Not-so-small representations of SL_2 in the finite Morley rank category

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In this article we consider representations of SL_2 which are interpretable in finite Morley rank theories, meaning that inside a universe of finite Morley rank we shall study the following definable objects: a group G isomorphic to SL_2, an abelian group V, and an action of G on V; V is thus a definable G-module on which G acts definably. Our goal will be to identify V with a standard G-module, under an assumption on its Morley rank. (A word on this notion of rank will be said shortly, after we have stated the results.)

It will be convenient to work with a faithful representation, possibly replacing SL_2 by the quotient PSL_2, and we shall write $G \simeq (\mathrm{P})\mathrm{SL}_2$ to cover both cases.

Theorem. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a faithful action of G on V for which V is G-minimal. Assume $\operatorname{rk} V \leq 3 \operatorname{rk} \mathbb{K}$. Then V bears a structure of $\mathbb{K}$-vector space such that:*

- *either $V \simeq \mathbb{K}^2$ is the natural module for $G \simeq \mathrm{SL}_2(\mathbb{K})$, or*
- *$V \simeq \mathbb{K}^3$ is the irreducible 3-dimensional representation of $G \simeq \mathrm{PSL}_2(\mathbb{K})$ with $\operatorname{char} \mathbb{K} \neq 2$.*

On the way we shall establish the following interesting results.

Proposition 2.3. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a non-trivial action of G on V. Then for v generic in V, $C_G^\circ(v)$ is semi-simple or unipotent (possibly trivial).*

Proposition 2.5. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$* of characteristic 0*, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a faithful action of G on V for which V is G-minimal. Assume that for v generic in V, $C_G^\circ(v)$ is a non-trivial unipotent subgroup. Then V is a natural module for G. In particular $G \simeq \mathrm{SL}_2(\mathbb{K})$.*

All statements above involve the Morley rank of a structure; the reader should bear in mind that it is an abstract analog of the Zariski dimension, which can be axiomatized by some natural properties [2]. The Morley rank is however not related *a priori* to any geometry or topology, being a purely model-theoretic notion. Yet in general if a field $\mathbb{K}$ has Morley rank k and V is an algebraic variety of (Zariski-)dimension d over $\mathbb{K}$, then its Morley rank is dk. The rank hypothesis in the Theorem would thus amount, if the configuration were known to be algebraic, to assuming that $\dim V \leq \dim G$; but of course the possibility for a field to have a finite Morley rank $k > 1$ makes algebraic geometry less general than our frame.

The latter is a ranked universe in the sense of [2]. Indeed, the semi-direct product $V \rtimes G$ is a ranked group in the sense of Borovik and Poizat [7, Corollaire 2.14 and Théorème 2.15]. In a word, we shall not go too deep into purely model-theoretic arguments but will merely use the natural, intuitive properties, which can be taken as the definition of the Morley rank.

Let us now say a word about the proof of the Theorem. As we have mentioned, there is no geometry a priori on $V \rtimes G$, and our efforts will be devoted to retrieving a suitable vector space structure on V which arises from the action of G. Model-theoretically speaking, the main tool is Zilber's so-called Field Theorem [7, Théorème 3.7], which enables one to find an (algebraically closed) field inside a solvable, non-nilpotent, infinite group of finite Morley rank. A major difficulty is that there is no immediate reason why the action of an algebraic torus of G should induce such a structure on all of V. And even if a good structure exists, this does not mean that G itself is linear on V. The 2-dimensional case relies on a theorem by Timmesfeld (Fact 1.1 below); as for dimension 3, we rather manually extend the field action and some miraculous computations will, in the end, prove linearity of G.

Now that we have said what the present paper is, let us say what it is not: it does not relate directly to the classification project for simple groups of finite Morley rank.

1 Preparatory Remarks

We shall use throughout a characterization of the natural module which is due to Timmesfeld.

Fact 1.1 ([9, Theorem 3.4]). *Let $\mathbb{K}$ be a field and let $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$. Let V be a faithful G-module. Suppose the following:*

(i). $C_V(G) = 0$

(ii). $[U, U, V] = 1$*, where U is a maximal unipotent subgroup of G.*

Let $0 \neq v \in C_V(U)$ and $W = \langle v^G \rangle$. Then there exists a field action of $\mathbb{K}$ on W such that W is the natural G-module. In particular $G \simeq \mathrm{SL}_2(\mathbb{K})$.

We shall use the non-standard notation $(+)$ to denote quasi-direct sum, i.e. the sum of two submodules (of a fixed module) which have a finite, possibly non-trivial, intersection.

1.1 On Malcev's Theorem

Fact 1.2 ([7, Théorème 3.18])**.** *Let G be a connected, solvable group of finite Morley rank acting definably and faithfully on a definable, abelian group A. If a definable subgroup $B \leq A$ is G- or G'-minimal, then B is centralized by G'.*

Lemma 1.3. *In a universe of finite Morley rank, consider the following definbale objects: a reductive algebraic group G, a nilpotent group V, and an action of G on V. Let U be a unipotent subgroup of G. Then $V \rtimes U$ is nilpotent.*

Proof. We may assume that U is a maximal unipotent subgroup. In this case, and by reductivity of G, U is the commutator subgroup of the Borel subgroup $B = N_G(U)$ [1, p. 65]. Now consider $H = V \rtimes B$ and write $F^\circ(H) = V \rtimes K$. Clearly $K \leq F^\circ(B)$, and $H/F^\circ(H) \simeq B/K$ is abelian by [2, Theorem 9.21], so $U = B' \leq K$. □

Corollary 1.4. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a quasi-simple algebraic group G over $\mathbb{K}$, an abelian group V, and a non-trivial action of G for which V is G-minimal. Then V has the same characteristic as $\mathbb{K}$.*

Proof. Let U be a maximal unipotent subgroup of G. By Lemma 1.3, $V \rtimes U$ is nilpotent. If $p = 0$ and V is torsion or if $p \neq 0$ and $pV = V$, then Burdges' structure theorem for nilpotent groups [3, Theorem 2.31] yields $[V, U] = 0$. As conjugates of U generate G, the action is trivial, a contradiction. □

1.2 Algebraicity in characteristic 0

We specialize [6] to our context.

Fact 1.5 (special case of [6, Theorem 4])**.** *In a universe of finite Morley rank, consider the following definable objects: an abelian, torsion-free group A, an infinite group S, and a faithful action of S on A for which A is S-minimal. Then there is a subgroup $A_1 \leq A$ and a field $\mathbb{K}$ such that $A_1 \simeq \mathbb{K}_+$ definably, and S embeds into $\mathrm{GL}_n(\mathbb{K})$ for some n.*

Lemma 1.6. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a simple algebraic group G over $\mathbb{K}$, a torsion-free abelian group V, and a faithful action of G on V for which V is G-minimal. Then $V \rtimes G$ is algebraic. Moreover, any definable subgroup of V has rank a multiple of* $\mathrm{rk}\,\mathbb{K}$.

Proof. The assumptions imply that G is interpretable in $\mathbb{K}$ as a pure field. By Fact 1.5, there is a field structure $\mathbb{L}$ such that $V \simeq \mathbb{L}_+^n$ and $G \hookrightarrow \mathrm{GL}_n(\mathbb{L})$ definably. $\mathbb{L}$ has of course characteristic 0. By a result of Macpherson and Pillay (see [8, Theorem 3]), G is Zariski-closed in $\mathrm{GL}_n(\mathbb{L})$; so far G and $V \rtimes G$ are algebraic groups over $\mathbb{L}$. In particular G *as a pure group* interprets $\mathbb{L}$, so $\mathbb{K}$ *as a pure field* interprets $\mathbb{L}$. It follows that $\mathbb{K} \simeq \mathbb{L}$ by [7, Théorème 4.15].

