# DEFINABLE $C^\infty G$ MANIFOLD STRUCTURES OF DEFINABLE $C^r G$ MANIFOLDS

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Abstract. Let $G$ be a compact definable $C^\infty$ group and $2 \le r < \infty$. Let $X$ be a noncompact affine definable $C^rG$ manifold and $X_1, \dots, X_k$ noncompact codimension one definable $C^rG$ submanifolds of $X$ such that $X_1, \dots, X_k$ are in general position in $X$ and $(X; X_1, \dots, X_k)$ satisfies the frontier condition. We prove that $(X; X_1, \dots, X_k)$ admits a unique definable $C^\infty G$ manifold structure $(Y; Y_1, \dots, Y_k)$.

## 1. Introduction[1]

Let $\mathcal{M} = (\mathbb{R}, +\cdot, <, e^x, \dots)$ denote an exponential o-minimal expansion of the standard structure $\mathcal{R} = (R, +\cdot, <)$ of the field $\mathbb{R}$ of real numbers such that $\mathcal{M}$ admits the $C^\infty$ cell decomposition and has piecewise controlled derivatives. The term "definable" means "definable with parameters in $\mathcal{M}$". General references on o-minimal structures are [4], [7], see also [23]. The Nash category is a special case of definable $C^\infty$ categories and it coincides with the definable $C^\infty$ category based on $\mathcal{R}$ [24]. Further properties and constructions of them are studied in [5], [6], [8], [21] and there are uncountably many o-minimal expansions of $\mathcal{R}$ [22]. Equivariant definable $C^r$ cases are studied in [12], [13], [14], [15] when $0 \le r < \infty$. Everything is considered in $\mathcal{M}$ and each manifold does not have boundary unless otherwise stated.

In this paper we consider simultaneous definable $C^\infty G$ manifold structures of definable $C^rG$ manifolds and their definable $C^rG$ submanifolds when $2 \le r < \infty$.

Let $X$ be a $C^r$ manifold, $X_1, \dots, X_k$ $C^r$ submanifolds of $X$ and $r \ge 1$. We say that $\{X_i\}_{i=1}^k$ *are in general position in* $X$ if for each $i \in I$ and $J \subset I - \{i\}$, $X_i$ intersects transverse to $\cap_{j \in J} X_j$.

Let $G$ be a compact definable $C^r$ group, $X$ a noncompact affine definable $C^r$ manifold, $X_1, \dots, X_k$ noncompact definable $C^r$ submanifolds of $X$ and $1 \le r \le \infty$. By 2.10 [14] and 1.3 [17], we may assume that $X$ is a bounded definable $C^rG$ submanifold of some representation $\Omega$ of $G$. We say that $(X; X_1, \dots, X_k)$ satisfies *the frontier condition* if each $\overline{X_i} - X_i$ is contained in $\overline{X} - X$, where $\overline{X_i}$ (resp. $\overline{X}$) denotes the closure of $X_i$ (resp. $X$) in $\Omega$.

**Theorem 1.1.** *Let $G$ be a compact definable $C^\infty$ group and $2 \le r < \infty$. Let $X$ be an affine definable $C^rG$ manifold and $X_1, \dots, X_k$ codimension one definable $C^rG$ submanifolds of $X$ such that $X_1, \dots, X_k$ are in general position in $X$. If either $X, X_1, \dots, X_k$ are compact, or $X, X_1, \dots, X_k$ are noncompact and $(X; X_1, \dots, X_k)$ satisfies the frontier condition, then there exist an affine definable $C^\infty G$ manifold $Y$, definable $C^\infty G$ submanifolds $Y_1, \dots, Y_k$*

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*of $Y$ and a definable $C^rG$ diffeomorphism $f : X \to Y$ such that for each $i$ $f(X_i) = Y_i$. Moreover if $Z$ is an affine definable $C^\infty G$ manifold, $Z_1, \dots, Z_k$ are definable $C^\infty G$ submanifolds of $Z$ and $(X; X_1, \dots, X_k)$ is definably $C^rG$ diffeomorphic to $(Z; Z_1, \dots, Z_k)$, then $(Y; Y_1, \dots, Y_k)$ is definably $C^\infty G$ diffeomorphic to $(Z; Z_1, \dots, Z_k)$.*

A non-equivariant definable non-relative version of Theorem 1.1 is proved in 1.5 [9] and a locally definable non-relative version of it is proved in [16] when $G$ is a finite abelian group.

