On dp-minimal ordered structures

Pierre Simon

September 21, 2009

Abstract

We show basic facts about dp-minimal ordered structures. The main results are : dp-minimal groups are abelian-by-finite-exponent, in a divisible ordered dp-minimal group, any infinite set has non-empty interior, and any theory of pure tree is dp-minimal.

Introduction

One of the latest topic of interest in abstract model theory is the study of dependent, or NIP, theories. The abstract general study, was initiated by Shelah in [Sh715], and pursued by him in [Sh783], [Sh863] and [Sh900]. One of the questions he addresses is the definition of *super-dependent* as an analog of superstable for stable theories. Although, as he writes, he has not completely succeeded, the notion he defines of strong-dependence seems promising. In [Sh863] it is studied in details and in particular, ranks are defined. Those so-called dp-ranks are used to prove existence of an indiscernible sub-sequence in any long enough sequence. Roughly speaking, a theory is strongly dependent if no type can fork infinitely many times, each forking being independent from the previous one. (Stated this way, it is naturally a definition of "strong-NTP_2"). Also defined in that paper are notions of minimality, corresponding to the ranks being equal to 1 on 1-types. In [OnsUsv], Onshuus and Usvyatsov extract from this material the notion of dp-minimality which seems to be the relevent one. A dp-minimal theory is a theory where there cannot be two independent witnesses of forking for a 1-type. It is shown in that paper that a stable theory is dp-minimal if and only if every 1-type has weight 1. In general, unstable, theories, one can link dp-minimality to *burden* as defined by H. Adler ([Adl]).

Dp-minimality on ordered structures can be viewed as a generalization of weak-o-minimality. In that context, there are two main questions to address : what do definable sets in dimension 1 look like, (*i.e.* how far is the theory from being o-minimal), and what theorems about o-minimality go through. J. Goodrick has started to study those questions in [Goo], focussing on groups. He proves that definable functions are piecewise locally monotonous extending a similar result from weak-o-minimality.

In the first section of this paper, we recall the definitions and give equivalent formulations. In the second section, we make a few observations on general linearly ordered inp-minimal theories showing in particular that, in dimension 1, forking is controlled by the ordering. The lack of a cell-decomposition theorem makes it unclear how to generalize results to higher dimensions.

In section 3, we study dp-minimal groups and show that they are abelian-by-finite-exponent. The linearly ordered ones are abelian. We prove also that an infinite definable set in a dp-minimal ordered divisible group has non-empty interior, solving a conjecture of Alf Dolich.

Finally, in section 4, we give examples of dp-minimal theories. We prove that colored linear orders, orders of finite width and trees are dp-minimal.

1 Preliminaries on dp-minimality

Definition 1.1. (Shelah) An independence (or inp-) pattern of length κ is a sequence of pairs$(\phi^\alpha(x,y),k^\alpha)_{\alpha<\kappa}$ of formulas such that there exists an array $\langle a_i^\alpha : \alpha<\kappa, i<\lambda\rangle$ for some $\lambda \geq \omega$ such that :

- Rows are k^α-inconsistent : for each $\alpha < \kappa$, the set $\{\phi^\alpha(x,a_i^\alpha) : i<\lambda\}$ is k^α-inconsistent,
- paths are consistent : for all $\eta \in \lambda^\kappa$, the set $\{\phi^\alpha(x,a^\alpha_{\eta(\alpha)}) : \alpha<\kappa\}$ is consistent.

Definition 1.2.
- (Goodrick) A theory is inp-minimal if there is no inp-pattern of length two in a single free variable x.
- (Onshuus and Usvyatsov) A theory is dp-minimal if it is NIP and inp-minimal.

A theory is NTP_2 if there is no inp-pattern of size ω for which the formulas $\phi^\alpha(x,y)$ in the definition above are all equal to some $\phi(x,y)$. It is proven in [Che] that a theory is NTP_2 if this holds for formulas $\phi(x,y)$ where x is a single variable. As a consequence, any inp-minimal theory is NTP_2.

We now give equivalent definitions (all the ideas are from [Sh863], we merely adapt the proofs there from the general NIP context to the dp-minimal one).

Definition 1.3. Two sequences $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ are *mutually indiscernible* if each one is indiscernible over the other.

Lemma 1.4. *Consider the following statements :*

1. T *is inp-minimal.*
2. *For any two mutually indiscernible sequences* $A=(a_i : i<\omega)$, $B=(b_j : j<\omega)$ *and any point* c*, one of the sequences* $(tp(a_i/c) : i<\omega)$, $(tp(b_i/c) : i<\omega)$ *is constant.*
3. *Same as above, but change the conclusion to : one the sequences* A *or* B *stays indiscernible over* c.
4. *For any indiscernible sequence* $A=(a_i : i\in I)$ *indexed by a dense linear order* I*, and any point* c*, there is* i_0 *in the completion of* I *such that the two sequences* $(tp(a_i/c) : i<i_0)$ *and* $(tp(a_i/c) : i>i_0)$ *are constant.*

5. *Same as above, but change the conclusion to : the two sequences* $(a_i : i < i_0)$ *and* $(a_i : i > i_0)$ *are indiscernible over* c.

6. T *is dp-minimal.*

Then for any theory T*, (2), (3), (4), (5), (6) are equivalent and imply (1). If* T *is* NIP*, then they are all equivalent.*

Proof. (2) $\Rightarrow$ (1) : In the definition of independence pattern, one may assume that the rows are mutually indiscernible. This is enough.

