ON A CLASSIFICATION OF THEORIES WITHOUT THE INDEPENDENCE PROPERTY

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Abstract. A theory is stable up to Δ if any Δ-type over a model has a few extensions up to complete types. I prove that a theory has no the independence property iff it is stable up to some Δ, where each $\varphi(x;\bar{y}) \in \Delta$ has no the independence property. Definability of one-types over a model of a stable up to Δ theory is investigated.

1. Introduction

Stable theories have a few types. This fact has a very important consequent: each type has a finite local rank and is definable, so it is possible to classify all types. Unstable theories have too many types to classify them. What can we do? Each time when a problem is too complicated to be solved, we can partition this problem into several more simple subproblems. Let us apply this general principle to unstable theories. We can partition the set of all 1-types into small pieces. If some piece has a few types we obtain stability inside this piece. And if a chosen piece will be quite tame, then types from this piece will have locally a local rank[1]. Possibly, a partition can be of any origin, but the most natural is the following: two types are equivalent iff their Δ-parts are equal, where Δ is a collection of formulae. Thus we obtain a notion of a stable up to Δ theory: each equivalence class is small. Sufficiently closed notion was suggested by D. Macpherson and Ch. Steinhorn in [4]. Suppose $\mathcal{L} \subset \mathcal{L}^+$ are languages, and $\mathcal{K}$ is an elementary class of $\mathcal{L}$-structures. We say that an $\mathcal{L}^+$-structure $\mathcal{M}$ is $\mathcal{K}$-minimal if the reduct $\mathcal{M} \upharpoonright \mathcal{L} \in \mathcal{K}$ and every $\mathcal{L}^+$-definable subset of M is definable by a quantifier-free $\mathcal{L}$-formula. A complete $\mathcal{L}^+$-theory is $\mathcal{K}$-minimal if all its models are. Indeed, let Δ consist of atomic $\mathcal{L}$-formulae. Then any Δ-type over a model has a unique extension up to a complete type. Moreover, a Δ-type is definable in $\mathcal{L}^+$ iff it is definable in $\mathcal{L}$. So, if we replace 'a unique extension' with 'a few extensions' we obtain stability up to Δ. If we add the second property: a Δ-type is definable iff it is definable in $\mathcal{L}$, then a notion of strong stability up to Δ arises. O-stability, which is a partial case of stability up to Δ, has been developed in [2, 7].

In the paper I proved that a theory T does not have the independence property iff it is stable up to some Δ, where each formula in Δ does not have the independence propertry (Theorem 1.12). In the second part of the paper I investigate definability of 1-types over a model with a stable up to Δ theory and introduce a variant of the φ-rank (Definition 2.2). In particular, I prove that a one-type over an $\aleph_0$-saturated model with a stable up to Δ theory is definable iff its Δ-part is definable (Theorem 2.6). If a theory does not have the independence property, then the condition of $\aleph_0$-saturatedness can be omitted.

[1]That is φ-rank inside this piece of a type from this piece is finite.

Notation 1.1. Let s be a partial n-type, A a set, Δ a collection of formulae in n free variables. Then

$$S^n_{\Delta,s}(A) \triangleq \{p \in S^n_\Delta(A) : p \cup s \text{ is consistent}\}$$

If $\Delta = \mathcal{L}$ I omit it and write S^n_s. Note, s need not be a type over A.

Definition 1.2. Let $\mathcal{M}$ be an arbitrary structure, $A \subseteq M$. Let Δ and ∇ be sets of formulae of the form $\varphi(x; \bar{y})$.

(1) The model $\mathcal{M}$ is *stable up to* Δ in (λ, ∇) if for all $A \subseteq M$ with $|A| \leq \lambda$, for any Δ-type p over M there are at most λ ∇-types over A which are consistent with p, i.e. $|S^1_{\nabla,p}(A)| \leq \lambda$.
(2) The theory T is *stable up to* Δ in (λ, ∇) if every model of T is. Sometimes I write T is (λ, ∇)-stable up to Δ.
(3) If $\nabla = \mathcal{L}$ I omit it and write that T is stable in λ or λ-stable up to Δ.
(4) T is *stable up to* Δ if there exists a λ in which T is stable up to Δ. I write T is *stable up to* φ meaning that T is stable up to $\Delta = \{\varphi\}$.
(5) T is *superstable up to* Δ if there exists a λ such that T is stable up to Δ in all $\mu \geq \lambda$.
(6) Let $\varphi(x; y) \triangleq x < y$ and T contain axioms saying that $<$ is a total order. If T is stable up to φ, then T is said to be *o-stable*.