**Corollary 1.2.** *Let $G$ be a compact definable $C^\infty$ group and $2 \le r < \infty$. Let $X$ be an affine definable $C^rG$ manifold. Then $X$ admits a unique definable $C^\infty G$ manifold structure up to definable $C^\infty G$ diffeomorphism.*

If $G$ is a finite abelian group, then every definable $C^\infty G$ manifold is affine [19]. By a way similar to the proof of it proves every definable $C^rG$ manifold is affine when $r$ is a non-negative integer. Thus we have the following theorem as a corollary of Theorem 1.1.

**Theorem 1.3.** *Let $G$ be a finite abelian group and $2 \le r < \infty$. Let $X$ be a definable $C^rG$ manifold and codimension one $X_1, \dots, X_k$ definable $C^rG$ submanifolds of $X$ such that $X_1, \dots, X_k$ are in general position in $X$. If either $X, X_1, \dots, X_k$ are compact, or $X, X_1, \dots, X_k$ are noncompact and $(X; X_1, \dots, X_k)$ satisfies the frontier condition, then there exist a definable $C^\infty G$ manifold $Y$, definable $C^\infty G$ submanifolds $Y_1, \dots, Y_k$ of $Y$ and a definable $C^rG$ diffeomorphism $f : X \to Y$ such that for each $i$ $f(X_i) = Y_i$. Moreover if $Z$ is a definable $C^\infty G$ manifold, $Z_1, \dots, Z_k$ are definable $C^\infty G$ submanifolds of $Z$ and $(X; X_1, \dots, X_k)$ is definably $C^rG$ diffeomorphic to $(Z; Z_1, \dots, Z_k)$, then $(Y; Y_1, \dots, Y_k)$ is definably $C^\infty G$ diffeomorphic to $(Z; Z_1, \dots, Z_k)$.*

As a corollary of Theorem 1.3, we have the following corollary.

**Corollary 1.4.** *Let $G$ be a finite abelian group and $2 \le r < \infty$. Let $X$ be a definable $C^rG$ manifold. Then $X$ admits a unique definable $C^\infty G$ manifold structure up to definable $C^\infty G$ diffeomorphism.*

## 2. Proof of Theorem 1.1

Suppose that $r$ is a positive integer, $\infty$ or $\omega$. A definable $C^r$ manifold $G$ is a *definable $C^r$ group* if the group operations $G \times G \to G$ and $G \to G$ are definable $C^r$ maps.

Let $G$ be a definable $C^r$ group. A *representation map* of $G$ is a group homomorphism from $G$ to some $O_n(\mathbb{R})$ which is a definable $C^r$ map. A *representation* means the representation space of a representation map of $G$. In this paper, we assume that every representation of $G$ is orthogonal. A *definable $C^rG$ submanifold* of a representation $\Omega$ of $G$ is a $G$ invariant definable $C^r$ submanifold of $\Omega$. A *definable $C^rG$ manifold* is a pair $(X, \phi)$ consisting of a definable $C^r$ manifold $X$ and a group action $\phi : G \times X \to X$ which is a definable $C^r$ map. We simply write $X$ instead of $(X, \phi)$. A definable $C^rG$ manifold is *affine* if it is definably $C^rG$ diffeomorphic to a definable $C^rG$ submanifold of some representation of $G$. Definable $C^rG$ manifolds and affine definable $C^rG$ manifolds are introduced in [14].

By a way similar to the proof of 1.3 [18], we have the following theorem.

