Noncommutative Tori, Real Multiplication and Line Bundles

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Abstract

This thesis explores an approach to Hilbert's twelfth problem for real quadratic number fields, concerning the determination of an explicit class field theory for such fields. The basis for our approach is a paper by Manin proposing a theory of *Real Multiplication* realising such an explicit theory, analogous to the theory of Complex Multiplication associated to imaginary quadratic fields. Whereas elliptic curves play the leading role in the latter theory, objects known as *Noncommutative Tori* are the subject of Manin's dream.

In this thesis we study a family of topological spaces known as Quantum Tori that arise naturally from Manin's approach. Our aim throughout this thesis is to show that these non-Hausdorff spaces have an "algebraic character", which is unexpected through their definition, though entirely consistent with their envisioned role in Real Multiplication.

Chapter 1 is a general introduction to the problem, providing a historical and technical background to the motivation behind this thesis. Chapter 2 deals with the problem of defining continuous maps between Quantum Tori using ideas from Nonstandard Analysis, culminating in a description of the action of a Galois group on certain isomorphism classes of these spaces. Chapter 3 concerns the problem of defining a nontrivial notion of line bundles over Quantum Tori, while Chapter 4 concerns the existence of sections of these line bundles. We show that such sections have applications to the null values of the derivatives of L-functions attached to real quadratic fields, which in the context of Stark's conjectures is seen to be relevant to Hilbert's problem.

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Contents

Chapter 1

Introduction

Class Field Theory attempts to classify the abelian extensions of a field K, in terms of data intrinsic to K, namely the idele class group. When K is a number field, the celebrated "Existence Theorem" asserts a bijection between the finite abelian extensions over K, and open subgroups of finite index in the idele class group. Fields arising in this way are known as *class fields* for K. Despite ensuring the existence of such fields for K, the proof of this result is in general nonconstructive[1], not providing a set of generators for the abelian extension. In the early twentieth century Hilbert listed the need for such an *explicit class field theory* as the twelfth in a list of twenty three problems he deemed to be of importance to mathematics.

A century later Hilbert's twelfth problem remains unanswered, except in a few special circumstances. In 1896 Hilbert himself gave the first complete answer to the case when K is the field $\mathbb{Q}$ of rational numbers following the work of Kronecker and Weber. By the end of the nineteenth century a solution was known for the case when K is an imaginary quadratic field, fulfilling Kronecker's *Jugendtraum*, or "dream of youth". This was achieved by generating abelian extensions of K by adjoining spe-

[1]The proof is constructive for a certain type of extensions, known as Kummer type extensions. Such extensions can be characterised as being abelian extensions of a field K over which the polynomial $X^n - 1$ splits, with abelian Galois group of exponent n. For more details see Chapter IV §3 of [37].

cial values of certain functions on elliptic curves with Complex Multiplication. Much later in the 1970's, this result inspired the construction of the Lubin-Tate formal group, which was used to establish a solution when K is a local field [22].

The simplest class of number fields for which the problem remains unsolved is the case when K is a real quadratic field.

Some progress has been made in obtaining solutions for isolated classes of real quadratic fields by Shimura [51, 52] and Shintani [55]. Shimura uses a similar philosophy to that of the theory of Complex Multiplication, generating abelian extensions of certain real quadratic fields (of class number one) by the torsion points of certain abelian varieties. Shintani's work is motivated by the ability to express the L-function of a real quadratic field in terms of certain special functions studied by Barnes in [4]. However, neither of these provides a systematic way of giving a solution for a general real quadratic field.

The quest for such a solution is the subject of Manin's paper "Real Multiplication and Noncommutative Geometry" [32] where he poses his *Alterstraum* - a theory of Real Multiplication. The theory laid out in Manin's paper is connected to the development of Noncommutative Geometry studied by A.Connes in the early 1980's. His book [9], considered one of the milestones in mathematics, studied the analytical theory of non-Hausdorff spaces using C*-algebra and operator theory. Such "Noncommutative spaces" have a noncommutative C*-algebra associated to them which is an analogue to the algebra of $\mathbb{C}$-valued functions on a Hausdorff space.

The ideas contained within Manin's paper form the basis for this thesis. In this chapter we will state the problem of obtaining an *explicit class field theory* in more detail, and explain the problems which arise when attempting to derive such a theory for real quadratic fields. The framework of Manin's proposed theory of Real Multiplication is introduced, as well as some basic concepts from Noncommutative Geometry, which is fundamental to his approach. Finally we explain how this is related to our work, and give a brief description of the structure of this thesis.

1.1 Explicit Class Field Theory

Let K be a number field, and let v be a valuation on K corresponding to a place of K. The completion of K at v is a discrete valuation field which we denote by K_v. The adele ring of K is defined to be

$$\mathbb{A}_K := \hat{\prod_v} K_v$$

where v ranges over all places of K. The product $\hat{\prod}$ implies that if $(a_v) \in \mathbb{A}_K$ then the following condition is satisfied:

For all but finitely of the finite places v, we have $\mathrm{ord}_v(a_v) \geq 0$, where

$$\mathrm{ord}_v(x) = -\frac{\log(v(x))}{\log(N_{K/\mathbb{Q}}\mathfrak{p})}$$

and $\mathfrak{p}$ is the prime corresponding to the valuation v.

There is a natural embedding of K^* in to $\mathbb{A}_K^*$, and the *idele class group* of K is defined to be the quotient $C_K := \mathbb{A}_K^*/K^*$. When L is a finite extension of K, and w is a place of L lying above v in K, we have a natural norm map between local fields

$$N_{L_w/K_v} : L_w \longrightarrow K_v.$$

These induce a global norm on the idele class group of L, which can be defined by

$$\begin{array}{rcl} N_{L/K} : C_L & \to & C_K \\ (a_w)_w & \mapsto & \left(\prod_{w|v} N_{L_w/K_v}(a_w)\right)_v . \end{array}$$

Central to Class Field Theory is the *reciprocity map*, which exhibits a homomorphism:

$$\psi_K : C_K \longrightarrow G_K^{ab},$$

where G_K^{ab} denotes the Galois group $\mathrm{Gal}(K^{ab}/K)$ and K^{ab} is the maximal abelian extension of K over K. When L is a finite abelian extension of K, this map supplies an isomorphism

$$\psi_{L/K} : \frac{C_K}{N_{L/K}(C_L)} \longrightarrow Gal(L/K).$$

The groups $N_{L/K}(C_L)$ are open in a certain natural topology on C_K. The following result uses these ideas to give a classification of the abelian extensions of a number field:

Theorem 1.1.1 (Existence Theorem of Global Class Field Theory, Theorem 6.1 of [37][2]). *Let K be a number field. The map*

$$L \mapsto N_{L/K}(C_L)$$

establishes a one-to-one correspondence between the finite abelian extensions L/K and the open subgroups of finite index in C_K. The field L_U corresponding to the subgroup U of C_K is called the class field *of U, and we have*

$$Gal(L_U/K) \cong C_K/U.$$

[2] In Neukirch's treatment, he uses the complementary topology on the ideles to the one we refer to here, so in his terminology the subgroups $N_{L/K}(C_L)$ are *closed* in C_K. Neukirch's treatment describes *all* abelian extensions (possibly of infinite degree) while in Theorem 1.1.1 we concentrate on those of finite degree.

As mentioned previously, this construction is not explicit in general, since it does not explicitly define the field L_U corresponding to an open subgroup U of C_K. A complete solution to Hilbert's twelfth problem requires

1. A set of generators $\{a_1, \ldots, a_n\}$ for the field L_U for each open subgroup U of C_K;
2. An explicit action of C_K on the generators $\{a_1, \ldots, a_n\}$.

To illustrate these concepts let us consider those number fields for which Hilbert's problem has been solved.

When K is equal to the field $\mathbb{Q}$ of rationals, it is proved that every abelian extension of $\mathbb{Q}$ is contained within the cyclotomic field $K_n := \mathbb{Q}(\zeta_n)$ for some nth root of unity ζ_n. Considering the finite extension K_n it can be shown that

$$\frac{C_{\mathbb{Q}}}{N_{K_n/\mathbb{Q}}(C_{K_n})} \cong (\mathbb{Z}/n\mathbb{Z})^*.$$

If $\sigma \in Gal(K_n/\mathbb{Q})$, then by the above isomorphism we have $\psi_{K_n/\mathbb{Q}}(\sigma) \in (\mathbb{Z}/n\mathbb{Z})^*$. Let m be an integer such that $m \equiv \psi_{K_n/\mathbb{Q}}(\sigma) \mod n$. The reciprocity law is given explicitly on roots of unity by

$$\zeta_n^\sigma = \zeta_n^m.$$

Since the field K_n is generated over $\mathbb{Q}$ by roots of unity this serves to provide an explicit reciprocity law on the whole of K_n.

The maximal abelian extension of $\mathbb{Q}$ is the extension generated by all the roots of unity. Let S denote the algebraic subvariety of $\mathbb{R}^2$ defined by

$$S : x^2 + y^2 - 1 = 0.$$

This has an embedding in to the complex plane by $(x, y) \mapsto x + iy$ whose image we also denote by S. This gives S an abelian group structure induced by the law of multiplication in $\mathbb{C}$, allowing us to view S as an abelian variety. We let S_{tors} denote the torsion points of S with respect to this group law and observe that we can write

$$\mathbb{Q}^{ab} = \mathbb{Q}(S_{tors}).$$

We notice that S_{tors} is isomorphic to the quotient $K/\mathcal{O}_K = \mathbb{Q}/\mathbb{Z}$ via the injective homomorphism

$$\begin{aligned} \exp : K/\mathcal{O}_K &\longrightarrow S \\ x + \mathcal{O}_K &\mapsto \exp(2\pi i x). \end{aligned}$$

Hence we may write $\mathbb{Q}^{ab} = \mathbb{Q}(\exp(K/\mathcal{O}_K))$.

Let $\sigma : \mathbb{Q} \hookrightarrow \mathbb{R}$ denote the infinite (real) place of $\mathbb{Q}$, and let K_σ denote the completion of K at σ. Then $K_\sigma \simeq \mathbb{R}$, and the map exp extends to one on $K_\sigma/\mathcal{O}_K$ whose image is S.

When K is an imaginary quadratic field, the torsion points of of an abelian variety play a fundamental role in describing the maximal abelian extension of K, as the role of S does in the cyclotomic theory. In this case the algebraic variety considered is an elliptic curve E whose endomorphism ring satisfies a certain property. An elliptic curve over $\mathbb{C}$ can be defined by those points of $\mathbb{C}^2$ satisfying the equation

$$E : y^2 = 4x^3 - g_2 x - g_3 \tag{1.1}$$

for some $g_2, g_3 \in \mathbb{C}$, together with a specified "point at infinity" O. These points form a group with the point at infinity acting as the identity.

An important result concerning such curves is given by the Uniformization Theorem. To each elliptic curve E defined by an equation as in (1.1), there exists a lattice Λ in $\mathbb{C}$ such that E is isomorphic as a complex Lie group to the complex torus $\mathbb{C}/\Lambda$. This construction underlies the following:

Theorem 1.1.2 (Uniformisation Theorem). *We have an equivalence between the following categories:*

$\mathcal{EC}$ *The category whose objects are elliptic curves over* $\mathbb{C}$*, and whose morphisms are isogenies (rational homomorphisms);*

$\mathcal{L}$ *The category whose objects are lattices in* $\mathbb{C}$*, and whose morphisms between lattices* Λ_1 *and* Λ_2 *are nonzero complex numbers* α *such that* $\alpha\Lambda_1 \subseteq \Lambda_2$*.*

As a corollary we deduce that every elliptic curve is isomorphic (as a complex Lie group) to a torus $\mathbb{C}/(\mathbb{Z}+\tau_E\mathbb{Z})$ for some representative τ_E of a unique element of $\mathfrak{H}/SL_2(\mathbb{Z})$. Here the matrix group $SL_2(\mathbb{Z})$ acts on the upper half plane $\mathfrak{H}$ by möbius transformations. The j-invariant of the elliptic curve E is defined to be $j(\tau_E)$ where j denotes the modular j-function [26]. This is well defined since the function j is invariant under the action of $SL_2(\mathbb{Z})$.

An elliptic curve E is said to have *Complex Multiplication* if there exist non-integral scalars $\alpha \in \mathbb{C}^*$ such that $\alpha\Lambda_{\tau_E} \subseteq \Lambda_{\tau_E}$ where $\Lambda_{\tau_E} = \mathbb{Z} + \tau_E\mathbb{Z}$. This is equivalent to stating that the endomorphism ring of E contains a *proper* subring isomorphic to $\mathbb{Z}$. In this case it can be shown that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(E)$ is isomorphic to an imaginary quadratic field K, and we say that E has *Complex Multiplication by* K. In this situation the field $K(j(E))$ is shown to be equal to *Hilbert Class field* of K - the maximal abelian unramified extension of K [56]. The culminating result in the theory of Complex Multiplication is that if E has Complex Multiplication by K

then

$$K^{ab} = K(j(E), E_{tors}).$$

The elliptic curve may be chosen such that it corresponds (under the association of Theorem 1.1.2) to the lattice $\mathcal{O}_K$ - the ring of integers of K. The group of torsion points E_{tors} of E is isomorphic to the quotient $K/\mathcal{O}_K$ via the injective homomorphism:

$$\begin{aligned} \phi : \mathbb{C}/\mathcal{O}_K &\longrightarrow E \\ x + \mathcal{O}_K &\mapsto \begin{cases} (\wp(x), \wp'(x)) & \text{if } x \neq 0 \\ O & \text{if } x = 0, \end{cases} \end{aligned}$$

where $\wp$ denotes the *Weierstrass $\wp$-function.*

Let $\sigma : K \hookrightarrow \mathbb{C}$ denote a complex place of K, and let K_σ denote the completion of K at σ. The isomorphism ϕ extends to one on $K_\sigma/\mathcal{O}_K$ whose image is E.

Now let K be a real quadratic field. The previous two examples serve to provide a philosophy we keep in mind when attempting to solve Hilbert's problem for K. We may naïvely expect that in this case abelian extensions of K are generated by the torsion points of a suitable abelian variety N, isomorphic to $K_\sigma/\mathcal{O}_K$ for some infinite place $\sigma : K \hookrightarrow \mathbb{R}$ of K. We would expect an analogous function to the j-invariant for elliptic curves to exist for the variety N, which classified its isomorphism type, and provided a generator for the Hilbert Class Field of K. In summary, using the perspective gained by the functions exp and $\wp$, we would be interested in finding a function J with domain $\mathbb{R}/SL_2(\mathbb{Z})$ and for each real quadratic field K a function W_K with domain $\mathbb{R}/\mathcal{O}_K$ such that:

1. For each $k \in K$, $J(k)$ is algebraic and generates the Hilbert Class field of K over K;

2. $W_K : K/\mathcal{O}_K \to \mathbb{C}^2$ and $K^{ab} = K(J(k), W_K(K/\mathcal{O}_K))$ for some $k \in K$;

3. W_K extends to an isomorphism $W_K : K_\sigma/\mathcal{O}_K \to N$ where $\sigma : K \hookrightarrow \mathbb{R}$ is an infinite place of K and N is an abelian variety.

Since K is a real field this leads us to consider functions on the space $\mathbb{R}/\mathcal{O}_K$.

An immediate problem which faces us is that in the natural quotient topology induced from the Euclidean one on $\mathbb{R}$, the quotient $\mathbb{R}/\mathcal{O}_K$ is non-Hausdorff. This arises because the group $\mathcal{O}_K$ is dense in $\mathbb{R}$, and presents problems when we try to use usual techniques for studying topological spaces. For example, there are no nontrivial continuous functions on this space. The study of such pathological spaces using analytic methods was initiated by Connes [9], who pioneered the subject that has been termed *Noncommutative Geometry* .

1.2 Real Multiplication and Noncommutative Geometry

The philosophy underlying Noncommutative Geometry is to generalise notions valid on Hausdorff spaces to yield nontrivial results when applied to non-Hausdorff spaces. For example, the Gelfand-Naimark theorem establishes a bijection between compact Hausdorff topological spaces and commutative C*-algebras (Theorem I.9.12 of [10]), associating to such a space X the C*-algebra $C(X)$ of continuous $\mathbb{C}$-valued functions on X. Hence compact Hausdorff topological spaces may be defined to be commutative C*-algebras. Noncommutative Geometry generalises this notion by associating noncommutative C*-algebras to non-Hausdorff *commutative* spaces. These noncommutative C*-algebras are then viewed as the function space of a virtual "noncommutative space" associated to the non-Hausdorff space [30].

For another example, we consider vector bundles over a compact Hausdorff space X. A vector bundle over X is a topological space E together with a projection $\pi : E \to X$ satisfying certain local conditions. A section of a vector bundle is a map $\sigma : X \to E$ such that $\pi \circ E = 1_X$. The set of sections of E over X is denoted by $\Gamma(X, E)$ and is a finitely generated projective C(X)-module. A theorem of Swan [61] states that the map $E \mapsto \Gamma(X, E)$ establishes a bijection between vector bundles E over X, and finitely generated projective $C(X)$-modules. Hence given a noncommutative C*-algebra A, the vector bundles over the "noncommutative space" associated to A are defined as finitely generated projective right (or left) modules over A.

Definition 1.2.1 (Noncommutative Tori). Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Define A_θ to be the universal C*-algebra generated by unitary operators U and V such that

$$VU = e^{2\pi i\theta}UV. \tag{1.2}$$

In the context of Noncommutative Geometry this is known as the Noncommutative, or Quantum Torus. Throughout this thesis we will call A_θ a *Noncommutative Torus*, reserving the term Quantum Torus for a topological space to be defined shortly.

In Noncommutative Geometry [9], the C*-algebra A_θ is viewed as the algebra of continuous $\mathbb{C}$-valued functions on a certain non-Hausdorff "commutative" space $\mathbb{T}_\theta$. The space $\mathbb{T}_\theta$ is the quotient of the standard torus by a foliation consisting of a line of constant slope known as the *Kronecker foliation.*

From our previous discussion, when investigating Hilbert's twelfth problem for a real quadratic field K we expect to study the space $\mathbb{R}/\mathcal{O}_K$. If K is a real quadratic field then there exists $\theta \in K$ such that $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\theta$, reducing our study to those spaces $\mathbb{R}/L_\theta$ where $L_\theta = \mathbb{Z} + \mathbb{Z}\theta$.

Definition 1.2.2 (Quantum Tori). For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, define Z_θ to be the quotient $\mathbb{R}/(\mathbb{Z} + \mathbb{Z}\theta)$. We say Z_θ is a *Quantum Torus* with parameter θ. More generally we say that Z is a Quantum Torus if $Z = \mathbb{R}/L$ where L is a dense additive subgroup of $\mathbb{R}$ of rank two.

It can be shown (as discussed in §3.6.1 of Chapter 3) that the space of leaves of the Kronecker foliation $\mathbb{T}_\theta$ is isomorphic to Z_θ. Hence the algebra A_θ also serves to act as the space of "continuous functions" on Z_θ.

In [32], Manin describes a framework in which Noncommutative Tori serve as an analogue to elliptic curves in a potential solution to Hilbert's twelfth problem for real quadratic fields. His approach relies on the existence of Noncommutative Tori whose endomorphism rings are isomorphic to orders in a real quadratic field F. Such Noncommutative Tori are said to have Real Multiplication by F, and the (as yet undeveloped) theory of these relating to a solution of Hilbert's problem for real quadratic fields is known as the theory of Real Multiplication. One important result described in his paper is a duality between Noncommutative Tori and dense additive subgroups of $\mathbb{R}$ of rank two, mirroring the relationship between elliptic curves and complex lattices given by the Uniformization Theorem. We will give a short account of this relationship, and how it relates to the approach we take in this thesis.

Let A be a C*-algebra, and for each $n \in \mathbb{N}$ let $M_n(A)$ denote the algebra of $n \times n$ matrices with coefficients in A. We let $\mathcal{M}A$ denote the union of $M_n(A)$ for all n. Classical theory gives a bijection between idempotents (elements $e \in \mathcal{M}A$ such that $e^2 = e$) and finitely projective right (or left) A-modules. A projection is an idempotent e such that $e^* = e$, where $*$ denotes the convolution induced on $\mathcal{M}A$ by that on the C*-algebra A. In Manin's paper, only those right (or left) A-modules which correspond to projections are considered.