Now consider a definable subgroup V_1 of V. Then the setwise stabilizer of V_1 in $\mathbb{K}$ is a definable, non-trivial subgroup of $\mathbb{K}$, whence equal to $\mathbb{K}$ by [7, Corollaire 3.3]. Hence V_1 is a vector space on $\mathbb{K}$, which proves that its rank is a multiple of $\operatorname{rk}\mathbb{K}$. □

As a consequence, one can drastically simplify certain identification results *in characteristic* 0. For example, the following simplification of part of [5] results.

Theorem 1.7 ([5, Theorem A in char. 0]). *Let G be a connected, non-solvable group of finite Morley rank acting definably and faithfully on a* torsion-free *connected abelian group V of Morley rank* 2*. Then there is an algebraically closed field $\mathbb{K}$ of Morley rank* 1 *and characteristic* 0 *such that $V \simeq \mathbb{K}_+^2$, and G is isomorphic to* $\mathrm{GL}_2(\mathbb{K})$ *or* $\mathrm{SL}_2(\mathbb{K})$ *in its natural action.*

Proof. V is clearly G-minimal. By Fact 1.5, there is an interpretable field structure $\mathbb{K}$ such that $G \hookrightarrow \mathrm{GL}_n(\mathbb{K})$ with $V \simeq \mathbb{K}^n$. Clearly the dimension must be 2, making the rank of the field 1. So there is a field $\mathbb{K}$ of rank 1 such that $V \simeq \mathbb{K}_+^2$ and $G \hookrightarrow \mathrm{GL}_2(\mathbb{K})$. But definable subgroups of $\mathrm{GL}_2(\mathbb{K})$, especially over a field of rank 1, are known: [8, Theorem 5] together with connectedness and non-solvability of G forces either $G \simeq \mathrm{GL}(V)$ or $G \simeq \mathrm{SL}(V)$. □

1.3 A Theorem of Frank Wagner

Fact 1.8 ([10, Corollary 9]). *Let $\mathbb{K}$ be a field of finite Morley rank of characteristic $p > 0$. Then $\mathbb{K}^\times$ has no definable torsion-free section.*

A *good torus* is a definable, abelian, divisible group with no torsion-free definable section; the latter condition being equivalent to: every definable subgroup is the definable hull of its torsion subgroup. Wagner's Theorem states that in finite Morley rank, the multiplicative group of a field of characteristic p is a good torus.

Lemma 1.9. *In a universe of finite Morley rank, consider the following definable objects: two infinite, abelian groups K and H, and a faithful action of K on H for which H is K-minimal. Suppose that H has exponent p and that K contains a non-trivial q-torus for each $q \neq p$. Then $\operatorname{rk} H = \operatorname{rk} K$.*

Proof. By Zilber's Field Theorem, there is a field structure $\mathbb{L}$ such that $K \hookrightarrow \mathbb{L}^\times$ and $H \simeq \mathbb{L}_+$. In particular, $\operatorname{char}\mathbb{L} = p$. Now $\mathbb{L}^\times/K$ is torsion-free, so by Wagner's Theorem, K cannot be proper in $\mathbb{L}^\times$. Hence $\operatorname{rk} K = \operatorname{rk}\mathbb{L} = \operatorname{rk} H$. □

Lemma 1.10. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$ of characteristic p, a connected subgroup Θ of $\mathbb{K}^\times$, an abelian group V, and an action of Θ on V. Then there is $\theta \in \operatorname{Tor}\Theta$ such that $C_V(\Theta) = C_V(\theta)$ and $[V, \Theta] = [V, \theta]$.*

Proof. By Wagner's Theorem, $\mathbb{K}^\times$ is a good torus, hence $\Theta = d(\operatorname{Tor}\Theta)$. By the descending chain condition on centralizers, $C_V(\Theta) = C_V(\operatorname{Tor}\Theta) = C_V(\theta_1, \ldots, \theta_n)$ for torsion elements, and we take a generator θ_0 of the finite cyclic group $\langle \theta_1, \dots, \theta_n \rangle$: one has $C_V(\Theta) = C_V(\theta_0)$, and this holds true of any root of θ_0.

Now the group $[V, \operatorname{Tor}\Theta]$ is definable, so $\Sigma = \{t \in \Theta : [V,t] \leq [V, \operatorname{Tor}\Theta]\}$ is a definable subgroup of Θ containing $\operatorname{Tor}\Theta$. Again, as $\Theta = d(\operatorname{Tor}\Theta)$, it follows $\Sigma = \Theta$, that is $[V, \Theta] = [V, \operatorname{Tor}\Theta]$. By the ascending chain condition on the lattice of connected groups $[V, t]$ ($t \in \operatorname{Tor}\Theta$), there is $\theta \in \operatorname{Tor}\Theta$ such that $[V, \theta] = [V, \operatorname{Tor}\Theta] = [V, \Theta]$. We may assume that θ is a root of θ_0, and we are done. □

1.4 Cohomological computations

Fact 1.11. *Let A be a connected, abelian group of finite Morley rank of bounded exponent and α a definable automorphism of finite order coprime to the exponent of A. Then $A = C_A(\alpha) \oplus [A, \alpha]$. Moreover, if $A_0 < A$ is a definable, connected, α-invariant subgroup, then $[A, \alpha] \cap A_0 = [A_0, \alpha]$.*

Proof. Let ad_α and Tr_α be the adjoint and trace maps, that is

$$\operatorname{ad}_\alpha(x) = x^\alpha - x \qquad \text{and} \qquad \operatorname{Tr}_\alpha(x) = x + \cdots + x^{\alpha^{n-1}}$$

where n is the order of α. It is easily seen, as A has no n-torsion, that $\ker \operatorname{ad}_\alpha \cap \ker \operatorname{Tr}_\alpha = 0$. In particular, $\operatorname{rk} A \geq \operatorname{rk}(\ker \operatorname{ad}_\alpha) + \operatorname{rk}(\ker \operatorname{Tr}_\alpha)$. Moreover, $\operatorname{im} \operatorname{ad}_\alpha \leq \ker \operatorname{Tr}_\alpha$ and $\operatorname{im} \operatorname{Tr}_\alpha \leq \ker \operatorname{ad}_\alpha$. It follows therefore that $\operatorname{rk} A \geq \operatorname{rk}(\ker \operatorname{ad}_\alpha) + \operatorname{rk}(\ker \operatorname{Tr}_\alpha) \geq \operatorname{rk}(\ker \operatorname{ad}_\alpha) + \operatorname{rk}(\operatorname{im} \operatorname{ad}_\alpha) = \operatorname{rk} A$, so $\operatorname{im} \operatorname{ad}_\alpha = \ker \operatorname{Tr}_\alpha$. Hence $A = \ker \operatorname{ad}_\alpha + \ker \operatorname{Tr}_\alpha = \ker \operatorname{ad}_\alpha \oplus \operatorname{im} \operatorname{ad}_\alpha = C_A(\alpha) \oplus [A, \alpha]$.

Let $a_0 \in A_0$; then $a_0 \in \operatorname{ad}_\alpha(A_0)$ iff $\operatorname{Tr}_\alpha(a_0) = 0$ iff $a_0 \in \operatorname{ad}_\alpha(A)$. □

Corollary 1.12. *Let A be a connected elementary abelian p-group of finite Morley rank and T a $p^\perp$ good torus acting definably on A. Then $A = C_A(T) \oplus [A, T]$. Let $A_0 < A$ be a definable, connected, T-invariant subgroup. Then $C_A(T)$ covers $C_{A/A_0}(T)$. Moreover, $C_T(A) = C_T(A_0, A/A_0)$.*

Proof. Since T is a good torus we may work as in Lemma 1.10 to find a torsion element $t \in T$ such that $C_A(T) = C_A(t)$ and $[A, T] = [A, t]$. We apply Fact 1.11 and deduce that $A = C_T(A) \oplus [A, T]$.

