DEPENDENCE AND ISOLATED EXTENSIONS

VINCENT GUINGONA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND

Abstract. In this paper, we show that if $\varphi(x; y)$ is a dependent formula, then all φ-types p have an extension to a φ-isolated φ-type, p'. Moreover, we can choose p' to be a *elementary φ-extension* of p (see Definition 2.3 below) and so that $|\mathrm{dom}(p') - \mathrm{dom}(p)| \leq 2 \cdot \mathrm{ID}(\varphi)$. We show that this characterizes φ being dependent. Finally, we give some corollaries of this theorem and draw some parallels to the stable setting.

1. Introduction

There is a characterization of the stability of a formula $\varphi(\overline{x}; \overline{y})$ in terms of the definability of all φ-types. A partitioned formula $\varphi(\overline{x}; \overline{y})$ is stable if and only if all φ-types are definable by a formula over their domain [Sh]. We create an analogous result for dependent formulas (that is, formulas without the independence property, sometimes referred to as "NIP" formulas). Since dependence is a strictly weaker notion than stability, we cannot hope to have definability of φ-types over their domain for general dependent formulas, φ. However, we change the conclusion slightly, in two separate ways, and get a characterization of dependent formulas.

First, we weaken the requirement that a φ-type p be definable over $\mathrm{dom}(p)$. Instead, we take a model M containing $\mathrm{dom}(p)$, take an elementary extension $(N; B)$ of the pair structure $(M; \mathrm{dom}(p))$, and demand that p be definable over B. Second, we strengthen the method by which the φ-type p is definable. Instead of being merely definable over this expanded set B, we demand that there exists an extension of p to a φ-type p' such that $\mathrm{dom}(p') \subseteq B$ and p' is φ-isolated. From all of this, we construct an analogous result to the characterization of stable formulas, the Isolated Extension Theorem (Theorem 2.4 below). The proof of this theorem is loosely based on a paper by Shelah [Sh900].

In Section 2 we discuss definitions, state the main theorem, and list some consequences of that theorem. The main theorem, Theorem 2.4, is proved in Section 3. Finally, in the Section 4, we discuss the implications of this theorem to the stable case. Even in the stable case, Theorem 2.4 provides new information.

Date: November 6, 2009.
Special thanks to Chris Laskowski.

2. Definitions and The Isolated Extension Theorem

Fix a complete, first-order theory T in a language L. We include the case where L is multi-sorted, so we need to keep track of the sorts of variables. For convenience, if $\psi(\overline{x})$ is any formula, then let $\psi(\overline{x})^0 = \neg\psi(\overline{x})$ and let $\psi(\overline{x})^1 = \psi(\overline{x})$.

For the first three definitions, fix $\varphi(\overline{x}; \overline{y})$ a partitioned formula of L. By a φ-type, we mean a consistent set of formulas $p(\overline{x}) = \{\varphi(\overline{x}; \overline{b})^{s(\overline{b})} : \overline{b} \in B\}$ for some set B of elements of the same sort as $\overline{y}$ and some $s \in {}^B2$ (the set of functions from B to $2 = \{0, 1\}$). We say that $\mathrm{dom}(p) = B$ and the space of all φ-types over B is denoted

$$S_\varphi(B) = \{p(x) \text{ a } \varphi\text{−type} : \mathrm{dom}(p) = B\} \tag{1}$$

For any model $M \models T$, for any $\overline{a}$ from M and any B a set of elements of the same sort as $\overline{y}$ from M, let $\mathrm{tp}_\varphi(\overline{a}/B)$ be the φ-type over B given by:

$$\mathrm{tp}_\varphi(\overline{a}/B) = \{\varphi(\overline{x}; \overline{b})^t : \overline{b} \in B, t < 2 \text{ such that } M \models \varphi(\overline{a}; \overline{b})^t\} \tag{2}$$

The above notions can be defined for sets of formulas $\Gamma(\overline{x}; \overline{y})$ (instead of a single formula) in the obvious way. Throughout this section, when we mention a φ-type over B, look at $\mathrm{tp}_\varphi(\overline{a}/B)$, or consider the set $S_\varphi(B)$, we want B to be a set of elements of the same sort as $\overline{y}$ (that is, if $\overline{y} = (y_0, ..., y_{n-1})$, then B is a set of n-tuples $\overline{b} = (b_0, ..., b_{n-1})$ such that b_i is of the same sort as y_i for all $i < n$). In Section 3 when we consider Δ-types, we will alter this notation slightly for simplification. When we consider the set of formulas $\Delta(y; z_0, ..., z_{n-1})$ where all the z_i's are of the same sort and B is a set of elements of that sort, we will abuse notation and say that a Δ-type is *over* B when it is actually over B^n and we will write $\mathrm{tp}_\Delta(c/B)$ when we mean $\mathrm{tp}_\Delta(c/B^n)$.