Definition 1.2.3 (Morita Category of Noncommutative Tori). Let $\mathcal{NT}$ denote the category such that:

- The objects of $\mathcal{NT}$ are Noncommutative Tori A_θ.
- A morphism between two Noncommutative Tori A_{θ_1} and A_{θ_2} is an $A_{\theta_1} - A_{\theta_2}$-bi-module $M(A_{\theta_1}, A_{\theta_2})$ such that

  1. As a left A_{θ_1}-module, $M(A_{\theta_1}, A_{\theta_2})$ corresponds to a projection in $\mathcal{M}A_{\theta_1}$;
  2. As a right A_{θ_2}-module, $M(A_{\theta_1}, A_{\theta_2})$ corresponds to a projection in $\mathcal{M}A_{\theta_2}$.

  The composition of the morphisms $M(A_{\theta_1}, A_{\theta_2})$ and $M(A_{\theta_2}, A_{\theta_3})$ is the tensor product $M(A_{\theta_1}, A_{\theta_2}) \otimes_{A_{\theta_2}} M(A_{\theta_2}, A_{\theta_3})$.

The Uniformization Theorem is fundamental in many calculations concerning elliptic curves, reducing those calculations to ones on lattices, which are often easier to handle. This is crucial in the development of the theory of Complex Multiplication, which forms the cornerstone to the solution of Hilbert's problem for imaginary quadratic fields. In the proposed theory of Real Multiplication, the Morita category of Noncommutative Tori assumes the role that the category $\mathcal{EC}$ (as defined in Theorem 1.1.2) plays in Complex Multiplication. A candidate for the Real Multiplication analogue of the category $\mathcal{L}$ is given by the following:

Definition 1.2.4 (Category of Pseudolattices). Let $\mathcal{PL}$ denote the category such that:

- The objects of $\mathcal{PL}$ are dense additive subgroups L of $\mathbb{R}$ of rank two.
- A morphism between two pseudolattices L_1 and L_2 is a nonzero positive real number β such that $\beta L_1 \subseteq L_2$.

To describe the relationship between the categories $\mathcal{NT}$ and $\mathcal{PL}$ we recall some basic elements of the K-theory of C*-algebras.

Given a semi-group S, there exists a construction (known as the Grothendieck construction) of completing S to form a group G_S. Given a C*-algebra A, the set of isomorphism classes of finitely generated projective right (or left) A-modules forms a semi-group [47]. The group obtained by the Grothendieck construction is denoted $K_0(A)$. Due to the nature of the construction, elements of $K_0(A)$ can be represented as the differences of certain classes of matrices. The trace of the matrix is independent of its class, allowing the existence of a canonical trace on $K_0(A)$ induced by the canonical trace on matrices. This trace is unique if we insist that $\mathrm{Tr}(1) = 1$.

In the case of Noncommutative Tori, we have $K_0(A_\theta) \cong \mathbb{Z}^2$ and $\mathrm{Tr}(K_0(A_\theta)) = \mathbb{Z} + \mathbb{Z}\theta$ [44]. Those elements of $K_0(A_\theta)$ which have positive trace are precisely those elements arising from finitely generated projective right (or left) A_θ-modules as opposed to those "virtual" elements which are added in the Grothendieck construction. Via Swan's result in [61], we can view these elements as representing isomorphism classes of vector bundles over the noncommutative space attached to A_θ. Observe that we can recover the pseudolattice $\mathbb{Z} + \mathbb{Z}\theta$ from the K-theory of A_θ as the image of $K_0(A_\theta)$ under the canonical trace.

The following is the result of many different peoples efforts, and provides a weak analogy to the Uniformization Theorem for elliptic curves:

Theorem 1.2.5. [3] *Let $K : \mathcal{NT} \to \mathcal{PL}$ be the functor defined in the following way:*

- *On objects $K(A_\theta) = Tr(K_0(A_\theta))$. Hence by the comments above, $K(A_\theta) = L_\theta$.*

[3] This is Theorem 1.7.1 of [32]. Manin's statement differs slightly from the one we give here, since he considers pairs (L, s) where L is a pseudolattice and s is an *orientation* of L.

- *To define K on morphisms let $M(A_{\theta_1}, A_{\theta_2})$ be a morphism in $\mathcal{NT}$. The trace gives a bijection between $K_0(A_{\theta_1})$ and L_{θ_1}. This bijection allows us to assign to each $l \in L_{\theta_1}$ a unique element $[N^l_{A_{\theta_1}}] \in K_0(A_{\theta_1})$. We define*

$$K(M(A_{\theta_1}, A_{\theta_2}))(l) := Tr([N^l_{A_{\theta_1}} \otimes_{A_{\theta_1}} M(A_{\theta_1}, A_{\theta_2})]).$$

  Since $N^l_{A_{\theta_1}} \otimes_{A_{\theta_1}} M(A_{\theta_1}, A_{\theta_2})$ is a finitely generated projective right A_{θ_2}-module, the right hand side of the above equation lies in L_{θ_2}. This map is a morphism $L_{\theta_1} \to L_{\theta_2}$.

Then the functor K is essentially surjective[4] *on objects and morphisms.*

Among those bi-modules which give rise to morphisms in $\mathcal{NT}$ are those which are isomorphisms. Such a bi-module is said to be a Morita equivalence between the corresponding Quantum Tori. A bi-module M is a Morita equivalence between two C*-algebras A and B if it is finitely generated as a module over A and B separately. For such an $A - B$-bi-module the map

$$N \mapsto N \otimes_A M$$

defines a bijection between finitely generated projective right modules over A and finitely generated projective right modules over B. The following result of Rieffel describes the moduli space of Noncommutative Tori up to Morita equivalence.

Theorem 1.2.6. *Let A_{θ_1} and A_{θ_2} be Noncommutative Tori. Then A_{θ_1} and A_{θ_2} are Morita equivalent if and only if there exist integers a, b, c, d such that $ad - bc = \pm 1$ and*

$$\theta_2 = \frac{a\theta_1 + b}{c\theta_1 + d}.$$

[4]The term *essentially surjective* in this context implies that every element of $\mathcal{PL}$ is isomorphic to an object in the image of K.

The space $\mathbb{P}^1(\mathbb{R})/SL_2(\mathbb{R})$ therefore plays an analogous role in the theory of Noncommutative Tori to that of the modular curve $\mathfrak{H}/SL_2(\mathbb{R})$ in the theory of elliptic curves. The following result constructs a map in the reverse direction to K which respects the notion of isomorphism.

Theorem 1.2.7 (Schwarz, Dieng [11]). *Consider the map from $\mathcal{PL}$ to $\mathcal{NT}$ defined on objects by $E : L_\theta \mapsto A_\theta$. This map can be extended to be defined on isomorphisms between pseudolattices. If $L_{\theta_1} \cong L_{\theta_2}$ then $E(L_{\theta_1})$ and $E(L_{\theta_2})$ are Morita equivalent. Moreover the composition $K \circ E$ is the identity on isomorphism classes of pseudolattices.*

1.3 The structure of this Thesis

One of the main aims of this thesis is to show that we can attach meaning to certain objects associated with Noncommutative Tori, which one would normally associate with the theory of abelian varieties. If Noncommutative Tori are to play an instrumental role in a solution to Hilbert's twelfth problem for real quadratic fields, this should not be surprising given the roles of the circle and elliptic curves in the solutions for the rational and imaginary quadratic fields. In our approach we will focus on viewing Noncommutative Tori as Quantum Tori - topological spaces $\mathbb{R}/L$ where L is a pseudolattice. Indeed, in [32] Manin comments that Theorems 1.2.5 and 1.2.7 should provide a strong enough duality between Noncommutative Tori and pseudolattices to "suffice for the envisioned applications to Real Multiplication".

Many "difficult" problems in number theory involve describing the relationship between algebraic, analytic and topological objects. Examples include the Uniformization Theorem, the Selberg-Trace formula[5] and the explicit reciprocity laws for the

[5]The Selberg-Trace formula describes the dimension of the space of cusp forms (analytic data) in terms of the volume of a fundamental domain (topological data) [15].

rational and imaginary quadratic fields. Recently, a branch of logic known as Model Theory has been used to great effect to prove number theoretic problems requiring just such a bridge between analysis and algebra. In [18] Hrushovski proves the Mordell-Lang conjecture[6] for function fields using these techniques, and used Model Theory in his proof of the Manin-Mumford conjecture[7] in 2001 [19].

Model theory provides a way to "algebraize" analytic theories. The concept of Nonstandard Analysis is an example of this, seeming to supply purely algebraic interpretations of analytic concepts such as continuity, differentiation and integration. A model theoretical interpretation of Noncommutative Geometry would potentially provide an algebraic interpretation to Connes' analytical theory. This algebraic theory may be more readily applicable for applications in number theory. The application of Model theoretic techniques to the study of Quantum Tori has been the subject of a recent study by Zilber [64, 66], whose work has inspired many of the ideas behind our study.

This thesis is split in to three chapters, each of which highlights an aspect of Quantum Tori that one finds in the theory of abelian varieties.

Chapter 2 can be thought of as having two parts, the first developing a notion of morphism between Quantum Tori, and the second the application of these ideas in §2.6.

As we have already remarked, the non-Hausdorff nature of Quantum Tori implies

[6] The Mordell-Lang conjecture describes the intersection of a closed subvariety X of an abelian variety A with certain subgroups $\Gamma \leq A$ in terms of the intersection of Γ with a finite number of *translates* of subvarieties X_i of A.

[7] The Manin-Mumford conjecture describes the intersection of a closed subvariety X of an abelian variety A over a number field with the torsion points $A_{tors}(\overline{K})$ of A in terms of the intersection of $A(\overline{K})$ with a finite number of *translates* of subvarieties X_i of A.

that any continuous function defined on such an object is constant. We introduce and demonstrate the use of Nonstandard Analysis to introduce a Hausdorff topological space T_L, which we aim to use to define a suitable notion of continuous functions on Quantum Tori yielding nontrivial results. Section 2.2 describes the terminology we will need to define what we mean by Nonstandard Analysis, and in §2.2.3 we introduce the topological space T_L, known as a Hyper Quantum Torus. We aim to induce nontrivial morphisms between Quantum Tori by continuous morphisms between Hyper Quantum Tori.

Section 2.3 defines the category $\mathcal{LIQ}$ of locally internal quotient spaces, which Hyper Quantum Tori can be viewed as objects in. Our definition of morphisms in this category insists that morphisms lift to certain maps between covering spaces of objects in $\mathcal{LIQ}$ and allows us to compute the fundamental group of a Hyper Quantum Torus. We observe that Hyper Quantum Tori can be viewed as objects in a a subcategory $\mathcal{LIQ}^{lim}$ of $\mathcal{LIQ}$, and Proposition 2.3.13 shows that in this category we can recover the pseudolattice L from the fundamental group of the Hyper Quantum Torus T_L. The final part of §2.3 is a detailed investigation in to our definition of morphisms, and whether we can weaken our definition and maintain the lifting properties we require.

The work of the previous sections allows us to define the category $\mathcal{HQT}$ of Hyper Quantum Tori in §2.4, and Lemma 2.4.2 explicitly computes the morphisms between these objects in this category. Our motivation for studying Hyper Quantum Tori was to use morphisms between these objects to define nontrivial maps between Quantum Tori. This is the subject of §2.5, in which we formulate a philosophy as to what the morphisms between topological spaces with Hausdorff covers should be, and show how this is related to the work of §2.3. In particular we calculate the possible homomorphisms between Quantum Tori in Corollary 2.4.2 and in Proposition 2.5.5 classify those continuous morphisms between Quantum Tori.

The final section of Chapter 2 applies the definitions of the previous sections, proving a structure result for the endomorphism ring of a Quantum Torus in Theorem 2.4. This asserts the existence of a class of Quantum Tori whose endomorphisms rings are strictly greater than $\mathbb{Z}$. Such Tori are said to have *Real Multiplication*, and are the subject of the main result of this section in Theorem 2.6.6. This is a simple yet remarkable result showing that we have an action of a Galois group on isomorphism classes of certain Quantum Tori with Real Multiplication. The action of such a group is common place when we are considering algebraic varieties over a number field, but there is no reason to expect such an action on Quantum Tori. We discuss the possible ramifications of this result, including a mysterious relationship between isomorphism classes of such Tori and the Hilbert class field of a real quadratic field.

In Chapter 3 we investigate and develop a notion of line bundles over a Quantum Torus. A line bundle over a topological space is usually defined to be a topological space X subject to some "local conditions". As a consequence of this definition, if X is non-Hausdorff then the only line bundles over X are so called *trivial* line bundles. The introduction of nontrivial line bundles over Quantum Tori assigns to these objects a concept which is usually only associated to Hausdorff spaces.

Like Chapter 2, this chapter has two main themes. The first three sections consider the problem of defining line bundles in terms of algebraic data, before proving a structure theorem for the group of isomorphism classes of such bundles. In the second half of this chapter we view line bundles as topological spaces, and study various objects arising from this viewpoint in relation to the algebraic definition given previously.

In §3.2, motivated by the theory of line bundles over elliptic curves, we define line bundles over Quantum Tori as elements of a group of cocycles. Isomorphic line bundles correspond to cocycles that have the same image in the associated cohomology group. Section 3.3 concerns a structure result for this cohomology group, which is proved in Theorem 3.3.12. The proof of this result relies on the introduction of the Chern class of a line bundle, arising from the long exact sequence of cohomology. By examining the kernel and image of the map taking a line bundle to its Chern class we prove the split exactness of a certain short exact sequence. Using the theory of such sequences our structure theorem is deduced. This result is used in §3.3.4 to prove Theorem 3.3.18 - a result analogous to the Appel-Humbert Theorem for Complex Tori, describing isomorphism classes of line bundles over Quantum Tori in terms of pairs of alternating forms and characters.

In §3.4 we consider a different approach to defining line bundles over Quantum Tori, viewing them as topological spaces. The philosophy behind our approach is underpinned by the existence of a surjection

$$\pi : \mathbb{R} \to Z_L,$$

and that if Z_L was a Hausdorff space, any line bundle over Z_L would pull back to one on $\mathbb{R}$. In order to make sense of this notion for Quantum Tori, we introduce the Heisenberg group $H(\mathcal{L})$ associated to a line bundle $\mathcal{L}$ over a topological space. This leads us to a *topological* definition of a line bundle over Quantum Tori in Definition 3.4.9, which we see agrees with the previous definition of §3.2 in Proposition 3.4.10.

Viewing line bundles as topological spaces leads us to consider a pairing $e^{\mathcal{L}}$ defined on a certain subgroup $K(\mathcal{L})$. Section 3.5 is concerned with the relationship between these objects and the Chern class of a line bundle introduced in §3.3. The main result of this section is Theorem 3.5.7, which shows that the behaviour of $e^{\mathcal{L}}$ is governed by the Chern class of $\mathcal{L}$, and gives an explicit formula for $e^{\mathcal{L}}$ when the Chern class is nontrivial. The proof of this result demonstrates that we can view line bundles over Quantum Tori as the limit (in an appropriate sense) of line bundles over a sequence of Complex Tori. We discuss the significance of this result in §3.6.2.

Throughout our research, we have drawn on the work of many others for inspiration. In §3.6 we focus on two aspects of our work which have links to the approaches to Real Multiplication of other mathematicians. We described previously how Model Theory has become a new force in tackling problems in Number Theory. This extends to other areas of Mathematics, and Zilber's research in to Quantum Tori promises to shed new light on Noncommutative Geometry using Model Theory. His work views Quantum Tori as definable groups in a certain class of structures. In §3.6.1 we ask whether the objects $e^{\mathcal{L}}$ and $K(\mathcal{L})$ associated to line bundles over Quantum Tori can be viewed as lying in such structures, and the current obstructions to this being the case. Section 3.6.2 explores how Theorem 3.3.18 hints at a relationship between Quantum Tori and elliptic curves, and discusses how this phenomenon is already present in the work of Manin [32], Nikolaev [40] and Zilber.

In [60], Stark proved a special case of Hilbert's twelfth problem for imaginary quadratic fields. Although at the time of publication a complete solution already existed for such fields, Stark's case is unique in that it provides generators for abelian extensions in terms of holomorphic *theta functions*, which can be viewed as *sections* of line bundles over elliptic curves. Chapter 4 is concerned with the existence of sections of the line bundles which formed the subject of Chapter 3, and their potential application to Real Multiplication. The existence of such functions for line bundles over Quantum Tori is the subject of §4.2 and §4.3, where we conclude that any such holomorphic function is a scalar multiple of the function $z \mapsto e^{2\pi i \alpha z}$ for some $\alpha \in \mathbb{R}$. However, drawing on the work of Barnes [4] and Shintani [54] we are able to exhibit a *meromorphic* theta function corresponding to a section over a Quantum Torus.

The result of Stark mentioned above appears in the last of a series of papers [57–60] in which he formulates a series of conjectures. These conjectures propose a precise relationship between the L-function attached to a number field K, and units lying in an abelian extension of K. We discuss Stark's conjectures in §4.4, as well as Shintani's work, which provides a solution to these conjectures in specific cases for real quadratic fields in terms of the double gamma function introduced in [4] by Barnes. In [43], Ramachandra gives an expression for the L-function associated to an imaginary quadratic field at $s = 1$ in terms of theta functions associated to elliptic curves, which is used by Stark.

Ramachandra's result, together with Shintani's work motivates us in the second half of this chapter to investigate the values at $s = 0$ of the higher derivatives of L-functions associated to real quadratic field F. In §4.5 we define two families of functions G_2^r and $H^{k,q}$ indexed by $r, k, q \in \mathbb{N}$. We show that the first of these defines a family of meromorphic theta functions for a certain pseudolattice, and are a direct generalisation of the double gamma function. The functions $H^{k,q}$ seem to have no such analogue in the literature, and having proved their analyticity in some half plane, we conjecture that they too are meromorphic theta functions for a specific pseudolattice.

The main result of chapter 4 is Theorem 4.6.1, which states that the value when

$s = 0$ of the derivatives of an L-function attached to a real quadratic field F lie in a certain field over F, whose generators include the special values of certain meromorphic theta functions for pseudolattices contained within F. The proof of this result is the content of §4.7. Shintani's results of [54] supply the basis to our induction argument, and the inductive step is proved via the calculation of various contour integrals.

Although Theorem 4.6.1 specifies a field in which the null values of derivatives of L-functions associated to a real quadratic field F lie in, it has no immediate application to a verification of Stark's conjectures and Hilbert's twelfth problem. The field is not algebraic over F, containing transcendental generators such as $2\pi i$ and γ. Theorem 4.8.1 uses a technique from Shintani's paper [55], introducing an expression concerning the null values of derivatives of L-functions, which lies in a field that may be algebraic over F. This field is generated by special values of the theta functions introduced in §4.5. Via the proof of Theorem 4.6.1, this result could be used to construct terms, which according to Stark's conjectures are units in some abelian extension of F.

Chapter 2

Hyper Quantum Tori

2.1 Introduction

The central object in the proposed theory of Real Multiplication is the Quantum Torus $Z_\theta := \mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z})$ where θ is an irrational element of a real quadratic field F. The non-Hausdorff nature of this space implies that the space of continuous $\mathbb{C}$-valued functions on Z_θ is trivial. In [13] and [14], Fesenko suggests the use of Model Theory, and more specifically Nonstandard Analysis as a tool for studying such topological spaces.

The potential relevance of these techniques to Real Multiplication is apparent when we observe that we can write Z_θ as the limit of complex tori

$$\mathbb{C}/(\mathbb{Z}+\tau_n\mathbb{Z})$$

for some sequence $\tau_n \in \mathbb{C}$ with positive imaginary part tending to θ. Further observe that the sequence may be chosen such that each element τ_n of it lies within a fixed imaginary quadratic field K. Via the Uniformization Theorem (Theorem 1.1.2) we may view each Complex Torus in this sequence as an elliptic curve with Complex

Multiplication by K. The construction of Nonstandard Models by Robinson [45] also concerns the behaviour of objects in the tail end of a limit, and enables us to "algebraize" analytic concepts. It is conceivable that the explicit class field theory of K supplied by Complex Multiplication may induce a theory of Real Multiplication for the Quantum Torus.