In a similar way it is possible to define stability up to, say, $\varphi(x_1, x_2; \bar{y})$, but I am not sure that starting investigating types from two-types (not from one-types) is a good idea.

Obviously, if T is λ-stable up to Δ and (λ, Δ)-stable, then T is λ-stable. If Δ consists of a single formula $\varphi(x; y) \triangleq (x = y)$, then stability up to Δ is equivalent to stability.

Recall that a formula $\varphi(\bar{x}, \bar{y})$ has *the order property* if there are sequences $\bar{a}_n$ and $\bar{b}_m$ for $n, m \in \omega$ such that $\varphi(\bar{a}_n, \bar{b}_m)$ holds iff $n \leq m$.

Definition 1.3. A formula $\varphi(\bar{x}; \bar{y})$ has *the order property over B inside a partial type $s(\bar{x})$* over a set A if the following formula (in an extended language)

$$\varphi(\bar{x}; \bar{y}) \wedge \bigwedge_i S(x_i) \wedge \bigwedge_j B(y_j)$$

has the order property where the unary predicate S names the set of all realization of the type s in some $|A|^+$-saturated model $\mathcal{N}$.

A theory T has *the order property in spite of* Δ if there is a model $\mathcal{M}$ of T and a Δ-type $s(\bar{x})$ over M such that some formula $\varphi(\bar{x}; \bar{y})$ has the order property over M inside the type s.

Similarly one can define *the strict order property inside a partial type* and *the strict order property in spite of* Δ, *the independence property inside a partial type*, and *the independence order property in spite of* Δ.

Lemma 1.4. *A theory T is stable up to Δ iff T has no the order property in spite of Δ.*

Proof. Let a theory T of a language $\mathcal{L}$ is not stable up to Δ. Then for any cardinality λ there is a model $\mathcal{M}$ of T and a Δ-type s over M which has more than λ extensions up to complete types. Let $\lambda = 2^{|\mathcal{L}|}$. Then there is a formula $\psi(\bar{x}, \bar{y})$ such that more than λ different ψ-types over M are consistent with s. Indeed, otherwise the number of types consisting with s would be restricted by $(2^{|\mathcal{L}|})^{|\mathcal{L}|} = 2^{|\mathcal{L}| \times |\mathcal{L}|} = 2^{|\mathcal{L}|} = \lambda$.

Let $\mathcal{N}$ be an $|M|^+$-saturated elementary extension of $\mathcal{M}$, a predicate S name the set of all realizations of the type s and P name the set M. Let

$$\psi^+(\bar{x}, \bar{y}) \triangleq \psi(\bar{x}; \bar{y}) \wedge \bigwedge_i S(x_i) \wedge \bigwedge_j P(y_j)$$

Obviously there are more than λ ψ^+-types over M which are consistent with the type s. Then ψ^+ has the order property and T has the order property in spite of Δ.

Now let T have the order property in spite of Δ. Then there are a pair $(\mathcal{N}, \mathcal{M})$ of models of T, a Δ-type s over M and a formula ψ such that ψ has the order property over M inside s. Let T^+ be the elementary theory of $(\mathcal{N}, P, S)$, where P names the set M and S names $s(\mathcal{N})$. If s is not definable in T I extend the language so that s in T^+ becomes definable. The formula ψ^+ has the order property in T^+. Then for any cardinality λ there is a model $(\mathcal{N}_1, P, S)$ with more than λ ψ^+-types and with $P(\mathcal{N}_1)$ of cardinality λ. Let $M_1 = p(\mathcal{N}_1)$. Then $\mathcal{M}_1 \prec \mathcal{N}_1$. In each ψ^+-type I can replace the formula ψ^+ with the formula ψ. So there are more than λ ψ-types over M which are consistent with $S(\mathcal{N}_1)$, and $S(\mathcal{N}_1)$ implies some Δ-type s_1 over M_1, because s_1 is definable. Hence $\mathcal{M}_1$ is not λ-stable up to Δ. Since λ is arbitrary, the theory T is not stable up to Δ. $\square$

By other words Lemma 1.4 says that if T is stable up to Δ then for any formula ψ the formula ψ^+ in the extended language is stable. This provides a regular way of constructing stable formulae in the extended language. Claim that if a theory T has no the independence property, then the formula ψ^+ from Lemma 1.4 has no the independence property for any formula ψ.