Definition 2.1. We say that a set B of elements of the same sort as $\overline{y}$ is **φ-independent** if, for all $s \in {}^B2$, the set of formulas $\{\varphi(\overline{x}; \overline{b})^{s(\overline{b})} : \overline{b} \in B\}$ is consistent. We say that φ has **independence dimension** $n < \omega$, denoted $\mathrm{ID}(\varphi) = n$, if n is maximal such that, for some (equivalently any) model $M \models T$, there exists a set B of elements of the same sort as $\overline{y}$ from M with $|B| = n$ such that B is φ-independent. If such an n exists, then we say that φ is **dependent**. If no such n exists, then we say that φ is **independent**.

Notice that when B is finite, B is φ-independent if and only if $|S_\varphi(B)| = 2^{|B|}$.

Definition 2.2. We say that a φ-type $p(\overline{x})$ is **φ-isolated** if there exists a finite φ-subtype $p_0(\overline{x}) \subseteq p(\overline{x})$ such that $p_0(\overline{x}) \vdash p(\overline{x})$. We say that a formula $\psi(\overline{x})$ is a **φ-formula** if it is of the form $\psi(\overline{x}) = \bigwedge_{i<n} \varphi(\overline{x}; \overline{b}_i)^{s(i)}$ for some $n < \omega$, some elements $\overline{b}_i$ of the same sort as $\overline{y}$, and some $s \in {}^n2$.

We see that a φ-type $p(\overline{x})$ is φ-isolated if and only if there exists a φ-formula, $\psi(\overline{x})$ over $\mathrm{dom}(p)$ such that $p(\overline{x})$ is equivalent to $\psi(\overline{x})$. This φ-formula is simply the conjunction of the finite φ-subtype $p_0(\overline{x})$ given in Definition 2.2.

For a model $M \models T$ and a set B of elements of same sort as $\overline{y}$ from M, consider the language $L_B = L \cup \{P_B\}$ an expansion of L by adding a single predicate, $P_B(\overline{y})$. Let $(M; B)$ be the obvious L_B-structure. By "$(N; B') \succeq (M; B)$" we mean that $(N; B')$ is an elementary extension of $(M; B)$ in the language L_B.

Definition 2.3. Fix $M \models T$ and a set B of elements of the same sort as $\overline{y}$ from M. We say that a φ-type p' is an **elementary φ-extension** of the φ-type $p \in S_\varphi(B)$ if p' extends p and $\mathrm{dom}(p') \subseteq B'$ for some $(N; B') \succeq (M; B)$.

Now we are ready to state the main theorem of the paper. We will give the proof in Section 3 below.

Theorem 2.4 (The isolated extension theorem)**.** *For any partitioned formula $\varphi(\overline{x}; \overline{y})$, the following are equivalent:*

(i) *φ is dependent;*
(ii) *For all φ-types p, there exists a φ-isolated elementary φ-extension of p.*

Moreover, if the above conditions hold, we can choose p' a φ-isolated elementary φ-extension of $p \in S_\varphi(B)$ such that $|\mathrm{dom}(p') - B| \leq 2 \cdot \mathrm{ID}(\varphi)$.

We remark on some consequences of the theorem.

Definition 2.5. Fix a partitioned formula $\varphi(\overline{x}; \overline{y})$, a φ-type $p(\overline{x})$, and a formula $\psi(\overline{y})$. We say that ψ **defines** p if, for all $\overline{b} \in \mathrm{dom}(p)$, $\varphi(\overline{x}; \overline{b}) \in p(\overline{x})$ if and only if $\psi(\overline{b})$ holds. We say that ψ **φ-defines** p if it defines p and it is of the form $\psi(\overline{y}) = \forall \overline{x}(\gamma(\overline{x}) \to \varphi(\overline{x}; \overline{y}))$ for some φ-formula $\gamma(\overline{x})$.

Merely requiring that a φ-type has a defining formula has no content. Indeed, for any type $p \in S_\varphi(B)$, p is defined by the formula $\varphi(\overline{a}; \overline{y})$ for any realization $\overline{a}$ of p. The strength of having a defining formula is to have one with a controlled domain, preferably over $\mathrm{dom}(p)$. It is known, for example, that for stable formulas φ, all φ-types p have a defining formula over $\mathrm{dom}(p)$ [Sh], but, when $\mathrm{dom}(p)$ is an arbitrary set, it does not necessarily have a φ-defining formula over $\mathrm{dom}(p)$.