If $x \in A$ maps to an element in $C_{A/A_0}(t)$, then denoting the canonical projection by π one has $\pi \operatorname{ad}_t(x) = \operatorname{ad}_t \pi(x) = 0$. Hence $\operatorname{ad}_t(x) \in A_0$ and by Fact 1.11 there is $x_0 \in A_0$ such that $\operatorname{ad}_t(x) = \operatorname{ad}_t(x_0)$, whence $x \in x_0 + \ker \operatorname{ad}_t$, and $\ker \operatorname{ad}_t = C_A(t)$.

Now let $\Theta = C_T(A_0, A/A_0)$; as T is a good torus, Θ is the definable hull of its torsion subgroup. Let $t \in \operatorname{Tor}\Theta$. Then as above $C_A(t)$ covers $C_{A/A_0}(t) = A/A_0$; it follows that $A = C_A(t) + A_0 \leq C_A(t)$. Hence $t \in C_T(A)$ and $\Theta = d(\operatorname{Tor}\Theta) \leq C_T(A)$. □

1.5 Automorphisms of semi-direct products

Lemma 1.13. *In a universe of finite Morley rank, let A, T be definable, abelian, infinite groups such that A is T-minimal and the action is faithful. Let K be a definable group normalizing A and T. Then K centralizes T.*

Proof. We let K act on $\operatorname{End} A$ by:

$$s^{\varphi}(a) := (s(a^{\varphi^{-1}}))^{\varphi}$$

But by assumption, K normalizes the image of T in $\operatorname{End} A$, which additively generates an algebraically closed field (this is the proof of Zilber's field theorem). In particular, as there are no definable groups of automorphisms of a field of finite Morley rank [2, Theorem 8.3], K acts trivially on T. □

2 Actions of $(\mathrm{P})\mathrm{SL}_2$

The present section is devoted to general actions of $(\mathrm{P})\mathrm{SL}_2$ in the finite Morley rank category, with no assumption on the rank itself. Proposition 2.3 is our main result. The following notations will be adopted in §§2 and 3.

Notation 2.1. Let $G \simeq (\mathrm{P})\mathrm{SL}_2$. Fix a Borel subgroup B of G and let $U = B'$ be its unipotent subgroup. Let T be an algebraic torus such that $B = U \rtimes T$. Let i be the involution in T, and $\zeta \in N(T)$ a 2-element inverting T (the order of ζ depends on the isomorphism type of G).

Let us start with a classical observation.

Lemma 2.2. *A definable, connected subgroup of* $(\mathrm{P})\mathrm{SL}_2$ *is semi-simple, unipotent, or contains a maximal unipotent subgroup of* $(\mathrm{P})\mathrm{SL}_2$.

Proof. Let K be a definable, connected subgroup. We may assume that K is proper; as K is then solvable (see for instance [8, Théorème 4]), up to conjugacy $K \leq B$. It follows that $K = U_1 \rtimes T_1$ with obvious notations. If $U_1, T_1 > 1$ then T_1 additively generates $\mathbb{K}$, so U_1 must be stable under $\mathbb{K}^\times$, and $U_1 = U$. □

2.1 Actions of $(\mathrm{P})\mathrm{SL}_2$ and centralizers

Proposition 2.3. *In a universe of finite Morley rank, consider the following definable objects: a field* $\mathbb{K}$*, a group* $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$*, an abelian group* V*, and a non-trivial action of* G *on* V*. Then for* v *generic in* V*,* $C_G^\circ(v)$ *is semi-simple or unipotent (possibly trivial).*

Proof. Let V be as in the statement; we first show that we may assume $C_V(G) = 0$. Assume the result holds when $C_V(G) = 0$ and let V as in the statement. Let $V_0 = C_V(G) < V$. Since G is perfect, one has $C_{V/V_0}(G) = 0$, and the action of G on V/V_0 is non-trivial. By assumption, the result holds for V/V_0. Now let $v \in V$ be generic. Then $\bar{v} \in V/V_0$ is generic too, and in particular $C_G^\circ(\bar{v})$ is either semi-simple or unipotent. As $C_G^\circ(v) \leq C_G^\circ(\bar{v})$, we are done.

So from now on we suppose $C_V(G) = 0$. In Notation 2.1 we had fixed a maximal unipotent subgroup $U \leq G$; $B = N(U)$ its normalizer; T an algebraic torus such that $B = U \rtimes T$; and a 2-element ζ inverting T.

Let $v \in V$ be generic. $C_G^\circ(v)$ is proper in $(\mathrm{P})\mathrm{SL}_2$, hence solvable [8, Théorème 4]; up to conjugacy, $C_G^\circ(v) \leq B$. Assume that $C_G^\circ(v)$ is neither unipotent nor semi-simple. Then by Lemma 2.2, $C_G^\circ(v)$ contains U.

So up to conjugacy, $C_G^\circ(v) = U \rtimes T_v$ for some non-trivial $T_v \leq T$. The family $\{C_G^\circ(v)/U : v \in V : U \leq C_G^\circ(v) \leq B\}$ of subgroups of T is uniformly definable; as $T \simeq \mathbb{K}^\times$ is a good torus, the family is finite [4, Rigidity II]. It follows that there is a *common* $T_0 \leq T$ such that generically, $C_G^\circ(v)$ is conjugate to $U \rtimes T_0$.

Now let $V_1 = C_V(U)$. Clearly V_1 is infinite, taking a B-minimal subgroup of V and applying Malcev's Theorem (Fact 1.2). As any two distinct conjugates of U generate G and $C_V(G) = 0$, V_1 must be disjoint from V_1^g for $g \notin B$. It follows that $N_G(V_1) = B$ and that V_1 is disjoint from its distinct conjugates. One therefore has

$$\mathrm{rk}\, V_1^G = \mathrm{rk}\, V_1 + \mathrm{rk}\, G - \mathrm{rk}\, B = \mathrm{rk}\, V_1 + \mathrm{rk}\, \mathbb{K}.$$

Notice that by assumption, the generic element of V is centralized by a conjugate of $U \rtimes T_0$. Thus V_1^G is generic in V. But furthermore, for v generic in V_1, $C_G^\circ(v)$ is a conjugate of $U \rtimes T_0$ containing U; conjugacy is therefore obtained by an element of $N(U) = B$. As $B' = U$, $U \rtimes T_0$ is normal in B; hence $C_G^\circ(v) = U \rtimes T_0$. This means that T_0 centralizes a generic subset X of V_1; as $X + X = V_1$ it follows that $V_1 = C_V(U \rtimes T_0)$.

Let $W = V_1 \oplus V_1^\zeta$ and $\check{W} = W \setminus (V_1 \cup V_1^\zeta)$. The generic element of W is in $\check{W}$. Let $v \in \check{W}$. Clearly $T_0 \leq C_G^\circ(v)$. Moreover, if $C_G^\circ(v)$ is not semi-simple, then it must meet a unipotent subgroup which can only be either U or U^ζ as $1 \neq T_0 \leq C_G^\circ(v)$. In that case, $C_G^\circ(v)$ contains either U or U^ζ by Lemma 2.2, against the definition of $\check{W}$. This means that for $v \in \check{W}$, one has $T_0 \leq C_G^\circ(v) \leq T$. In particular, $\check{W}^G$ is *not* generic in V.

It follows that $W < V$. As V_1^G is generic in V, W cannot be G-invariant. Therefore $T \cdot \langle\zeta\rangle \leq N_G(W) < G$, and equality follows from maximality of $T \cdot \langle\zeta\rangle$. As $T \cdot \langle\zeta\rangle$ also normalizes $V_1 \cup V_1^\zeta$, one sees that $N_G(\check{W}) = T \cdot \langle\zeta\rangle$.

Let $w \in \check{W}$. Assume that $w \in \check{W}^g$ for some $g \in G$. Then $C_G^\circ(v)$ is a non-trivial subgroup of T, so $C_G(C_G^\circ(v)) = T = T^g$, and $g \in N(T) = T \cdot \langle\zeta\rangle = N(\check{W})$. This implies that

$$\mathrm{rk}\, \check{W}^G = \mathrm{rk}\, \check{W} + \mathrm{rk}\, G - \mathrm{rk}\, T = 2\,\mathrm{rk}\, V_1 + 2\,\mathrm{rk}\, \mathbb{K} = 2\,\mathrm{rk}\, V_1^G.$$

But V_1^G is already generic in V which is infinite: this is a contradiction. □

Corollary 2.4. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a non-trivial action of G on V. Then $\mathrm{rk}\, V \geq 2\,\mathrm{rk}\, \mathbb{K}$.*

2.2 The characteristic 0 case

We continue in the vein of Proposition 2.3.