Although this was the motivation behind Fesenko's application of Nonstandard Analysis to Real Multiplication, this is not the idea we explore in this chapter. We concentrate on another aspect of the theory, which allows us to view Quantum Tori as the images of a Hausdorff space. Let L be a pseudolattice, and Z_L the corresponding Quantum Torus. Due to the non-Hausdorff property of Z_L, the space of continuous functions on Z_L is trivial. Using Nonstandard Analysis, in §2.2.3 we define a Hausdorff space T_L together with a surjection

$$p_L : T_L \to Z_L. \tag{2.1}$$

One of the first aims of this chapter is to define morphisms on the spaces T_L in such a way so that they "push forward" continuous functions on T_L to nontrivial functions on Z_L.

Nonstandard Analysis was originally a construction of A.Robinson in [45] in the 1960's. Robinson's work serves to give a logical foundation to the idea of an *infinitesimal* - a number which is smaller in magnitude than any real number. The notion of infinitesimals was used extensively by both Leibniz and Newton in their formulations of calculus. However, their arguments would not be considered rigorous by today's standards, and it was left to Weierstrass to formulate the epsilon-delta notion of calculus we are now familiar with. Robinson's work provides alternative (and some would say conceptually simpler) definitions for analytical ideas such as limits and continuity.

Nowadays, Nonstandard Analysis is viewed as a branch of logic, and Robinson's ideas are incorporated in a subject known as Model Theory. We begin this chapter in §2.2, defining some of the basic objects in mathematical logic with a view to defining a *Nonstandard model* in Definition 2.2.13. Nonstandard Analysis is the process of doing mathematical analysis within such Nonstandard models. Many properties of Nonstandard Analysis follow from those of "standard" analysis by a property known as $*$-transform. The introduction of this property leads on to the idea that there are some subsets of a Nonstandard model which are intrinsically nonstandard. These are known as *internal* sets, and will go on to play a crucial role in our study of Quantum Tori.

When we consider the application of these ideas to Quantum Tori, we will be concerned with a Nonstandard model of the real numbers, and in §2.2.2 we exhibit some basic properties of such a model. In particular we demonstrate the existence of infinitesimals, and their use in an "algebraic" description of limits and continuity.

In §2.3 we define the category $\mathcal{LIQ}$ of *locally internal quotient spaces.* Hyper Quantum Tori can be viewed as objects in this category, and we use our definitions to calculate the fundamental group of T_L in §2.3.4. Morphisms are defined in this category, motivated by the desire that we should be able to lift such morphisms to internal maps of various covering spaces. This allows us to define paths in Hyper Quantum Tori, and in Proposition 2.3.11 we show that the fundamental group of T_L is isomorphic to ${}^*\mathbb{Z} \oplus \mathbb{Z}$ where ${}^*\mathbb{Z}$ denotes a Nonstandard model of the integers. This motivates the introduction of a certain subcategory $\mathcal{LIQ}^{\lim}$ of $\mathcal{LIQ}$ and of the *limited* fundamental group, which for a Hyper Quantum Torus T_L is shown to be isomorphic to the pseudolattice L.

Section 2.3.5 is a discussion concerning the definition of morphisms in $\mathcal{LIQ}$ we gave in Definition 2.3.3. In making this definition we imposed some strong conditions concerning the lifts of morphisms to internal covering spaces. We investigate whether we can obtain an equivalent definition which does not refer to an ambient internal covering space. We consider the specific problem of defining paths in Hyper Quantum Tori, provisionally allowing paths which are defined on every infinitesimal neighbourhood of a point. Through our discussion we conclude that this definition does not suffice to give the property of path lifting to internal covers that we require.

In §2.4 we calculate the homomorphisms between Hyper Quantum Tori when viewed as objects in $\mathcal{LIQ}$, showing that the subcategory $\mathcal{HQT}$ of these objects is equivalent to the category $\mathcal{PL}$ of pseudolattices of Definition 1.2.4.

Our motivation for using Nonstandard Analysis and the study of Hyper Quantum Tori T_L was to provide a nontrivial notion of continuous morphisms on these objects, which using the map of (2.1) we could "push forward" to define nontrivial functions between Quantum Tori. If f is a morphism between Hyper Quantum Tori satisfying a property known as *S-continuity*, then we show in §2.5 that using the projection p_L we can give a well defined map between Quantum Tori. Generalising this notion we define the *standardisation* of a morphism in $\mathcal{LIQ}^{\mathrm{lim}}$, which leads us to pose a philosophy as to what the morphisms "should be" between two standard topological spaces with Hausdorff covers. In Lemma 2.5.2 we show that when two standard topological spaces are obtained as the shadow images of objects in $\mathcal{LIQ}^{\mathrm{lim}}$ then this philosophy gives rise to the same set of morphisms as that obtained by the standardisation of morphisms between the associated objects in $\mathcal{LIQ}^{\mathrm{lim}}$. Based on this result, Definition 2.5.3 defines a notion of continuous maps between general (Hausdorff or non-Hausdorff) standard topological spaces which we use for the remainder of the thesis. These ideas allow us to define a classification result for continuous maps between Quantum Tori in Proposition 2.5.5.

The work of the preceding two sections allows us to define a category $\mathcal{QT}$, whose objects are Quantum Tori, and whose morphisms are continuous homomorphisms. This is entirely analogous to the category $\mathcal{EL}$ of elliptic curves with isogenies between them in the statement of Theorem 1.1.2. It is an easy consequence of our definitions that the category $\mathcal{QT}$ is equivalent to both the category $\mathcal{PL}$ in Definition 1.2.4, and the category $\mathcal{HQT}$ of Definition 2.2.3. We use this fact to prove a structure theorem for the endomorphism ring of an Quantum Torus in Theorem 2.6.1. The final section of this chapter is dedicated to those Quantum Tori for which this ring is strictly larger than $\mathbb{Z}$. In this case the endomorphism ring is isomorphic to an order in a real quadratic field F, and we say such Quantum Tori have *Real Multiplication by F*.

Motivated by Silverman's approach to Complex Multiplication in [56], the main object we study in this section is the set $\mathcal{QT}(\mathcal{O}_F)$ of isomorphism classes of Quantum Tori which have endomorphism ring isomorphic to the ring of integers of a predefined real quadratic field F. We show in Lemma 2.6.5 that Quantum Tori whose endomorphism ring is isomorphic to the ring over integers of such a field F correspond (in the equivalence between $\mathcal{QT}$ and $\mathcal{PL}$) to fractional ideals[1] of F. In Theorem 2.6.6 we use this correspondence to define a simply transitive action of the *class group of* F on $\mathcal{QT}(\mathcal{O}_F)$. As a corollary we find that the cardinality of $\mathcal{QT}(\mathcal{O}_F)$ is equal to the class number of F.

[1] A fractional ideal $\mathfrak{a}$ is an $\mathcal{O}_F$-module of finite rank.

These last two results represent an important step in the use of Quantum Tori in Real Multiplication. In Chapter 1 we noted that many difficult problems in number theory arise from a need to unify algebraic and analytical theories. The reciprocity map of class field theory induces an isomorphism between the class group of a number field K, and the Galois group of the Hilbert class field H_K of K over K. The result of Theorem 2.6.6 can be interpreted as describing an action of $\mathrm{Gal}(H_F/F)$ on $\mathcal{QT}(\mathcal{O}_F)$, and its corollary links $\mathcal{QT}(\mathcal{O}_F)$ to the arithmetic of the field F. In the final part of §2.6 we discuss how these results hint that Quantum Tori have an "algebraic character", and how this may be of use in Real Multiplication.

2.2 Nonstandard Models

The use of Nonstandard Analysis to give an algebraic description of analytic concepts such as those of a limit, continuity and differential relies on the existence of so called *infinitesimals*. As we mentioned before, the notion of such "infinitely small" numbers were used in a non-rigorous manner by Newton and Leibniz in their formulation of the differential and integral calculus. Their approach was formalised by Weierstrass who introduced the epsilon-delta definition of limits, establishing the branch of mathematics we now know as *Analysis*. In Nonstandard Analysis, tools from mathematical logic are used to give rigorous definitions of analytical concepts.

In this section we give a very brief introduction to the notation and terminology used in basic logic, for which our main reference has been [34]. This will enable us to define what we mean by a Nonstandard model, and the notion of Nonstandard analysis. The language of logic is also that used by Zilber, whose work has been an inspiration to many of the ideas contained within this thesis.

2.2.1 Elements of Model Theory

We begin with some basic definitions and examples:

Definition 2.2.1. A *language* $\mathcal{L}$ is given by the following data:

- a set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$;
- a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$;
- a set of constant symbols $\mathcal{C}$.

We write $\mathcal{L} = (\mathcal{F}; \mathcal{R}; \mathcal{C})$. The positive integers express the arity of the function or relation.

For example, the language of fields is $\mathcal{L}_f = \{+, \times; 0, 1\}$. The symbols $+$ and $\times$ are binary function symbols (so $n_+ = n_\times = 2$), and 0 and 1 are constants. If we wish to talk about the language of *ordered* fields we need to add the relation symbol $<$ (with $n_< = 2$). Hence the language of ordered fields is $\mathcal{L}_{of} = \{+, \times; <; 0, 1\}$.

Definition 2.2.2. Let $\mathcal{L}$ be a language. An $\mathcal{L}$-structure $\mathcal{M}$ is given by the following data:

- a nonempty set M called the *universe* of $\mathcal{M}$;
- a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$;
- a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$;
- an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$.

The objects $f^{\mathcal{M}}, R^{\mathcal{M}}$ and $c^{\mathcal{M}}$ are referred as the *interpretations* of the symbols f, R and c. We write $\mathcal{M} = (M; \mathcal{L})$.

Continuing with our previous example, the real numbers $\mathbb{R}$ are an $\mathcal{L}_f$-structure. We interpret $+$ and $\times$ as the operations of addition and multiplication respectively, and the symbols 0 and 1 as the additive and multiplicative identities. The real numbers are also an $\mathcal{L}_{of}$-structure, where $< (a, b)$ with $a, b \in \mathbb{R}$ is interpreted to mean $a < b$.

Definition 2.2.3. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures with universes M and N respectively. An $\mathcal{L}$-embedding $\eta : \mathcal{M} \to \mathcal{N}$ is a one-to-one map $\eta : M \to N$ that preserves the interpretation of all the symbols of $\mathcal{L}$.

Consider the language of groups $\mathcal{L}_g = \{\circ; e\}$. Then both $(\mathbb{R}; +; 0)$ and $(\mathbb{R}^*; \times; 1)$ are $\mathcal{L}_g$-structures. In $\mathbb{R}$ the composition $\circ$ is interpreted as addition, in $\mathbb{R}^*$ it is interpreted as multiplication. The map

$$\begin{aligned} \eta : \mathbb{R} &\rightarrow \mathbb{R}^* \\ x &\mapsto e^x \end{aligned}$$

defines an $\mathcal{L}_g$-embedding of $(\mathbb{R}; +; 0)$ in to $(\mathbb{R}^*; \times; 1)$.

Definition 2.2.4. The set of $\mathcal{L}$-terms is the smallest set $\mathcal{T}$ such that

- $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$;
- each variable $v_i \in \mathcal{T}$ for $i = 1, 2, \ldots$;
- if $t_1, \ldots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$ then $f(t_1, \ldots, t_{n_f}) \in \mathcal{T}$.

For example, consider $\mathcal{L}_f = \{+, \times; 1, 0\}$ - the language of fields. Then $+(1, +(1, 1))$ is an $\mathcal{L}_f$ term. We usually denote it by 3. Out of terms we build *formulae*, the simplest of which are the *atomic formula*.

Definition 2.2.5. ϕ is an atomic formula of $\mathcal{L}$ if ϕ is either

1. $t_1 = t_2$ where t_1 and t_2 are terms, or
2. $R(t_1, \ldots, t_{n_R})$, where $R \in \mathcal{R}$ and $t_1, \ldots, t_{n_R}$ are terms.

The set of $\mathcal{L}$-formals is the smallest set $\mathcal{W}$ containing the atomic formula's such that

1. if ϕ is in $\mathcal{W}$ then $\neg\phi$ is in $\mathcal{W}$;
2. if $\phi, \psi \in \mathcal{W}$ then $\phi \wedge \psi$ and $\phi \vee \psi$ are in $\mathcal{W}$;
3. if ψ is in $\mathcal{W}$ then $\exists v_i \phi$ and $\forall v_i \phi$ are in $\mathcal{W}$.

We say that a variable v occurs *freely* in a formula ϕ if it does not occur inside a $\exists v$ or $\forall v$ quantifier, otherwise we say it is *bound*.

For example, in the language $\mathcal{L}_{of} = \{+, \times; <; 0, 1\}$ of ordered fields, the following are atomic formulae:

$$< (1, +(1, 1,)) \qquad\qquad = (1, +(1, 1)) \qquad\qquad < (+(1, 1), 1).$$

We would usually write

$$1 < 2 \qquad\qquad 1 = 2 \qquad\qquad 2 < 1.$$

Formula are built up from terms using boolean operations and existential quantifiers. The following are examples of $\mathcal{L}_{of}$ formula's:

$$\exists v(3 < v) \qquad \forall v((1 < v) \wedge (v < 1)).$$

Note that if ϕ is an $\mathcal{L}$-formula, we know nothing about its validity. Indeed, the truth of a formula ϕ depends crucially on the structure in it is interpreted in. For example, consider the following formula in $\mathcal{L}_r = \{+, \times; 0, 1\}$ - the language of rings:

$$\phi : \quad \forall v_1 (= \ (v_1, 0) \vee \exists v_2 (= (1, \times(v_1, v_2)))). \tag{2.2}$$

This statement expresses the property that every non-zero element has a multiplicative inverse. While this is certainly true in the $\mathcal{L}_r$-structure $\mathcal{M} := (\mathbb{R}; +, \times; 0, 1)$, it does not hold in the $\mathcal{L}_r$-structure $\mathcal{N} := (\mathbb{Z}; +, \times; 0, 1)$.

Definition 2.2.6. Let ϕ be a formula with free variables $\bar{v} = (v_{i_1}, \ldots, v_{i_m})$, and let $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^{i_m}$. We inductively define $\mathcal{M} \models \phi(\bar{a})$ by

1. If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$;
2. If ϕ is $R(t_1, \ldots, t_{n_R})$, then $\mathcal{M} \models \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$;
3. If ϕ is $\neg\psi$, then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$;
4. If ϕ is $\psi \wedge \theta$, then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$;
5. If ϕ is $\psi \vee \theta$, then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ or $\mathcal{M} \models \theta(\bar{a})$;
6. If ϕ is $\exists v_j \psi(\bar{v}, v_j)$, then $\mathcal{M} \models \phi(\bar{a})$ if there is a $b \in M$ such that $\mathcal{M} \models \psi(\bar{a}, b)$; If ϕ is $\forall v_j \psi(\bar{v}, v_j)$, then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \phi(\bar{a}, b)$ for all $b \in M$.

If $\mathcal{M} \models \phi(\bar{a})$ we say that $\phi(\bar{a})$ is *true* in $\mathcal{M}$.

Looking back to (2.2) we note that ϕ has no free variables. We have $\mathcal{M} \models \phi$, but $\mathcal{N} \not\models \phi$. Now consider the $\mathcal{L}_r$-formula

$$\psi(v_1) : \exists v_2 (= (1, \times(v_1, v_2))).$$

The statement $\psi(a)$ expresses the notion that a has a multiplicative inverse. Since $\mathcal{M} \models \phi$ we have $\mathcal{M} \models \psi(r)$ for every $r \in \mathbb{R}$. More interestingly perhaps, we have

$$\mathcal{N} \models \psi(1) \qquad \text{but} \qquad \mathcal{N} \not\models \psi(2).$$

Definition 2.2.7. Let $\mathcal{L}$ be a language. An $\mathcal{L}$-theory is a set of $\mathcal{L}$-sentences. We say that $\mathcal{M}$ is a *model* of T if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. We write $\mathcal{M} \models T$.

For example, consider the set T of $\mathcal{L}_g$ sentences

$$\forall g(= (e, = (\circ(e,x), \circ(x,e))))$$
$$\forall g_1 g_2 g_3(\circ(g_1, \circ(g_2, g_3)) = \circ(\circ(g_1, g_2), g_3)$$
$$\forall g_1 \exists g_2(= (e, = (\circ(g_1, g_2), \circ(g_2, g_1))))$$

These express the group axioms of identity, associativity and inverses respectively. Given an group G, with law of composition $\star$ and identity 1 we have $(G; \star; 1) \models T$, where $\circ$ and e are interpreted in G to be $\star$ and 1 respectively. The addition of another binary operation $+$ and identity element extends $\mathcal{L}_g$ to the language of rings. Adding sentences to T expressing the notion that $\circ$ is distributive over $+$ gives a theory whose models are rings. We say that a theory T is satisfiable if there exists a model of T. For example, the $\mathcal{L}_r$ theory

$$T = \{(\exists x)(= (x, 0) \wedge \neg = (x, 0))\}$$

is not satisfiable. A basic result in model theory is the Compactness Theorem:

Theorem 2.2.8 (Compactness Theorem). *T is satisfiable iff every finite subset of T is satisfiable.*

If $\mathcal{M}$ is an $\mathcal{L}$-structure, we consider subsets of $\mathcal{M}$ which arise naturally from its description as an $\mathcal{L}$-structure:

Definition 2.2.9. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A set $A \subset M^n$ is *definable* if there exists an $\mathcal{L}$-formula $\phi(\bar{v}, \bar{w})$ and $\bar{b}$ such that

$$A = \{\bar{a} \in M^n : \mathcal{M} \models \phi(\bar{a}, \bar{b})\}. \tag{2.3}$$

For example, if $\mathcal{M} = (K; +, \times; 0, 1)$ is an $\mathcal{L}_f$-structure, then the set of units is definable in $\mathcal{M}$ by

$$U_K = \{x \in K : \mathcal{M} \models \psi(x)\}$$

where

$$\psi(v) : (\exists y)(= (\times(x, y), 1)).$$

Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure, and $A \subseteq M$. Let $\mathcal{L}_A$ be the language obtained by adjoining constant symbols for each $a \in A$. Then $\mathcal{M}$ is naturally a $\mathcal{L}_A$-structure. We let $Th_A(\mathcal{M})$ denote the set of $\mathcal{L}_A$-sentences ϕ such that $\mathcal{M} \models \phi$.

Definition 2.2.10 (Types). Let p be a set of $\mathcal{L}_A$-formula's in free variables $v_1, \ldots, v_n$. Then p is an *n-type* if $p \cup Th_A(\mathcal{M})$ is satisfiable. We say that p is *complete* if for all $\mathcal{L}_A$-formula's ϕ, either $\phi \in p$ or $\neg\phi \in p$. The set of all complete n-types is denoted by $S_n^{\mathcal{M}}(A)$.

If p is an n-type over A, then we say that $\bar{a} \in M^n$ *realises* p if $\mathcal{M} \models \phi(\bar{a})$ for all $\phi \in p$. We say that a type p is isolated if it can be described by a single formula.

Definition 2.2.11 (Saturated Models). Let κ be an infinite cardinal. A structure $\mathcal{M} \models T$ is *κ-saturated* if for all $A \subseteq M$, if $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then p is realised in $\mathcal{M}$. We say that $\mathcal{M}$ is saturated if it is $|M|$-saturated.

For example, consider the $\mathcal{L}_<$-structure $\mathcal{M} = (\mathbb{Q}; <)$, where $\mathcal{L}_< = \{<\}$. By Proposition 4.3.2 of [34], to show that $\mathcal{M}$ is $\aleph_0$-saturated it is sufficient to show that for every finite $A \subseteq \mathbb{Q}$, given $p \in S_1^{\mathcal{M}}(A)$, p is realised in $\mathcal{M}$.

Let $A \subseteq \mathbb{Q}$ be the finite set $\{a_1, \ldots, a_m\}$ with $a_1 < \ldots < a_m$, and suppose $p \in S_1^{\mathcal{M}}(A)$. Due to the completeness of p, for each $a \in A$ exactly one of $v = a$, $v < a$ and $v > a$ is in p. If p is realised in A, then p consists of the single formula $v = a$ for some $a \in A$. Otherwise each formula is either $v < a$ or $a < v$ for each $a \in A$. Hence $\left|S_1^{\mathcal{M}}(A)\right| = 2m + 1$.