Notice that if p is φ-isolated, then p has a φ-defining formula ψ over $\mathrm{dom}(p)$. Namely, take the φ-formula γ over $\mathrm{dom}(p)$ such that $p(\overline{x})$ is equivalent to $\gamma(\overline{x})$ and let $\psi(\overline{y}) = \forall \overline{x}(\gamma(\overline{x}) \to \varphi(\overline{x}; \overline{y}))$. It is clear that if ψ φ-defines p, then ψ defines p, but the converse does not necessarily hold. We immediately get the following corollary to Theorem 2.4.

Corollary 2.6 (Elementary φ-definability of types)**.** *If $M \models T$, $\overline{y}$ is a list of variables, and B is a set of elements of same sort as $\overline{y}$ from M, then there exists an elementary extension $(N; B') \succeq (M; B)$ such that, for all dependent formulas $\varphi(\overline{x}; \overline{y})$, for all $p(\overline{x}) \in S_\varphi(B)$, there exists $\psi(\overline{y})$ over B' such that ψ φ-defines p.*

Proof. Fix $M \models T$ and B from M of the appropriate sort, and fix $(N; B') \succeq (M; B)$ sufficiently saturated. Then, by Theorem 2.4, there exists p' a φ-isolated elementary φ-extension of p (with $\mathrm{dom}(p') \subseteq B'$). Since p' is φ-isolated, there exists ψ (over $\mathrm{dom}(p') \subseteq B'$) that φ-defines p'. Since $p \subseteq p'$, ψ φ-defines p. $\square$

Notice that Corollary 2.6 is, on the one hand, stronger than standard definability of types for stable formulas, and, on the other hand, weaker. We get that, for dependent formulas φ, φ-types are not only definable, but φ-definable. However, the formula doing the defining is not over $\mathrm{dom}(p)$, but over B' for some $(N; B') \succeq (M; \mathrm{dom}(p))$.

As in the stable case, this φ-definability of types leads to a notion of stable embeddability.

Corollary 2.7 (Elementary stable embeddability)**.** *If $M \models T$ for a dependent theory T, $\overline{y}$ is a list of variables, and B is a set of elements of same sort as $\overline{y}$ from M, then there exists an elementary extension $(N; B') \succeq (M; B)$ such that, for all formulas $\varphi(\overline{y})$ over any elementary supermodel of M, there exists a formula $\psi(\overline{y})$ over B' such that $\varphi(B) = \psi(B)$. Moreover, $\psi(\overline{y}) = \forall \overline{x}(\gamma(\overline{x}) \to \varphi(\overline{x}; \overline{y}))$ for some φ-formula γ.*

Proof. Fix $(N; B') \succeq (M; B)$ sufficiently saturated as above. For any fixed formula $\varphi(\overline{y})$, say φ is over $N' \succeq M$, let $\varphi(\overline{y}) = \varphi_0(\overline{a}; \overline{y})$ for $\varphi_0(\overline{x}; \overline{y})$ over $\emptyset$ and $\overline{a}$ from N', and let $p(\overline{x}) = \mathrm{tp}_{\varphi_0}(\overline{a}/B)$. As φ_0 is dependent, by Corollary 2.6, there exists $\psi(\overline{y})$ over B' that φ_0-defines p. Then, by definition, $\varphi(B) = \psi(B)$. $\square$

3. The Proof of the Isolated Extension Theorem

To aid notation, assume that the length of $\overline{x}$ and the length of $\overline{y}$ is 1. Other than having more complicated notation, the general case is identical.

First, to show (ii) implies (i), we will exhibit the contrapositive. Assume then that $\varphi(x; y)$ is independent. By compactness, there exists a model M with an infinite φ-independent set B. Let $(N; B') \succeq (M; B)$. By elementarity, it follows that all finite subsets of B' are φ-independent. Let p' be any extension of p to a φ-type such that $\mathrm{dom}(p') \subseteq B'$. Fix any finite subtype $p_0(x) \subseteq p'(x)$. Now, for any finite φ-type $p_1(x)$ with $p_0(x) \subsetneq p_1(x) \subseteq p'(x)$, since $\mathrm{dom}(p_1)$ is φ-independent, we cannot have that $p_0(x) \vdash p_1(x)$. Thus, $p_0(x) \nvdash p'(x)$. This shows that no elementary φ-extension of p is φ-isolated. Therefore, (ii) implies (i).