Corollary 1.5. *A theory T is stable up to Δ iff for any model $\mathcal{M}$ of T, for any formula $\psi(\bar{x}; \bar{y})$ and for any Δ-type s over M the number of ψ-types over M which are consistent with s is at most $|M|$.*

Proof. If there are a model $\mathcal{M}$ of T, a formula $\psi(\bar{x}; \bar{y})$ and a Δ-type s over M such that the number of ψ-types over M which are consistent with s is bigger than $|M|$, then ψ^+ has the order property and T is not stable up to Δ.

If T is not stable up to Δ then for some formula ψ the formula ψ^+ has the order property and the corollary follows. $\square$

Standard compactness's arguments allow to give a 'localization' of the order property: a formula $\varphi(\bar{x}, \bar{y}; \bar{z})$ has the local order property if for any natural number k there are a tuple $\bar{c}_k$ and sequences $\bar{a}_n$ and $\bar{b}_m$ for $n, m < k$ such that $\varphi(\bar{a}_n, \bar{b}_m; \bar{c}_k)$ holds iff $n \leq m$. But for stability up to Δ the equivalence of the local order property in spite of Δ and of the order property in spite of Δ is not so immediate. The problem is the following: realizing $\bar{c}_\omega$ which gives infinite sequences of $\bar{a}_n$ and $\bar{b}_m$ I should consider a Δ-type over a new model. The Δ-type over the old model can have several extensions, and probably, in each extension a given formula does not have the order property.

Lemma 1.6. *A theory T has the order property in spite of Δ if there is a formula $\psi_{\bar{z}}(\bar{x}; \bar{y}) = \psi(\bar{x}, \bar{y}, \bar{z})$ such that for some model $\mathcal{M}$ of T and for any natural number k there are a Δ-type s_k over M, a tuple $\bar{c}_k \in M$, and sequences $\bar{a}_n \models s_k$ and $\bar{b}_m \in M$ for $n, m < k$ such that $\psi_{\bar{c}}(\bar{a}_n; \bar{b}_m)$ holds iff $n \leq m$.*

Proof. Assume that there is a formula $\psi_{\bar{z}}(\bar{x};\bar{y}) = \psi(\bar{x},\bar{y},\bar{z})$ such that for some model $\mathcal{M}$ of T and for any natural number k there are a Δ-type s_k over M, a tuple $\bar{c}_k \in M$, and sequences $\bar{a}_n \models s_k$ and $\bar{b}_m \in M$ for $n, m < k$ such that $\psi_{\bar{c}}(\bar{a}_n;\bar{b}_m)$ holds iff $n \leq m$, and let $\mathcal{N}$ be an $|M^+|$-saturated elementary extension of $\mathcal{M}$. I shall consider a pair of models $(\mathcal{N},\mathcal{M})$, where $P(\mathcal{N}) = M$. Let the cardinality of M be λ. I extend the language $\mathcal{L}$ of T to $\mathcal{L}^+$ adding a new relation $R(\bar{x}, z, \bar{u})$ and making each type s_k definable. Let $\{\theta_{k,\alpha} : \alpha < \lambda\}$ be an enumeration of all finite conjunctions of formulae in s_k and $\{d_\alpha : \alpha < \lambda\}$ an enumeration of all elements in M. I define the predicate R as follows: $R(M, d_\alpha, \bar{c}_k) = \theta_{k,\alpha}(M)$ and if $\bar{c} \neq \bar{c}_k$ for any k then $\mathcal{M} \models \forall \bar{x}, z \neg R(\bar{x}, z, \bar{c})$. Then the defined below partial type $p(\bar{u}, \bar{x}_i, \bar{y}_i : i < \omega)$ is finitely realizable in $\mathcal{M}$. The type p consists of the following formulae: $\psi(\bar{x}_i, \bar{y}_j, \bar{u})$ holds iff $i \leq j$, $P(\bar{y}_i)$ for each $i < \omega$, $P(\bar{u})$, and $\forall t(P(t) \rightarrow R(\bar{x}_i, t, \bar{u}))$. The last scheme of formulae provides that all $\bar{x}_i$ realizes the same Δ-type over a small model in the elementary theory of the pair $(\mathcal{N},\mathcal{M})$. Thus $\psi(\bar{x},\bar{y},\bar{c}_\omega)$ has the order property inside some Δ-type, where c_ω realizes $(\exists \bar{x}_i \bar{y}_i : i < \omega) p(\bar{u}, \bar{x}_i, \bar{y}_i : i < \omega)$. $\square$

The inverse direction of Lemma 1.6 requires some more advanced analysis and it is still an open question, because standard compactness works with formulae and here it is necessary to find a formula whose intersection with a type(!) has the order property.