Each type is isolated by one of the following:

$$v = a_i \qquad v < a_0 \qquad a_i < v < a_{i+1} \qquad a_m < v \qquad i = 0, \ldots, m-1.$$

All these are realised in $\mathbb{Q}$. Hence $\mathcal{M}$ is ($\aleph_0$-) saturated.

Definition 2.2.12. Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $j : \mathcal{M} \to \mathcal{N}$ is called an *elementary embedding* if

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{N} \models \phi(j(a_1), \ldots, j(a_n)) \tag{2.4}$$

for all $\mathcal{L}$-formula's $\phi(v_1, \ldots, v_n)$ and all $a_1, \ldots, a_n \in M$. We say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$.

Definition 2.2.13 (Nonstandard Model). Let $\mathcal{M}$ be a $\mathcal{L}$-structure. A *Nonstandard model* of $\mathcal{M}$ is a saturated elementary extension of $\mathcal{M}$. We usually denote it by

$$^* : \mathcal{M} \to {}^*\mathcal{M}.$$

In [45], A.Robinson demonstrates the existence of Nonstandard Models of structures associated to a topological space, explicitly constructing such models using *ultrafilters*.

Since $^* : \mathcal{M} \to {}^*\mathcal{M}$ is an elementary embedding we have

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \Leftrightarrow {}^*\mathcal{M} \models {}^*\phi({}^*a_1, \ldots, {}^*a_n).$$

This property is known as $*$-transform. An essential concept in Nonstandard Analysis is that of an *internal* element of the Nonstandard Model:

Definition 2.2.14 (Internal sets and formulae, [6]). Define

$$\mathcal{I} := \bigcup_{A \in \mathcal{M} \setminus M} {}^*A \subseteq {}^*\mathcal{M}, \tag{2.5}$$

where the union ranges over those definable sets A in $\mathcal{M} \setminus M$. An *internal* element of ${}^*\mathcal{M}$ is one which is an element of $\mathcal{I}$. A set S is therefore internal if and only if it is an element of ${}^*A$, where A is a definable set in $\mathcal{M}$. Similarly, an internal formula is one which lies in the image of $\mathcal{L}$ under *.

Note that it is only possible to use $*$-transform to deduce properties of internal sets. As we shall see, those sets which are not internal may have properties which differ significantly from those that are. It is an easy consequence of the definitions that to show that internal formulae define internal sets:

Proposition 2.2.15 (Internal Set Definiton Principle). *Let $\phi(\bar{v}, \bar{w})$ be an internal formula. For every $m \in {}^*\mathbb{N}$ and $\bar{b} \in {}^*M^m$*

$$A_\phi := \{\bar{a} \in {}^*M^n : (n \in {}^*\mathbb{N}) \wedge ({}^*\mathcal{M} \models \phi(\bar{a}, \bar{b}))\}$$

is an internal set.

Proof. Robinson's' construction of Nonstandard models using ultrafilters provides a simple proof of this result. See §11.7 of [16]. □

Corollary 2.2.16 (Permenance). *Let ϕ be an internal statement in n variables. Then the set of $\bar{a} \in {}^*M^n$ for which ${}^*\mathcal{M} \models \phi(\bar{a})$ is an internal set.*

This property is important, since it shows that the set of elements satisfying an internal formula must be internal. We will use this fact in Proposition 2.3.16

It is natural to ask whether we obtain anything new by studying a Nonstandard model. If $\mathcal{M}$ is an $\mathcal{L}$ structure with universe M, does ${}^*M$ contain any elements which are not contained within M? The fact that ${}^*\mathcal{M}$ is saturated implies the following result, which if the cardinality of M is not finite provides us with the existence of "new elements":

Proposition 2.2.17. *Let I be a set of cardinality less than that of M. Let $\{U_i\}$ be a family of nonempty definable internal subsets of M^n with the* finite intersection property *- for all finite subsets $J \subseteq I$ the intersection $\bigcap_{j\in J} U_j$ is non-empty. Then*

$$\bigcap_{i\in I} {}^*U_i \neq \emptyset.$$

Proof. Let ψ_i be the formula $\bar{v} \in U_i$ then $p = \{\psi_i : i \in I\}$ is a complete n-type. (Because every finite subset is satisfiable, p is by Theorem 2.2.8). Consider the image of the type p under $*$:

$${}^*p = \{{}^*\psi_i : i \in I\}.$$

The saturation property of ${}^*\mathcal{M}$ implies that ${}^*p$ is realised in $\mathcal{M}$, and therefore

$$\bigcap_{i\in I} {}^*U_i$$

is nonempty. □

We will see how this gives rise to "infinitesimal" elements in a Nonstandard Model of the real numbers in the next section.

2.2.2 A Nonstandard model of $\mathbb{R}$

When we study Quantum Tori, we shall be concerned with a Nonstandard model ${}^*\mathbb{R}$ of the real numbers. In this section we introduce some terminology specific to the study of topological spaces using Nonstandard Analysis, and show how Nonstandard Analysis can be used to give alternative definitions to familiar analytic concepts. We assume we work with a suitably large language $\mathcal{L}$, and let $\mathcal{R}$ be an $\mathcal{L}$-structure with universe $\mathbb{R}$. We start by exhibiting the existence of *infinitesimal* elements:

Proposition 2.2.18. *There exists an element $\epsilon \in {}^*\mathbb{R}$ such that $\epsilon > 0$ and for all $r \in \mathbb{R}$ we have $0 < \epsilon < |r|$. Such an element is said to be an* infinitesimal *and we write $\epsilon \simeq 0$.*

Proof. In accordance with the notation of the previous section, we assume that our language $\mathcal{L}$ extends the language $\mathcal{L}_{or}$ of linear orders. For each $r \in \mathbb{R}$ let U_r be the set

$$U_r := \{x \in \mathbb{R} : 0 < x < r\}.$$

The family of sets $\{U_r\}_{r\in\mathbb{R}}$ has the finite intersection property. Therefore the intersection of their image in ${}^*\mathbb{R}$ is non-empty. $\square$

In the same way, by considering the family of sets $V_r := \{x \in \mathbb{R} : r < x\}$ we exhibit the existence of *infinitely large* elements. We can extend these ideas to a general topological space:

Definition 2.2.19. Let $\mathcal{L}$ be a language, and suppose $\mathcal{X}$ is an $\mathcal{L}$-structure whose universe is a topological space X. Let ${}^*X$ denote a Nonstandard model of X.

- An element of ${}^*X$ which lies in the image of ${}^* : X \to {}^*X$ is called *standard*.
- Let x be an element of ${}^*X$, and suppose that $y \in {}^*X$ is such that y lies in the image under * of every open set U of X containing x. Then we say y is *infinitesimally close* to x, and write $x \simeq y$.
- If $y \in {}^*X$ is infinitesimally close to a standard element we say y is *near standard* or *limited.*

Hence every real number can be viewed as a standard element of ${}^*\mathbb{R}$, but the existence of infinitely large elements ensures that not all elements of ${}^*\mathbb{R}$ are limited.

In fact, it is shown in [45] that a topological space X is compact if and only if every element of ${}^*X$ is near standard.

In (2.5) we defined an important class of sets in a nonstandard structure - the internal sets. Those sets which are not internal are called *external*.

Lemma 2.2.20. *Let $\mu(0) = \{\epsilon \in {}^*\mathbb{R} : \epsilon \simeq 0\}$. Then $\mu(0)$ is an external set.*

Proof. Let A be a set, and consider the sentence

$$\beta : (\forall x)((x \in A) \Rightarrow (\exists N)(x < N))$$

expressing the property that A has an upper bound. Now consider the sentence

$$\psi : (\exists v_1)(((x \in A) \Rightarrow (x < v_1))\wedge$$

$$(\forall v_2)((x \in A) \Rightarrow (x < v_2) \Rightarrow ((v_1 < v_2) \vee (v_1 = v_2))))$$

expressing the property that A has a least upper bound. Hence $\mathcal{R} \models (\beta \Rightarrow \psi)$. By (2.4) ${}^*\mathcal{R} \models ({}^*\beta \Rightarrow {}^*\psi)$. But observe that this statement only holds for sets lying in the image of * - the internal ones. The set $\mu(0)$ is bounded above by 1, but possesses no supremum, therefore it is not internal. □

Essentially the same proof is used to show that the set of *limited* elements of ${}^*\mathbb{R}$ is external. A popular use of Nonstandard Analysis is to provide conceptually simpler proofs of analytical results to those using the epsilon-delta method. The shadow, or standard part map provides the bridge between the Nonstandard and "standard" models:

Definition 2.2.21 (Shadow Map). Let X be a topological space, and let ${}^*X^{\text{lim}}$

denote the limited elements of a Nonstandard model ${}^*X$ of X. Let

$$sh : {}^*X^{\text{lim}} \to X$$

denote the map sending $x \in {}^*X^{\text{lim}}$ to the unique standard element x_0 of X such that $x \simeq x_0$. Following [16] we will call this map the *shadow map*. When $X = \mathbb{R}$ it can be shown that sh is an additive homomorphism, and if $x, y \in {}^*\mathbb{R}^{\text{lim}}$ then $sh(xy) = sh(x)sh(y)$.

To illustrate the use of Nonstandard Analysis, consider the problem of defining the limit of a sequence (a_n). The sequence (a_n) can be viewed as a function $a : \mathbb{N} \to \mathbb{R}$. We consider this as a map between two structures $\mathcal{N}$ and $\mathcal{R}$ over universes $\mathbb{N}$ and $\mathbb{R}$ respectively, and consider it as part of a single (two sorted) structure $\mathcal{M}$. Consider a Nonstandard model ${}^*\mathcal{M}$ of $\mathcal{M}$, and the image ${}^*a : {}^*\mathbb{N} \to {}^*\mathbb{R}$ of the function a. The statement "a_i converges" is equivalent to the statement (Theorem 6.1.1 of [16])

$$\text{“for all } n \in {}^*\mathbb{N} \setminus \mathbb{N},\ {}^*a_n \in {}^*\mathbb{R}^{\text{lim}}\text{”}\ .$$

If the limit exists, then it is equal to $sh(a_n)$ for some $n \in {}^*\mathbb{N} \setminus \mathbb{N}$.

With similar notation, to say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous is captured in the statement

$$(\forall x)(\forall y)(x \simeq y \Leftrightarrow {}^*f(x) \simeq {}^*f(y)).$$

It is tempting to look at this definition of continuity and think how much simpler it looks to the familiar one involving epsilons and deltas. However, the use of the shadow map hides much of the analysis and much of the "simplification" achieved in Nonstandard statements is done by using the shadow map to absorb the messy analysis.

Finally we note the existence of two natural topologies on ${}^*\mathbb{R}$. The first is simply the $*$-transform of the natural topology on $\mathbb{R}$ and has a basis of open sets given by

$$\mathcal{B}_Q := \{(x-r, x+r) : (x \in {}^*\mathbb{R}) \wedge (r \in {}^*\mathbb{R})\}.$$

The topology for which this is a basis is called the Q-topology on ${}^*\mathbb{R}$. The S-topology is a coarser topology on ${}^*\mathbb{R}$ which has a basis of open sets given by

$$\mathcal{B}_S := \{(x-r, x+r) : (x \in {}^*\mathbb{R}) \wedge (r \in \mathbb{R})\}.$$

Given a continuous map $f : \mathbb{R} \to \mathbb{R}$, its image ${}^*f$ in a nonstandard model over $\mathbb{R}$ will always be S-continuous.

2.2.3 Hyper Quantum Tori

Let $\mathcal{L}_{or}$ be the language of ordered rings, and let $\mathcal{L}$ be a language containing $\mathcal{L}_{or}$. Let $\mathcal{M}$ be an $\mathcal{L}$-structure that contains both $\mathbb{R}$ and $\mathbb{Z}$ as definable sets with the inclusion $\mathbb{Z} \subseteq \mathbb{R}$. We let ${}^*\mathbb{R}$ denote the Nonstandard model of $\mathbb{R}$ within a Nonstandard model ${}^*\mathcal{M}$ of $\mathcal{M}$.

Definition 2.2.22 (Hyper Quantum Torus). For a pseudolattice $L \subseteq \mathbb{R}$, define the Hyper Quantum Torus associated to L by $T_L := {}^*\mathbb{R}^{\lim}/L$. If $\theta \in \mathbb{R}$ then we let $T_\theta = T_{L_\theta}$ where $L_\theta = \mathbb{Z} + \mathbb{Z}\theta$.

Proposition 2.2.23. *Let T_L be a Hyper Quantum Torus endowed with the induced quotient topology from the Q-topology on ${}^*\mathbb{R}$. Then T is a Hausdorff space.*

Proof. We first observe that L is not dense in ${}^*\mathbb{R}^{\lim}$. Given $x \in {}^*\mathbb{R}^{\lim}$ consider the monad $\mu(x) = \{y \in {}^*\mathbb{R}^{\lim} : y \simeq x\}$. Suppose $z \in (x+L) \cap \mu(x)$. Then $z = x + l$ for some $l \in L$, but $x \simeq x + l$, and hence $l \simeq 0$. Since every element of L is standard we have $l = 0$.

Now suppose $x+L,\ y+L \in T_L$. Because ${}^*\mathbb{R}^{\lim}$ is Hausdorff and L is not dense in ${}^*\mathbb{R}^{\lim}$, there exist open sets U_x and U_y of ${}^*\mathbb{R}^{\lim}$ containing x and y respectively such that

$$U_x \cap (x+L) = \{x\}$$
$$U+y \cap (y+L) = \{y\}$$
$$U_x \cap U_y = \emptyset.$$

Now put $\mathcal{U}_x = U_x + L$ and $\mathcal{U}_y = U_y + L$. These are open disjoint subsets of T_L containing $x+L$ and $y+L$ respectively. $\square$

Observe that we have a natural projection

$$\begin{aligned} \pi_L^{\lim} : {}^*\mathbb{R}^{\lim} &\to T_L \\ x &\mapsto x+L. \end{aligned}$$

This induces a well defined map p_L on T_L whose image is a Quantum Torus:

$$\begin{aligned} p_L : T_L &\to Z_L \\ \pi_L^{\lim}(x) &\mapsto sh(x)+L. \end{aligned}$$

The rest of this section is motivated towards an appropriate definition of continuous morphism between Hyper Quantum Tori with the following property:

> Suppose $\alpha : T_L \to T_M$ is a map of Hyper Quantum Tori. Then α induces a well defined map $\bar{\alpha}$ on Quantum Tori such that the following diagram is commutative:
>
> $$\begin{tikzcd} T_L \arrow[r, "\alpha"] \arrow[d, "p_L"] & T_M \arrow[d, "p_M"] \\ Z_L \arrow[r, "\bar{\alpha}"] & Z_M. \end{tikzcd}$$

We will use this property in §2.5 to define morphisms between Quantum Tori.

2.3 Locally Internal Topological Spaces

The proof of Proposition 2.2.23 shows that any point of T_L has a neighbourhood which is isomorphic to an internal subset of ${}^*\mathbb{R}^{\lim}$ - if $x + L \in T_L$, then the set

$$\mathcal{V}_x^{\varepsilon} := \{y + L : (y \in {}^*\mathbb{R}^{\lim}) \wedge (|y - x| < \varepsilon)\}$$

is isomorphic to the open interval $(-\varepsilon, \varepsilon)$ for any infinitesimal ε. This inspires the following definition:

Definition 2.3.1 (Locally Internal Topological Space). Let S be a topological space in a Nonstandard structure such that

> For every $s \in S$ there exists an open neighbourhood V_s of S such that V_s is isomorphic to an internal topological space.

We say that S is a *locally internal* topological space.

The previous section shows that Hyper Quantum Tori are locally internal topological spaces. Our goal is to derive a notion of functions between these spaces with two aims in mind:

- A morphism α between Hyper Quantum Tori lifts to an internal function between internal covering spaces for Hyper Quantum Tori;
- We can recover the pseudolattice L from T_L as a "fundamental group" associated to T_L.

There are several ideas in these statements that need clarification. In this section we explore the concepts of covering space, and the fundamental group for a general locally internal space, before considering the special case of Hyper Quantum Tori.

2.3.1 Internal Covering Spaces

Suppose X is a standard topological space. Basic results in topology [1] imply that if γ is a path in X, and $\tilde{X}$ is a covering space for X, then γ lifts to a path $\tilde{\gamma}$ in $\tilde{X}$. This result implies that a continuous map between topological spaces X and Y lifts to a map between covers $\tilde{X}$ and $\tilde{Y}$ of these respective spaces.

We wish to have an analogous situation for locally internal topological spaces, where the covering space is an *internal* topological space.

Definition 2.3.2 (Internal Covering space). An internal cover of a locally internal space S is a pair $(\tilde{S}, p)$ such that

- $\tilde{S}$ is an internal topological space;
- p is a surjective map from $\tilde{S}$ to S satisfying the following condition:

  For every $s \in S$, there exists an open neighbourhood U_s of s; an isomorphism ψ_s of U_s on to an internal topological space $\mathfrak{U}_s$, and a decomposition of $p^{-1}(U_s)$ as a family $\{V_{s,i}\}$ of disjoint open internal subsets of $\tilde{S}$ such that the restriction of $\phi_s \circ p$ to $V_{s,i}$ is an internal homeomorphism from $V_{s,i}$ to $\mathfrak{U}_s$.

We say that S is a *locally internal quotient* space.

Internal covering spaces are not unique, as the following examples show:

1. Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a pseudolattice, and let ${}^*S^1 \cong {}^*\mathbb{R}^{\lim}/\mathbb{Z}\omega_1$ denote the unit circle. Consider the pair $({}^*S^1, p_1)$, where p_1 is the map

$$\begin{aligned} p_1 : {}^*S^1 &\to T_L \\ x + \mathbb{Z}\omega_1 &\mapsto x + L. \end{aligned}$$

   Then $({}^*S^1, p_1)$ is an internal covering space for T_L.

2. Suppose $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a pseudolattice. Let $\mathfrak{L} : {}^*\mathbb{R} \to {}^*\mathbb{R}^{\lim}$ be a map of the form

$$\mathfrak{L}(x) = x + m(x)\omega_1$$

where $m(x)$ is some integer (depending on x) such that $x + m(x)\omega_1$ is limited. Then the pair $({}^*\mathbb{R}, p_2)$ is an internal covering space for T_L, where

$$\begin{aligned} p_2 : {}^*\mathbb{R} &\to T_L \\ x &\mapsto \mathfrak{L}(x) + L. \end{aligned}$$

2.3.2 Morphisms between locally internal quotient spaces

Given a Hausdorff locally internal space S, in general S will be an external object in our Nonstandard structure. As a consequence the space of continuous functions on such a space can be very big. When S has an internal cover, we use this to restrict the space of such functions by the following definition:

Definition 2.3.3. Let S and T be locally internal quotient spaces. Let $p_S : \tilde{S} \to S$ and $p_T : \tilde{T} \to T$ denote the respective covering maps. A morphism between S and T is a map $f : S \to T$ such that there exists an internal function $\tilde{f} : \tilde{S} \to \tilde{T}$ such that

$$p_T \circ \tilde{f}(\tilde{s}) = f \circ p_S(\tilde{s})$$

for all $\tilde{s} \in \tilde{S}$. We say that a morphism f is Q-continuous if $\tilde{f}$ is Q-continuous.

From this definition it is clear that the composition of two morphisms is again a morphism.

Definition 2.3.4 (Category of Locally Internal Quotient spaces). Let $\mathcal{LIQ}$ be the category such that

- The objects of $\mathcal{LIQ}$ are locally internal quotient spaces;
- A morphism between locally internal quotient spaces S and T is as defined in Definition 2.3.3.

We are aware that in this definition we are giving morphisms precisely the property which is nontrivial to prove in the standard Hausdorff case - that we can lift morphisms of quotients to their covering spaces. It would be desirable to determine an equivalent definition of morphisms which does not refer to an ambient internal covering space. We discuss this possibility in §2.3.5.

In the next section we use the ideas developed to define the fundamental group of a locally internal quotient space. We apply these ideas to the Hyper Quantum Torus to show that we can recover the underlying pseudolattice as the fundamental group. We view this as analogous to the determination of the pseudolattice L_θ from the K-theory of the Noncommutative Torus A_θ in Noncommutative Geometry.