**Theorem 2.1.** *Let $G$ be a compact definable $C^r$ group and $1 \le r < \infty$. Let $X, Y$ be compact affine definable $C^rG$ manifolds, $X_1, \dots, X_n$ (reps. $Y_1, \dots, Y_n$) compact definable $C^rG$ submanifolds of $X$ (resp $Y$) such that $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$) are in general position. Suppose that $f : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ is a $C^rG$ map. Then $f$ is approximated by a definable $C^rG$ map $h : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ in the $C^r$ Whitney topology. Moreover if for $1 \le i_1 < \dots < i_k \le n$, $f|X_{i_1}, \dots, f|X_{i_k}$ are definable $C^rG$ maps, then we can take $h$ such that $h| \cup_{j=1}^k X_{i_j} = f| \cup_{j=1}^k X_{i_j}$.*

**Theorem 2.2** ([10]). *Let $X, Y$ be compact $C^r$ manifolds and $1 \le r \le \infty$ The subset of diffeomorphisms from $X$ to $Y$ is open in the set of $C^r$ maps from $X$ to $Y$ with respect to the $C^r$ Whitney topology.*

Let $f : U \to \mathbb{R}$ be a definable $C^\infty$ function on a definable open subset $U \subset \mathbb{R}^n$. We say that $f$ has *controlled derivatives* if there exist a definable continuous function $u : U \to \mathbb{R}$, real numbers $C_1, C_2, \dots$ and natural numbers $E_1, E_2, \dots$ such that $|D^\alpha f(x)| \le C_{|\alpha|} u(x)^{E_{|\alpha|}}$ for all $x \in U$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$, where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. We say that $\mathcal{M}$ has *piecewise controlled derivatives* if for every definable $C^\infty$ function $f : U \to \mathbb{R}$ defined in a definable open subset $U$ of $\mathbb{R}^n$, there exist definable open sets $U_1, \dots, U_l \subset U$ such that $\dim(U - \cup_{i=1}^l U_i) < n$ and each $f|U_i$ has controlled derivatives. G.O. Jones [20] proved the following theorem.

**Theorem 2.3** (1.2 [20]). *Every definable closed subset of $\mathbb{R}^n$ is the zero set of a definable $C^\infty$ function on $\mathbb{R}^n$.*

Using Theorem 2.3, we have the following two results [17].

**Theorem 2.4** ([17]). *Let $G$ be a compact definable $C^\infty$ group and $\Omega$ a representation of $G$. Every $G$ invariant definable closed subset of $\Omega$ is the zero set of a $G$ invariant definable $C^\infty$ function on $\Omega$.*

**Theorem 2.5** ([17]). *Let $G$ be a compact definable $C^\infty$ group and $X$ an affine definable $C^\infty G$ manifold. Suppose that $A, B$ are $G$ invariant definable disjoint closed subsets of $X$. Then there exists a $G$ invariant definable $C^\infty$ function $f : X \to \mathbb{R}$ such that $f|A = 1$ and $f|B = 0$.*

Using Theorem 2.4 and Theorem 2.5, By a way similar to the proof of 1.3 [18], we have the following theorem.

**Theorem 2.6.** *Let $G$ be a compact definable $C^\infty$ group. Let $X, Y$ be affine definable $C^\infty G$ manifolds and $X_1, \dots, X_n$ (reps. $Y_1, \dots, Y_n$) definable $C^\infty G$ submanifolds of $X$ (resp $Y$) such that $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$) are in general position. Suppose that $f : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ is a definable $C^rG$ map and $1 \le r < \infty$. Then $f$ is approximated by a definable $C^\infty G$ map $h : (X; X_1, \dots, X_n) \to (Y; Y_1, \dots, Y_n)$ in the definable $C^r$ topology. Moreover if for $1 \le i_1 < \dots < i_k \le n$, $f|X_{i_1}, \dots, f|X_{i_k}$ are definable $C^\infty G$ maps, then we can take $h$ such that $h| \cup_{j=1}^k X_{i_j} = f| \cup_{j=1}^k X_{i_j}$.*

**Theorem 2.7** ([23]). *Let $X, Y$ be definable $C^r$ submanifolds of $\mathbb{R}^n$ and $0 < r < \infty$. Let $f : X \to Y$ be a definable $C^r$ map. If $f$ is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image), then an approximation of $f$ in the definable $C^r$ topology* *is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image). Moreover if* $f$ *is a diffeomorphism, then* $h^{-1} \to f^{-1}$ *as* $h \to f$.