To show (i) implies (ii), we will first show that the following proposition holds:

Proposition 3.1. *For any dependent formula $\varphi(x; y)$ in a theory T, for any model $M \models T$, for any partial type $\Theta(y)$ over $\emptyset$, and for any $B \subseteq \Theta(M)$, there exists $N \succeq M$ and $C \subseteq \Theta(N)$ with $|C| \leq 2 \cdot \mathrm{ID}(\varphi)$ and an extension $p'(x) \in S_\varphi(B \cup C)$ of $p(x)$ that is φ-isolated.*

Fix $\varphi(x; y)$ a dependent formula in a theory T and $\Theta(y)$ any partial type over $\emptyset$. Let $n = \mathrm{ID}(\varphi)$, the independence dimension of $\varphi(x; y)$. Fix $M \models T$, $N \succeq M$ sufficiently saturated, $B \subseteq \Theta(M)$, and $p(x) \in S_\varphi(B)$. If B is finite, p is already isolated, so assume that B is infinite. Define a set of formulas $\Delta(y; z_0, ..., z_{n-1})$ as follows:

$$\Delta(y; z_0, ..., z_{n-1}) = \left\{ \exists x \left(\varphi(x;y)^t \wedge \bigwedge_{i<n} \varphi(x;z_i)^{s(i)} \right) : t < 2, s \in {}^n 2 \right\} \tag{3}$$

We will now define the notion of a good configuration. This will end up allowing us to build up the external C in at most $\mathrm{ID}(\varphi)$ steps (adding two elements at a time).

Definition 3.2. A **good configuration** of p of size K is a sequence $C = \{c_{i,t} : i < K, t < 2\}$ such that the following conditions hold:

(i) $c_{i,t} \models \Theta(y)$ for all $i < K$, $t < 2$;
(ii) $p(x) \cup \{\varphi(x; c_{j,t})^t : j < K, t < 2\}$ is consistent; and
(iii) For all $s \in {}^K 2$, all $j < K$, $c_{j,0}$ and $c_{j,1}$ have the same Δ-type over $B \cup \{c_{i,s(i)} : i \neq j\}$.

If C is a good configuration of p of size K, then let $p_C(x) = p(x) \cup \{\varphi(x; c_{j,t})^t : j < K, t < 2\}$.

The first thing to note is that these good configurations are used to extend the type p in a very specific way. These could, *a priori*, be arbitrarily large. However, the fact that φ is dependent forces good configurations to be of bounded size.

Lemma 3.3. *If $C = \{c_{i,t} : i < K, t < 2\}$, is a good configuration of p of size K, then $K \leq n = \mathrm{ID}(\varphi)$.*

Proof. Suppose not, i.e. $K > n$. Now, for each $s \in {}^{n+1} 2$, notice that

$$\models \exists x \bigwedge_{i<n+1} \varphi(x; c_{i,s(i)})^{s(i)} \tag{4}$$

because $\{\varphi(x; c_{i,s(i)})^{s(i)} : i < n+1\}$ is a consistent type. Now, notice that, for any $j \leq n$,

$$\exists x \left(\bigwedge_{i<j} \varphi(x; c_{i,0})^{s(i)} \wedge \varphi(x; c_{j,s(j)})^{s(j)} \wedge \bigwedge_{j<i<n+1} \varphi(x; c_{i,s(i)})^{s(i)} \right) \implies \exists x \left(\bigwedge_{i<j} \varphi(x; c_{i,0})^{s(i)} \wedge \varphi(x; c_{j,0})^{s(j)} \wedge \bigwedge_{j<i<n+1} \varphi(x; c_{i,s(i)})^{s(i)} \right) \tag{5}$$

because $c_{j,0}$ and $c_{j,1}$ have the same Δ-type over $\{c_{i,0} : i < j\} \cup \{c_{i,s(i)} : j < i < n+1\}$. Starting with (4), then using (5) and induction, we get that:

$$\models \exists x \bigwedge_{i<n+1} \varphi(x; c_{i,0})^{s(i)} \tag{6}$$

But this holds for any $s \in {}^{n+1}2$. This contradicts the fact that $n = \mathrm{ID}(\varphi)$. $\square$

Now that we have good configurations, we need a sufficient condition for taking a good configuration and building a larger one out of it. Clearly any new d_0 and d_1 we would like to add on must realize Θ and must be so that $\neg\varphi(x; d_0) \wedge \varphi(x; d_1)$ is consistent with $p_C(x)$. However, the third condition for a good configuration is a bit tricky. Not only do d_0 and d_1 have to have the same Δ-type over $B \cup \{c_{i,s(i)} : i < K\}$, but also each $c_{j,0}$ and $c_{j,1}$ have to have the same Δ-type over $B \cup \{c_{i,s(i)} : i \neq j\} \cup \{d_t\}$. We now give a sufficient condition for being able to add on to good configurations.