(2) $\Rightarrow$ (3) : Assume $A = \langle a_i : i < \omega \rangle$, $B = \langle b_i : i < \omega \rangle$ and c are a witness to $\neg$(3). Then there are two tuples $(i_1 < \ldots < i_n)$, $(j_1 < \ldots < j_n)$ and a formula $\phi(x; y_1, \ldots, y_n)$ such that $\models \phi(c; a_{i_1}, \ldots, a_{i_n}) \wedge \neg\phi(c; a_{j_1}, \ldots, a_{j_n})$. Take an $\alpha < \omega$ greater than all the i_k and the j_k. Then, exchanging the i_k and j_k if necessary, we may assume that $\models \phi(c; a_{i_1}, \ldots, a_{i_n}) \wedge \neg\phi(c; a_{n.\alpha}, \ldots, a_{n.\alpha+n-1})$. Define $A' = \langle (a_{i_1}, \ldots, a_{i_n}) \rangle {}^\frown \langle (a_{n.k}, \ldots, a_{n.k+n-1}) : k \geq \alpha \rangle$. Construct the same way a sequence B'. Then A', B', c give a witness of $\neg$(2).

(3) $\Rightarrow$ (2) : Obvious.

(3) $\Rightarrow$ (5) : Let $A = \langle a_i : i \in I \rangle$ be indiscernible and let c be a point. Then assuming (3) holds, for every i_0 in the completion of I, one of the two sequences $A_{<i_0} = \langle a_i : i < i_0 \rangle$ and $A_{>i_0} = \langle a_i : i > i_0 \rangle$ must be indiscernible over c. Take any such i_0 such that both sequences are infinite, and assume for example that $A_{>i_0}$ is indiscernible over c. Let $j_0 = \inf\{i \leq i_0 : A_{>i}$ is indiscernible over c $\}$. Then $A_{>j_0}$ is indiscernible over c. If there are no elements in I smaller than j_0, we are done. Otherwise, if $A_{<j_0}$ is not indiscernible over c, then one can find $j_1 < j_0$ such that again $A_{<j_1}$ is not indiscernible over c. By definition of j_0, $A_{>j_1}$ is not indiscernible over c either. This contradicts (3).

(5) $\Rightarrow$ (4) : Obvious.

(4) $\Rightarrow$ (2) : Assume $\neg$ (2). Then one can find a witness of it consisting of two indiscernible sequences $A = \langle a_i : i \in I \rangle$, $B = \langle b_i : i \in I \rangle$ indexed by a dense linear order I and a point c.

Now, we can find an i_0 in the completion of I such that for any $i_1 < i_0 < i_2$ in I, there are i, i', $i_1 < i < i_0 < i' < i_2$ such that $tp(a_i/c) \neq tp(a_{i'}/c)$. Find a similar point j_0 for the sequence B. Renumbering the sequences if necessary, we may assume that $i_0 \neq j_0$. Then the indiscernible sequence of pairs $\langle (a_i, b_i) : i \in I \rangle$ gives a witness of $\neg$ (4).

(6) $\Rightarrow$ (2) : Let A, B, c be a witness of $\neg$ (2). Assume for example that there is $\phi(x, y)$ such that $\models \phi(c, a_0) \wedge \neg\phi(c, a_1)$. Then set $A' = \langle (a_{2k}, a_{2k+1}) : k < \omega \rangle$ and $\phi'(x; y_1, y_2) = \phi(x; y_1) \wedge \neg\phi(x; y_2)$. Then by NIP, the set $\{\phi'(x, \bar{y}) : \bar{y} \in A'\}$ is k-inconsistent for some k. Doing the same construction with B we see that we get an independence pattern of length 2.

(5) $\Rightarrow$ (6) : Statement (5) clearly implies NIP (because IP is always witnessed by a formula $\phi(x, y)$ with x a single variable). We have already seen that it implies inp-minimality. $\square$

Standard examples of dp-minimal theories include :

- O-minimal or weakly o-minimal theories (recall that a theory is weakly-o-minimal if every definable set in dimension 1 is a finite union of convex sets),
- C-minimal theories,
- $\mathrm{Th}(\mathbb{Z}, +, \leq)$.

The reader may check this as an exercise or see [Goo].

More examples are given in section 4 of this paper.

2 Inp-minimal ordered structures

Little study has been made yet on general dp-minimal ordered structures. We believe however that there are results to be found already at that general level. In fact, we prove here a few lemmas that turn out to be useful for the study of groups.

We show that, in some sense, forking in dimension 1 is controlled by the order.

We consider $(M, <)$ an inp-minimal linearly ordered structure with no first nor last element. We denote by T its theory, and let $\mathbb{M}$ be a monster model of T.

Lemma 2.1. *Let $X = X_{\bar{a}}$ be a definable subset of $\mathbb{M}$, cofinal in $\mathbb{M}$. Then X is non-forking (over $\emptyset$).*

Proof. If $X_{\bar{a}}$ divides over $\emptyset$, there exists an indiscernible sequence $(\bar{a}_i)_{i<\omega}$, $\bar{a}_0 = \bar{a}$, witnessing this. Every $X_{\bar{a}_i}$ is cofinal in $\mathbb{M}$. Now pick by induction intervals I_k, $k < \omega$, with $I_k < I_{k+1}$ containing a point in each $X_{\bar{a}_i}$. We obtain an inp-pattern of length 2 by considering $x \in X_{\bar{a}_i}$ and $x \in I_k$.