Let $G$ be a compact definable $C^r$ group, $X$ a noncompact definable $C^rG$ manifold, $X_1, \dots, X_k$ noncompact definable $C^rG$ submanifolds of $X$ in general position in $X$ and $1 \leq r \leq \infty$. We say that $(X; X_1, \dots, X_k)$ is *simultaneously definably $C^rG$ compactifiable* if there exist a compact definable $C^rG$ manifold $Y$ with boundary $\partial Y$, compact definable $C^rG$ submanifolds $Y_1, \dots, Y_k$ of $Y$ with boundary $\partial Y_1, \dots, \partial Y_n$, respectively, and a definable $C^rG$ diffeomorphism $f : X \to Int\ Y$ such that for any $i$, $f(X_i) = Int\ Y_i$, each $\partial Y_i$ is contained in $\partial Y$, and $Y_1, \dots, Y_k$ and $\partial Y$ are in general position in $Y$. Here $Int\ Y$ (resp. $Int\ Y_i$) denotes the interior of $Y$ (resp. $Y_i$).

To prove Theorem 1.1, we have need the following theorem.

**Theorem 2.8** ([19])**.** *Let $G$ be a compact definable $C^r$ group, $X$ a noncompact affine definable $C^rG$ manifold, $X_1, \dots, X_k$ noncompact definable $C^rG$ submanifolds of $X$ in general position in $X$ such that $(X; X_1, \dots, X_k)$ satisfies the frontier condition and $1 \leq r < \infty$. Then $(X; X_1, \dots, X_k)$ is simultaneously definably $C^rG$ compactifiable.*

In Theorem 2.8, we can take $r = \infty$.

**Theorem 2.9** ([17])**.** *Let $G$ be a compact definable $C^\infty$ group, $X$ a noncompact affine definable $C^\infty G$ manifold and $X_1, \dots, X_k$ noncompact definable $C^\infty G$ submanifolds of $X$ in general position in $X$ such that $(X; X_1, \dots, X_k)$ satisfies the frontier condition. Then $(X; X_1, \dots, X_k)$ is simultaneously definably $C^\infty G$ compactifiable.*

A subset $V$ of $\mathbb{R}^n$ is an *algebraic subset* of $\mathbb{R}^n$ if it is the zeros of some polynomial function on $\mathbb{R}^n$. An *algebraic set* means an algebraic subset of some $\mathbb{R}^n$. A point $x$ in an algebraic set $V \subset \mathbb{R}^n$ is *nonsingular of dimension $d$ in $V$* if there exist polynomial functions $p_i : \mathbb{R}^n \to \mathbb{R}$, $(1 \leq i \leq n-d)$, and an open neighborhood $U$ of $x$ in $\mathbb{R}^n$ such that:

(1) $p_i(V) = 0$, $(1 \leq i \leq n-d)$.
(2) $V \cap U = U \cap (\cap_{i=1}^{n-d} p_i(0))$.
(3) The gradients $(\nabla p_i)_x$ $(1 \leq i \leq n-d)$ are linearly independent on $U$.

The *dimension* $\dim V$ of $V$ is $\max\{d |$there exists an $x \in V$ which is nonsingular of dimension $d\}$. $Nonsing\ V = \{x \in V | x$ is nonsingular of dimension $\dim V\}$ and $Sing\ V = V - Nonsing\ V$. An algebraic set is *nonsingular* if $Sing\ V = \emptyset$. Remark that $Sing\ V$ is an algebraic subset of $V$ with dim $Sing\ V <$ dim $V$. An algebraic subset $W$ of a nonsingular algebraic set $V$ is a *nonsingular algebraic subset* of $V$ if $W$ is nonsingular.