Question 1. Do the inverse direction of Lemma 1.6 hold?

By standard compactness arguments it is possible to prove a little bit stronger version of Lemma 1.6

Lemma 1.7. *A theory T has the order property in spite of Δ if there is a formula $\psi_{\bar{z}}(\bar{x};\bar{y}) = \psi(\bar{x},\bar{y},\bar{z})$ such that for some model $\mathcal{M}$ of T and for any natural number k and for any finite $\Delta_0 \subseteq \Delta$ there are a Δ_0-type s_{k,Δ_0} over M, a tuple $\bar{c}_{k,\Delta_0} \in M$, and sequences $\bar{a}_{n,\Delta_0} \models s_{k,\Delta_0}$ and $\bar{b}_{m,\Delta_0} \in M$ for $n, m < k$ such that $\psi_{\bar{c}}(\bar{a}_{n,\Delta_0};\bar{b}_{m,\Delta_0})$ holds iff $n \leq m$.*

Claim that by compactness a formula $\varphi(x;\bar{y})$ has no order property in spite of Δ iff for some finite $\Delta_\varphi \subseteq \Delta$ the formula $\varphi(x;\bar{y})$ has no order property in spite of Δ_φ. So the next property is an obvious corollary of Lemma 1.7 and compactness.

Lemma 1.8. *If a theory T is stable up to Δ then for any formula $\varphi(x;\bar{y})$ there is a finite $\Delta_\varphi \subseteq \Delta$ such that T is φ-stable up to Δ_φ.*

Definition 1.9 (S. Shelah)**.** Let T be an $\mathcal{L}$-theory and $\phi(\bar{x},\bar{y})$ a formula. The formula ϕ is said to have the *independence property* (relatively T) if for all $n < \omega$ there is a model $\mathcal{M} \models T$ and two sequences $(\bar{a}_i : i < n)$ and $(\bar{b}_J : J \subseteq n)$ in M such that $\mathcal{M} \models \phi(\bar{a}_i, \bar{b}_J)$ if and only if $i \in J$. A theory T has the independence property if some formula has the independence property.

I shall use the following well-known facts.

Fact 1.10. *Let T be a theory and $\phi(\bar{x},\bar{y})$ be a formula. Then the following are equivalent:*

(1) *$\phi(\bar{x},\bar{y})$ has the independence property relatively T.*
(2) *There exists an indiscernible sequence $(\bar{a}_i : i \in I)$ in some model $\mathcal{M}$ of T and some tuple $\bar{b} \in M$ such that $\mathcal{M} \models \phi(\bar{a}_i, \bar{b})$ if and only if i is even.*

Fact 1.11. *A theory T has the independence property, i.e. there is a formula $\psi(\bar{x}, \bar{y})$ with the independence property iff there is a formula $\phi(x, \bar{z})$ with the independence property.*

Our aim is to prove the following theorem:

Theorem 1.12. *A theory T has no the independence property iff T is stable up to some Δ, where each $\varphi(x;\bar{y}) \in \Delta$ has no the independence property.*

Proof. The direction '$\Rightarrow$' is simple. Let Δ consist of all formulae $\varphi(x;\bar{y})$ with the strict order property and let $\mathcal{M}$ be a model of T of cardinality $\lambda = 2^{\mu}$ for some $\mu \geq \max\{\aleph_0, |T|\}$. If some Δ-type s has more than λ extensions up to complete types then by the standard counting type procedure I obtain that there is a formula $\psi(x;\bar{y})$ such that there are more than λ ψ-types which are consistent with s. Then ψ has order property. Since T has no the independence property, ψ must have the strict order property. Then $\psi \in \Delta$, for a contradiction.

Now I prove the inverse direction. A partial case of Theorem 1.12 saying that o-stable theories have no the independence property has been proved in [2].

The main idea here is due to Bruno Poizat [5] (or one can see [6]: Theorem 12.28): a theory has the independence property iff there is a type over a model of cardinality λ which has $2^{(2^{\lambda})}$ coheirs.

From now on I assume that a theory T is stable up to Δ, where Δ is a collection of formulae of the form $\varphi(x;\bar{y})$.

I say that a Morley sequence $\langle a_i : i < \lambda\rangle$ is Δ-*indivisible* if for any $\varphi(x;\bar{y})$ from Δ and any $\bar{b}$ exactly one of the following two sets is cofinal in λ: $\{i < \lambda :\models \varphi(a_i, \bar{b})\}$, $\{i < \lambda :\models \neg\varphi(a_i, \bar{b})\}$. Claim that by Fact 1.10 for any formula φ without the independence property any indiscernible sequence is φ-indivisible.