Proposition 2.5. *In a universe of finite Morley rank, consider the following definable objects: a field* $\mathbb{K}$ of characteristic 0*, a group* $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$*, an abelian group* V*, and a faithful action of* G *on* V *for which* V *is* G*-minimal. Assume that for* v *generic in* V*,* $C_G^\circ(v)$ *is a non-trivial unipotent subgroup. Then* V *is a natural module for* G*. In particular* $G \simeq \mathrm{SL}_2(\mathbb{K})$*.*

Proof. Since V is torsion-free and G-minimal, $C_V(G) = 0$. Let $\bar{G} = G/C_G(V)$; $\bar{G}$ is still isomorphic to $(\mathrm{P})\mathrm{SL}_2(\mathbb{K})$ and now faithful on V. We apply Lemma 1.6 and retrieve a $\mathbb{K}$-vector space structure on V: in particular every definable subgroup has rank a multiple of $\operatorname{rk}\mathbb{K}$.

Let $v \in V$ be generic; as there are no non-trivial, proper, definable subgroups of $\mathbb{K}_+$ [7, Corollaire 3.3], $C_G^\circ(v)$ is a conjugate of a *maximal* unipotent subgroup.

Let $V_1 = C_V(U) < V$; as any two distinct conjugates of U generate G, V_1 is disjoint from V_1^g for $g \notin B$. Thus $N_G(V_1) = B$ and V_1 meets its distinct conjugates trivially. By our assumption, V_1^G is generic in V. So

$$\operatorname{rk} V = \operatorname{rk} V_1^G = \operatorname{rk} V_1 + \operatorname{rk} G - \operatorname{rk} N_G(V_1) = \operatorname{rk} V_1 + \operatorname{rk}\mathbb{K}.$$

In particular $\operatorname{rk} V/V_1 = \operatorname{rk}\mathbb{K}$. Now V_1 has rank a multiple of $\operatorname{rk}\mathbb{K}$, so $\operatorname{rk} V_1 \geq \operatorname{rk}\mathbb{K}$. It follows that $V_1 \oplus V_1^\zeta = V$, and $V/V_1 \simeq V_1^\zeta \simeq \mathbb{K}_+$ has no proper non-trivial definable subgroups.

Thus B acts on V/V_1 which is B-minimal, and by Malcev's Theorem (Fact 1.2) $B' = U$ centralizes V/V_1. So $[V, U, U] = 1$ and we may apply Timmesfeld's identification result, Fact 1.1. It follows that V is the natural module for $\bar{G} \simeq \mathrm{SL}_2(\mathbb{K})$, and this forces $G \not\simeq \mathrm{PSL}_2(\mathbb{K})$ but $G \simeq \mathrm{SL}_2(\mathbb{K})$ and $G = \bar{G}$. □

Unfortunately, nothing similar seems to be possible in characteristic p.

2.3 Four-groups of PSL_2

We finish this section with an easy but useful relation on ranks when the characteristic is not 2. Given a definable, involutive automorphism j of an abelian group W of finite Morley rank with no involutions, one has $W = W^{+_j} \oplus W^{-_j}$ with obvious notations [2, Exercise 14 p.73].

Lemma 2.6. *In a universe of finite Morley rank, consider the following definable objects: a field* $\mathbb{K}$*, a group* $G \simeq \mathrm{PSL}_2(\mathbb{K})$*, an abelian group* V of characteristic not 2*, and a non-trivial action of* G *on* V*. Then*

$$\operatorname{rk} V = \operatorname{rk}\left(V^{+_i+_\varsigma}\right) + \frac{3}{2}\operatorname{rk}\left(V^{-_i}\right)$$

Proof. Write $V = V^{+_i} \oplus V^{-_i}$, then $V^{+_i} = V^{+_i+_\varsigma} \oplus V^{+_i-_\varsigma}$ and $V^{-_i} = V^{-_i+_\varsigma} \oplus V^{-_i-_\varsigma}$. Let $a = \operatorname{rk} V^{+_i+_\varsigma}$ and $b = \operatorname{rk} V^{+_i-_\varsigma}$. Clearly, $b = \operatorname{rk} V^{-_i+_\varsigma} = \operatorname{rk} V^{-_i-_\varsigma}$. It follows $\operatorname{rk} V^{-_i} = 2b$ and $\operatorname{rk} V = a + 3b$. □

This lemma will yield a crucial estimate of ranks in Corollary 3.15.

3 Proof of the Theorem

We now attack our main result.

Theorem. *In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a faithful action of G on V for which V is G-minimal. Assume $\operatorname{rk} V \leq 3 \operatorname{rk} \mathbb{K}$. Then V bears a structure of $\mathbb{K}$-vector space such that:*

- *either $V \simeq \mathbb{K}^2$ is the natural module for $G \simeq \mathrm{SL}_2(\mathbb{K})$, or*
- *$V \simeq \mathbb{K}^3$ is the irreducible 3-dimensional representation of $G \simeq \mathrm{PSL}_2(\mathbb{K})$ with $\operatorname{char} \mathbb{K} \neq 2$.*

Notation 3.1. In a universe of finite Morley rank, consider the following definable objects: a field $\mathbb{K}$, a group $G \simeq (\mathrm{P})\mathrm{SL}_2(\mathbb{K})$, an abelian group V, and a non-trivial action of G on V for which V is G-minimal. Assume $\operatorname{rk} V \leq 3 \operatorname{rk} \mathbb{K}$.

One should also bear in mind Notation 2.1 which introduces the usual elements and subgroups of $(\mathrm{P})\mathrm{SL}_2$.

Notation 3.2. Let $k = \operatorname{rk} \mathbb{K}$ and write $\operatorname{rk} V = 2k + \nu$.

Notice that $0 \leq \nu \leq k$ by Corollary 2.4 and our assumption that $\operatorname{rk} V \leq \operatorname{rk} G$. Moreover, if $\nu = 0$, then by [5, Theorem B] (which is a consequence, in finite Morley rank, of Timmesfeld's identification result, Fact 1.1), we are done. So we suppose $\nu > 0$ throughout. Our goal is to show that the characteristic is not 2, $\nu = k$, and $G \simeq \mathrm{PSL}_2$ acts on $V \simeq \mathbb{K}^3$ in the usual irreducible way.

If V has characteristic 0, then by Lemma 1.6, $V \rtimes G$ or $V \rtimes G/Z(G)$ is algebraic; $\dim_{\mathbb{K}} V$ is 2 or 3, and as irreducible algebraic representations of $(\mathrm{P})\mathrm{SL}_2$ are well-known, the analysis already ends. From now on, we suppose $\operatorname{char} \mathbb{K}$ to be a prime number p. The proof will involve studying various submodules of V, defining an action piecewise, and eventually proving its linearity. On our way we shall prove $p \neq 2$, though a more direct attack could be possible.

Lemma 3.3. *We may suppose that $C_V(G) = 0$.*

Proof. Suppose our Theorem holds for modules with a trivial right-kernel. Notice that by G-minimality, $W = C_V(G)$ is finite. It follows that there is no right kernel for G on the G-minimal module $\bar{V} = V/W$; so the result holds for the action of G on $\bar{V}$. In particular, as we have assumed $\operatorname{rk} V > 2k$, we find that $\operatorname{char} \mathbb{K} \neq 2$ and $G \simeq \mathrm{PSL}_2(\mathbb{K})$, so that $\langle i, \zeta \rangle$ is a four-group. We also know that ζ inverts a set of rank $2k$ in $\bar{V}$.

