1, H_2 \dots$ be a family of subgroups defined respectively by formulae $f(x, a_1), f(x, a_2) \dots$ where $f(x, y)$ is a fixed formula. Any descending chain of definable intersections of H_i becomes stationary after finitely many steps, up to finite index.*

Recall that two subgroups of a given group are *commensurable* if the index of their intersection is finite in both of them.

Fact 5.2. (Schlichting [21, 25]) *Let G be a group and $\mathfrak{H}$ a family of uniformly commensurable subgroups. There is a subgroup N of G commensurable with members of $\mathfrak{H}$ and invariant under the action of the automorphisms group of G stabilising the family $\mathfrak{H}$ setwise. If the members of $\mathfrak{H}$ are definable, so is N. More precisely, N is a finite extension of a finite intersection of elements in $\mathfrak{H}$.*

We now turn to small simple groups. The first step is to appeal to Theorem 3.5. Note that in a stable group, every set of pairwise commuting elements is trivially contained in a definable abelian subgroup. Shelah showed that this also holds in a dependent group [22]. The second step is the following :

Proposition 5.3. *In a group with simple theory, every set of pairwise commuting elements is contained in a definable finite-by-abelian subgroup.*

Proof. As the hypothesis and conclusion are properties of the theory, we may assume the group to be sufficiently saturated. Let G be this group, and A this abelian set. By Fact 5.1, there exists a finite intersection H of centralisers of elements in A such that H be minimal up to finite index. The group H contains A, and the centraliser of every element in A has finite index in H. Consider the almost centre $Z^*(H)$ of H consisting of elements in H the centraliser of which has finite index in H. It is a subgroup containing A. According to [25, Lemma 4.1.15], a definable subgroup B of G has finite index in G if and only if the equality $D_G(B, \varphi, k) = D_G(G, \varphi, k)$ holds for every formula φ and integer k. So we have the following equality

$$Z^*(H) = \{h \in H : D_G(C_H(h), \varphi, k) \geq D_G(H, \varphi, k), \varphi \text{ formula}, k \text{ integer}\}$$

Recall that for a partial type $\pi(x, A)$, the sentence "$D_G(\pi(x, A), \varphi, k) \geq n$" is a type-definable condition on A [25, Remark 4.1.5], so the group $Z^*(H)$ is type-definable. By compactness and saturation, every centraliser has bounded index in H, and every conjugacy class of H is boundedly finite. So $Z^*(H)$ is definable. According to [14, Theorem 3.1], the derived subgroup of $Z^*(H)$ is finite. □

Corollary 5.4. *An infinite group the theory of which is small and simple has an infinite definable finite-by-abelian subgroup.*

Proof. Follows from Theorem 3.4 and Proposition 5.3. □

Remark 5.5. The result is best possible as there are omega-categorical simple groups without infinite abelian definable subgroups. For instance, infinite extraspecial groups of exponent p are omega-categorical [8, Felgner], supersimple of SU rank one as they can be interpreted in an infinite dimensional vector space over $\mathbf{F}_p$ endowed with a non degenerate skew-symmetric bilinear form. They have no infinite definable subgroup by [15, Plotkin].

Corollary 5.6. *A small supersimple group of SU rank one is finite-by-abelian-by-finite.*

As noticed by Aldama in his thesis [1], Shelah's result concerning abelian subsets of a dependent group extends to a nilpotent subset of a dependent group. In a group definable in a simple theory, we may say the following :

Definition 5.7. A group G is *almost solvable of class* n if there is a decreasing sequence of subgroups G_i so that $G_0 = G \trianglerighteq G' \trianglerighteq G_1 \trianglerighteq G_1' \cdots \trianglerighteq G_n' \trianglerighteq G_{n+1} = \{1\}$ and so that the indexes $[G_i' : G_{i+1}]$ be finite for all i.

An almost solvable group of class zero is a finite-by-abelian group.

Corollary 5.8. *In a group with simple theory, let n be an integer, and A be a solvable subgroup of class n. Then A is contained in a definable almost solvable group of class n.*

Proof. By induction on the integer n, let us show that there is some definable almost solvable group N of class n containing A. When n equals zero, this is Proposition 5.3. Suppose that the result hold until $n-1$. By induction hypothesis, the group A' is contained in a definable almost solvable group H of class $n-1$. We shall now use an argument of Wagner in [11]. Let I be an intersection of conjugates of H by elements in A, minimal up to finite index. I does exist by Fact 5.1. Let us write $\mathfrak{H}$ for the set of conjugates of I by elements in A. After Fact 5.2, there is a group N commensurable with I and invariant by conjugation under elements of A. Moreover, N is a finite extension of a finite intersection of elements in $\mathfrak{H}$, so it contains A', and is a finite extension of an almost solvable group of class $n-1$. The group NA/A is abelian : according to the preceeding Proposition, there is a definable group M so that

$$NA/A \leq M/N \leq N_G(N)/N$$

where M'/N is finite. Thus, M is almost solvable of class n. □

Question. We may define a group G to be *almost nilpotent of class* n if there exists a decreasing sequence of subgroups G_i so that $G_0 = G \trianglerighteq G_1 \trianglerighteq G_2 \cdots \trianglerighteq G_{n+1} = \{1\}$ and such that the indexes $[[G, G_i] : G_{i+1}]$ be finite for all i. In a group with simple theory, is any nilpotent subgroup of class n contained in a definable almost nilpotent group of class n?

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