Let $\mathcal{N}$ be an $|M|^{+}$-saturated elementary extension of $\mathcal{M}$, a type $p \in S_1(M)$ and a type $q \in S_1(N)$ extend the type p. Recall, that q is called *a special son of p* if for any formula $\varphi(x;\bar{y})$ and for any $\bar{a}$ and $\bar{b}$ realizing the same type over M if $q \models \varphi(x, \bar{a})$ then $q \models \varphi(x, \bar{b})$.

Lemma 1.13. *Let p be a type over M and p_1, p_2 two special sons of p, whose Morley sequences have the same type over M and are Δ-indivisible. Then their Δ-parts are equal: $p_1 \restriction \Delta = p_2 \restriction \Delta$.*

Proof. Let $\mathcal{N}$ be a sufficiently saturated elementary extension of $\mathcal{M}$ and p_1, p_2 defined over N. Let $\langle a_i : i < \omega\rangle$ be a Morley sequence of the type p_1 and $\langle b_i : i < \omega\rangle$ be a Morley sequence of the type p_2. Following Bruno Poizat I construct a third sequence $\langle c_i : i < \omega\rangle$ over N, using infinite definition of p_1 and p_2 alternately: c_{2n+1} realizes a unique M-special son of p_1 over $N \cup \{c_0, \dots, c_{2n}\}$, c_{2n+2} realizes a unique M-special son of p_2 over $N \cup \{c_0, \dots, c_{2n+1}\}$.

I claim that all of these three sequences have the same Δ-type over M. I prove this by induction. Let $\langle c_0, \dots, c_{2n}\rangle$, $\langle a_0, \dots, a_{2n}\rangle$, $\langle b_0, \dots, b_{2n}\rangle$ have the same type over M (for the last two it is a hypothesis of the lemma). If $\psi(c_0, \dots, c_{2n}, c_{2n+1})$ holds, where ψ is a formula with parameters in M, then $\langle c_0, \dots, c_{2n}\rangle$ satisfies the infinite definition of p_1 over M. Since $\langle c_0, \dots, c_{2n}\rangle$ and $\langle a_0, \dots, a_{2n}\rangle$ have the same type over M, $\psi(a_0, \dots, a_{2n}, a_{2n+1})$ also holds. Then $\langle c_0, \dots, c_{2n+1}\rangle$ and $\langle a_0, \dots, a_{2n+1}\rangle$ have the same type over M. On an even step I do similarly using the infinite definition of p_2.

Then the sequence $\langle c_i : i < \omega \rangle$ is Δ-indivisible (Δ-indivisibility is a property of the type of a sequence over $\emptyset$). Since this sequence realizes alternatively p_1 and p_2, so their Δ-parts are equal. $\square$

Corollary 1.14. *If each formula in Δ has no the independence property then the number of different Δ-parts of M-special sons over an arbitrary elementary extension of $\mathcal{M}$ is bounded by $|S_\omega(M)|$.*

Proof. By Lemma 1.13 the Δ-part of an M-special son q is defined by the type over M of the Morley sequence of q. $\square$

Lemma 1.15. *If a theory T has the independence property witnessed by a formula $\psi(x;\bar{y})$ and each formula $\varphi(x;\bar{y})$ in Δ has no the independence property, then for each cardinality λ there is a Δ-type over a model with cardinality 2^λ which is consistent with $2^{(2^\lambda)}$ ψ-types.*

Proof. I realize I_λ in the following way: there are a_α for $\alpha \in \lambda$ and $\bar{b}_w$ for $w \subset \lambda$ such that $\psi(a_\alpha, \bar{b}_w)$ holds iff $\alpha \in w$. Then there is a model $\mathcal{M}$ of cardinality λ which contains all a_α. Let $\mathcal{N} \succ \mathcal{M}$ contain all $\bar{b}_w$ and with cardinality 2^λ. Let each ultrafilter U over λ correspond to a ψ-type p_U over N in the following way: $\psi(x,\bar{c}) \in p_U$ iff $\{\alpha : \mathcal{N} \models \psi(a_\alpha, \bar{c})\} \in U$. Claim that each p_U is finitely realizable in M and is contained in some complete type q_U over N which coinherits its restriction to M.