It follows that ζ inverts a set of rank $\geq 2k$ in V. Hence $\operatorname{rk} V^{-\zeta} \geq 2k$, and Lemma 2.6 implies that $V^{+_i +_\zeta}$ is finite. As it is clearly connected, we deduce that $W \leq V^{+_i +_\zeta} = 0$. $\square$

Bear in mind on the contrary that when $\operatorname{char} \mathbb{K} = 2$, $\mathrm{SL}_2(\mathbb{K})$ acts on $\mathrm{sl}_2(\mathbb{K})$ by conjugacy with an infinite right-kernel. Characteristic 2 will be eliminated in Lemma 3.13 below.

3.1 T-invariant sections

Our finer study of submodules starts here. A word on terminology: if K is a group acting on an abelian group V, we shall call any definable, connected, K-invariant subgroup a K-submodule.

In this subsection only, we work with abstract T-modules (of finite Morley rank) which need not relate to our current representation V.

Definition 3.4. Call a T-module X *degenerate* if $C_T^\circ(X) \neq 1$.

We now consider $\operatorname{cork} C_T^o(X) = \operatorname{rk}(T/C_T^o(X))$.

Lemma 3.5.

(i). *Let X be a T-module. Then* $\operatorname{cork} C_T^\circ(X) \leq \operatorname{rk} X$.

If X is degenerate, then $\operatorname{cork} C_T^\circ(X) < \operatorname{rk} X$.

(ii). *Let X be a $T \cdot \langle\zeta\rangle$-module. Then* $\operatorname{cork} C_T^\circ(X) \leq \frac{\operatorname{rk} X}{2}$.

If X is degenerate, then $\operatorname{cork} C_T^\circ(X) < \frac{\operatorname{rk} X}{2}$.

Proof. Let $\Theta = C_T^\circ(X)$.

(i). Let $0 = X_0 < X_1 < \cdots < X_n = X$ be a maximal series of T-modules, and $\Theta_i = C_T^\circ(X_i/X_{i-1})$. As $\Theta = (\cap_i \Theta_i)^\circ$, one has $\operatorname{cork} \Theta \leq \sum_i \operatorname{cork} \Theta_i$. So we may assume that X itself is T-minimal.

By Zilber's Field Theorem, there is a field structure $\mathbb{L}$ such that T/Θ embeds into $\mathbb{L}^\times$ and $X \simeq \mathbb{L}_+$; the first claim follows. If in addition X is degenerate, that is if we know $\Theta \neq 1$, then by Wagner's Theorem Θ must contain torsion; as Θ is connected it follows that $T/\Theta \not\simeq \mathbb{L}^\times$, and the embedding is proper, whence the second claim.

(ii). Considering a maximal series of $T \cdot \langle\zeta\rangle$-modules, we may now assume that X is $T \cdot \langle\zeta\rangle$-minimal.

Let $Y \leq X$ be a T-minimal T-submodule. If $Y < X$, then $Y \cap Y^\zeta$ is finite, and $X = Y(+)Y^\zeta$. Moreover $C_T^\circ(Y) = \Theta$. Applying (i) we find $\operatorname{cork} \Theta \leq \operatorname{rk} Y = \frac{\operatorname{rk} X}{2}$, the inequality being strict if X is degenerate.

We now suppose that $Y = X$, that is X is T-minimal. But now Lemma 1.13 forces the action of ζ to be trivial on T/Θ, whence $\Theta = T$, and the claim is obvious. $\square$

3.2 Maximin

Proposition 3.6. *The largest degenerate T-submodule of V exists; it has rank $\leq \nu$.*

Proof.

Step 1. *There is a non-trivial degenerate T-submodule of V.*

Proof: Suppose not. Let $V_1 \leq V_2 \leq V$ be B-submodules, with V_1 and V_2/V_1 B-minimal. Notice that by Malcev's Theorem (Fact 1.2), both V_1 and V_2/V_1 are even T-minimal. Notice further that $V_2 < V$, as otherwise the action is quadratic, and Fact 1.1 yields a contradiction.

If $\operatorname{rk} V_1 \neq k$ then by Lemma 1.9 $T_1 = C_T^\circ(V_1)$ must be infinite; taking $C_V^\circ(T_1) \geq V_1$ we are done. So we may assume $\operatorname{rk} V_1 = k$.

Suppose $\operatorname{rk} V_2/V_1 \neq k$. As V_2/V_1 is T-minimal, the group $T_2 = C_T^\circ(V_2/V_1)$ is non-trivial by Lemma 1.9. Let $t_2 \in \operatorname{Tor} T_2$ be given by Lemma 1.10 (with respect to the action on V); $V_2 = C_{V_2}(t_2) \oplus [V_2, t_2]$. Now $C_{V_2}(T_2) = C_{V_2}(t_2)$ covers V_2/V_1 by Corollary 1.12, so $C_{V_2}(T_2)$ is non-trivial; in particular $C_V^\circ(T_2) \neq 1$: we are done.

So suppose $\operatorname{rk} V_2/V_1 = k$, that is $\operatorname{rk} V_2 = 2k$, and let $W_2 = (V_2 \cap V_2^\zeta)^\circ$. Clearly $\operatorname{rk} W_2 \geq 2k - \nu > 0$. If $(V_1 \cap W_2)^\circ \neq 0$, then by T-minimality of V_1, one has $V_1 \leq W_2$. By T-minimality of V_1 and V_2/V_1, one finds that W_2 is either V_1 or V_2, a contradiction as neither is ζ-invariant.

Therefore $(V_1 \cap W_2)^\circ = 0$, and in particular $V_2 = V_1(+)W_2$; whence W_2 is T-minimal, and ζ-invariant. As ζ inverts T, Lemma 1.13 then forces T to centralize W_2: we are done. $\Diamond$

Step 2. *Any degenerate T-submodule of V has rank $\leq \nu$.*

Proof: Let X be degenerate and $\Theta = C_T^\circ(X) \neq 1$. We first claim that for x generic in X, $C_G^\circ(x)$ is semi-simple. Otherwise, as $C_G^\circ(x)$ contains $\Theta \leq T$, it contains either U or U^ζ; we may assume that for x generic in X, U centralizes x. Thus U centralizes X. As the latter is ζ-invariant, it follows that $G = \langle U, U^\zeta \rangle$ centralizes X, a contradiction.

Hence, the centralizer in G of the generic element of X is semi-simple. Let $x \in X$ be generic, and suppose that there is $g \in G$ such that $x \in X^g$. Then $C_G^\circ(x) \geq \langle \Theta, \Theta^g \rangle$ which is semi-simple. Then $C_G^\circ(\langle \Theta, \Theta^g \rangle)$ is an algebraic torus, which can be only $C_G^\circ(\Theta) = T$, and only T^g for a similar reason. Hence $g \in N_G(T) = T \cdot \langle \zeta \rangle = N_G(X)$. So X is generically disjoint from its distinct conjugates; it follows that

$$\operatorname{rk} X^G = 2k + \operatorname{rk} X \leq \operatorname{rk} V = 2k + \nu$$

which proves the claim. $\Diamond$

Step 3. *The sum of two degenerate T-submodules is degenerate.*

Proof: Let X_1, X_2 be degenerate T-submodules, and $\Theta_i = C_T^\circ(X_i) \neq 1$. Considering $\hat{X}_i = C_V^\circ(\Theta_i) \geq X_i$, we may assume that the X_i are $T \cdot \langle \zeta \rangle$-modules.

By Lemma 3.5 (ii) $\operatorname{cork}_T \Theta_i < \frac{\operatorname{rk} X_i}{2}$, so using Step 2 $\operatorname{rk} \Theta_i > k - \frac{\nu}{2} \geq \frac{k}{2}$. It follows that $\Theta_{12} = (\Theta_1 \cap \Theta_2)^\circ$ is non-trivial. Now $X_{12} = C_V^\circ(\Theta_{12})$ contains $X_1 + X_2$. $\Diamond$

This concludes the proof of Proposition 3.6. $\square$

Notation 3.7. Let X be the largest degenerate T-submodule of V, and let $\Theta = C_T^\circ(X)$. Let $\theta_0 \in \operatorname{Tor}\Theta$ be given by Lemma 1.10 (for the action on V), so that $X = C_V(\theta_0)$ and $[V,\Theta] = [V,\theta_0]$.