If $X_{\bar{a}}$ forks over $\emptyset$, it implies a disjunction of formulas that divide, but one of these formulas must be cofinal : a contradiction. $\square$

A few variations are possible here. For example, we assumed that X was cofinal in the whole structure $\mathbb{M}$, but the proofs also works if X is cofinal in a $\emptyset$-definable set Y, or even contains an $\emptyset$-definable point in its closure. This leads to the following results.

For X a definable set, let $\mathrm{Conv}(X)$ denote the convex hull of X. It is again a definable set.

Porism 2.2. *Let X be a definable set of $\mathbb{M}$ (in dimension 1). Assume $\mathrm{Conv}(X)$ is A definable. Then X is non-forking over A.*

Porism 2.3. *Let $M \prec N$ and let p be a complete 1-type over N. If the cut of p over N is of the form $+\infty$, $-\infty$, a^+ or a^- for $a \in M$, then p is non-forking over M.*

Proposition 2.5 generalizes this.

Lemma 2.4. *Let X be an A-definable subset of $\mathbb{M}$. Assume that X divides over some model M, then :*

1. *We cannot find $(a_i)_{i<\omega}$ in M and points $(x_i)_{i<\omega}$ in $X(\mathbb{M})$ such that $a_0 < x_0 < a_1 < x_1 < a_2 < \dots$.*
2. *The set X can be written as a finite disjoint union $X = \bigcup X_i$ where the X_i are definable over $M \cup A$, and each $\mathrm{Conv}(X_i)$ contains no M-point.*

Proof. Easy ; (2) follows from (1). $\square$

Proposition 2.5. *Let $A \subset M$, with M, $|A|^+$-saturated, and let $p \in S_1(M)$. The following are equivalent :*

1. *The type p forks over A,*
2. *There exist $a, b \in M$ such that $p \vdash a < x < b$, and a and b have the same type over A,*
3. *There exist $a, b \in M$ such that $p \vdash a < x < b$, and the interval $I_{a,b} = \{x : a < x < b\}$ divides over A.*

Proof. (3) $\Rightarrow$ (1) is trivial.

For (2) $\Rightarrow$ (3), it is enough to show that if $a \equiv_A b$, then $I_{a,b}$ divides over A. Let σ be an A-automorphism sending a to b. Then the tuple $(b = \sigma(a), \sigma(b))$ has the same type as (a, b), and $a < b < \sigma(b)$. By iterating, we obtain a sequence $a_1 < a_2 < \dots$ such that (a_k, a_{k+1}) has the same type over A as (a, b). Now the sets $I_{a_{2k}, a_{2k+1}}$ are pairwise disjoint and all have the same type over A. Therefore each of them divides over M.

We now prove (1) $\Rightarrow$ (2)

Assume that (2) fails for p. Let $X_{\bar{a}}$ be an M-definable set such that $p \vdash X_{\bar{a}}$. Let $\bar{a}_0 = a, \bar{a}_1, \bar{a}_2, \dots$ be an A-indiscernible sequence. Note that the cut of p is invariant under all A-automorphisms. Therefore each of the $X_{\bar{a}_i}$ contains a type with the same cut over M as p. Now do a similar reasoning as in Lemma 2.1. $\square$

Corollary 2.6. *Forking equals dividing : for any $A \subset B$, any $p \in S(B)$, p forks over A if and only if p divides over A.*

Proof. By results of Chernikov and Kaplan ([CheKap]), it is enough to prove that no type forks over its base. And it suffices to prove this for one-types (because of the general fact that if $\mathrm{tp}(a/B)$ does not fork over A and $\mathrm{tp}(b/Ba)$ does not fork over Aa, then $\mathrm{tp}(a, b/B)$ does not fork over A).

Assume $p \in S_1(A)$ forks over A. Then by the previous proposition, p implies a finite disjunction of intervals $\bigcup_{i<n}(a_i, b_i)$ with $a_i \equiv_A b_i$. Assume n is minimal. Without loss, assume $a_0 < a_1 < \dots$. Now, as $a_0 \equiv_A b_0$ we can find points a_i', b_i', with $(a_i, b_i) \equiv_A (a_i', b_i')$ and $a_0' = b_0$.

Then p proves $\bigcup_{i<n}(a_i', b_i')$. But the interval (a_0, b_0) is disjoint from that union, so p proves $\bigcup_{0<i<n}(a_i, b_i)$, contradicting the minimality of n. $\square$

Note that this does not hold without the assumption that the structure is linearly ordered. In fact the standard example of the circle with a predicate $C(x, y, z)$ saying that y is between x and z (see for example [Wag], 2.2.4.) is dp-minimal.

Lemma 2.7. *Let E be a definable equivalence relation on M, we consider the imaginary sort $S = M/E$. Then there is on S a definable equivalence relation $\sim$ with finite classes such that there is a definable linear order on $S/\sim$.*

Proof. Define a partial order on S by $a/E \prec b/E$ if $\inf(\{x : xEa\}) < \inf(\{x : xEb\})$. Let $\sim$ be the equivalence relation on S defined by $x \sim y$ if $\neg(x \prec y \vee y \prec x)$. Then $\prec$ defines a linear order on $S/\sim$. The proof that $\sim$ has finite classes is another variation on the proof of 2.1. $\square$

From now until the end of this section, we also assume NIP.