It is easy to check that if $U \neq V$ then $p_U \neq p_V$. Since there are $2^{(2^\lambda)}$ ultrafilters over λ there are $2^{(2^\lambda)}$ types q_U which coinherit its restriction to M. The number of types over M is at most 2^λ. Since the confinality of 2^λ is strictly less than $2^{(2^\lambda)}$ there are $2^{(2^\lambda)}$ types q_U whose restrictions r_U to M are equal. Since for r_U there are at most 2^λ coheirs whose Δ-parts are not equal, there is a type q_U whose Δ-part is consistent with $2^{(2^\lambda)}$ different q_V. By the choice of q_V's it is clear that the Δ-part of q_U is consistent with $2^{(2^\lambda)}$ ψ-types p_V. $\square$

Proof of Theorem 1.12. Assume that T has the independence property, then there is a formula $\psi(x;\bar{y})$ with the independence property. By Lemma 1.15 there is a model $\mathcal{N}$ of T and a subset A of N with cardinality 2^λ such that some Δ-type p over N is consistent with $2^{(2^\lambda)}$ ψ-types over A. This contradicts to Corollary 1.5. $\square$

Theorem 1.12 gives a good measure of complexity of a dependent theory. The most simple dependent theories are that which are stable up to $\varphi(x;y)$, where y is a single variable. For instance, an o-stable theory. A theory is more complex if Δ consists of boundedly many formulae of the form $\varphi(x,y)$. The elementary theory of $\mathbb{Q}^n$ with $n!$ different lexicographical orderings is the most simple example of a theory which is stable up to $n!$ formulae of the form $\varphi(x;y)$. The next level of complexity is described by $\Delta = \{\varphi(x;y_1,y_2)\}$. And so on.

Another applications of this notion one can see in [2, 7].

I am not sure that the notion of stability up to Δ is useful for simple theories. For instance, let T_0 be the restriction of a theory T to a sublanguage $\mathcal{L}_0$. Let T_0 be simple and T stable up to $\mathcal{L}_0$. Does it imply that T is simple? It seems to be that there must be a counterexample to the following question, but it is an open problem.

Question 2. Is some dual of Theorem 1.12 holds? That is, let a theory be stable up to Δ, where each formula in Δ has the independence property. Does this imply that the theory has no the strict order property?

2. Definability of one-types

In a stable theory each type is definable. If a theory is not stable in Δ, then there are Δ-types which are not definable. But if I assume that the Δ-part p_Δ of a type p over a model with a stable up to Δ theory is definable, does it imply that the type p is definable? In other words using Algorithm Theory terminology, is each 1-type over a model with a stable up to Δ theory is definable with an oracle which makes each Δ-type over the model definable?

From now on consider a model $\mathcal{M}$ of a stable up to Δ theory T. Let $s \in S^1_\Delta(M)$ be definable, a type $p \in S_1(M)$ contain s, and $\varphi(x;\bar{y})$ be some formula. My aim is to find the definition $d_\varphi(\bar{y})$ of the formula $\varphi(x;\bar{y})$ for the type p.

By Lemma 1.8 without loss of generality I may assume that Δ is finite.

The most trivial case is the type s is realized in $\mathcal{M}$. Since s is definable and Δ is finite, the set of realizations $s(\mathcal{M})$ is definable and I obtain that the formula $\varphi^+(x,\bar{y}) \triangleq \varphi(x,\bar{y}) \wedge s(x) \wedge \bar{y} \in M$ is definable in the old language. As I claimed above φ^+ is a stable formula and it is possible to write $d_\varphi(\bar{y})$ as in stability theory.

Another trivial case is φ-rank of the type s is finite. Thus I assume that the type s is omitted in $\mathcal{M}$ and φ-rank of s is infinitity.

The following definition is closed to the definition of converging formula in the context of weak o-minimality given by B. Baizhanov in [1].

Definition 2.1. Let $\psi(x,\bar{z})$ and $\theta(\bar{z})$ be formulae, and $s(x)$ a partial type. I say that the formula $\psi(x,\bar{z})$ *converges* on $\theta(\bar{z})$ to $s(x)$ if $s(x) \models \psi(x,\bar{a})$ for any $\bar{a} \models \theta(\bar{z})$, and for any finite part s_0 of s there is $\bar{a} \models \theta(\bar{z})$ such that $\psi(x,\bar{a}) \models s_0(x)$.

I say that a partial type $s(x)$ over a set A is *approximatizable over a set B* if there are B-definable formulae $\psi(x,\bar{z})$ and $\theta(\bar{z})$, such that $\psi(x,\bar{z})$ converges on $\theta(\bar{z})$ to $s(x)$.