Lemma 3.8. *U does not centralize X.*

Proof. Assume it does. Then since X is ζ-invariant, so does U^ζ. It follows that $\langle U, U^\zeta\rangle = G$ centralizes $X \neq 0$, a contradiction. □

We shall eventually prove that $T = \Theta$ centralizes X (Proposition 3.14 below). But let us first introduce some useful objects.

3.3 Minimax and a commutator subgroup

Notation 3.9. Let $M = [V,\Theta] = [V,\theta_0]$ (see Notation 3.7).

One has $V = M \oplus X$ by Corollary 1.12.

Lemma 3.10. *M has rank $2k$. Moreover, non-trivial proper T-submodules of M are T-minimal and have rank k.*

Proof. As $V = C_V(\theta_0) \oplus M$ and $X = C_V(\theta_0)$, we know $\operatorname{cork} M \leq \nu$ by Proposition 3.6, so $\operatorname{rk} M \geq 2k$. Let $V_1 \leq M$ be a T-minimal T-submodule. If $\operatorname{rk} V_1 \neq k$ then by Lemma 1.9, $C_T^\circ(V_1)$ is infinite, and it follows from Proposition 3.6 that $V_1 \leq X$ by definition of X, a contradiction to $X \cap M = 0$. Hence V_1 has rank k.

Now let $V_1 < V_2 \leq M$ be a T-submodule such that V_2/V_1 is T-minimal. If $\operatorname{rk} V_2/V_1 \neq k$, then by Lemma 1.9 again, $T_2 = C_T^\circ(V_2/V_1)$ is infinite. $C_{V_2}(T_2)$ covers V_2/V_1, but by Proposition 3.6 it lies in X. Since $X \cap M = 0$, we have a contradiction.

So $\operatorname{rk} V_2 = 2k$. If $V_2 < M$, then there is $V_3 \leq M$ such that V_3/V_2 is T-minimal. But $M < V$, so $\operatorname{rk} V_3/V_2 < k$, and we argue as before to deduce that $V_3 \leq V_2 + X$, whence $V_3 = V_2$, a contradiction. Hence $M = V_2$ has rank $2k$. It is now clear that any non-trivial proper T-submodule of M is actually T-minimal, and therefore has rank k. □

Notation 3.11. Let $Y = [X,U]$.

In the representation of PSL_2 in its Lie algebra, this should capture the set of upper-triangular matrices.

Lemma 3.12. *Y is a subgroup of M of rank k. It is B-minimal and $C_T(Y) \leq Z(G)$. Moreover, $M = Y \oplus Y^\zeta$; U centralizes Y, $(X+Y)/Y$, and $V/(X+Y)$.*

Proof. Let $u \in U$, $x \in X$, and $\theta \in \Theta$. We denote by $\cdot$ the action of T on U (isomorphic to the action of $\mathbb{K}^\times$ on $\mathbb{K}_+$). One has

$$\begin{aligned} [[x,u],\theta]] &= [x,u]^\theta - [x,u] \\ &= [x,\theta\cdot u] - [x,u] \\ &= [x,(\theta-1)\cdot u] \end{aligned}$$

So $[x, (\theta - 1) \cdot u] \in [V, \theta] \leq M$. But $(\Theta - 1) \cdot U = U$, ,so $[x, U] \leq M$ and $Y = [X, U] \leq M$. We claim that Y is neither 0 nor M. By Lemma 3.12, $Y > 0$, and Y is clearly definable, connected, and T-invariant. If $Y = M$, then Y is $\langle U, \zeta \rangle = G$-invariant, which contradicts G-minimality of V. So $0 < Y < M$.

By Lemma 3.10, Y is T-minimal and has rank k. Now by construction, U normalizes Y; so does T, hence Y is B-invariant. In particular, Y is B-minimal. By Lemma 1.3 $[Y, U] < Y$, so $[Y, U] = 0$, that is U centralizes Y. If Y were ζ-invariant, we would have that $G = \langle U, U^\zeta \rangle$ centralizes Y, a contradiction. By T-minimality and rank estimates, it follows that $M = Y + Y^\zeta$; if there is a non-trivial intersection then T must centralize this intersection, against $Y \cap X \leq M \cap X = 0$. So $M = Y \oplus Y^\zeta$.

We now consider $C_T(Y)$. Suppose that there exists $t \in C_T(Y)$ which is not central in G. Let $\check{M} = M \setminus (Y \cup Y^\zeta)$; t centralizes M and $\check{M}$. Notice that $N_G(\check{M}) = T \cdot \langle \zeta \rangle$. Now let $g \notin N_G(\check{M})$. If there is $m \in \check{M} \cap \check{M}^g$, then $C_G(m)$ contains t and t^g, so it can't be semi-simple; it must therefore contain a maximal unipotent subgroup, which can only be U or U^ζ. So one has for instance $U \leq C_G(m)$, which proves $m \in C_M(U) = Y$ by Lemma 3.10; this violates $m \notin Y$ and the other case is similar. So distinct conjugates of $\check{M}$ are disjoint, which blows up the rank of V. Hence $C_T(Y) \leq Z(G)$.

It remains to show that U centralizes $V/(X + Y)$. As $X + Y$ is B-invariant, $V/(X + Y)$ is a B-module. Now $V/(X + Y) \simeq M/Y$ as a T-module, so $V/(X + Y)$ is T-minimal, whence B-minimal. Therefore U centralizes $V/(X + Y)$ by Malcev's Theorem, Fact 1.2. □

By Zilber's Field Theorem, there is a field structure $\mathbb{L}_1$ such that $Y \simeq (\mathbb{L}_1)_+$ and $T/C_T(Y) \hookrightarrow \mathbb{L}_1^\times$. But $C_T(Y)$ is finite, so $T/C_T(Y) \simeq \mathbb{K}^\times$; by Wagner's Theorem it follows that $T/C_T(Y) \simeq \mathbb{L}_1^\times$. Hence Y is a $\mathbb{K}$-vector space.

We eventually get rid of the characteristic 2 case.

Lemma 3.13. *$p \neq 2$.*

Proof. Suppose $p = 2$. We show first that $[V, U, U] = 0$.

Fix $u \in U^\#$. Then since Θ additively generates $\mathbb{K}$, we have $C_X(u) = C_X(u, \Theta) = C_X(U) = 0$. Since $Y \leq C_V(U)$ we find $C_{(X+Y)}(u) = Y + C_X(u) = C_{(X+Y)}(U)$. Now $[V, u] \leq X + Y$ by Lemma 3.12; also, $[V, u] \leq C_V(u)$ since $p = 2$. So $[V, u] \leq C_{X+Y}(u) \leq C_V(U)$. Hence $[V, U, U] = 0$.

Since $C_V(G) = 0$ (Lemma 3.3) and $C_G(V) = 1$, Fact 1.1 forces V to contain an isomorphic copy W of the natural representation. Using indecomposability, W must clearly be definable; as it is G-invariant, it follows $V = W$ has rank $2k$, a contradiction. □

From now on the characteristic is an odd prime p. This will play a crucial role in the rank computation of Corollary 3.15. But we shall now make a first incursion into linear structures.

3.4 Finer study of X

As promised, we prove the following.

Proposition 3.14. *$\Theta = T$ centralizes X.*

Proof. If $[T, X] \leq C_X(U)$, then since $[T, X]$ is ζ-invariant one even has $[T, X] \leq C_X(U, U^\zeta) \leq C_V(G) = 0$: we are done.

So we assume that $\bar{X} = ([T, X] + C_X(U))/C_X(U)$ is not 0 and shall prove a contradiction.

Notation 1.