Lemma 3.4. *If $C = \{c_{i,t} : i < K, t < 2\}$ is a good configuration of p, and there exists d_0, d_1 such that:*

(i) $d_0, d_1 \models \Theta(y)$;
(ii) $p_C(x) \cup \{\varphi(x; d_t)^t : t < 2\}$ *is consistent;*
(iii) $\mathrm{tp}_\Delta(d_0/B \cup C) = \mathrm{tp}_\Delta(d_1/B \cup C)$*; and*
(iv) $\mathrm{tp}_\Delta(d_0/B \cup C)$ *is finitely satisfiable in* B.

Then, $C \cup \{d_0, d_1\}$ *is a good configuration of p (of size $K + 1$).*

Proof. Clearly all conditions for $C \cup \{d_0, d_1\}$ to be a good configuration of p are met except perhaps the condition that $c_{j,0}$ and $c_{j,1}$ have the same Δ-type over $B \cup \{c_{i,s(i)} : i \neq j\} \cup \{d_t\}$ for all $s \in {}^K2$, $t < 2$. So suppose this fails, and fix the $s \in {}^K2$ and $t < 2$ where this fails.

Then there exists δ either an element of Δ or the negation of an element of Δ such that $N \models \delta(c_{j,0}, \overline{e}) \wedge \neg\delta(c_{j,1}, \overline{e})$ for some $\overline{e}$ from $B \cup \{c_{i,s(i)} : i \neq j\} \cup \{d_t\}$. Since $c_{j,0}$ and $c_{j,1}$ have the same Δ-type over $B \cup \{c_{i,s(i)} : i \neq j\}$, we must have that $\overline{e} = d_t{}^\frown\overline{e}'$ for some $\overline{e}'$ from $B \cup \{c_{i,s(i)} : i \neq j\}$. Therefore, we get that:

$$N \models \delta(c_{j,0}, d_t, \overline{e}') \wedge \neg\delta(c_{j,1}, d_t, \overline{e}') \tag{7}$$

By condition (iv) of the hypothesis, there exists $b \in B$ such that:

$$N \models \delta(c_{j,0}, b, \overline{e}') \wedge \neg\delta(c_{j,1}, b, \overline{e}') \tag{8}$$

But, as $b^\frown\overline{e}'$ is from $B \cup \{c_{i,s(i)} : i \neq j\}$, this contradicts the fact that $c_{j,0}$ and $c_{j,1}$ have the same Δ-type over $B \cup \{c_{i,s(i)} : i \neq j\}$. $\square$

Fix C a maximal good configuration of p, so $p_C(x)$ is a φ-type over $B \cup C$. Let $s(x)$ be any extension of $p_C(x)$ to a complete type over $B \cup C$. Define $r_s(y)$ as follows:

$$r_s(y) = \{\exists x(\varphi(x; y)^t \wedge \psi(x)) : \psi \in s, t < 2\} \cup \Theta(y) \tag{9}$$

Lemma 3.5. *r_s is not finitely satisfied in B.*

Proof. Suppose, by means of contradiction, that r_s is finitely satisfied in B. Let $\mathcal{D}$ be an ultrafilter on B such that for all $\delta(y) \in r_s(y)$, $\delta(B) \in \mathcal{D}$ (this exists by finite satisfiability of r_s in B). Let $q(y) = \mathrm{Av}(\mathcal{D}, B \cup C)$, the average type of $\mathcal{D}$ over $B \cup C$. That is, for any formula $\delta(y)$ over $B \cup C$, $\delta(y) \in q(y)$ if and only if $\delta(B) \in \mathcal{D}$. Then $q \in S(B \cup C)$, q extends r_s, and q is finitely satisfied in B. Let $q' = q \restriction_\Delta$.