**Theorem 2.10** ([1])**.** *Let $X$ be a compact $C^\infty$ manifold and $X_1, \dots, X_n$ compact $C^\infty$ submanifolds of $X$ in general position. Then there exist a nonsingular algebraic set $Y$ and a $C^\infty$ diffeomorphism $\phi : X \to Y$ such that each $\phi(X_i)$ is a nonsingular algebraic subset $Y_i$ of $Y$. In particular, $(X; X_1, \dots, X_n)$ admits a simultaneous Nash manifold structure $(Y; Y_1, \dots, Y_n)$.*

The equivariant version of Theorem 2.10 is called the relative equivariant algebraic realization problem and it is not known its complete answer. The following is its partial answer.

**Theorem 2.11** ([3])**.** *Let $G$ be a compact definable $C^\infty$ group and $X$ a compact $C^\infty G$ manifold. Then the disjoint union $X \amalg X$ of two copies of $X$ is $C^\infty G$ diffeomorphic to a nonsingular algebraic $G$ set.*

Using their argument and Exercise in P58 [2], we have the following theorem.

**Theorem 2.12.** *Let $G$ be a compact definable $C^\infty$ group, $X$ a compact $C^\infty G$ manifold and $X_1, \dots, X_k$ codimension one compact $C^\infty G$ submanifolds of $X$ in general position. Then there exist a nonsingular algebraic set $Y$ and nonsingular algebraic $G$ subsets $Y_1, \dots, Y_k$ such that $(X \amalg X; X_1 \amalg X_1, \dots X_k \amalg X_k)$ is $C^\infty G$ diffeomorphic to $(Y; Y_1, \dots Y_k)$.*

**Corollary 2.13.** *Let $G$ be a compact definable $C^\infty$ group, $X$ a compact $C^\infty G$ manifold and $X_1, \dots, X_k$ codimension one compact $C^\infty G$ submanifolds of $X$ in general position. Then there exist an affine definable $C^\infty G$ manifold $Y$ and definable $C^\infty G$ submanifolds $Y_1, \dots, Y_k$ such that $(X; X_1, \dots X_k)$ is $C^\infty G$ diffeomorphic to $(Y; Y_1, \dots Y_k)$.*

Using some refinement of the proof of 2.2.9 [10] and [11], we have the following theorem.

**Theorem 2.14.** *Let $G$ be a compact Lie group, $X$ a compact $C^s G$ manifold, $X_1, \dots, X_k$ compact $C^s G$ submanifolds of $X$ in general position and $1 \le s < \infty$. Then there exist a compact $C^\infty G$ manifold $Y$ and its compact $C^\infty G$ submanifolds $Y_1, \dots, Y_k$ of $Y$ such that $(X; X_1, \dots, X_k)$ is $C^s G$ diffeomorphic to $(Y; Y_1, \dots, Y_k)$.*

**Theorem 2.15** (1.2 [19])**.** *Let $G$ be a compact definable $C^r$ group and $2 \le r < \infty$. Let $X$ be a compact affine definable $C^r G$ manifold with boundary $\partial X$, and $X_1, \dots, X_k$ compact definable $C^r G$ submanifolds of $X$ with boundary $\partial X_1, \dots, \partial X_k$, respectively, such that $X_1, \dots, X_k, \partial X$ are in general position, every $\partial X_i$ is contained in $\partial X$. Then there exists a relative definable $C^r G$ collar $\phi : (\partial X \times [0,1]; \partial X_1 \times [0,1], \dots, \partial X_k \times [0,1]) \to (X; X_1, \dots, X_n)$ of $(\partial X; \partial X_1, \dots, \partial X_k)$.*