- Let W be a T-minimal T-subgroup of $\bar{X}$. (Notice that T does not centralize W, since Corollary 1.12 shows that $C_{\bar{X}}(T) = 0$.)
- Let $\mathbb{L}$ be the definable field structure such that $W \simeq \mathbb{L}_+$ and $T/C_T(W) \hookrightarrow \mathbb{L}^\times$. We use $*$ to denote multiplication in $\mathbb{L}$.

There is such a structure indeed, as we have observed that T does not centralize W. We shall embed $\mathbb{L}$ into $\mathbb{K}$ definably. This will eventually force $\operatorname{rk} \mathbb{L} = \operatorname{rk} \mathbb{K}$, which will be very close to a conclusion. Let us work on constructing such a definable field embedding.

Fix $u_0 \in U^\#$, $w_0 \in W^\#$. Recall that W is made of classes modulo $C_X(U)$, so $[w_0, u_0]$ denotes a well-defined element of Y. Moreover, this element is non-trivial, as otherwise $\langle \Theta \cdot u_0 \rangle = U$ centralizes w_0, and $w_0 = 0$ in W. So $[w_0, u_0]$ is a well-defined, *non-trivial* element of $Y \simeq \mathbb{K}_+$. We introduce the following operation.

Notation 2. Fix $u_0 \in U^\#$ and $w_0 \in W^\#$. For $w' \in W^\#$ there is a unique $k \in \mathbb{K}^\times$ such that
$$[w', u_0] = k \cdot [w_0, u_0]$$

Step 3. *k depends on w_0 and w', but not on u_0.*

Proof: Since $T \simeq \mathbb{K}^\times$, there is $t \in T$ such that $k \cdot [w_0, u_0] = [w_0, u_0]^t$. Let u' be any element of $U^\#$. As Θ additively generates $\mathbb{K}$, there are elements $\theta_1, \dots, \theta_n \in \Theta$ such that $u' = \theta_1 \cdot u_0 + \cdots + \theta_n \cdot u_0$. Since Θ centralizes W, it follows that:

$$\begin{array}{rclcll}
[w', u'] & = & \sum_i [w', \theta_i \cdot u_0] & = & \sum_i [w', u_0]^{\theta_i} & \\
& = & \sum_i (k \cdot [w_0, u_0])^{\theta_i} & = & \sum_i [w_0, u_0]^{t\theta_i} & \\
& = & \sum_i [w_0, u_0]^{\theta_i t} & = & (\sum_i [w_0, u_0]^{\theta_i})^t & \\
& = & k \cdot \sum_i [w_0, u_0]^{\theta_i} & = & k \cdot [w_0, \sum_i \theta_i \cdot u_0] & = \; k \cdot [w_0, u']
\end{array}$$

which proves the claim. $\Diamond$

Notation 4. For $\ell \in \mathbb{L}$, let $f(\ell) \in \mathbb{K}$ be defined as the unique k such that:

$$[\ell * w_0, u_0] = k \cdot [w_0, u_0]$$

Step 5. *f is a definable field embedding.*

Proof: f is clearly additive. We now show that it is multiplicative. Let $\ell_1, \ell_2 \in \mathbb{L}$; let k_1, k_2 be their images through f. First assume that ℓ_1 and ℓ_2 are in the image of $T/C_T(W)$: there exist elements $t_1, t_2 \in T$ which map to ℓ_1, ℓ_2. Then bearing in mind that f does not depend on the base point u_0 (Step 3), one has:

$$\begin{array}{rclcl} f(\ell_1 * \ell_2) \cdot [w_0, u_0] & = & [(\ell_1 * \ell_2) * w_0, u_0] & = & [w_0^{t_1 t_2}, u_0] \\ & = & [w_0^{t_1}, u_0^{t_2^{-1}}]^{t_2} & = & [\ell_1 * w_0, u_0^{t_2^{-1}}]^{t_2} \\ & = & (k_1 \cdot [w_0, u_0^{t_2^{-1}}])^{t_2} & = & [w_0, u_0^{t_2^{-1}}]^{t_1 t_2} \\ & = & [w_0, u_0^{t_2^{-1}}]^{t_2 t_1} & = & k_1 \cdot ([w_0, u_0^{t_2^{-1}}])^{t_2} \\ & = & k_1 \cdot [w_0^{t_2}, u_0] & = & k_1 \cdot [\ell_2 * w_0, u_0] \\ & = & k_1 \cdot k_2 \cdot [w_0, u_0] & = & (f(\ell_1) \cdot f(\ell_2)) \cdot [w_0, u_0] \end{array}$$

It follows that $f(\ell_1 * \ell_2) = f(\ell_1) \cdot f(\ell_2)$. By additivity, f is actually multiplicative on the set additively generated by the image of T in $\mathbb{L}$, that is on all of $\mathbb{L}$. $\lozenge$

Now we have a definable field embedding from $\mathbb{L}$ to $\mathbb{K}$. As there are no extensions of infinite fields in a universe of finite Morley rank, this forces

$$\operatorname{rk} \mathbb{L} = \operatorname{rk} \mathbb{K} = \operatorname{rk} Y = k \geq \nu \geq \operatorname{rk} X \geq \operatorname{rk} W = \operatorname{rk} \mathbb{L}$$

In particular $\nu = k$, and W is a quotient of X by a finite subgroup; so X is T-minimal. As it is ζ-invariant and ζ inverts T, Lemma 1.13 implies $\Theta = T$.

This is a contradiction, and in particular T does centralize X. All our construction collapses: the field structure $\mathbb{L}$ vanishes. $\square$

Proposition 3.14 is now proved. Remember that we have lost the field stucture $\mathbb{L}$ on which the argument relied.

3.5 Consequences

Before moving to the identification argument we draw important consequences of Proposition 3.14.

Corollary 3.15. *$G \simeq \mathrm{PSL}_2$ and ζ inverts X; $\nu = k$ is the rank of X; $C_V(U) = Y$ is disjoint from X.*

Proof. By Proposition 3.14 T centralizes X, so the involution i of T cannot invert V; we may suppose $G \simeq \mathrm{PSL}_2$. As $G \simeq \mathrm{PSL}_2(\mathbb{K})$, there is no central element; by Lemma 3.12, T is faithful on Y. In particular, i inverts Y and Y^ζ, so it inverts $M = Y \oplus Y^\zeta$. It follows that $\operatorname{rk} V^{-i} \geq 2k$, and by Lemma 2.6, one finds $\nu = k$. Moreover $V^{+i+\zeta} = 0$ and this means that ζ inverts X. Also, since $V = M \oplus X$ (see Notation 3.9) and $\operatorname{rk} M = 2k$ (Lemma 3.10), one has $\operatorname{rk} X = k$.

We now let $\hat{Y} = C_V(U)$; if $Y < \hat{Y}$, let $Y \leq Z \leq \hat{Y}$ be such that Z/Y is B-minimal. As $\hat{Y}$ is disjoint from $\hat{Y}^\zeta$, it has rank at most $\frac{3k}{2}$, so $\operatorname{rk} Z/Y < k$. By Lemma 1.9, $C_T(Z/Y) \neq 1$; Z contains a non-trivial degenerate module, that is $C_X(U) \neq 0$. As ζ inverts X, $C_X(U)$ is also centralized by U^ζ while $C_V(G) = 0$. Hence $\hat{Y} = Y$ and $X \cap Y = 0$. $\square$

This is indeed the case in the three-dimensional representation of PSL_2. We now work towards understanding the scalar action on X.