Lemma 2.8. *(NIP). Let $p \in S_1(\mathbb{M})$ be a type inducing an M-definable cut, then p is definable over M.*

Proof. We know that p does not fork over M, so by NIP, p is M-invariant. Let M_1 be an $|M|^+$-saturated model containing M. Then the restriction of p to M_1 has a unique global extension inducing the same cut as p. In particular p has a unique heir. Being M-invariant, p is definable over M. $\square$

The next lemma states that members of a uniformly definable family of sets define only finitely many "germs at $+\infty$".

Lemma 2.9. *(NIP). Let $\phi(x, y)$ be a formula with parameters in some model M_0, x a single variable. Then there are $b_1, \ldots, b_n$ such that for every b, there is $\alpha \in \mathbb{M}$ and k such that the sets $\phi(x, b) \wedge x > \alpha$ and $\phi(x, b_k) \wedge x > \alpha$ are equal.*

Proof. Let E be the equivalence relation defined on tuples by bEb' iff $(\exists\alpha)(x > \alpha \rightarrow (\phi(x, b) \leftrightarrow \phi(x, b')))$. Let b, b' having the same type over M_0. By NIP, the formula $\phi(x, b)\triangle\phi(x, b')$ forks over M_0. By Lemma 2.1, this formula cannot be cofinal, so b and b' are E-equivalent. This proves that E has finitely many classes. $\square$

If the order is dense, then this analysis can be done also locally around a point a with the same proof :

Lemma 2.10. *(NIP + dense order). Let $\phi(x, y)$ be a formula with parameters in some model M_0, x a single variable. Then there exists n such that : For any point a, there are $b_1, \ldots, b_n$ such that for all b, there is $\alpha < a < \beta$ and k such that the sets $\phi(x, b) \wedge \alpha < x < \beta$ and $\phi(x, b_k) \wedge \alpha < x < \beta$ are equal.*

3 Dp-minimal groups

We study inp-minimal groups. Note that by an example of Simonetta, ([Sim]), not all such groups are abelian-by-finite. It is proven in [MacSte] that C-minimal groups are abelian-by-torsion. We generalize the statement here to all inp-minimal theories.

Proposition 3.1. *Let* G *be an inp-minimal group. Then there is a definable normal abelian subgroup* H *such that* G/H *is of finite exponent.*

Proof. Let A, B be two definable subgroups of G. If $a \in A$ and $b \in B$, then there is $n > 0$ such that either $a^n \in B$ or $b^n \in A$. To see this, assume $a^n \notin B$ and $b^n \notin A$ for all $n > 0$. Then, for $n \neq m$, the cosets $a^m B$ and $a^n B$ are distinct, as are $A.b^m$ and $A.b^n$. Now we obtain an independence pattern of length two by considering the sequences of formulas $\phi_k(x) =$ "$x \in a^k B$" and $\psi_k(x) =$ "$x \in A.b^k$".

For $x \in G$, let $C(x)$ be the centralizer of x. By compactness, there is k such that for $x, y \in G$, for some $k' \leq k$, either $x^{k'} \in C(y)$ or $y^{k'} \in C(x)$. In particular, letting $n = k!$, x^n and y^n commute.

Let $H = C(C(G^n))$, the bicommutant of the nth powers of G. It is an abelian definable subgroup of G and for all $x \in G$, $x^n \in H$. Finally, if H contains all n powers then it is also the case of all conjugates of H, so replacing H by the intersection of its conjugates, we obtain what we want. □

Now we work with ordered groups.

Lemma 3.2. *Let* G *be an inp-minimal ordered group. Let* H *be a definable sub-group of* G *and let* C *be the convex hull of* H*. Then* H *is of finite index in* C*.*

Proof. We may assume that H and C are $\emptyset$-definable. So without loss, assume $C = G$.

If H is not of finite index, there is a coset of H that forks over $\emptyset$. All cosets of H are cofinal in G. This contradicts Lemma 2.1. □

Proposition 3.3. *Let* G *be an inp-minimal ordered group, then* G *is abelian.*

Proof. Note that if $a, b \in G$ are such that $a^n = b^n$, then $a = b$, for if for example $0 < a < b$, then $a^n < a^{n-1}b < a^{n-2}b^2 < \ldots < b^n$.

For $x \in G$, let $C(x)$ be the centralizer of x. We let also $D(x)$ be the convex hull of $C(x)$. By 3.2, $C(x)$ is of finite index in $D(x)$. Now take $x \in G$ and $y \in D(x)$. Then xy is in $D(x)$, so there is n such that $(xy)^n \in C(x)$. Therefore $(yx)^n = x^{-1}(xy)^n x = (xy)^n$. So $xy = yx$ and $y \in C(x)$. Thus $C(x) = D(x)$ is convex.

Now if $0 < x < y \in G$, then $C(y)$ is a convex subgroup containing y, so it contains x, and x and y commute. □

This answers a question of Goodrick ([Goo] 1.1).

Now, we assume NIP, so G is a dp-minimal ordered group. We denote by G^+ the set of positive elements of G.

Let $\phi(x)$ be a definable set (with parameters). For $\alpha \in G$, define $X_\alpha = \{g \in G^+ : (\forall x > \alpha)(\phi(x) \leftrightarrow \phi(x+g))\}$. Let H_α be equal to $X_\alpha \cup -X_\alpha \cup \{0\}$. Then H_α is a definable subgroup of G and if $\alpha < \beta$, H_α is contained in H_β. Finally, let H be the union of the H_α for $\alpha \in G$, it is the subgroup of *eventual periods* of $\phi(x)$.