2.3.3 The fundamental group of a locally internal space

For a standard topological space X, a path in X is a continuous map $\gamma : I \to X$ where I is the unit interval. Taking the $*$-transform of this definition, and internal path in an internal topological space Y in a Nonstandard structure is a Q-continuous map $\gamma : {}^*I \to Y$ where

$$^*I := \{x \in {}^*\mathbb{R} : 0 \leq x \leq 1\}.$$

We note that $^*I$ is trivially a locally internal quotient space (covered by itself) and extend the above notion to define paths in objects in $\mathcal{LIQ}$.

Definition 2.3.5. Let S be a locally internal topological space. A path in S is a Q-continuous morphism γ in $\mathcal{LIQ}$ from the hyper-unit interval $^*I$ to S. We say that a path γ is a loop based at $s \in S$ if $\gamma(0) = \gamma(1) = s$.

Similarly we can extend the notion of homotopies between paths in locally internal quotient spaces:

Definition 2.3.6. Let γ_1 and γ_2 be paths in a locally internal quotient space S. A homotopy F between γ_1 and γ_2 is a Q-continuous morphism in $\mathcal{LIQ}$ from ${}^*I^2$ to S such that

- $F(t, 0) = \gamma_1(t)$ for all $t \in {}^*I$;
- $F(t, 1) = \gamma_2(t)$ for all $t \in {}^*I$.

We say the two paths γ_1 and γ_2 are homotopic and write $\gamma_1 \simeq \gamma_2$. If we wish to refer explicitly to the homotopy F we may write $\gamma_1 \simeq_F \gamma_2$. If γ_1 and γ_2 agree on some subset A of ${}^*I$, we say that F is a homotopy between γ_1 and γ_2 relative to A if we have the additional condition

- $F(a, s) = \gamma_1(a) = \gamma_2(a)$ for all $a \in A$.

It is easily shown that the relation $\simeq$ is an equivalence relation. If γ is a path in S we let $\langle\gamma\rangle$ denote the equivalence (or homotopy) class of γ. Given $s \in S$, we denote the set of homotopy classes relative to $\{0, 1\}$ of loops in S based at s by $\pi_1(S, s)$. We define a law of composition on $\pi_1(S, s)$ by

$$\langle\gamma_1\rangle.\langle\gamma_2\rangle = \langle\gamma_1 \star \gamma_2\rangle$$

where

$$\gamma_1 \star \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

One can prove using exactly the same methods as for standard topological spaces that $\pi_1(S, s)$ is a group under this operation [1]. The identity element is the constant loop at s.

Note: Observe that if S is an internal topological space then this agrees with the natural definition (the $*$-transform of the standard definition) of $\pi_1(S)$. At first it may seem to be a stronger definition since we have the property that paths and homotopies lift to internal covers, but these results follow for internal spaces by $*$-transform of the standard results.

Suppose $f : S \to T$ is a morphism of locally internal quotient spaces. Then f induces a homomorphism

$$\begin{array}{rcl} f_* : \pi_1(S, s) & \longrightarrow & \pi_1(T, f(s)) \\ \langle\gamma\rangle & \mapsto & \langle f \circ \gamma\rangle. \end{array}$$

Because of the way morphisms are defined between locally internal quotient spaces, all of the classical results concerning the lifting of paths to covering spaces have an analogue in this context. Although (since the spaces may be external) the proofs do not follow by $*$-transform, they are almost identical to the standard proofs in the techniques which they employ. As an example we will prove the following:

Proposition 2.3.7. *Let $(\tilde{S}, p)$ be an internal covering space for a locally internal space S. If $\tilde{S}$ is path connected then for any $\tilde{s} \in \tilde{S}$ the induced map $p_* : \pi_1(\tilde{S}, \tilde{s}) \to \pi_1(S, s)$ is injective, where $s = p(\tilde{s})$.*

Proof. Suppose $\tilde{\gamma}$ is a loop in $\tilde{S}$ such that $\gamma := p \circ \tilde{\gamma}$ is null homotopic. Choose a specific homotopy $1_s \simeq_F \gamma$, where 1_s denotes the constant loop at s. Choose a Q-continuous lift $\tilde{F}$ of F such that $p \circ \tilde{F} = F$. We may assume that $\tilde{F}(0) = \tilde{s}$ since if not, chose a path p from $\tilde{F}(0)$ to $\tilde{s}$ and replace $\tilde{F}$ by the homotopy G such that for each $s \in {}^*I$

$$\tilde{G}(t, s) = p^{-1} \star \tilde{F}(\ , s) \star p(t).$$

Note that once we have fixed $\tilde{F}(0)$ the function $\tilde{F}$ is unique. Suppose there were two such lifts $\tilde{F}_1$ and $\tilde{F}_2$. Then we would have $\tilde{F}_1(t)-\tilde{F}_2(t) \in \ker(p)$. The kernel of p is a Q-discrete set since for each point s, p identifies an internal neighbourhood of s homeomorphically with an internal subset of $\tilde{S}$. Since both the lifts are Q-continuous and internal maps of the connected set ${}^*I$ and agree at 0, we have $\tilde{F}_1 = \tilde{F}_2$.

We need to show that $\tilde{F}$ gives a homotopy from $\tilde{\gamma}$ to the constant loop at $\tilde{s}$. Let P denote the internal path connected set

$$\{(t,0) \in {}^*I^2 : 0 \leq t \leq 1\} \cup \{(0,t) \in {}^*I^2 : 0 \leq t \leq 1\}.$$

Since F is a homotopy relative to $\{0,1\}$ we see that $F(P) = s$. Since $p \circ \tilde{F} = F$ we have $\tilde{F}(P) \in p^{-1}(s)$, which as we have seen is a Q-discrete set. Since $\tilde{F}$ is internal and Q-continuous we have $\tilde{F}(P) = \tilde{s}$. This shows that the path $F(t,0)$ is the constant loop at $\tilde{s}$. The path $F(t,1)$ is a lift of γ which starts at $\tilde{s}$. There is a unique such path by an analogous argument to the uniqueness of homotopy lifting in the above paragraph. Hence $F(t,1) = \tilde{\gamma}(t)$.

□

We stress that despite working within a nonstandard model, the proof of the previous proposition does not require any new ingredients mathematically. Once we have the properties of path and homotopy lifting we are working with internal functions on the covering space and the proofs carry through by $*$-transform of the standard results.

Proposition 2.3.8. *Let $f : S \longrightarrow T$ be a morphism between monadically internal spaces, let $s \in S$ and suppose that $\tilde{S}$ and $\tilde{T}$ are path connected. There is a lift $\tilde{f} : S \longrightarrow \tilde{T}$ such that $f(s) = \tilde{t}$ if and only if $f_*(\pi_1(S,s)) \subseteq \pi_*(\pi_1(\tilde{T},\tilde{t}))$. This lift is* *unique.*

Proof. The result is proved for standard spaces in [1]. It is easy to adapt these techniques to obtain the result for locally internal quotient spaces. □

Definition 2.3.9 (Covering Transformation). A covering transformation for an internal covering space $(\tilde{S}, \pi)$ is an internal homeomorphism $h : \tilde{S} \to \tilde{S}$ such that $\pi \circ h = \pi$.

The set of all covering transformations forms a group $\mathrm{Cov}(\tilde{S}/S)$ under composition, and we have an action of $\mathrm{Cov}(\tilde{S}/S)$ on $\tilde{S}$ by

$$\begin{aligned} \mathrm{Cov}(\tilde{S}/S) \times \tilde{S} &\to \tilde{S} \\ (h, \tilde{s}) &\mapsto h(\tilde{s}). \end{aligned}$$

Note that if h_1 and h_2 are covering transformations which agree on a point $\tilde{s}$, then both $h_1(x) - h_2(x)$ and the constant map $K(x) = 0$ take the value 0 at $x = \tilde{s}$ and lift π. Hence by the uniqueness part of Proposition 2.3.8 we have $h_1 = h_2$.

Utilising methods from the proof of the corresponding standard result, it is easy to show that the following is true:

Proposition 2.3.10. *Let $(\tilde{S}, p)$ be a path connected internal covering space. Suppose $p_*(\pi_1(\tilde{S}, \tilde{s}))$ is a normal subgroup of $\pi_1(S, s)$. Then $Cov(\tilde{S}/S)$ is isomorphic to the quotient $\pi_1(S, s)/p_*(\pi_1(\tilde{S}, \tilde{s}))$.*

We now use this result to describe various fundamental groups associated to Hyper Quantum Tori.

2.3.4 Covering spaces for Hyper Quantum Tori

The definitions of the previous sections enable us to calculate the fundamental group associated to Hyper Quantum Tori:

Proposition 2.3.11. *Let L be a pseudolattice. For any $z \in T_L$, we have an isomorphism $\pi_1(T_L, z) \cong {}^*\mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Consider the internal covering space $({}^*\mathbb{R}, p_2)$ considered in §2.3.1 :

$$\begin{aligned} p_2 : {}^*\mathbb{R} &\rightarrow T_L \\ x &\mapsto \mathfrak{L}(x) + L \end{aligned}$$

Since $\pi_1({}^*\mathbb{R}, r)$ is trivial for any $r \in {}^*\mathbb{R}$, Proposition 2.3.10 gives an isomorphism $\mathrm{Cov}({}^*\mathbb{R}/T_L) \cong \pi_1(T_L, z)$. Suppose $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. For each $l \in {}^*\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ let h_l be the covering transformation defined by $h_l(r) = r + l$, and consider the homomorphism

$$\begin{aligned} {}^*\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 &\rightarrow \mathrm{Cov}({}^*\mathbb{R}/\mathfrak{F}_\theta) \\ l &\mapsto h_l. \end{aligned}$$

By the previous discussion we see that elements $f \in \mathrm{Cov}({}^*\mathbb{R}/T_L)$ are determined by their value at 0, hence this map is injective. Given $f \in \mathrm{Cov}({}^*\mathbb{R}/T_L)$ we have $f(0) \in {}^*\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and hence $f = h_{f(0)}$. Hence the above map defines an isomorphism ${}^*\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \cong \mathrm{Cov}({}^*\mathbb{R}/\mathfrak{F}_\theta)$. Finally note that ${}^*\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \cong {}^*\mathbb{Z} \oplus \mathbb{Z}$. □

One of the purposes for studying Hyper Quantum Tori was to use morphisms between these objects to determine an appropriate notion of morphism between Quantum Tori. Recall the natural projection

$$\begin{aligned} \pi : \mathbb{R} &\rightarrow T_L \\ x &\mapsto x + L. \end{aligned}$$

With the above philosophy in mind we may hope that covering transformations of the universal covering space ${}^*\mathbb{R}$ for T_L may induce maps on $\mathbb{R}$ using the shadow map. However, by Proposition 2.3.11 such transformations are translations by elements of ${}^*\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and as such there exist some covering transformations which do not map ${}^*\mathbb{R}^{\mathrm{lim}}$ to itself. We therefore make the following definition

Definition 2.3.12. Let $\mathcal{LIQ}^{\mathrm{lim}}$ be the subcategory of $\mathcal{LIQ}$ such that

- The objects of $\mathcal{LIQ}^{\mathrm{lim}}$ are objects of $\mathcal{LIQ}$ such that

  The restriction of the projection $p : \tilde{S} \to S$ to those limited elements of $\tilde{S}$ is an S-continuous surjection on to S.

- Morphisms in $\mathcal{LIQ}^{\mathrm{lim}}$ are those morphisms in $\mathcal{LIQ}$ which map limited elements to limited elements.

For $z \in T_L$, we let $\pi_1^{\mathrm{lim}}(T_L, z)$ denote the fundamental group of T_L based at z where all the paths and homotopies are required to be morphisms in $\mathcal{LIQ}^{\mathrm{lim}}$.

Proposition 2.3.13. *For any $z \in T_L$, we have an isomorphism $\pi_1^{\mathrm{lim}}(T_L, z) \cong L$.*

Proof. Proposition 2.3.11 implies that the following injection is an isomorphism:

$$\begin{aligned} \Phi : \mathrm{Cov}({}^*\mathbb{R}/T_L) &\to \pi_1(T_L, z) \\ h &\mapsto \gamma_h \end{aligned}$$

where

$$\gamma_h(t) := \pi \circ (\tilde{z} + t(h(1) - h(0)).$$

Such a transformation preserves ${}^*\mathbb{R}^{\mathrm{lim}}$ if and only if it is a translation by an element of $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = L$. □

2.3.5 Another look at morphisms in $\mathcal{LIQ}$

Recall that in Definition 2.3.3 we defined morphisms between locally internal quotient spaces to possess the property that they lifted to internal maps between their internal covering spaces. In this section we look at whether it is possible to obtain an equivalent definition without reference to such a cover. Rather than consider the general case of maps between locally internal quotient spaces, we shall consider the specific problem of defining paths. This would enable us to generalise the ideas we have previously discussed, defining the fundamental group of locally internal spaces which were not obviously quotients of internal spaces. If this were possible then it may be possible to construct a universal cover for such a space - see §10 of [48]. We begin with the following definition:

Definition 2.3.14. Let S be a locally internal space. A *locally internal* path in S is a map $\gamma : {}^*I \to S$ such that for all $t \in {}^*I$, there exists an internal neighbourhood U_t of t with the following properties:

1. The image of U_t lies in an open neighbourhood $V_{\gamma(t)}$ of $\gamma(t)$;
2. There exists an isomorphism $\phi_{\gamma(t)} : V_{\gamma(t)} \cong \mathfrak{V}_{\gamma(t)}$ for some internal topological space $\mathfrak{V}_{\gamma(t)}$;
3. The composition $\phi_{\gamma(t)} \circ \gamma : U_t \to \mathfrak{V}_{\gamma(t)}$ is an internal map.

Hyper Quantum Tori posses a slightly stronger property than local internality. Instead of the condition of Definition 2.3.1 we have

> For every $z \in T_L$, for every infinitesimal ε the set $(z-\varepsilon, z+\varepsilon)+L \in T_L$ is an open neighbourhood of z isomorphic to an internal topological space.

In this case we strengthen the notion of path in a suitable way:

Definition 2.3.15. A *monadically internal* path in T_L is a map $\gamma : {}^*I \to T_L$ such that (with the notation of Definition 2.3.14), for every $t \in {}^*I$, and every infinitesimal ε, there exist $V_{\gamma(t)}$, $\mathfrak{V}_{\gamma(t)}$, $\phi_{\gamma(t)}$ corresponding to $U_t = (t-\varepsilon, t+\varepsilon)$.

When γ is an S-continuous monadically internal path in T_L, we can deduce some information about the lift of γ to ${}^*\mathbb{R}$:

Proposition 2.3.16. *Fix $t \in {}^*I$. Let γ be an S-continuous monadically internal path in T_L, and let $\tilde{x} \in {}^*\mathbb{R}$ be such that $\pi(\tilde{x}) = \gamma(t)$. Then there exists a unique internal function f and $r \in \mathbb{R}$ such that $f : [t-r, t+r] \cap {}^*I \to {}^*\mathbb{R}$, $f(0) = \tilde{x}$ and on $\mu(t)$ we have $\pi \circ f = \gamma$.*

Proof. This is a consequence of the property of *permanence* introduced in Corollary 2.2.16. We first suppose that $t \notin \mu(0) \cup \mu(1)$.

For each $\eta \simeq 0$, there exist internal sets $\mathfrak{V}^{\eta}_{\gamma(t)} \subseteq {}^*\mathbb{R}$ and functions $\phi^{\eta}_{\gamma(t)}$ such that $\pi \circ \phi^{\eta}_{\gamma(t)}(t) = \gamma(t)$ and

$$\phi^{\eta}_{\gamma(t)} : (t-\eta, t+\eta) \to \mathfrak{V}^{\eta}_{\gamma(t)}$$

is an internal function.

Fix $\eta_0 \simeq 0$. Then for each $\eta \simeq 0$ with $\eta > \eta_0$ we obtain an internal function $\phi^{\eta}_{\gamma(t)}$ such that on $(t-\eta_0, t+\eta_0)$ we have $\phi^{\eta}_{\gamma(t)} = \phi^{\eta_0}_{\gamma(t)}$. By permanence there exists $r \in \mathbb{R}$ and an internal function $\phi^{r}_{\gamma(t)}$ and an internal open set $\mathfrak{V}^{r}_{\gamma(t)}$ such that $\phi^{r}_{\gamma(t)} : (t-r, t+r) \to \mathfrak{V}^{r}_{\gamma(t)}$ and the restriction of $\phi^{r}_{\gamma(t)}$ to $(t-\eta_0, t+\eta_0)$ is $\phi^{\eta_0}_{\gamma(t)}$. Since this holds for any $\eta_0 \simeq 0$ we see that $\phi^{r}_{\gamma(t)}$ agrees with $\phi^{\eta_0}_{\gamma(t)}$ for all $\eta_0 \simeq 0$.

Now suppose we had two such lifts $\phi^{r}_{\gamma(t)}$ and $\psi^{r}_{\gamma(t)}$ defined on $(t-r, t+r)$. Consider the internal set

$$S := \{t' \in (t-r, t+r) : \phi^{r}_{\gamma(t)}(t) = \psi^{r}_{\gamma(t)}(t)\}.$$

Consider the internal statement

$$\varphi(s) : \quad t' \in (t-s, t+s) \Rightarrow t' \in S.$$

Then $\psi(\varepsilon)$ is valid for all $\varepsilon \simeq 0$, hence by permanence there exists $r' \in \mathbb{R}$ such that

$\psi(r')$ holds. Hence $\phi^r_{\gamma(t)}$ is unique on $(t-r', t+r')$.

If $t \in \mu(0)$ then we apply the above techniques to the interval $[0, \eta)$, and similarly if $t \in \mu(1)$ we consider the interval $(\eta, 1]$. □

The previous result and its proof poses the following question:

Question 1. *With the notation of Proposition 2.3.16 do we have* $\pi \circ f(t') = \gamma(t')$ *for all* $t' \in (t-r, t+r)$*?*

Note that in the proof we do not show that the function f we obtain is a lift of γ on the whole of the interval $(t-r, t+r)$. One may hope that it follows from applying the permanence principal to the statement

$$\phi(r): \quad \pi \circ f(t') = \gamma(t') \qquad \forall t' \in (t-r, t+r).$$

However for this to be successful we would require that π and γ were internal functions on ${}^*[0, r]$

Answer to Question 1: No. We can give an example of an S-continuous monadically internal path in T_L which does not lift to an internal S-continuous path in ${}^*\mathbb{R}$. Consider the following monadically internal path in T_L:

$$\begin{aligned} \gamma : {}^*I &\rightarrow T_L \\ t &\mapsto \pi(sh(t) + L). \end{aligned} \tag{2.6}$$

By Proposition 2.3.16 there exists an $r \in {}^*\mathbb{R}^{\lim}$ and a unique continuous function f on $\left(\frac{1}{2} - r, \frac{1}{2} + r\right)$ such that f lifts γ on $\mu\left(\frac{1}{2}\right)$. Since the path γ is constant on $\mu\left(\frac{1}{2}\right)$ we see that $f(t) = k$ for some constant $k \in {}^*\mathbb{R}^{\lim}$. If f lifted γ then we would have $k + L = sh(t) + L$ for all $t \in \left(\frac{1}{2} - r, \frac{1}{2} + r\right)$. But there exist real t' in this interval such that $t' - k \notin L$, so f cannot lift γ.

This shows that Definition 2.3.14 does not provide a notion of paths in locally internal spaces which lift to internal paths in internal covering spaces. The problem essentially lies in the fact that despite having internal neighbourhoods of each point of T_L, we do not know how to "glue" these neighbourhoods together.

Problems with gluing the fundamental region

Let us first consider a standard example of the gluing together of a quotient space. Consider the circle S^1 as $\mathbb{R}/\mathbb{Z}$. Let $F := F_1 \cup F_2$ where $F_1 := [0, \frac{1}{2})$ and $F_2 := [1\frac{1}{2}, 2)$. Then F is a fundamental region for the action of $\mathbb{Z}$ on $\mathbb{R}$. Those continuous functions $f : I \to F$ which define continuous functions in S^1 are precisely those satisfying the following conditions:

$$\begin{aligned} &\lim_{t \to \sup F_1} f(t) = f(\inf F_2) \\ &\lim_{t \to \sup F_2} f(t) = f(\inf F_1) \end{aligned} \tag{2.7}$$

where $\sup(F_1)$ and $\inf(F_2)$ denote the supremum and infimum of F_1 (equal to $\frac{1}{2}$ and 0 respectively), and similarly for F_2.