A partial type $s(x)$ over a set A is *approximatizable* if in addition $B = A$.

Below I prove that if a Δ-type s over a model is approximatizable then it is possible to express φ^+-rank by means of the old language.

Definition 2.2. Let $\psi(x,\bar{z})$ and $\theta(\bar{z})$ be formulae. Now I define (φ,ψ,θ)-rank of $\rho(x)$.

1. (φ,ψ,θ)-rank$(\rho) = -1$ if $\exists \bar{z}(\theta(\bar{z}) \wedge \neg\exists x(\psi(x,\bar{z}) \wedge \rho(x)))$.
2. (φ,ψ,θ)-rank$(\rho) \geq 0$ if $\forall \bar{z}(\theta(\bar{z}) \rightarrow \exists x(\psi(x,\bar{z}) \wedge \rho(x)))$.
3. (φ,ψ,θ)-rank$(\rho) \geq 1$ if there is $\bar{a}$ such that (φ,ψ,θ)-rank of each of the formulae $\rho(x) \wedge \varphi(x,\bar{a})$ and $\rho(x) \wedge \neg\varphi(x,\bar{a})$ is non-negative, that is
$$\exists \bar{y} \forall \bar{z}(\theta(\bar{z}) \rightarrow \exists x_0, x_1(\varphi(x_0,\bar{y}) \wedge \neg\varphi(x_1,\bar{y}) \wedge \bigwedge_{i<2} \psi(x_i,\bar{z}) \wedge \rho(x_i))$$
4. (φ,ψ,θ)-rank$(\rho) \geq n+1$ if there is $\bar{a}$ such that (φ,ψ,θ)-rank of each of the formulae $\rho(x) \wedge \varphi(x,\bar{a})$ and $\rho(x) \wedge \neg\varphi(x,\bar{a})$ is at least n.

As usually, (φ,ψ,θ)-rank$(s) = \min\{(\varphi,\psi,\theta)\text{-rank}(\rho) : \rho \in s\}$.

Observe, that as in stability theory there is a formula which says that (φ,ψ,θ)-rank of a formula (or a type) is equal to n.

Lemma 2.3. *Let s be a Δ-type over a model $\mathcal{M}$ with a stable up to Δ theory. Let $\varphi(x;\bar{y})$ be a formula and $\varphi^+(x;\bar{y}) \triangleq \varphi(x,\bar{y}) \wedge s(x) \wedge (\bar{y} \in M)$. If some formula $\psi(x;\bar{z})$ converges on $\theta(\bar{z})$ to the type $s(x)$, then φ^+-rank of s is equal to (φ,ψ,θ)-rank of s.*

Proof. It is easy to see that φ^+-rank of s is less than or equal to (φ,ψ,θ)-rank of s (roughly speaking, because $\exists x\forall\bar{z}P(x,\bar{z})$ implies $\forall\bar{z}\exists xP(x,\bar{z})$). The inverse inequality follows by compactness. □

If each Δ-type over a model is approximatizable then by Lemma 2.3 and by standard stability theory technique one can prove that each one-type over a model of T is definable iff its Δ-part is definable. If $\Delta = \{x <_1 y, \ldots, x <_n y\}$ where each $<_i$ is an ordering, then it is clear that each Δ-type over a model (which is the intersection of cuts relatively $<_i$) is approximatizable. Indeed, let, for simplicity, $\Delta = \{x < y, y < x\}$ and (C,D) be a cut. Consider a Δ-type $s(x) = \{c < x : c \in C\} \cup \{x < d : d \in D\}$. Obviously, the formula $y < x < z$ converges to s on the formula $(y \in C) \wedge (z \in D)$. Since neither of $x <_i y$ has the independence property, any stable up Δ theory T has no the independence property. Taking into account the strict order property, the following question is quite natural:

Question 3. Let a theory T have no the independence property and be stable up to (finite) Δ. Is each Δ-type over a model of T is approximatizable?[2]

In general it is not clear if each Δ-type is approximatizable (and, possibly, it is not true). Nevertheless it is possible to express φ^+-rank by means of the old language under the additional supposition that a considered model is ω-saturated. But first I prove some auxiliary lemma.