Now apply Lemma 2.9 to the formula $\psi(x, y) = \phi(x-y)$. It gives n points $b_1, \ldots, b_n$ such that for all $b \in G$, there is k such that $b - b_k$ is in H. This implies that H has finite index in G.

If furthermore G is densely ordered, then we can do the same analysis locally. This yields a proof of a conjecture of Alf Dolich : in a dp-minimal divisible ordered group, any infinite set has non empty interior. As a consequence, a dp-minimal divisible definably complete ordered group is o-minimal.

As before, $I_{a,b}$ denotes the open interval (a, b), and τ_b is the translation by $-b$. We will make use of two lemmas from [Goo] that we recall here for convenience.

Lemma 3.4 ([Goo], 3.3)**.** *Let G be a divisible ordered inp-minimal group, then any infinite definable set is dense in some non trivial interval.*

In the following lemma, $\overline{M}$ stands for the completion of M. By a definable function f into $\overline{M}$, we mean a function of the form $a \mapsto \inf \phi(a; M)$ where $\phi(x; y)$ is a definable function. So one can view $\overline{M}$ as a collection of imaginary sorts (in which case it naturally contains only *definable* cuts of M), or understand $f : M \to \overline{M}$ simply as a notation.

Lemma 3.5 ([Goo], 3.19)**.** *Let $f : M \to \overline{M}$ be a definable partial function such that $f(x) > 0$ for all x in the domain of f. Then for every interval I, there is a sub-interval $J \subseteq I$ and $\epsilon > 0$ such that for $x \in J \cap \mathrm{dom}(f)$, $|f(x)| \geq \epsilon$.*

Theorem 3.6. *Let G be a divisible ordered dp-minimal group. Let X be an infinite definable set, then X has non-empty interior.*

Proof. Let $\phi(x)$ be a formula defining X.

By Lemma 3.4, there is an interval I such that X is dense in I. By Lemma 2.10 applied to $\psi(x; y) = \phi(y + x)$ at 0, there are $b_1, \ldots, b_n \in M$ such that for all $b \in M$, there is $\alpha > 0$ and k such that $|x| < \alpha \to (\phi(b + x) \leftrightarrow \phi(b_k + x))$.

Taking a smaller I and X, if necessary, assume that for all $b \in I \cap X$, we may take $k = 1$.

Define $f : x \mapsto \sup\{y : I_{-y,y} \cap \tau_{b_1} X = I_{-y,y} \cap \tau_x X\}$, it is a function into $\overline{M}$, the completion of M. By Lemma 3.5, there is $J \subset I$ such that, for all $b \in J$, we have $|f(b)| \geq \epsilon$.

Fix $\nu < \frac{\epsilon}{2}$ and $b \in J$ such that $I_{b-2\epsilon,b+2\epsilon} \subseteq J$ (taking smaller ϵ if necessary). Set $L = I_{b-\nu,b+\nu}$ and $Z = L \cap X$. Assume for simplicity $b = 0$. Easily, if $g_1, g_2 \in Z$, then $g_1 + g_2 \in Z \cup L^c$ and $-g_1 \in Z$ (because any two points of Z have isomorphic neighborhoods of size ϵ). So Z is a group interval : it is the intersection with $I_{b-\nu,b+\nu}$ of some subgroup H of G. Now if $x, y \in L$ satisfy that there is $\alpha > 0$ such that $I_{-\alpha,\alpha} \cap \tau_x X = I_{-\alpha,\alpha} \cap \tau_y X$, then $x \equiv y$ modulo H. It follows that points of L lie in finitely many cosets modulo H. Assume Z is not convex, and take $g \in L \setminus Z$. Then for each $n \in \mathbf{N}$, the point g/n is in L and the points g/n define infinitely many different cosets; a contradiction.

Therefore Z is convex and X contains a non trivial interval. □

Corollary 3.7. *Let G be a dp-minimal ordered group. Assume G is divisible and definably complete, then G is o-minimal.*

Proof. Let X be a definable subset of G. By 3.6, the (topological) border Y of X is finite.

Let $a \in X$, then the largest convex set in X containing a is definable. By definable completeness, it is an interval and its end-points must lie in Y.

This shows that G is o-minimal. □

4 Examples of dp-minimal theories

We give examples of dp-minimal theories, namely : linear orders, order of finite width and trees.

We first look at linear orders. We consider structures of the form $(M, \leq, C_i, R_j)$ where $\leq$ defines a linear order on M, the C_i are unary predicates ("colors"), the R_j are binary monotone relations (that is $x_1 \leq x R_j y \leq y_1$ implies $x_1 R_j y_1$).

The following is a (weak) generalization of Rubin's theorem on linear orders (see [Poi]).

Proposition 4.1. *Let* $(M, \leq, C_i, R_j)$*be a colored linear order with monotone relations. Assume that all* $\emptyset$*-definable sets in dimension 1 are coded by a color and all monotone* $\emptyset$*-definable binary relations are represented by one of the* R_j*. Then the structure eliminates quantifiers.*

Proof. The result is obvious if M is finite, so we may assume (for convenience) that this is not the case.

We prove the theorem by back-and-forth. Assume that M is ω-saturated and take two tuples $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$ from M having the same quantifier free type.