Now notice that $\{\exists x(\varphi(x; y)^t \wedge \psi(x))\} \cup q(y)$ is consistent for each $\psi \in s$ and each $t < 2$. Since s is closed under conjunction, by compactness we get that $s(x) \cup \{\varphi(x; y)^t\} \cup q(y)$ is consistent for each $t < 2$. Therefore, $s(x) \cup \{\varphi(x; y)^t\} \cup q'(y) \cup \{\theta(y)\}$ is consistent for each $t < 2$ and each $\theta(y)$ a finite conjunction of formulas from $\Theta(y)$ (as $q'(y) \cup \Theta(y) \subseteq q(y)$). This means that $s(x) \cup \{\exists y(\varphi(x; y)^t \wedge \theta(y) \wedge \psi(y))\}$ is consistent for each $\psi(y)$ a finite conjunction of formulas from $q'(y)$ and each $\theta(y)$ a finite conjunction of formulas from $\Theta(y)$. But, since s is a complete type in the x variable, s decides all formulas of the form $\exists y(\varphi(x; y)^t \wedge \theta(y) \wedge \psi(y))$. Therefore, we get that:

$$\exists y(\varphi(x; y)^t \wedge \theta(y) \wedge \psi(y)) \in s(x) \tag{10}$$

Choose $\psi_t(x)$ a finite conjunction of formulas from $q'(y)$ and $\theta_t(y)$ a finite conjunction of formulas from $\Theta(y)$ for both $t < 2$. Then $\exists y_t(\varphi(x; y_t)^t \wedge \theta_t(y_t) \wedge \psi_t(y_t)) \in s(x)$ for both $t < 2$. Therefore, we get that:

$$s(x) \cup \{\exists y_0(\neg\varphi(x; y_0) \wedge \theta_0(y_0) \wedge \psi_0(y_0))\} \cup \{\exists y_1(\varphi(x; y_1) \wedge \theta_1(y_1) \wedge \psi_1(y_1))\} \tag{11}$$

is consistent. Now, by compactness,

$$u(x, y_0, y_1) = s(x) \cup \{\neg\varphi(x; y_0) \wedge \varphi(x; y_1)\} \cup q'(y_0) \cup q'(y_1) \cup \Theta(y_0) \cup \Theta(y_1) \tag{12}$$

is consistent. So, taking any realization (a, d_0, d_1) of $u(x, y_0, y_1)$ from N, we see that $d_0, d_1 \models \Theta(y)$, $d_0, d_1 \models q'(y)$, and $p_C(x) \cup \{\varphi(x; d_t)^t : t < 2\}$ is consistent. So conditions (i), (ii), and (iii) of Lemma 3.4 are met. However, since q is finitely satisfied in B, q' is finitely satisfied in B. Therefore, condition (iv) of Lemma 3.4 is met, so $C \cup \{d_0, d_1\}$ is a good configuration of p. This contradicts the maximality of C. $\square$

We will now show how the non-finite-satisfiability of r_s in B leads to a formula definition of $p_C(x)$.

Lemma 3.6. *For any C a maximal good configuration of p and any $s(x) \in S(B \cup C)$ an extension of $p_C(x)$, there exists a formula $\gamma(x) \in s(x)$ such that $\gamma(x) \vdash p_C(x)$.*

Proof. Consider r_s as given above. Then, since r_s is not finitely satisfiable in B, there exists $m < \omega$ and $\psi_\ell(x) \in s(x)$ for each $\ell < m$ such that, for all $b \in B$, $N \models \neg\exists x(\varphi(x;b)^t \wedge \psi_\ell(x))$ for some $\ell < m$ and some $t < 2$ (notice here that $b \models \Theta(y)$ for all $b \in B$, so that the formulas in $\Theta(y) \subseteq r_s(y)$ are always realized in B). Let $\gamma(x)$ be defined as follows:

$$\gamma(x) = \bigwedge_{\ell<m} \psi_\ell(x) \wedge \bigwedge_{i<K,u<2} \varphi(x;c_{i,u})^u. \tag{13}$$

Since s is closed under conjunction, s extends p_C, and $\psi_\ell(x) \in s(x)$, we get that $\gamma(x) \in s(x)$. To prove that $\gamma(x) \vdash p_C(x)$, notice that, for all $b \in B$, there exists $t < 2$ such that $N \models \forall x(\bigwedge_{\ell<m} \psi_\ell(x) \to \varphi(x;b)^t)$. Therefore, $s(x) \vdash \gamma(x) \vdash \varphi(x;b)^t$, hence $\varphi(x;b)^t \in s(x)$. But s extends p_C, so we get that $\varphi(x;b)^t \in p_C(x)$. Similarly, $\gamma(x) \vdash \varphi(x;c_{i,u})^u$ for all $i < K$ and $u < 2$. Therefore, $\gamma(x) \vdash p_C(x)$. ☐

Now that we have a formula definition for $p_C(x)$ for each $s \in S(B \cup C)$, we will see that a single formula is equivalent to $p_C(x)$ using compactness. After that, we will show that this means a finite φ-subtype of $p_C(x)$ is equivalent to the whole of $p_C(x)$.