Lemma 2.4. *Let T be stable up to Δ, and $\mathcal{M}$ a model of T. Let also $p \in S_1(M)$ and s be its Δ-part. If s is definable and there are formulae $\psi(x,\bar{z})$ and $\theta(\bar{z})$ such that (φ,ψ,θ)-rank of s for some formula $\varphi(x;\bar{y})$ is equal to φ^+-rank of s then the φ-part of p is definable.*

Proof. Obviously, (φ,ψ,θ)-rank$(p) \leq (\varphi,\psi,\theta)$-rank$(s) < \omega$. Then I write the definition $d_\varphi(\bar{y})$ for $\varphi(x,\bar{y})$ as in stability theory replacing φ-rank with (φ,ψ,θ)-rank. □

Theorem 2.5. *Let a theory T be stable up to Δ, and $\mathcal{M}$ an $\aleph_0$-saturated model of T. Let $s \in S_\Delta(M)$ be definable. Then for any formula $\varphi(x;\bar{y})$ there are formulae $\psi(x;\bar{z})$ and $\theta(\bar{z})$ such that φ^+-rank of s is equal to (φ,ψ,θ)-rank of s.*

Proof. Since the inequality φ^+-rank$(s) \leq (\varphi,\psi,\theta)$-rank$(s)$ is obvious, I prove the inverse ineqaulity. By our agreement Δ is assumed to be finite, so it is possible to say that Δ consists of one formula $\mu(x;\bar{u})$. Since s is definable, there is a definition $d_\mu(\bar{u})$ of $\mu(x;\bar{u})$. Let n and k be natural numbers and let

$$\psi_{n,k}(x,\bar{u}_0,\ldots,\bar{u}_{n-1},\bar{v}_1,\ldots,\bar{v}_{k-1}) \triangleq \bigwedge_{i<n}\mu(x,\bar{u}_i) \wedge \bigwedge_{j<k}\neg\mu(x,\bar{v}_j)$$

[2]Recently, Vincent Guingona in his paper "Dependence and isolated extensions" (preprint, # 212 on Modnet preprint server) gave a positive answer to this question. So the following theorem holds: Let a theory T do not have the independence property and be stable up to Δ. then any one-type over a model of T is definable iff its Δ-part is definable.

$$\theta_{n,k}(\bar{u}_0,\dots,\bar{u}_{n-1},\bar{v}_1,\dots,\bar{v}_{k-1}) \triangleq \bigwedge_{i<n} d_\mu(\bar{u}_i) \wedge \bigwedge_{j<k} \neg d_\mu(\bar{v}_j)$$

Consider $(\varphi, \psi_{n,k}, \theta_{n,k})$-rank of s. Let φ^+-rank of s be equal to m. Assume that for each naturals n and k $(\varphi, \psi_{n,k}, \theta_{n,k})$-rank of s is at least $m+1$. Then by compactness a binary tree of depth $m+1$ representing $(\varphi, \psi_{n,k}, \theta_{n,k})$-rank of s is consistent with s. This can be written as the following partial type q over a finite set, which consists of parameters I need to define the formulae φ, ψ, θ. Below $\tau \restriction i \in 2^i$ is the restriction to the first i elements of $\tau \in 2^{m+1}$, where 2^{m+1} is the set of functions $\pi : \{1,\dots,m+1\} \to \{0,1\}$.

$$q(\bar{y}_{\langle\rangle}, \bar{y}_{\langle 0\rangle}, \bar{y}_{\langle 1\rangle}, \bar{y}_{\langle 0,0\rangle}, \bar{y}_{\langle 0,1\rangle}, \bar{y}_{\langle 1,0\rangle}, \bar{y}_{\langle 1,1\rangle}, \dots, \bar{y}_{\langle 1,1,\dots,1\rangle}) =$$
$$\left\{ \forall \bar{z} \left(\theta_{n,k}(\bar{z}) \to \exists \{x_\tau : \tau \in 2^{m+1}\} \left(\bigwedge_{\tau \in 2^{m+1}} \psi_{n,k}(x_\tau, \bar{z}) \wedge \bigwedge_{i=1}^{m+1} \varphi^{\tau(i)}(x, \bar{y}_{\tau \restriction (i-1)}) \right) \right) : \right.$$
$$\left. : n, k < \omega \right\}$$

Since $\mathcal{M}$ is assumed to be $\aleph_0$-saturated, this type is realized in $\mathcal{M}$. Then φ^+-rank of s is at least $m+1$, for a contradiction. □

The following theorem is an immediate corollary of Theorem 2.5 and Lemma 2.4.

Theorem 2.6. *Let a theory T be stable up to Δ, and $\mathcal{M}$ an $\aleph_0$-saturated model of T. Then a one-type over M is definable iff its Δ-part is definable.*

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