Can we do a similar thing for T_L? Choose a set of representatives A for the action of L in $\mathbb{R}$. Then it is easily shown that a fundamental region for the action of L on ${}^*\mathbb{R}^{\lim}$ is given by

$$F_L := \{\mu(a_k) : a_k \in A\}.$$

Let $P : T_L \to F_L$ be the map which send each element of T_L to its unique representative in F_L. Suppose γ is a monadically internal path in T_L. Let us try to impose conditions on γ analogous to those in (2.7). Let $F_k := \mu(a_k)$. Since the intervals F_k

are all open, a natural generalisation of these conditions is

$$\lim_{t \to \sup F_k} P \circ \gamma(t) = \lim_{t \to \inf F_{\sigma(k)}} P \circ \gamma(t)$$

where σ is some permutation of the elements of A. However, despite being bounded the infimum and supremum of the sets F_k do not exist. We conclude that it is not sufficient to define γ on every infinitesimal neighbourhood of a point of T_L - we need more information on how γ behaves outside each F_k. In light of Proposition 2.3.16 we see that an equivalent definition of a monadically internal path is the following:

Definition 2.3.17. A monadically internal path in T_L is a function $\gamma : {}^*I \to F_L$ such that

> For each $x \in {}^*I \subset {}^*\mathbb{R}^{\lim}$, for every $\varepsilon \simeq 0$ there exists a Q-continuous internal function $\tilde{\gamma}_x : [x - \varepsilon, x + \varepsilon] \cap {}^*I \to {}^*\mathbb{R}$ such that for all $t \in [x - \varepsilon, x + \varepsilon] \cap {}^*I$, $\pi \circ \tilde{\gamma}(t) = P^{-1} \circ \gamma(t)$.

We know that this is not enough to give us the property of path lifting. A natural weakening of this notion is given by

Definition 2.3.18. An appreciably internal path in T_L is a function $\gamma : {}^*I \to F_L$ such that

> For each $x \in {}^*I \subset {}^*\mathbb{R}^{\lim}$, there exists $r_x \in \mathbb{R}$ and a Q-continuous internal function $\tilde{\gamma}_x : [x - r_x, x + r_x] \cap {}^*I \to {}^*\mathbb{R}$ such that for all $t \in [x - \varepsilon, x + \varepsilon] \cap {}^*I$, $\pi \circ \tilde{\gamma}(t) = P^{-1} \circ \gamma(t)$.

This is a failure to define paths in T_L with the path lifting property without reference to the covering space ${}^*\mathbb{R}$. However, a simple argument shows that this property implies that we have the property of path lifting:

Proposition 2.3.19. *Let γ be an appreciably internal path in T_L. Then γ is a path in T_L.*

Proof. We will show that $P^{-1} \circ \gamma : {}^*I \to T_L$ is a path in T_L in the sense that is is a morphism in $\mathcal{LIQ}$. We need to show that there exists a unique internal Q-continuous map $f : {}^*I \to {}^*\mathbb{R}$ such that $\pi \circ f(t) = P^{-1} \circ \gamma(t)$ for all $t \in {}^*I$. With the notation of Definition 2.3.18 for each $x \in I$, let $U_x := {}^*(x - r_x, x + r_x)$. Since I is compact we may chose finitely many $x_0, x_1 \ldots, x_n$ such that the U_{x_i} cover ${}^*I$. We may assume that $x_0 = 0$ and $x_n = 1$. We label the corresponding lifts of γ to on these intervals $\tilde{\gamma}_{x_i} : U_{x_i} \to {}^*\mathbb{R}$.

Choose $\tilde{x} \in {}^*\mathbb{R}^{\lim}$ such that $\pi(\tilde{x}) = P^{-1} \circ \gamma(0)$. On $[0, r_0]$ define $f(t) := \tilde{\gamma}_{x_0}(t)$. We define f recursively. Suppose we have defined f on $U_{x_{i-1}}$ such that for $t \in U_{x_{i-1}} \cap U_{x_i}$ we have $f(t) - \tilde{\gamma}_{x_i}(t) = l_i$ for some $l_i \in L$. On U_{x_i} define $f(t) := \tilde{\gamma}_{x_i}(t) + l_i$.

This also shows that f is Q-continuous. At each stage of the recursion f is an internal function. Since there are only finitely many steps we see that f is internal. □

Summarising this section, we conclude that we cannot define paths in a monadically internal quotient space without reference to a covering space, possessing the property that they can be lifted to internal paths in an internal covering space. From now on, when referring to morphisms between such spaces we will use the notion defined in Definition 2.3.3.

2.4 Morphisms between Hyper Quantum Tori

Our motivation for defining and studying locally internal quotient spaces arose from the observation that a Hyper Quantum Torus T_L can be viewed as an object in the category of such spaces. We have developed a notion of morphism in between such spaces, which we now apply specifically to Hyper Quantum Tori:

Definition 2.4.1 (Category of Hyper Quantum Tori). Let $\mathcal{HQT}$ denote the category such that

- The objects of $\mathcal{HQT}$ are Hyper Quantum Tori;
- The morphisms between Hyper Quantum Tori correspond to those homomorphisms of the universal internal covering space of Hyper Quantum Tori which map limited elements to limited elements.

Lemma 2.4.2. *We have a bijection between the set of morphisms $T_L \to T_M$ in $\mathcal{HQT}$, and the set of nonzero real numbers α such that $\alpha(L) \subseteq M$.*

Proof. Proposition 2.3.8 shows that a covering space for T_L is universal if it has trivial fundamental group. Hence ${}^*\mathbb{R}$ is the universal internal covering space for Hyper Quantum Tori. Definition 2.3.3 implies that the morphisms between Hyper Quantum Tori are homomorphisms $\phi : {}^*\mathbb{R} \to {}^*\mathbb{R}$ which map ${}^*\mathbb{R}^{\lim}$ to itself and satisfy the following condition:

$$\phi(r) + M = \phi(r + L) \qquad \forall r \in {}^*\mathbb{R}^{\lim}. \tag{2.8}$$

Since ϕ is an internal homomorphism of ${}^*\mathbb{R}$ it is equal to multiplication by α for some $\alpha \in {}^*\mathbb{R}^*$. By (2.8) we have $\alpha(L) \subseteq M$, which since L and M are standard imply that $\alpha \in \mathbb{R}^*$. Conversely, multiplication by any such element induces a morphism in $\mathcal{HQT}$.

□

Corollary 2.4.3. *The categories $\mathcal{PL}$ and $\mathcal{HQT}$ are equivalent.*

Proof. This following from the above result, and the definition of $\mathcal{PL}$ in Definition 1.2.4. □

2.5 Removing nonstandard analysis

Recall how in §2.2.3 we saw how the shadow map provides us with a surjection $p_L : T_L \to Z_L$, rendering Quantum Tori as the shadow image of Hyper Quantum Tori. We now look at how morphisms in $\mathcal{HQT}$ can be used to induce a notion of morphisms between Quantum Tori.

Definition 2.5.1. Let $f : S \to T$ be an S-continuous morphism in $\mathcal{LIQ}^{\mathrm{lim}}$. Define the *standardisation* of f to be

$$\begin{aligned} \bar{f} : sh(S) &\to sh(T) \\ s &\mapsto sh(f(s)). \end{aligned}$$

Given $S, T \in \mathcal{LIQ}^{\mathrm{lim}}$ let $\mathrm{Std}(\mathcal{LIQ}^{\mathrm{lim}}(S,T))$ denote the standardisation of the morphisms between S and T in $\mathcal{LIQ}^{\mathrm{lim}}$.

Motivated by our failure in §2.3.5 to define morphisms between locally internal spaces without reference to an ambient covering space, we postulate the following philosophy:

> Suppose X and Y are topological spaces, such that there exist Hausdorff spaces $\tilde{X}$, $\tilde{Y}$, together with surjective morphisms $q_X : \tilde{X} \to X$ and $q_Y : \tilde{Y} \to Y$. Then the morphisms from $(\tilde{X}, q_X, X)$ to $(\tilde{Y}, q_Y, Y)$ should be those maps $f : \tilde{X} \to \tilde{Y}$ such that $f(x+y) = f(x)$ for all $x \in X$, $y \in \ker(q_X)$.

Given our definition of morphisms in $\mathcal{LIQ}^{\mathrm{lim}}$ refers to the lifts of morphisms to their covering spaces, it is unsurprising that the standardisations of such morphisms and those morphisms of the standard spaces according to the above philosophy are related:

Lemma 2.5.2. *Let S and T be objects of $\mathcal{LIQ}^{\mathrm{lim}}$. Suppose*

1. *The internal covers* $\tilde{S}$ *and* $\tilde{T}$ *of* S *and* T *are equal to* ${}^*A$ *and* ${}^*B$ *respectively for some standard spaces* A *and* B*;*

2. *The kernels of the restricted projections* $p_S : \tilde{S}^{\lim} \to S$ *and* $p_T : \tilde{T}^{\lim} \to T$ *are standard.*

Then $Std(\mathcal{LIQ}^{\lim}(S,T))$ *is equal to the set of continuous morphisms between* $(A, sh(p_S), sh(S))$ *and* $(B, sh(p_T), sh(T))$ *according to the above philosophy.*

Proof. First of all note that $sh(\tilde{S}^{\lim}) = sh({}^*A^{\lim}) = A$, and similarly $sh(\tilde{T}^{\lim}) = B$.

Let $f : S \to T$ be an S-continuous morphism in $\mathcal{LIQ}^{\lim}$. By definition f lifts to an S-continuous map $\tilde{f} : \tilde{S} \to \tilde{T}$ of the internal covering spaces which preserves limited elements. Take the standardisation of this lift to obtain a continuous map $\bar{f} : A \to B$ of the covering spaces. Hence if $y \in \ker(sh(p_S)) = \ker(p_S)$ then $f(x+y) = f(x)$ for all $x \in \tilde{S}^{\lim}$, and hence $\bar{f}(x+y) = \bar{f}(x)$.

Conversely suppose $f : A \to B$ is a morphism between $(A, sh(p_S), sh(S))$ and $(B, sh(p_T), sh(T))$ according to the above philosophy. Then let ${}^*f$ denote its image in a nonstandard structure containing A and B. Then ${}^*f$ is a morphism in $\mathcal{LIQ}^{\lim}$ such that its standardisation is equal to f. □

Based on this analysis we make the following definition

Definition 2.5.3. Let X and Y be topological spaces, together with Hausdorff topological spaces $\tilde{X}$ and $\tilde{Y}$ with projections $q_X : \tilde{X} \to X$ and $q_Y : \tilde{Y} \to Y$. A morphism from $(\tilde{X}, q_X, X)$ to $(\tilde{Y}, q_Y, Y)$ is a map $f : \tilde{X} \to \tilde{Y}$ such that $f(x+y) = f(x)$ for all $x \in \tilde{X}$, $y \in \ker(q_X)$. We say that f is continuous if it is continuous as a map between $\tilde{X}$ and $\tilde{Y}$.

We wish to consider the case when X and Y be Quantum Tori. In this scenario we define a morphism between Quantum Tori Z_L and Z_M to be a continuous morphism between $(\mathbb{R}, \pi_L, Z_L)$ and $(\mathbb{R}, \pi_M, Z_M)$ as defined above. Note that an equivalent definition would be to consider Quantum Tori as quotients of S^1 as discussed in §2.3.1, since $\mathbb{R}$ is a cover for S^1.

Corollary 2.5.4. *Let Z_L and Z_M be Quantum Tori. The continuous homomorphisms between Z_L and Z_M are nonzero real numbers α such that $\alpha L \subseteq M$.*

Proof. By explicit calculation based on Definition 2.5.3 or using Lemma 2.4.2 together with Lemma 2.5.2. □

When we require that the morphisms are merely continuous we obtain the following classification of continuous maps between Quantum Tori:

Proposition 2.5.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous morphism of Quantum Tori Z_L and Z_M. Then there exists $\alpha \in \mathbb{R}$ such that $\alpha L \subseteq M$ and*

$$g(x + l) - g(x) = \alpha l \qquad \forall x \in \mathbb{R}, l \in L.$$

Proof. Since g is a morphism of Quantum Tori we have

$$g(x + l) - g(x) = \lambda(l, x)$$

for some function $\lambda : L \times \mathbb{R} \to M$. First note that since g is continuous, the left hand side is a continuous function in x. Fix $l \in L$, and consider the function $\lambda(l, -) : \mathbb{R} \to M$. This is continuous, and hence maps compact connected subsets of $\mathbb{R}$ to compact connected subsets of M. But the only such sets of the latter are singletons. Hence $\lambda(l, x)$ is independent of x. We will write $\lambda(l) := \lambda(l, x)$ for some

$x \in \mathbb{R}$. Note that λ is a homomorphism since

$$\begin{aligned}\lambda(l_1 + l_2) &= g(x + l_1 + l_2) - g(x) \\ &= g(x + l_1 + l_2) - g(x + l_2) + g(x + l_2) - g(x) \\ &= \lambda(l_1) + \lambda(l_2).\end{aligned}$$

Now fix x and consider the function $g(x + r) - g(x) : \mathbb{R} \to \mathbb{R}$. This is a continuous function which agrees with λ on L. Hence λ extends to a continuous homomorphism of $\mathbb{R}$, and is therefore equal to multiplication by α for some $\alpha \in \mathbb{R}$. Since $\lambda(L) \subseteq M$ we have $\alpha(L) \subseteq M$. □

Definition 2.5.6. Let Z_L and Z_M be quantum tori. An α-morphism between Z_L and Z_M is a continuous map $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x + l) - g(x) = \alpha l \qquad \forall x \in \mathbb{R}, l \in L.$$

2.6 Quantum Tori with Real Multiplication

The theory of Complex Multiplication that forms the basis for the solution of Kronecker's Jugendtraum relies on the existence of elliptic curves over $\mathbb{C}$ whose endomorphism ring is strictly greater than $\mathbb{Z}$. We let $\mathcal{QT}$ denote the category whose objects are Quantum Tori, and whose morphisms are continuous homomorphisms as described in Corollary 2.5.4. Together with the work of the previous section, Theorem §2.6.1 informs us of the existence of Quantum Tori with endomorphism ring isomorphic to an order in a real quadratic field:

Theorem 2.6.1. *The endomorphism ring of a Quantum Torus is either isomorphic to $\mathbb{Z}$, or an order of a real quadratic field.*

Proof. Let Z_L be a Hyper Quantum Torus. By Corollary 2.5.4, the endomorphism ring of Z_L is isomorphic to the set of those $\alpha \in \mathbb{R}^*$ such that $\alpha L \subseteq L$. Suppose there exists such an α such that $\alpha \notin \mathbb{Z}$, and let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Then there exist a, b, c and $d \in \mathbb{Z}$ such that

$$\begin{aligned} \alpha\omega_1 &= a\omega_1 + b\omega_2 \\ \alpha\omega_2 &= c\omega_1 + d\omega_2. \end{aligned} \tag{2.9}$$

Dividing the second of these equations by ω_2, we observe that since $\alpha \notin \mathbb{Z}$, $c \neq 0$. We observe that $\theta := \omega_1/\omega_2$ satisfies the quadratic equation

$$cX^2 + (d-a)X - b = 0.$$

Hence $[\mathbb{Q}(\theta) : \mathbb{Q}] = 2$, and $\mathbb{Q}(\theta)$ is a real quadratic field.

Hence $\mathrm{End}(T_L)$ is an integral extension of $\mathbb{Z}$. Eliminating α from (2.9) we see that α satisfies the equation

$$X^2 - (a+d)X + ad - bc = 0.$$

Hence α is integral over $\mathbb{Z}$ and therefore contained in the ring of integers $\mathcal{O}_F$ of $F = \mathbb{Q}(\theta)$.

We can therefore identify the ring of endomorphisms as a subring of the ring of integers of F. Hence $\mathrm{End}(Z_L)$ is finitely generated as a $\mathbb{Z}$-module and satisfies $\mathrm{End}(Z_L) \otimes \mathbb{Q} \cong F$. These are the precisely the requirements for $\mathrm{End}(Z_L)$ to be an order in F. □

Definition 2.6.2. Let Z_L be a Quantum Torus such that $\mathrm{End}(Z_L)$ is isomorphic to an order in a real quadratic field F. We say that Z_L has *Real Multiplication* (by F). We sometimes abbreviate this to say that Z_L has RM.

In this section we consider those Quantum Tori Z such that $\mathrm{End}(Z)$ is isomorphic to the *maximal* order in a predefined real quadratic field F. From a number theoretic point of view, such orders have a special significance - being the ring of integers of F. We show that there exists an action of the Class Group of F on isomorphism classes of such Quantum Tori, and discuss how this can be interpreted to give Quantum Tori an *algebraic* character.

2.6.1 Isomorphism Classes of Quantum Tori with RM

When discussing isomorphism classes of Quantum Tori the following definition is important:

Definition 2.6.3 (Homothety). Let L and M be pseudolattices. We say that L and M are homothetic if there exists $\alpha \in \mathbb{R}$ such that $\alpha L = M$.

The relation of pseudolattices being homothetic is an equivalence relation and it follows immediately from the definitions that quantum tori Z_L and Z_M are isomorphic if and only if the associated pseudolattices are homothetic.

Remark. Note that if $Z_L \cong Z_M$, then $\mathrm{End}(Z_L) \cong \mathrm{End}(Z_M)$. This follows because if $L = \alpha M$ then:

$$\begin{aligned} \mathrm{End}(Z_L) &\cong \{x \in \mathbb{R} : xL \subseteq L\} \\ &= \{x \in \mathbb{R} : x\alpha L \subseteq \alpha L\} \\ &= \{x \in \mathbb{R} : xM \subseteq M\} \\ &\cong \mathrm{End}(Z_M). \end{aligned}$$

Definition 2.6.4. Let F be a real quadratic field, and denote by $\mathcal{O}_F$ its ring of integers. Set

$$\mathcal{QT}(\mathcal{O}_F) := \frac{\text{Quantum Tori } Z \text{ with } \mathrm{End}(Z) \cong \mathcal{O}_F}{\text{Isomorphism}}.$$

Note. Note that the above object is well defined since $\mathcal{O}_F$ is an (in fact the *maximal*) order of F, and if Z_L and Z_M are isomorphic, then by the above remark they have the same endomorphism ring.

Lemma 2.6.5. *Let L be a pseudolattice in $\mathbb{R}$. Then End$(Z_L) \cong \mathcal{O}_F$ if and only if L is homothetic to a fractional ideal in F.*

Proof. Suppose $\mathrm{End}(Z_L) \cong \mathcal{O}_F$. Then

$$\mathcal{O}_F \cong \{x \in \mathbb{R} : xL \subseteq L\}.$$

Suppose $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Then $L = \omega_1 L_\theta$, where $\theta = \omega_2/\omega_1$, so

$$\mathcal{O}_F \cong \{x \in \mathbb{R} : xL_\theta \subseteq L_\theta\}.$$

By the proof of Theorem 2.6.1 we deduce that $\theta \in F$, and that we can identify $\mathcal{O}_F$ precisely with the set $\{x \in \mathbb{R} : xL_\theta \subseteq L_\theta\}$. Hence L is homothetic to a rank two $\mathcal{O}_F$-module contained in F. The latter object is precisely the definition of a fractional ideal in F.

Conversely let $\mathfrak{a}$ be a fractional ideal of F. This is a rank two abelian subgroup of $\mathbb{R}$, and therefore a pseudolattice, so we may consider the quantum torus $Z_\mathfrak{a}$. Every element of $\mathrm{End}(Z_\mathfrak{a})$ lifts to multiplication by $\alpha_\mathfrak{a}$ on $\mathbb{R}$ such that $\alpha_\mathfrak{a}\mathfrak{a} \subseteq \mathfrak{a}$ for some $\alpha_\mathfrak{a} \in \mathbb{R}$. So $\mathrm{End}(Z_\mathfrak{a})$ contains $\mathcal{O}_F$ as a suborder. We know that $\mathrm{End}(Z_\mathfrak{a})$ is isomorphic to an order of F, and that $\mathcal{O}_F$ is the maximal order. Hence $\mathrm{End}(Z_\mathfrak{a}) \cong \mathcal{O}_F$. $\square$

2.6.2 The algebraic nature of Quantum Tori

Let σ be a generator of the Galois group of F over $\mathbb{Q}$, and denote by I_F the group of fractional ideals of F. We have a natural inclusion $\iota : F^* \to I_F$ which sends x to the principal ideal (x) generated by x. The class group of F is a finite group [37] defined to be the quotient

$$C(F) := \frac{I_F}{\iota(F^*)}.$$

The equivalence class of $\mathfrak{a} \in I_F$ in $C(F)$ is denoted by $[\mathfrak{a}]$.