*Proof of Theorem* 1.1. Assume that $X, X_1, \dots, X_k$ are compact. By Theorem 2.14, there exist compact $C^\infty G$ manifold $X'$ and compact $C^\infty G$ submanifolds $X'_1, \dots, X'_k$ of $X'$ such that $(X; X_1, \dots, X_k)$ is $C^r G$ diffeomorphic to $(X'; X'_1, \dots, X'_k)$. Thus by Corollary 2.13, we can find an affine definable $C^\infty G$ manifold $Y$ and definable $C^\infty G$ submanifolds $Y_1, \dots, Y_k$ of $Y$ such that $(X'; X'_1, \dots, X'_k)$ is $C^\infty G$ diffeomorphic to $(Y; Y_1, \dots, Y_k)$. Hence $(X; X_1, \dots, X_k)$ is $C^r G$ diffeomorphic to $(Y; Y_1, \dots, Y_k)$. By Theorem 2.1 and Theorem 2.2 and since $X, X_1, \dots, X_k$ are compact, $(X; X_1, \dots, X_k)$ is definably $C^r G$ diffeomorphic to $(Y; Y_1, \dots, Y_k)$.

Assume that $X, X_1, \dots, X_k$ are noncompact and $(X; X_1, \dots, X_k)$ satisfies the frontier condition. By Theorem 2.8, there exist a compact definable $C^r G$ manifold $\tilde{X}$ with boundary $\partial \tilde{X}$, compact definable $C^r G$ submanifolds $\tilde{X}_1, \dots, \tilde{X}_k$ of $\tilde{X}$ with boundary $\partial \tilde{X}_1, \dots, \partial \tilde{X}_k$, respectively, and a definable $C^r G$ diffeomorphism $\phi : X \to Int\ \tilde{X}$ such that $\phi(X_i) = Int\ \tilde{X}_i$, each $\partial \tilde{X}_i$ is contained in $\partial \tilde{X}$, and $\tilde{X}_1, \dots, \tilde{X}_k$ and $\partial \tilde{X}$ are in general position in $\tilde{X}$. Thus by Theorem 2.15, $(\tilde{X}; \tilde{X}_1, \dots, \tilde{X}_k)$ admits a relative definable $C^r G$ collar. Hence we have the relative definable $C^r G$ double $(D; D_1, \dots, D_k)$ of

$(\tilde{X}; \tilde{X}_1, \dots, \tilde{X}_k)$. Note that $D, D_1, \dots, D_k$ and $\partial \tilde{X}$ are compact and $D_1, \dots, D_k, \partial \tilde{X}$ are in general position.

By the argument in the first case, there exist an affine definable $C^\infty G$ manifold $W$ and definable $C^\infty G$ submanifolds $W_1, \dots, W_k, U$ of $W$ such that $(D; D_1, \dots, D_k, \partial \tilde{X})$ is definably $C^r G$ diffeomorphic to $(W; W_1, \dots, W_k, U)$. Therefore we can find some unions $Y, Y_1, \dots, Y_k$ of connected components of $W - U, W_1 - U, \dots, W_k - U$, respectively, such that $Y$ is an affine definable $C^\infty G$ manifold, each $Y_i$ is a definable $C^\infty G$ submanifold of $Y$ and $(X; X_1, \dots, X_k)$ is definably $C^r G$ diffeomorphic to $(Y, Y_1, \dots, Y_k)$.

Let $Z$ be an affine definable $C^\infty G$ manifold and $Z_1, \dots, Z_k$ definable $C^\infty G$ submanifolds of $Z$ such that $(Y; Y_1, \dots, Y_k)$ is definably $C^r G$ diffeomorphic to $(Z; Z_1, \dots, Z_k)$ Then there exists a definable $C^r G$ diffeomorphism $F : (Y; Y_1, \dots, Y_k) \to (Z; Z_1, \dots, Z_k)$. Applying Theorem 2.6 and Theorem 2.7, we have a definable $C^\infty G$ diffeomorphism $H : (Y; Y_1, \dots, Y_k) \to (Z; Z_1, \dots, Z_k)$. □

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