Lemma 3.7. *If $C = \{c_{i,t} : i < K, t < 2\}$ is a maximal good configuration of p, then there exists a formula $\psi(x)$ over $B \cup C$ such that $\psi(x)$ is equivalent to $p_C(x)$.*

Proof. For each such $s(x) \in S(B \cup C)$ extending $p_C(x)$, define $\gamma_s(x)$ to be the formula such that $\gamma_s(x) \in s(x)$ and $\gamma_s(x) \vdash p_C(x)$ as given in Lemma 3.6.

Consider the following partial type over $B \cup C$:

$$\Sigma(x) = \{\neg\gamma_s(x) : s \in S(B \cup C) \text{ and } s(x) \supseteq p_C(x)\} \cup p_C(x) \tag{14}$$

Now $\Sigma(x)$ is inconsistent, since otherwise we would have $a \models p_C(x)$ yet $a \not\models \gamma_s(x)$ for any $s(x)$ extending $p_C(x)$. In particular, $a \not\models \gamma_{s_0}(x)$ for $s_0 = \mathrm{tp}(a/B \cup C)$. This contradicts the fact that $s_0(x) \vdash \gamma_{s_0}(x)$. Therefore, by compactness, there exists some finite set $S_0 \subseteq S(B \cup C)$ of types extending p_C so that $\Sigma_0(x) = \{\neg\gamma_s(x) : s \in S_0\} \cup p_C(x)$ is inconsistent. Let $\psi(x) = \bigvee_{s \in S_0} \gamma_s(x)$.

Certainly $\psi(x) \vdash p_C(x)$ as $\gamma_s(x) \vdash p_C(x)$ for all $s \in S_0$. Conversely, if $a \models p_C(x)$, then $a \not\models \{\neg\gamma_s(x) : s \in S_0\}$ (by the inconsistency of $\Sigma_0(x)$). Therefore, $a \models \psi(x)$. Hence, $p_C(x) \vdash \psi(x)$, as desired. ☐

Lemma 3.8. *If $C = \{c_{i,t} : i < K, t < 2\}$ is a maximal good configuration of p, then there exists a finite φ-subtype $p_0(x) \subseteq p_C(x)$ so that $p_0(x) \vdash p_C(x)$.*

Proof. First let $\psi(x)$ be a formula over $B \cup C$ that is equivalent to $p_C(x)$, given by Lemma 3.7. Then consider $\{\neg\psi(x)\} \cup p_C(x)$, a partial type over $B \cup C$. This is clearly inconsistent.

Therefore, there exists a finite subset $p_0(x) \subseteq p_C(x)$ such that $\{\neg\psi(x)\} \cup p_0(x)$ is inconsistent. That is, $p_0(x) \vdash \psi(x)$ and, therefore, we get that $p_0(x) \vdash p_C(x)$. □

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Take $C = \{c_{i,t} : i < K, t < 2\}$ any maximal good configuration of p. By definition, $C \subseteq \Theta(N)$. By Lemma 3.3, $K \leq n$, hence $|C| \leq 2 \cdot n$. Let $p'(x) = p_C(x) = p(x) \cup \{\varphi(x; c_{i,t})^t : i < K, t < 2\}$. By Lemma 3.8, there exists a finite $p_0(x) \subseteq p'(x)$ so that $p_0(x) \vdash p'(x)$. Therefore, $p'(x)$ is φ-isolated. □

From here we can conclude that (i) implies (ii) holds for Theorem 2.4.

Let $\varphi(x; y)$ be dependent, fix $M \models T$, $B \subseteq M$ of elements of the same sort as y, and any φ-type $p \in S_\varphi(B)$. Let $\Theta(y) = \{P_B(y)\}$ in the language L_B (notice that $\varphi(x; y)$ is still dependent with the same independence dimension in the theory $Th_{L_B}(M; B)$). Therefore, by Proposition 3.1, there exists $(N; B') \succeq (M; B)$ and $C \subseteq \Theta((N; B')) = B'$ with $|C| \leq 2 \cdot \mathrm{ID}(\varphi)$ and a type $p'(x) \in S_\varphi(B \cup C)$ extending $p(x)$ such that $p'(x)$ is φ-isolated. Notice then that p' is an elementary φ-extension of p that is φ-isolated, so condition (ii) holds. Moreover, we get that $|\mathrm{dom}(p') - B| = |C| \leq 2 \cdot \mathrm{ID}(\varphi)$, as desired. □