Let

$$F^+ := \{x \in F : (x > 0) \wedge (x^\sigma > 0)\}$$

denote the subgroup of totally positive elements of F. The narrow class group of F is defined to be

$$C(F)^+ := \frac{I_F}{\iota(F^+)}.$$

The equivalence class of $\mathfrak{a} \in I_F$ in $C(F)^+$ is denoted by $[\mathfrak{a}]^+$. There is a canonical surjection

$$C(F)^+ \longrightarrow C(F). \tag{2.10}$$

Theorem 2.6.6. *There is a well defined action of $C(F)$ on $\mathcal{QT}(\mathcal{O}_F)$. This action is simply transitive.*

Proof. Let Z_L be a Quantum Torus with endomorphism ring isomorphic to $\mathcal{O}_F$. Let $\mathfrak{a}$ be a fractional ideal of F, and define $\mathfrak{a} * Z_L := Z_{\mathfrak{a}L}$. I claim this induces a well defined action of I_F on $\mathcal{QT}(\mathcal{O}_F)$.

- $\mathfrak{a}L$ is a pseudolattice. By Lemma 2.6.5 $L = \lambda\mathfrak{c}$ for some $\lambda \in \mathbb{R}^*$ and fractional ideal $\mathfrak{c}$ of F, so $\mathfrak{a}L = \lambda\mathfrak{a}\mathfrak{c}$. The fractional ideal $\mathfrak{a}\mathfrak{c}$ is a pseudolattice since it is a rank two abelian subgroup of $\mathbb{R}$, and hence $\mathfrak{a}L$ is.

- $\mathrm{End}(Z_{\mathfrak{a}L}) \cong \mathcal{O}_F$. With the notation of the last paragraph, $\mathfrak{a}L$ is homothetic to the fractional ideal $\mathfrak{a}\mathfrak{c}$. The statement follows from Lemma 2.6.5.

- If $[\mathfrak{a}] = [\mathfrak{b}]$ then $\mathfrak{a} * Z_L \cong \mathfrak{b} * Z_L$. If $\mathfrak{a}$ and $\mathfrak{b}$ represent the same elements in the class group, there exists $\alpha \in F^*$ such that $\mathfrak{a} = \alpha\mathfrak{b}$. Hence $\mathfrak{a}L = \alpha\mathfrak{b}L$, and $Z_{\mathfrak{a}L}$ and $Z_{\mathfrak{b}L}$ are isomorphic.

- The action is simple. If $Z_{\mathfrak{a}L}$ and $Z_{\mathfrak{b}L}$ are isomorphic, there exists $\alpha \in \mathbb{R}$ such that

$$\mathfrak{a}L = \alpha\mathfrak{b}L. \tag{2.11}$$

  Recall that $L = \lambda\mathfrak{c}$ for some $\mathfrak{c} \in I_F$, and multiply both sides of (2.11) by the pseudolattice $\lambda^{-1}\mathfrak{b}^{-1}\mathfrak{c}^{-1}$. This gives $\mathfrak{a}\mathfrak{b}^{-1} = \alpha\mathcal{O}_F$. The left hand side is contained in F, so we have $\alpha \in F$. Hence $[\mathfrak{a}] = [\mathfrak{b}]$.

- The action is transitive. Let Z_M be a quantum torus with endomorphism ring isomorphic to $\mathcal{O}_F$. Then $M = \mu\mathfrak{d}$ for some $\mu \in \mathbb{R}^*$, $\mathfrak{d} \in I_F$. Put $\mathfrak{a} := \mathfrak{d}\mathfrak{c}^{-1}$. Then $\mathfrak{a}L = \lambda\mathfrak{a}\mathfrak{c} = \lambda\mathfrak{d} = (\lambda/\mu)M$. Hence $\mathfrak{a} * Z_L \cong Z_M$.

□

Corollary 2.6.7.

$$|C(F)| = |\mathcal{QT}(\mathcal{O}_F)|\,.$$

Remarks:

1. Via (2.10) the narrow class group acts on $\mathcal{QT}(\mathcal{O}_F)$. This is transitive, but is only faithful when $C(F)^+ = C(F)$. This occurs precisely when both infinite primes of F are unramified in the narrow ray class field of F. Class Field Theory gives us an equation for the size of $C(F)^+$:

$$\left|C(F)^+\right| = \frac{4h_F}{[\mathcal{O}_F^* : F^+ \cap \mathcal{O}_F^*]}.$$

   Hence the action of $C(F)^+$ on $\mathcal{QT}(\mathcal{O}_F)$ is faithful precisely when

$$[\mathcal{O}_F^* : F^+ \cap \mathcal{O}_F^*] = 4.$$

2. Let E_Λ be an elliptic curve corresponding to a complex lattice Λ by the Uniformization Theorem, and let σ be an automorphism of $\mathbb{C}$. We have a natural action of σ on E_Λ by letting σ act on the coefficients of the equation for E_Λ. Analogous to the proof of Theorem 2.6.6, we have an action of the group of fractional ideals in K on the set of elliptic curves E with $\mathrm{End}(E) \simeq \mathcal{O}_K$, where K is some fixed quadratic imaginary field. This action is defined and denoted by

$$(\mathfrak{a}, E_\Lambda) \mapsto \mathfrak{a} * E_\Lambda := E_{\mathfrak{a}\Lambda}.$$

If $\mathfrak{a}$ is a fractional ideal of K, then the reciprocity map supplies a homomorphism

$$\psi_{H_K/K} : C(K) \to \mathrm{Gal}(H_K/K) \tag{2.12}$$

where H_K is the Hilbert class field of K. It is a fundamental result in the theory of Complex Multiplication that the following identity holds [56]:

$$\mathfrak{a}^{-1} * E_\Lambda = E_\Lambda^{\psi_{H_K/K}([\mathfrak{a}])}.$$

Since (2.12) exhibits an isomorphism between the class group and the Galois group of the Hilbert class field of K over K, this result is strongly linked to the following:

Proposition 2.6.8. *Let E be an elliptic curve with Complex Multiplication by an order in an imaginary quadratic field K. Then there exists an elliptic curve E' such that E and E' are isomorphic, and E' is defined over the Hilbert class field of K.*

By Theorem 2.6.6 we are able to describe an action of $\mathrm{Gal}(H_F/F)$ on isomorphism classes of Quantum Tori with Real Multiplication. There is no reason *a priori* why we should be able to do this. The objects Z_L are purely analytic constructions, associated to which there is no natural algebraic object for the automorphisms to act upon. However, this simple result shows that Quantum Tori with Real Multiplication do possess algebraic characteristics. Moreover, with respect to the algebraic property highlighted by this result, the suggestion is that Quantum Tori with Real Multiplication by F are somehow “defined up to isomorphism over the Hilbert Class field of F”.

Chapter 3

Line Bundles over Quantum Tori

3.1 Introduction

Central to the theory of Complex Multiplication is the existence of meromorphic elliptic[1] functions on the complex plane. The *Weierstrass $\wp$-function* is such a function, which provides the isomorphism between Complex Tori $\mathbb{C}/\Lambda$ and elliptic curves, where Λ is a lattice in $\mathbb{C}$. This isomorphism of complex Lie groups forms the basis of the Uniformization Theorem (Theorem 1.1.2), simplifying many calculations on elliptic curves to calculations on the associated lattices. The importance of this function is further emphasised in the context of the solution of Hilbert's twelfth problem for imaginary quadratic fields, where abelian extensions of the base field are generated over the Hilbert class field by special values of $\wp$ and its derivative.

We cannot hope for an obvious analogy for Quantum Tori:

Proposition 3.1.1. *Let L be a pseudolattice. Then any meromorphic function periodic with respect to L is constant.*

Proof. Let f be a meromorphic function such that $f(z+l) = f(z)$ for all $z \in \mathbb{C}$,

[1]An elliptic function is a complex valued function f on $\mathbb{C}$ such that $f(z+\omega_1) = f(z+\omega_2) = f(z)$ for all $z \in \mathbb{C}$, where ω_1 and ω_2 are complex numbers linearly independent over $\mathbb{R}$.

$l \in L$. If f has a pole at z_p then the set $\{z_p + l : l \in L\}$ has an accumulation point of poles of f, but the condition that f is meromorphic forbids this. Hence f has no poles and is holomorphic. For fixed $z_0 \in \mathbb{C}$ consider the function $f(z) - f(z_0)$. This has an accumulation point of zeros contained within the set $\{z_0 + l : l \in L\}$. Since f is holomorphic it is therefore constant. $\square$

Elliptic functions can be viewed as quotients of *theta functions* on the Complex Torus - functions satisfying certain periodicity conditions with respect to the lattice Λ. These functions can be viewed as sections of line bundles over complex tori, and due to a theorem of Swan [61] certain classes of these functions characterise line bundles up to isomorphism. It is this observation which motivates the study of line bundles over Quantum Tori, which forms the subject of this chapter.

In §3.2 we are concerned with giving a definition of a line bundle over the Quantum Torus that does not yield a trivial theory. We examine how line bundles over complex tori $\mathbb{C}/\Lambda$ have various descriptions in terms of holomorphic functions satisfying the *cocycle* condition with respect to the lattice Λ. Using these results as a guide we examine the notion of defining *holomorphic* line bundles over Quantum Tori to be holomorphic functions satisfying the cocycle condition with respect to the pseudolattice, and show that this does indeed yield a nontrivial definition.

In viewing line bundles as cocycles we have a natural definition of what it means for two line bundles to be isomorphic supplied by the theory of cohomology. We say that two line bundles are isomorphic if their image in the associated cohomology group is the same. Section 3.3 is concerned with proving a structure theorem for the space $H^1(L, \mathcal{H}^*)$ of line bundles over Z_L modulo isomorphism. This is an analogous result to the *Appel-Humbert Theorem* (Theorem 1.5 of [23]), which proves a similar result for line bundles over Complex Tori. The proof of this result has three main stages.

First we introduce the *Chern class* of a line bundle, which arises from the connecting map of cohomology. The Chern class of a line bundle is an alternating form on the pseudolattice L which takes values in $\mathbb{Z}$, and is the image of a homomorphism

$$Ch : H^1(L, \mathcal{H}^*) \longrightarrow \mathrm{Alt}^2(L, \mathbb{Z}).$$

Sections 3.3.2 and 3.3.3 describe the image and kernel of this homomorphism respectively. It is found that Ch is surjective, with kernel isomorphic to $\mathrm{Hom}(L, U(1))$, where $U(1) := \{z \in \mathbb{C}^* : |z| = 1\}$. Using these two results we use the snake lemma to show that $H^1(L, \mathcal{H}^*)$ is isomorphic to a certain group $P(L)$ whose elements are pairs (E, χ) where

- E is an integral valued alternating form on L;
- $\chi : L \to U(1)$ is a *semi-character* for E - for any $l_1, l_2 \in L_\theta$ we have

$$\chi(l_1 + l_2) = \chi(l_1)\chi(l_2)e^{\pi i E(l_1, l_2)}.$$

In §3.4 we consider the possibility of defining line bundles over Quantum Tori as topological spaces. In order to do this we introduce the *Heisenberg group* associated to a line bundle over a topological space. The philosophy behind our approach is that line bundles over Quantum Tori Z_L should "pull back" to trivial line bundles over $\mathbb{R}$. We define what we mean by a *topological* line bundle $\mathcal{L}$ over the Quantum Torus and show in Lemma 3.4.10 that this agrees with the algebraic definition in §3.2. We define a notion of morphisms between topological line bundles, and show that this corresponds to our previous definition of two line bundles being isomorphic in Proposition 3.4.12. In §3.4.4 we introduce the *translation* of a line bundle

$\mathcal{L}$ by an element $x \in Z_L$, motivated by the topological definition of line bundles we have developed. If $f : Z_L \to Z_M$ is a 1-morphism (see Proposition 2.5.5 of Chapter 2), given a line bundle $\mathcal{L}$ over Z_M we define the pullback $f^*(\mathcal{L})$ of $\mathcal{L}$ with respect to f.

The topological description of line bundles gives rise to an alternating form $e^{\mathcal{L}}$ on a certain subgroup $K(\mathcal{L}) \subseteq Z_L$ for each line bundle $\mathcal{L}$. Section 3.5 is concerned with describing the relationship between these topological constructions and the algebraic characteristics associated to $\mathcal{L}$, such as its Chern class. The main result of this section is Theorem 3.5.7 which exhibits a relationship between the subgroup $K(\mathcal{L})$, the alternating pairing $e^{\mathcal{L}}$ and the Chern class of a line bundle $\mathcal{L}$. The substance of the proof involves showing that if the Chern class of $\mathcal{L}$ is nontrivial, then $K(\mathcal{L})$ is finite. The proof of this result is interesting in its own right, since it shows that line bundles of Quantum Tori can be computed as the "limit" of line bundles over Complex Tori.

The problem of defining line bundles over Quantum and Noncommutative Tori has been studied by many others (Astashkevich, Schwarz [2, 50], Manin [31], Polischuck [41, 42], Zilber). In the final section of this chapter we examine how the results we have obtained from our methods are related to the approaches of others. The work of Zilber features in our analysis, who has identified a class of structures that represent a variation from the structures arising from the Zariski topology on an algebraic curve. It can be shown that Quantum Tori are definable in such *Analytic-Zariski* structures, and in §3.6.1 we examine whether it may be possible to define the alternating pairing $e^{\mathcal{L}}$ together with the subgroup $K(\mathcal{L})$ in such a structure. From our work in §3.5 we can show that $K(\mathcal{L})$ is indeed an Analytic-Zariski set, but problems arise in the definability of $e^{\mathcal{L}}$ in such a structure.

Finally in §3.6.2 we discuss a phenomenon which has previously brought the attention of numerous people. From our structure result for $H^1(L, \mathcal{H}^*)$ (Theorem 3.3.18), we observe that the group of isomorphism classes of line bundles over Quantum Tori is isomorphic to the corresponding group for Complex Tori. This hints at a deep relationship between Quantum and Complex Tori, which has previously been recorded by Nikolaev, Manin and Zilber among others. This notion is further expressed in the proof of Theorem 3.5.7 which exhibits a close relationship between cocycles defining line bundles over Quantum and Complex Tori. We examine how this is related to the result of Schwarz concerning *noncommutative theta functions*, and how Zilber's approach openly exhibits a duality between these two objects. This idea is present in the work of Nikolaev [38–40], who in [39] makes some precise conjectures concerning generators of the Hilbert class field of a real quadratic field.

3.2 Defining Line Bundles over Quantum Tori

Let $L \subseteq \mathbb{R}$ be a pseudolattice, and let Z_L denote the associated Quantum Torus. Our aim is to define the notion of a line bundle over Z_L. The classical definition is given as follows:

Definition 3.2.1. Let X be a topological space. A complex line bundle $\mathcal{L}$ over X is a topological space $\mathcal{L}$ equipped with a projection

$$\Pi : \mathcal{L} \to X$$

such that

- For each $x \in X$, $\Pi^{-1}(x)$ is a one dimensional $\mathbb{C}$-vector space;
- For each $x \in X$ there exists an open neighbourhood U of x such that $\Pi^{-1}(U) \cong U \times \mathbb{C}$.

When we try to apply this definition to Quantum Tori, we run in to problems when we try to impose the second criterion due to the fact that Z_L is not Hausdorff.

In the following section we discuss various characterisations of line bundles over Complex Tori, and use these to motivate an analogous definition for line bundles over Quantum Tori in terms of *cocycles*. We show in §3.2.2 that this does indeed give rise to a nontrivial definition of line bundle.

3.2.1 Line Bundles over Complex Tori

Let Λ denote a complex lattice, so $X_\Lambda := \mathbb{C}/\Lambda$ is a Complex Torus isomorphic to an elliptic curve E_Λ. The torus X_Λ admits nontrivial line bundles, and it is possible to define a notion of isomorphism between line bundles. If $\mathcal{L}$ is a line bundle over X_Λ, we let $[\mathcal{L}]$ denote the isomorphism class of $\mathcal{L}$. It can be shown that the set of such classes forms a group which we denote by $\mathrm{Pic}(X_\Lambda)$. The group law is given by the tensor product

$$[\mathcal{L}_1][\mathcal{L}_2] = [\mathcal{L}_1 \otimes \mathcal{L}_2].$$

Through the theory of Cartier Divisors, we can identify a line bundle $\mathcal{L}$ over X_Λ with an element of the group $Z^1(X, \mathcal{O}^*_{X_\Lambda})$ of 1-cocycles[2] [36]. It is through this identification that line bundles are sometimes referred to as *invertible sheaves*. Isomorphic line bundles differ by a coboundary in $Z^1(X_\Lambda, \mathcal{O}^*_{X_\Lambda})$, which yields an isomorphism

$$\mathrm{Pic}(X_\Lambda) \cong H^1(X, \mathcal{O}^*_{X_\Lambda}).$$

[2] $\mathcal{O}^*_{X_\Lambda}$ is a functor which assigns to each open subset U of X_Λ the ring of nonvanishing $\mathbb{C}$-valued functions on U.

The natural projection

$$\begin{aligned} P : \mathbb{C} &\rightarrow X_\Lambda \\ v &\mapsto v + \Lambda \end{aligned}$$

allows us to pull back line bundles over X_Λ to line bundles over $\mathbb{C}$. Given a line bundle $\mathcal{M}$ over X_Λ its pullback to one on $\mathbb{C}$ is trivial,[3] so there exists an isomorphism

$$\chi : P^*(\mathcal{L}) \cong \mathbb{C} \times \mathbb{C}.$$

Let $\mathcal{H}^*$ denote the multiplicative group of nowhere vanishing holomorphic functions on $\mathbb{C}$. The trivial action of Λ on $\mathcal{M}$ pulls back to an action on $\mathbb{C} \times \mathbb{C}$ given by

$$\lambda(v, z) = (v + \lambda, A_\lambda(v)z) \tag{3.1}$$

for some function $A_\lambda \in \mathcal{H}^*$. The condition that Λ *acts* on the trivial line bundle implies that $A_\lambda(v)$ satisfies the *cocycle condition:*

$$A_{\lambda_1+\lambda_2}(v) = A_{\lambda_1}(v + \lambda_2)A_{\lambda_2}(v).$$

This implies that we can view $A_\lambda(v)$ as an element of the group of 1-cocycles with coefficients in $\mathcal{H}^*$, which we denote by $Z^1(\Lambda, \mathcal{H}^*)$.

The homology group $H^1(\Lambda, \mathcal{H}^*)$ is defined to be the quotient

$$\frac{Z^1(\Lambda, \mathcal{H}^*)}{B^1(\Lambda, \mathcal{H}^*)}$$

where $B^1(\Lambda, \mathcal{H}^*)$ is the subgroup of $Z^1(\Lambda, \mathcal{H}^*)$ of those $A_\lambda(v)$ such that there exists $h \in \mathcal{H}^*$ such that

$$B_\lambda(v) = \frac{h(v + \lambda)}{h(v)}$$

[3] A line bundle $\mathcal{M}$ over X is trivial if there exists an isomorphism $\mathcal{M} \cong X \times \mathbb{C}$.

for all $v \in \mathbb{C}$. Such an element is called a *coboundary*.

If we change the isomorphism χ by a nowhere vanishing holomorphic function, then the image of a line bundle $\mathcal{L}$ in $H^1(\Lambda, \mathcal{H}^*)$ remains the same.

Conversely, given a 1-cocycle $A_\lambda(v)$ with coefficients in $\mathcal{H}^*$ the quotient of $\mathbb{C} \times \mathbb{C}$ by the action of Λ in (3.1) describes a line bundle over X. This yields an isomorphism $\mathrm{Pic}(X_\Lambda) \cong H^1(\Lambda, \mathcal{H}^*)$.

3.2.2 Line Bundles over Quantum Tori

Given a complex torus $X_\Lambda := \mathbb{C}/\Lambda$ we have two descriptions of line bundles in terms of cohomology:

- A line bundle $\mathcal{L}$ over X_Λ can be represented by an element of $Z^1(X_\Lambda, \mathcal{O}^*_{X_\Lambda})$;
- A line bundle $\mathcal{L}$ over X_Λ can be represented by an element of $Z^1(\Lambda, \mathcal{H}^*)$.