Take $x_0 \in M$; we look for a corresponding y_0. Notice that $\leq$ is itself a monotone relation, a finite boolean combinations of colors is again a color, a positive combination of monotone relations is again a monotone relation, and if xRy is monotone $\phi(x, y) = \neg y R x$ is monotone. By compactness, it is enough to find a y_0 satisfying some finite part of the quantifier-free type of x_0; that is, we are given

- One color C such that $M \models C(x_0)$,
- For each k, monotone relations R_k and S_k such that $M \models x_0 R_k x_k \wedge x_k S_k x_0$.

Define $U_k(x) = \{t : t R_k x_k\}$ and $V_k(x) = \{t : x S_k t\}$. The $U_k(x)$ are initial segments of M and the $V_k(x)$ final segments. For each k, k', either $U_k(x_k) \subseteq U_{k'}(x_{k'})$ or $U_{k'}(x_{k'}) \subseteq U_k(x_k)$. Assume for example $U_k(x_k) \subseteq U_{k'}(x_{k'})$, then this translates into a relation $\phi(x_k, x_{k'})$, where $\phi(x, y) = (\forall t)(t R_k x \to t R_{k'} y)$. Now $\phi(x, y)$ is a monotone relation itself. The assumptions on $\bar{x}$ and $\bar{y}$ therefore imply that also $U_k(y_k) \subseteq U_{k'}(y_{k'})$.

The same remarks hold for the final segments V_k.

Now, we may assume that $U_1(x_1)$ is minimal in the $U_k(x_k)$ and $V_l(x_l)$ is minimal in the $V_k(x_k)$. We only need to find a point y_0 satisfying $C(x)$ in the intersection $U_1(y_1) \cap V_l(y_l)$.

Let $\psi(x, y)$ be the relation $(\exists t)(C(t) \wedge t R_1 y \wedge x R_l t)$. This is a monotone relation. As it holds for (x_0, x_l), it must also hold for (y_0, y_l), and we are done. □

The following result was suggested, in the case of pure linear orders, by John Goodrick.

Proposition 4.2. *Let* $\mathcal{M} = (M, \leq, C_i, R_j)$ *be a linearly ordered infinite structure with colors and monotone relations. Then* $Th(\mathcal{M})$ *is dp-minimal.*

Proof. By the previous result, we may assume that $T = Th(\mathcal{M})$ eliminates quantifiers. Let $(x_i)_{i \in I}$, $(y_i)_{i \in I}$ be mutually indiscernible sequences of n-tuples, and let $\alpha \in M$ be a point. We want to show that one of the following holds :

- For all $i, i' \in I$, x_i and $x_{i'}$ have the same type over α, or
- for all $i, i' \in I$, y_i and $y_{i'}$ have the same type over α.

Assume that I is dense without end points.

By quantifier elimination, we may assume that $n = 1$, that is the x_i and y_i are points of M. Without loss, the (x_i) and (y_i) form increasing sequences. Assume there exists $i < j \in I$ and R a monotone definable relation such that $M \models \neg\alpha R x_i \wedge \alpha R x_j$. By monotonicity of R, there is a point i_R of the completion of I such that $i < i_R \to \neg\alpha R x_i$ and $i > i_R \to \alpha R x_i$.

Assume there is also a monotone relation S and an i_S such that $i < i_S \to \neg\alpha S y_i$ and $i > i_S \to \alpha S y_i$.

For points x, y define $I(x, y)$ as the set of $t \in M$ such that $M \models \neg tRx \wedge tRy$. This is an interval of M. Furthermore, if $i_1 < i_2 < i_3 < i_4$ are in I, then the intervals $I(x_{i_1}, x_{i_2})$ and $I(x_{i_3}, x_{i_4})$ are disjoint. Define $J(x, y)$ the same way using S instead of R.

Take $i_0 < i_R < i_1 < i_2 < \ldots$ and $j_0 < i_S < j_1 < j_2 < \ldots$. For $k < \omega$, define $I_k = I(x_{i_{2k}}, x_{i_{2k+1}})$ and $J_k = J(y_{j_{2k}}, y_{j_{2k+1}})$. The two sequences (I_k) and (J_k) are mutually indiscernible sequences of disjoint intervals. Furthermore, we have $\alpha \in I_0 \wedge J_0$. By mutual indiscernibility, $I_i \wedge J_j \neq \emptyset$ for all indices i and j, which is impossible.

We treated the case when α was to the left of the increasing relations R and S. The other cases are similar. □

An ordered set $(M, \leq)$ is of *finite width*, if there is n such that M has no antichain of size n.

Corollary 4.3. *Let $\mathcal{M} = (M, \leq)$ be an infinite ordered set of finite width, then $\mathrm{Th}(\mathcal{M})$ is dp-minimal.*

Proof. We can define such a structure in a linear order with monotone relations : see [Shm]. More precisely, there exists a structure $P = (P, \prec, R_j)$ in which $\prec$ is a linear order and the R_j are monotone relations. There is a definable relation $O(x, y)$ such that the structure (P, O) is isomorphic to $(M, \leq)$.

The result therefore follows from the previous one. □

We now move to trees. A tree is a structure $(T, \leq)$ such that $\leq$ defines a partial order on T, and for all $x \in T$, the set of points smaller than x is linearly ordered by $\leq$. We will also assume that given $x, y \in T$, the set of points smaller than x and y has a maximal element $x \wedge y$ (and set $x \wedge x = x$). This is not actually a restriction, since we could always work in an imaginary sort to ensure this.