Remark 3.9. Finally, we remark that this C, hence p', depends only on a type over B with enough information to guarantee that C is a good configuration of p of maximal size. For example, if we take $\overline{c} = (c_{i,t} : i < K, t < 2)$ for $C = \{c_{i,t} : i < K, t < 2\}$ a good configuration of p of maximal size, and let

(i) $q'(\overline{y}) = \{P_B(y_{i,t}) : i < K, t < 2\}$,
(ii) $q''(\overline{y}) = \left\{\exists x \left(\psi(x) \wedge \bigwedge_{i<K,t<2} \varphi(x; c_{i,t})^t\right) : \psi(x) \text{ a finite conjunction from } p(x)\right\}$,
(iii) $q'''(\overline{y}) = \mathrm{tp}_\Delta(\overline{c}/B)$, and
(iv) $q(\overline{y}) = q'(\overline{y}) \cup q''(\overline{y}) \cup q'''(\overline{y})$,

then, for any $\overline{c}' \models q(\overline{y})$, the type $p_{\overline{c}'}(x) = p(x) \cup \{\varphi(x; c'_{i,t})^t : i < K, t < 2\}$ is φ-isolated (and an elementary φ-extension of p). Notice here that $q(\overline{y}) \subseteq \mathrm{tp}(\overline{c}/B)$, the complete type of $\overline{c}$ over B. Therefore, so long as we choose $(N; B') \succeq (M; B)$ so that $(N; B')$ is $|B|^+$-saturated, $q(\overline{y})$ is realized in N. This allows us to pick $(N; B') \succeq (M; B)$ uniformly so that all φ-types over B have extensions to φ-isolated φ-types with domain contained in B'.

4. φ-Isolated Elementary φ-Extensions for Stable φ

Since stable formulas are, in particular, dependent, all stable formulas have the property of Theorem 2.4 (ii). But what is the φ-isolated elementary φ-extension $p'(x)$ of a given φ-type $p(x)$? In the interesting case when $p(x)$ is not already φ-isolated, $p'(x)$ is a *forking* extension of $p(x)$. This follows from the Open Mapping Theorem (i.e. the fact that the restriction map from non-forking φ-extensions of $S_\varphi(A)$ to $S_\varphi(A)$ is open) as, if p has a non-forking φ-isolated extension, then it is already φ-isolated.

On the issue of uniformity, the results of Theorem 2.4 differ strongly from the standard definability of φ-types in the stable case. In the case where φ is stable, we can use a compactness argument to get a uniform definition of φ-types. Note, however, that this uniform definition is not necessarily a φ-definition. One cannot, in general, get a uniform φ-definition of all φ-types, even in the case where φ is stable.

As an example, let T be the theory, in the language $L = \{E\}$ with a single binary relation E, stating that E is an equivalence relation with infinitely many E-equivalence classes all of infinite size. This theory is certainly stable, and even $\aleph_0$-stable. Fix $M \models T$ and let $B \subset M$ be a set containing one element from one class, two from another, three from a third class, and so on. Finally, let $\varphi(x; y, z, w)$ be the formula given by:

$$\varphi(x; y, z, w) = [(z = w \to x = y) \wedge (z \neq w \to E(x, y))] \tag{15}$$

(so φ encodes the two formulas "$x = y$" and $E(x, y)$ into a single formula). Now let $n \in \omega$ be arbitrary and let $a \in M - B$ be in the E-equivalence class with exactly n elements of B in it; call this class $[a]_E$. Finally, let $p_n(x) = \mathrm{tp}_\varphi(a/B)$. Now, for any $(N; B') \succeq (M; B)$, notice that the E-equivalence class with exactly n elements from B still has exactly n elements from B', so $[a]_E \cap B' = [a]_E \cap B$. However, this shows that any φ-extension of p_n to some p' with $\mathrm{dom}(p') \subseteq B'$ is φ-isolated only by a finite subtype whose domain contains $[a]_E \cap B$ (this is because we need the full set $[a]_E \cap B$ to say that $x \neq b$ for each $b \in ([a]_E \cap B)$ yet $E(x, b)$ for some (all) $b \in ([a]_E \cap B)$). As $|[a]_E \cap B| = n$ and $n < \omega$ was arbitrary, we see from this example that there is no *uniform* bound on the size of the φ-isolating φ-subtype of the elementary φ-extension given by Theorem 2.4, even in the stable case.

References

[Sh] Shelah, S.: Classification theory and the number of non-isomorphic models, *Studies in Logic and the Foundations of Mathematics, Volume 92, North-Holland Publishing Company, Amsterdam-New York* 1978.

[Sh900] Shelah, S.: Dependent theories and the generic pair conjecture, submitted