Corollary 3.16. *Let $x \in X$, $t \in T$, $u \in U^{\#}$. Then there is a unique $x' \in X$ such that $[x', u] = [x, u]^t = [x, t \cdot u]$; x' depends on x and t, but not on u.*

Proof. Fix $u_1 \in U^{\#}$ and consider the definable morphism from X to Y which maps x to $[x, u_1]$. This is injective, as the kernel lies in $C_X(u_1) = C_X(T, u_1) \leq C_X(U) = 0$. By equality of ranks, the map is a bijection. Now suppose another $u_2 \in U^{\#}$ is given, and we have elements x'_1, x'_2 such that $[x'_i, u_i] = [x, u_i]^t$. Then there is $\tau \in T$ such that $u_2 = u_1^\tau$, and it follows that:

$$\begin{array}{rcccccccc} [x'_2, u_2] & = & [x, u_2]^t & = & [x, u_1^\tau]^t & = & [x, u_1]^{\tau t} & & \\ & = & [x, u_1]^{t\tau} & = & [x'_1, u_1]^\tau & = & [x'_1, u_1^\tau] & = & [x'_1, u_2] \end{array}$$

whence $x'_1 = x'_2$, as claimed. □

3.6 Identification

Let us serve some refreshments.

- ζ has order 2 (Corollary 3.15)
- $V = Y \oplus X \oplus Y^\zeta$ (Notation 3.9 and Lemma 3.12).
- $Y = [X, U]$ is B-minimal (Lemma 3.12)
- $X = C_V(T)$ is inverted by ζ (Corollary 3.15)
- X and Y have rank k.

We define a $\mathbb{K}$-scalar action on each component:

Notation 3.17.

- On Y, $k \cdot y$ is given by the action of T.
- On Y^ζ, we let $k \cdot y^\zeta = (k \cdot y)^\zeta$.
- On X, we let $k \cdot x$ be the unique $x' \in X$ such that $[x', u] = k \cdot [x, u]$ (Corollary 3.16; this does not depend on the choice of u).

We shall check that G acts linearly. We do it piecewise; notice that when we claim that U acts linearly on X, we mean that the operation induced by elements of U from X *to* V is linear, without claiming anything about invariance under the action.

Lemma 3.18. *$T \cdot \langle \zeta \rangle$ acts linearly on V. U acts linearly on $Y \oplus X$.*

Proof. By construction, T is linear on Y and Y^ζ. It is linear on X, as it acts trivially! By construction, ζ is linear on $Y \oplus Y^\zeta$. As it inverts X, it is also linear on X. So $T \cdot \langle\zeta\rangle$ is linear on V.

As U acts trivially on Y, it is linear on Y. It remains to see that U is linear on X. Let $u \in U$, $x \in X$, and $k \in \mathbb{K}$. By definition of the action on X, one has $[k \cdot x, u] = k \cdot [x, u]$, and therefore:

$$k \cdot x^u - k \cdot x = k \cdot [x, u] = [k \cdot x, u] = (k \cdot x)^u - k \cdot x$$

Linearity follows. $\square$

It remains to prove that U is linear on Y^ζ. As T is, and since it acts transitively on $U^\#$, it suffices to exhibit one non-trivial element of U which is linear on Y^ζ.

Notation 3.19 (Bryant Park element)**.** Let $w = \zeta$ (it is an involution, after all). Let $u \in U$ be such that (wu) has order 3.

Such an element exists (this may be viewed as a special case of the Steinberg relations). We shall prove that this particular u is linear on Y^ζ.

Lemma 3.20. *For any $y \in Y$, there is a unique $x \in X$ such that $y^{wu} = y + x + y^w$.*

Proof. A priori, one has
$$y^{wu} = y_1 + x + y_2^w$$
for elements $y_1, y_2 \in Y$ and $x \in X$. But U centralizes Y, $(X+Y)/Y$, and $V/(X+Y)$ by Lemma 3.12. So $y_2 = y$. We push further, using the fact that w inverts X (Corollary 3.15).

$$\begin{array}{rcl} y^{(wu)^2} & = & y_1^{wu} + x^{wu} + y^{wwu} \\ & = & y_1^{wu} - x^u + y \end{array}$$

and
$$y = y^{(wu)^3} = y_1^{wuwu} - x^{uwu} + y^{wu}$$
whence applying u^{-1},
$$y = y_1^{wuw} - x^{uw} + y^w$$

Now U^w centralizes Y^w, $X + Y^w/Y^w$, and $V/(X + Y^w)$ (Lemma 3.12 again), so $[u^w, y_1] \in X + Y^w$. It follows that y_1 is the projection on Y of y_1^{wuw}. On the other hand, $x^u \in X + Y$, so $x^{uw} \in X + Y^w$. Taking projections on Y modulo $X + Y^w$, one has $y_1 = y$. $\square$

Lemma 3.21. *Let $y \in Y$ and $x \in X$ be as in Lemma 3.20. Then $[x, u] = 2y$.*

Proof. By definition,
$$y^{wu} = y + x + y^w$$

Let us iterate:

$$\begin{aligned} y^{(wu)^2} &= y^{wu} + x^{wu} + y^{wwu} \\ &= (y + x + y^w) - x^u + y \\ &= 2y + x - x^u + y^w \end{aligned}$$

and

$$\begin{aligned} y^{(wu)^3} &= 2y^{wu} + x^{wu} - x^{uwu} + y^{wwu} \\ &= 2(y + x + y^w) - x^u - x^{uwu} + y \\ &= 3y + 2x - x^u - x^{uwu} + 2y^w \end{aligned}$$

As wu has order three, one has:

$$2y + 2x - x^u - x^{uwu} + 2y^w = 0$$

Now u centralizes $(Y + X)/Y$, so there is $y_1 \in Y$ such that $x^u = x + y_1$. Let x_1 be associated to y_1 by Lemma 3.20: one has $y_1^{wu} = y_1 + x_1 + y_1^w$. Hence

$$\begin{aligned} x^{uwu} &= x^{wu} + y_1^{wu} \\ &= -x^u + (y_1 + x_1 + y_1^w) \\ &= -x - y_1 + y_1 + x_1 + y_1^w \\ &= x_1 - x + y_1^w \end{aligned}$$

It follows that

$$2y + 2x - (x + y_1) - (x_1 - x + y_1^w) + 2y^w = 0,$$

and projecting onto Y modulo $X + Y^w$,

$$y_1 = 2y$$

so that $[x, u] = y_1 = 2y$. □

Notation 3.22. For $y \in Y$, let $x(y)$ be the element x given by Lemma 3.20.

Lemma 3.23. *The function $x(y)$ is $\mathbb{K}$-linear.*

Proof. Let $k \in \mathbb{K}$. Then

$$[x(k \cdot y), u] = 2(k \cdot y) = k \cdot (2y) = k \cdot [x(y), u] = [k \cdot x(y), u]$$

And we are done. □

Corollary 3.24. *u is linear on Y^w.*

Proof. Let $y \in Y$ and $k \in \mathbb{K}$; let $y_2 = k \cdot y$, and $x_2 = x(y_2)$. Then

$$(k \cdot y^w)^u = y_2^{wu} = y_2 + x_2 + y_2^w = k \cdot y + x_2 + k \cdot y^w$$

On the other hand,

$$k \cdot y^{wu} = k \cdot (y + x + y^w) = k \cdot y + k \cdot x + k \cdot y^w$$

As x is $\mathbb{K}$-linear, both expressions are equal: u is linear on Y^w. □

It follows that $G = \langle T, \zeta, u\rangle$ is linear on V. We may now finish the proof. For clarity let us denote by $\mathbb{K}'$ the isomorphic copy of $\mathbb{K}$ which is definable in V. We have a definable embedding ι of $G \simeq \mathrm{PSL}_2(\mathbb{K}')$ into $\mathrm{GL}_3(\mathbb{K}')$. The algebraic torus T of G maps to an algebraic torus of $\mathrm{GL}_3(\mathbb{K}')$; hence $\iota(T)$ is Zariski-closed in $\mathrm{GL}_3(\mathbb{K}')$. Now $\iota(G) = \langle \iota(T^g) : g \in G\rangle$; it follows that $\iota(G)$ is Zariski-closed in $\mathrm{GL}_3(\mathbb{K}')$. So G, which is definably isomorphic to $\iota(G)$, is algebraic over $\mathbb{K}'$. This concludes the identification together with the proof of our Theorem.

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