Given $a, b \in T$, we define the open ball $B(a; b)$ of center a containing b as the set $\{x \in T : x \wedge b > a\}$, and the closed ball of center a as $\{x \in T : x \geq a\}$.

Notice that two balls are either disjoint or one is included in the other.

Lemma 4.4. *Let $(T, \leq)$ be a tree, $a \in T$, and let D denote the closed ball of center a. Let $\bar{x} = (x^1, \ldots, x^n) \in (T \setminus D)^n$ and $\bar{y} = (y^1, \ldots, y^m) \in D^m$. Then $\mathrm{tp}(\bar{x}/a) \cup \mathrm{tp}(\bar{y}/a) \vdash \mathrm{tp}(\bar{x} \cup \bar{y}/a)$.*

Proof. A straightforward back-and-forth, noticing that $\mathtt{tp}(\bar{x}/a) \cup \mathtt{tp}(\bar{y}/a) \vdash \mathtt{tp}_{qf}(\bar{x} \cup \bar{y}/a)$ (quantifier-free type). □

We now work in the language $\{\leq, \wedge\}$, so a sub-structure is a subset closed under $\wedge$.

Proposition 4.5. *Let* $A = (a_1, \ldots, a_n)$, $B = (b_1, \ldots, b_n)$ *be two sub-structures from* T. *Assume :*

1. A *and* B *are isomorphic as sub-structures,*
2. *for all* i, j *such that* $a_i \geq a_j$, $\mathtt{tp}(a_i, a_j) = \mathtt{tp}(b_i, b_j)$.

Then $\mathtt{tp}(A) = \mathtt{tp}(B)$.

Proof. We do a back-and-forth. Assume $\mathcal{T}$ is ω-saturated and A, B satisfy the hypothesis. We want to add a point a to A. We may assume that $A \cup \{a\}$ forms a sub-structure (otherwise, if some $a_i \wedge a$ is not in $A \cup \{a\}$, add first this element).

We consider different cases :

1. The point a is below all points of A. Without loss a_0 is the minimal element of A (which exists because A is closed under $\wedge$). Then find a b such that $\mathtt{tp}(a_0, a) = \mathtt{tp}(b_0, b)$. For any index i, we have : $\mathtt{tp}(a_i, a_0) = \mathtt{tp}(b_i, b_0)$ and $\mathtt{tp}(a, a_0) = \mathtt{tp}(b, b_0)$. By Lemma 4.4, $\mathtt{tp}(a_i, a) = \mathtt{tp}(b_i, b)$.

2. The point a is greater than some point in A, say a_1, and the open ball $\mathfrak{a} := B(a_1; a)$ contains no point of A.

   Let $\mathcal{A}$ be the set of all open balls $B(a_1; a_i)$ for $a_i > a_1$. Let n be the number of balls in $\mathcal{A}$ that have the same type p as $\mathfrak{a}$. Then $\mathtt{tp}(a_1)$ proves that there are at least $n + 1$ open balls of type p of center a_1. Therefore, $\mathtt{tp}(b_1)$ proves the same thing. We can therefore find an open ball $\mathfrak{b}$ of center b_1 of type p that contains no point from B. That ball contains a point b such that $\mathtt{tp}(b_1, b) = \mathtt{tp}(a_1, a)$. Now, if a_i is smaller than a_1, we have $\mathtt{tp}(a_i, a_1) = \mathtt{tp}(b_i, b_1)$ and $\mathtt{tp}(a_1, a) = \mathtt{tp}(b_1, b)$, therefore by Lemma 4.4, $\mathtt{tp}(a, a_i) = \mathtt{tp}(b, b_i)$.

   The fact that we have taken b in a new open ball of center b_1 ensures that $B \cup \{b\}$ is again a sub-structure and that the two structures $A \cup \{a\}$ and $B \cup \{b\}$ are isomorphic.

3. The point a is between two points of A, say a_0 and a_1 ($a_0 < a_1$), and there are no points of A between a_0 and a_1.

   Find a point b such that $\mathtt{tp}(a_0, a_1, a) = \mathtt{tp}(b_0, b_1, b)$. Then if i is such that $a_i > a$, we have $a_i \geq a_1$ and again by Lemma 4.4, $\mathtt{tp}(a_i, a) = \mathtt{tp}(b_i, b)$. And same if $a_i < a$.

□

Corollary 4.6. *Let* $A \subset T$ *be any subset. Then* $\bigcup_{(a,b,c) \in A^3} \mathtt{tp}(a, b, c) \vdash \mathtt{tp}(A)$.

Proof. Let A_0 be the substructure generated by A. By the previous theorem the following set of formulas implies the type of A_0 :

- the quantifier-free type of A_0,

- the set of 2-types $\mathrm{tp}(a,b)$ for $(a,b) \in A_0^2$, $a < b$.

We need to show that those formulas are implied by the set of 3-types of elements of A. We may assume A is finite.

First, the knowledge of all the 3-types is enough to construct the structure A_0. To see this, start of example with a point $a \in A$ maximal. Knowing the 3-types, one knows in what order the $b \wedge a, b \in A$ are placed. Doing this for all such a, enables one to reconstruct the tree A_0.

Now take $m_1 = a \wedge b$, $m_2 = c \wedge d$ for $a,b,c,d \in A$ such that $m_1 \leq m_2$. The points m_1 and m_2 are both definable using only 3 of the points a,b,c,d, say a,b,c. Then $\mathrm{tp}(a,b,c) \vdash \mathrm{tp}(m_1,m_2)$. □

The previous results are also true, with the same proofs, for colored trees.

It is proven in [Par] that theories of trees are NIP. We give a more precise result.

Proposition 4.7. *Let $\mathcal{T} = (T, \leq, C_i)$ be a colored tree. Then $\mathrm{Th}(\mathcal{T})$ is dp-minimal.*

Proof. We will use criterium (5) of 1.4 : if $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ are mutually indiscernible sequences and $\alpha \in T$ is a point, then one of the sequences (a_i) and (b_j) is indiscernible over α.

We will always assume that the index sets (I and J) are dense linear orders without end points.

1) We start by showing the result assuming the a_i and b_j are points (not tuples).

We classify the indiscernible sequence (a_i) in 4 classes depending on its quantifier-free type.

I The sequence (a_i) is monotonous (increasing or decreasing).

II The a_i are pairwise incomparable and $a_i \wedge a_j$ is constant equal to some point β.

III The a_i are incomparable and $a_i \wedge a_j$, $i < j$ depends only on i. Then let $a'_i = a_i \wedge a_j$ (for some $i < j$). The a'_i form an increasing indiscernible sequence.

IV The a_i are incomparable and $a_i \wedge a_j$, $i < j$ depends only on j. Then the $a'_j = a_i \wedge a_j$ $(i < j)$ form a decreasing indiscernible sequence.

Assume (a_i) lands in case **I**. Consider the set $\{x : x < \alpha\}$. If that set contains a non-trivial subset of the sequence (a_i), we say that α *cuts* the sequence. If this is not the case, then the sequence (a_i) stays indiscernible over α. To see this, assume for example that (a_i) is increasing and that α is greater that all the a_i. Take two sets of indices $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ and a $k \in I$ greater that all those indices. Then $\mathrm{tp}(a_{i_1}, \ldots, a_{i_n}/a_k) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_n}/a_k)$. Therefore by Lemma 4.4, $\mathrm{tp}(a_{i_1}, \ldots, a_{i_n}/\alpha) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_n}/\alpha)$.

In case **II**, note that if (a_i) is not α-indiscernible, then there is $i \in I$ such that α lies in the open ball $B(\beta; a_i)$ (we will also say that α *cuts* the sequence (a_i)). This follows easily from Proposition 4.5.

In the last two cases, if (a_i) is α-indiscernible, then it is also the case for (a_i'). Conversely, if (a_i') is α-indiscernible, then α does not cut the sequence (a_i'). From 4.5, it follows easily that (a_i) is also α-indiscernible. We can therefore replace the sequence (a_i) by (a_i') which belongs to case **I**.

Going back to the initial data, we may assume that (a_i) and (b_j) are in case **I** or **II**. It is then straightforward to check that α cannot cut both sequences. For example, assume (a_i) is increasing and (b_j) is in case **II**. Then define β as $b_i \wedge b_j$ (any i, j). If α cuts (b_j), then $\alpha > \beta$. But (a_i) is β-indiscernible. So β does not cut (a_i). The only possibility for α to cut (a_i) is that β is smaller that all the a_i and the a_i lie in the same open ball of center β as α. But then the a_i lie in the same open ball of center β as one of the b_j. This contradicts mutual indiscernability.

2) Reduction to the previous case. We show that if $(a_i)_{i \in I}$ is an indiscernible sequence of n-tuples and $\alpha \in T$ such that (a_i) is not α-indiscernible, then there is an indiscernible sequence $(d_i)_{i \in I}$ of points of T in $dcl((a_i))$ such that (d_i) is not α-indiscernible.

First, by 4.6, we may assume that $n = 2$. Write $a_i = (b_i, c_i)$ and define $m_i = b_i \wedge c_i$. We again study different cases :

1. The m_i are all equal to some m.

   As (a_i) is not α-indiscernible, necessarily, $\alpha > m$ and the ball $B(m; \alpha)$ contains one b_i (resp. c_i). Then take $d_i = b_i$ (resp. $d_i = c_i$) for all i.

2. The m_i are linearly ordered by $<$ and no b_i nor c_i is greater then all the m_i.

   Then the balls $B(m_i; b_i)$ and $B(m_i; c_i)$ contain no other point from $(b_i, c_i, m_i)_{i \in I}$. Then, α must cut the sequence (m_i) and one can take $d_i = m_i$ for all i.

3. The m_i are linearly ordered by $<$ and, say, each b_i is greater than all the m_i.

   Then each ball $B(m_i; a_i)$ contains no other point from $(b_i, c_i, m_i)_{i \in I}$. If α cuts the sequence m_i, than again one can take $d_i = m_i$. Otherwise, take a point γ larger than all the m_i but smaller than all the d_i. Applying 4.4 with a there replaced by γ, we see that (b_i) cannot be α-indiscernible. Then take $d_i = b_i$ for all i.

4. The m_i are pairwise incomparable.

   The the sequence (m_i) lies in case **II**, **III** or **IV**. The open balls $B(m_i; b_i)$ and $B(m_i; c_i)$ cannot contain any other point from $(b_i, c_i, m_i)_{i \in I}$. Considering the different cases, one sees easily that taking $d_i = m_i$ will work.

This finishes the proof.

□

Remark 4.8. If we define dp-minimal$^+$ analogously to strongly$^+$-dependent (see [Sh863]), all theories studied in this section are dp-minimal$^+$.

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