This suggests two possible definitions for line bundles over a Quantum Torus Z_L:

- A line bundle $\mathcal{L}$ over Z_L can be represented by an element of $Z^1(Z_L, \mathcal{O}^*_{Z_L})$;
- A line bundle $\mathcal{L}$ over Z_L can be represented by an element of $Z^1(L, \mathcal{H}^*)$.

The first of these definitions yields only trivial line bundles. An element of $Z^1(Z_L, \mathcal{O}^*_{Z_L})$ assigns to each $z \in Z_L$ a nonvanishing holomorphic function on some open neighbourhood U of z. This statement does not make sense because we have no complex structure on Z_L so we cannot talk about holomorphic functions on it. We could get around this problem by allowing the restriction of holomorphic functions to open subsets of Z_L, but since the only such subsets are the whole of Z_L and the empty set we only obtain constant functions.

Definition 3.2.2 (Line Bundles over Quantum Tori). Let Z_L be a Quantum Torus. A line bundle $\mathcal{L}$ over Z_L is an element of $Z^1(L, \mathcal{H}^*)$. We say that two line bundles are isomorphic if they have the same image in $H^1(L, \mathcal{H}^*)$, and denote by $[\mathcal{L}]$ the isomorphism class of $\mathcal{L}$. We denote the law of composition both in $Z^1(L, \mathcal{H}^*)$ and $H^1(L, \mathcal{H}^*)$ by $\otimes$.

Proposition 3.2.3. *Let Z_L be a Quantum Torus. Then there exist nontrivial line bundles on Z_L.*

Proof. Suppose $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Consider the function

$$\begin{array}{rcl} A : L \times \mathbb{C} & \longrightarrow & \mathbb{C}^* \\ (l, v) & \mapsto & e^{-\frac{\pi i}{\omega_1}[b^2\omega_2 - 2bv]}. \end{array}$$

where $l = a\omega_1 + b\omega_2$. I claim that the function $A_l(v)$ is a 1-cocycle. Suppose $l_1, l_2 \in L$ and $l_k = a_k\omega_1 + b_k\omega_2$.

$$\begin{aligned} A_{l_1+l_2}(v) &= e^{-\frac{\pi i}{\omega_1}[(b_1+b_2)^2\omega_2 - 2(b_1+b_2)v]} \\ &= e^{-\frac{\pi i}{\omega_1}[b_1^2\omega_2 + 2b_1b_2\omega_2 + b_2^2\omega_2 - 2(b_1+b_2)v]} \\ &= e^{-\frac{\pi i}{\omega_1}[b_1^2\omega_2 - 2b_1(v + a_2\omega_1 + b_2\omega_2)]} e^{-\frac{\pi i}{\omega_1}[b_2^2\omega_2 - 2b_2v]} \\ &= A_{l_1}(v + l_2)A_{l_2}(v). \end{aligned} \tag{3.2}$$

We now show that the class of this function on $H^1(L, \mathcal{H}^*)$ is nontrivial.

Suppose $A_l(v)$ does represent a trivial line bundle. Then there exists $h \in \mathcal{H}^*$ such that

$$A_l(v) = \frac{h(v + l)}{h(v)}$$

for all $v \in \mathbb{C}, l \in L$. Since h is nonvanishing we may write $h(v) = e^{\pi i g(v)}$ for some holomorphic function $g(v)$ which satisfies the following periodicity relations

$$\begin{aligned} g(v+\omega_1) - g(v) &= 2m \\ g(v+\omega_2) - g(v) &= -\frac{\omega_2 + 2v}{\omega_1} \end{aligned}$$

for some $m \in \mathbb{Z}$. Define the holomorphic function $k(v) := g(v) - 2mv/\omega_1$, which satisfies the following periodicity conditions:

$$k(v+\omega_1) - k(v) = 0 \tag{3.3}$$

$$k(v+\omega_2) - k(v) = -\frac{(2m+1)\omega_2 + 2v}{\omega_1} \tag{3.4}$$

Suppose such a function existed. Consider the continued fraction expansion of $\theta := \omega_2/\omega_1$, and let p_n/q_n be the convergents (see [7]). Then the sequence $q_n \to \infty$, but

$$|p_n\omega_1 - q_n\omega_2| < \frac{|\omega_1|}{q_n} \to 0 \tag{3.5}$$

as $n \to \infty$. Consider the sequence $x_n = p_n\omega_1 - q_n\omega_2$. By the continuity of k, and (3.5) we have $k(x_n) \to k(0)$. Hence by (3.4) we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} |k(x_n) - k(0)| \\ &= \lim_{n\to\infty} |k(q_n\omega_2) - k(0)| \\ &= \lim_{n\to\infty} q_n\, |\theta|\, |(2m+1) + (q_n+1)|\,. \end{aligned} \tag{3.6}$$

The last line is obtained by using (3.4) to note that for $d \in \mathbb{N}$ we have

$$k((d+1)\omega_2) - k(d\omega_2) = -\theta(2m + 2d + 1)$$

and hence

$$k(q_n\omega_2) - k(0) = \sum_{d=0}^{q_n} k((d+1)\omega_2) - k(d\omega_2) = q_n\theta(2m+1) + \theta q_n(q_n+1).$$

The expression in the final line of (3.6) tends to ∞ as $n \to \infty$, which is a contradiction.

Hence $A_l(v)$ represents a nontrivial line bundle. □

Note that the techniques we used to prove the nonexistence of a nonconstant coboundary function are very different from the ones used for the Complex Torus. For Complex Tori the main tool used for this purpose is Louisville's Theorem which implies that every bounded 2-periodic function is constant. When considering the analogous situation for Quantum Tori our main tool is the following:

Theorem 3.2.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function which is not identically zero. Then the set of zeros of f has no accumulation points.*

This has an obvious corollary:

Corollary 3.2.5. *Let f be a function for which $f(\omega)$ is known for each $l \in L$. Suppose there exists a holomorphic function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ which interpolates f on L. Then $\tilde{f}$ is unique.*

Proof. Suppose there existed two such functions $\tilde{f}_1$ and $\tilde{f}_2$. Then their difference F would be a holomorphic function with zeros at every $\omega \in L$. Since L is dense in $\mathbb{R}$ every element of $\mathbb{R}$ is an accumulation point of zeros of F. □

This distinction between the theory used for Complex and Quantum Tori suggests that there is no reason *a priori* to expect any relationship between the cohomology groups $H^1(L, \mathcal{H}^*)$ and $H^1(\Lambda, \mathcal{H}^*)$ where L is a pseudolattice and Λ a complex lattice.

3.3 The Appel-Humbert Theorem for Quantum Tori

The goal of this section is to prove a structure result for $H^1(L, \mathcal{H}^*)$. For Complex Tori this is achieved by the Appel-Humbert Theorem [23], which classifies isomorphism classes of line bundles by a hermitian form and a *semi-character*. The main result we prove is Theorem 3.3.18 which proves a similar result characterising isomorphism classes of line bundles in terms of alternating forms and semi-character.

3.3.1 The Chern Class of a Line Bundle

We now introduce the notion of the *Chern class* of a line bundle. The cohomology group $H^1(L, \mathcal{H}^*)$ is one of a family of such groups $H^i(L, \mathcal{H}^*)$ where the index i ranges over the natural numbers. The general construction of these groups is a routine operation in cohomology theory [28], and can be applied to any situation where we have a group G, and a G-module M. In our case the group L acts on $\mathcal{H}^*$ by translation

$$l.f(v) = f(v + l). \tag{3.7}$$

The Chern class of a line bundle can be viewed as an element of the cohomology group $H^2(L, \mathcal{H}^*)$. The starting point for its construction is the following exact sequence of L-modules:

$$0 \to \mathbb{Z} \to \mathcal{H} \to \mathcal{H}^* \to 0$$

where the action on $\mathbb{Z}$ is trivial and the action on $\mathcal{H}$ is defined as in (3.7). The cohomology theory implies that we have a long exact sequence, involving the cohomology groups of these modules. The *connecting* map is one map in this sequence and supplies a homomorphism

$$\partial : H^1(L, \mathcal{H}^*) \longrightarrow H^2(L, \mathbb{Z}).$$

Definition 3.3.1 (Chern class). Let $\mathcal{L}$ be a line bundle. The Chern class of $\mathcal{L}$ is given by $\partial(\mathcal{L}) \in H^2(L, \mathcal{H}^*)$.

The map ∂ can be defined on cocycles, and shown to map coboundaries to coboundaries. We let $\hat{\partial}$ denote the map on cocycles which induces the map ∂ on cohomology groups. A line bundle $\mathcal{L}$ is given by a cocycle $A_l(v) \in Z^1(L, \mathcal{H}^*)$. The cohomology theory provides us with an explicit formula for the image of $A_l(v)$ under $\hat{\partial}$. If $A_l(v) = e^{2\pi i a(l,v)}$, then $\hat{\partial}$ is a function on $L \times L$ taking values in $\mathbb{Z}$ and has the explicit formula (see the proof of Theorem 2.1.2 of [5])

$$\hat{\partial}(A)(l_1, l_2) = a(l_1 + l_1, v) - a(l_1, v) - a(l_2, v + l_1)$$

for some $v \in \mathbb{C}$. This is well defined since the cocycle condition satisfied by $A_l(v)$ implies that this is independent of the choice of v.

Define a map

$$\begin{aligned} \alpha : Z^2(L, \mathbb{Z}) &\rightarrow \mathrm{Alt}^2(L, \mathbb{Z}) \\ P &\mapsto \alpha(P), \end{aligned}$$

where $\alpha(P)$ is defined by

$$\alpha(P)(\omega_1, \omega_2) = P(\omega_1, \omega_2) - P(\omega_2, \omega_1).$$

This induces a well defined map (also denoted by α) from $H^2(L, \mathbb{Z})$ to the space $\mathrm{Alt}^2(L, \mathbb{Z})$ of alternating forms on L.

Lemma 3.3.2. *The map $\alpha : H^2(L, \mathbb{Z}) \rightarrow Alt^2(L, \mathbb{Z})$ is an isomorphism.*

Proof. See Lemma 2.1.3 of [5]. □

The composition of $\hat{\partial}$ with α establishes a homomorphism

$$\begin{aligned} Ch : H^1(L, \mathcal{H}^*) &\rightarrow \mathrm{Alt}^2(L, \mathbb{Z}) \\ [A] &\mapsto \alpha \circ [\hat{\partial}(A)] \end{aligned} \tag{3.8}$$

where [ ] denotes the cohomology class of the appropriate cocycle. An explicit expression for this map is given by (Theorem 2.1.2 of [5])

$$Ch(A)(l_1, l_2) = a(l_2, v + l_1) - a(l_1, v) + a(l_2, v) - a(l_1, v + l_2), \tag{3.9}$$

where we recall that for each $l \in L$, $a(l, v)$ is the holomorphic function such that $A_l(v) = e^{2\pi i a(l,v)}$. Analysing this map in more detail will enable us to probe the structure of $H^1(L, \mathcal{L})$. In the following sections we examine the image and kernel of Ch.

3.3.2 Surjectivity of $\mathbf{Ch} : \mathbf{H^1}(\mathbf{L}, \mathcal{H}^*) \rightarrow \mathbf{Alt^2}(\mathbf{L}, \mathbb{Z})$.

In this section we study the image of the homomorphism Ch defined in (3.8). We will prove the following result:

Proposition 3.3.3. *The map $Ch : H^1(L, \mathcal{H}^*) \rightarrow Alt^2(L, \mathbb{Z})$ is surjective. Furthermore, there exists a map $\sigma : Alt^2(L, \mathbb{Z}) \rightarrow H^1(L, \mathcal{H}^*)$ such that $Ch \circ \sigma$ is the identity on $Alt^2(L, \mathbb{Z})$.*

Proof. The proof relies on the construction of elements of $Z^1(L, \mathcal{H}^*)$ which are similar to that introduced in the proof of Proposition 3.2.3.

We first observe that we have an isomorphism $\mathrm{Alt}^2(L, \mathbb{Z}) \cong \mathbb{Z}$. This arises since every $\eta \in \mathrm{Alt}^2(L, \mathbb{Z})$ is determined by a skew-symmetric 2×2 matrix S_η with integral coefficients. Such a matrix has the form

$$S_\eta = \begin{pmatrix} 0 & s_\eta \\ -s_\eta & 0 \end{pmatrix}$$

for some $s_\eta \in \mathbb{Z}$. The form η is determined by S_η in the following way:

If $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and $l_1 = a\omega_1 + b\omega_2$, $l_2 = c\omega_1 + d\omega_2$ then

$$\eta(l_1, l_2) = \begin{pmatrix} a \\ b \end{pmatrix}^T S_\eta \begin{pmatrix} c \\ d \end{pmatrix} = s_\eta(ad - bc). \tag{3.10}$$

The assignment $\eta \mapsto s_\eta$ gives a bijection between $\mathrm{Alt}^2(L, \mathbb{Z})$ and $\mathbb{Z}$.

For each $\eta \in \mathrm{Alt}^2(L, \mathbb{Z})$ define

$$\hat{\sigma}(\eta)_l(v) := e^{s_\eta \frac{\pi i}{\omega_1}[b^2\omega_2 + 2bv]} \tag{3.11}$$

where $l = a\omega_1 + b\omega_2$. By a similar calculation to (3.2) we find that $\hat{\sigma}(\eta)_l(v) \in Z^1(L, \mathcal{H}^*)$, and hence we can define

$$\sigma(\eta) := [\hat{\sigma}(\eta)_l(v)] \in H^1(L, \mathcal{H}^*).$$

By the definition above we have $\hat{\sigma}(\eta)_l(v) = e^{2\pi i \Sigma_\eta(l,v)}$ where

$$\Sigma_\eta(l, v) = s_\eta \frac{1}{2\omega_1}[b^2\omega_2 + 2bv].$$

We calculate $Ch(\sigma(\eta))$ using (3.9). Let $l_1 = a\omega_1 + b\omega_2$, $l_2 = c\omega_1 + d\omega_2 \in L$. Then

$$\begin{aligned} Ch(\sigma(\eta))(l_1, l_2) &= \Sigma_\eta(l_2, v + l_1) + \Sigma_\eta(l_1, v) - \Sigma_\eta(l_2, v) - \Sigma_\eta(l_1, v + l_2) \\ &= \tfrac{s_\eta}{2\omega_1}[d^2\omega_2 + 2d(v + a\omega_1 + b\omega_2) - d^2\omega_2 - 2dv] \\ &\qquad - \tfrac{s_\eta}{2\omega_1}[b^2\omega_2 + 2b(v + c\omega_1 + d\omega_2) - b^2\omega_2 - 2bv] \\ &= s_\eta(ad - bc) \\ &= \eta(l_1, l_2). \end{aligned}$$

□

3.3.3 The kernel of $\mathbf{Ch} : \mathbf{H^1}(\mathbf{L}, \mathcal{H}^*) \to \mathbf{Alt^2}(\mathbf{L}, \mathbb{Z})$.

In the previous section we showed that the homomorphism $Ch : H^1(L, \mathcal{H}^*) \to \mathrm{Alt}^2(L, \mathbb{Z})$ defined in (3.8) is surjective. We showed furthermore that we have a homomorphism

$$\sigma : \mathrm{Alt}^2(L, \mathbb{Z}) \to H^1(L, \mathcal{H}^*)$$

such that $Ch \circ \sigma$ is the identity on $\mathrm{Alt}^2(L, \mathbb{Z})$. The significance of this result is apparent when we observe that if K denotes the kernel of Ch, then we have a split exact sequence

$$0 \to K \to H^1(L, \mathcal{H}^*) \to \mathrm{Alt}^2(L, \mathbb{Z}) \to 0.$$

We can then apply the theory of split exact sequences to give a description of $H^1(L, \mathcal{H}^*)$ in terms of K and $\mathrm{Alt}^2(L, \mathbb{Z})$.

Traditionally the kernel of the Chern map is given the following definition:

Definition 3.3.4. Let $\mathrm{Pic}^0(Z_L) = \{x \in H^1(L, \mathcal{H}^*) : x \in \ker(Ch)\}$.

Proposition 2.2.2 of [23] shows that every line bundle with trivial Chern class over a Complex Torus can be represented by a cocycle which is *constant*. We use the same techniques to prove the analogous result for line bundles over Quantum Tori.

Proposition 3.3.5. *Let $A_l(v) \in Z^1(L, \mathcal{H}^*)$ be such that $Ch([A]) = 0 \in Alt^2(L, \mathbb{Z})$. Then there exists $K_l(v) \in Z^1(L, \mathcal{H}^*)$ such that $A_l(v)K_l^{-1}(v) \in B^1(L, \mathcal{H}^*)$ and for each $l \in L$, $K_l(v)$ is constant.*

Proof. The proof of this result involves unravelling the definition of the map Ch. Let $A_l(v) = e^{2\pi i a(l,v)}$ for some function $a : L \times \mathbb{C} \to \mathbb{C}$ holomorphic for fixed $l \in L$. Since $A_l(v)$ satisfies the cocycle condition we have

$$a(l_1 + l_2, v) - a(l_1, v + l_2) - a(_2, v) \equiv 0 \pmod{\mathbb{Z}}. \tag{3.12}$$

The image of A_l under the map $\hat{\partial} : Z^1(L, \mathcal{H}^*) \to Z^2(L, \mathbb{Z})$ is trivial. By (3.9) this implies that

$$a(l_2, v + l_1) - a(l_1, v) + a(l_2, v) - a(l_1, v + l_2) = 0 \tag{3.13}$$

for every $v \in \mathbb{C}$ and for all $l_1, l_2 \in L$. Define $h(v) := a(0, v)$. Then

$$\begin{aligned} a(l, v) + h(v + l) - h(v) &= a(l, v) + a(0, v + l) - a(0, v) \\ &\equiv a(l, v) - a(0, v) \pmod{\mathbb{Z}} \\ &\equiv a(l, 0) - a(0, l) \pmod{\mathbb{Z}}. \end{aligned} \tag{3.14}$$

The second line follows from (3.12) by putting $l_1 = 0$, $l_2 = l$. Using these same substitutions in (3.13), together with $v = 0$ yields the final line. This is valid since (3.13) is independent of v. Put $H(v) = e^{2\pi i h(v)}$. Then $K_l(v) := A_l(v)H(v+l)H(v)^{-1}$ is independent of v and lies in the same cohomology class as $A_l(v)$. □

Hence each element of $\mathrm{Pic}^0(Z_L)$ gives rise to an element ϕ of $\mathrm{Hom}(L, \mathbb{C}^*)$, such that

$$h(v + l) = \phi(l)h(v) \qquad \forall l \in L$$

for some non vanishing holomorphic function h. This defines a homomorphism

$$C : \mathrm{Pic}^0(Z_L) \to \mathrm{Hom}(L, \mathbb{C}^*).$$

We aim to show that C is in fact an isomorphism of $\mathrm{Pic}^0(Z_L)$ on to $\mathrm{Hom}(L, U(1))$ where $U(1) = \{z \in \mathbb{C}^* : |z| = 1$. Our first task is to show that the image of C lies within $\mathrm{Hom}(L, U(1))$. We will need the following lemma:

Lemma 3.3.6. *Let $\theta \in \mathbb{R}$ be greater than 1, and let p_n^+/q_n^+ and p_n^-/q_n^- denote the convergents in the continued fraction of θ and $-\theta$ respectively - see chapter 14 of [7] for an account of the theory of continued fractions. Then as $n \to \infty$:*

1. $p_n^\pm \to \pm\infty$;
2. $q_n^\pm \to \infty$;
3. $d_n^\pm := p_n^\pm - q_n^\pm \to \pm\infty$.
4. $e_n^- := p_n^- + q_n^- \to -\infty$.

Proof. Let $x \in \mathbb{R}$. The continued fraction expansion for x is given by an infinite sequence $[a_0, a_1, a_2, a_3 \ldots]$, of which all the $a_i > 0$ with perhaps the exception of a_0. The recursion formula's for p_n and q_n read:

$$p_n = a_n p_{n-1} + p_{n-2},$$
$$q_n = a_n q_{n-1} + q_{n-2}.$$

The first few terms are given by:

$$p_{-2} = 0 \quad q_{-2} = 1$$
$$p_{-1} = 1 \quad q_{-1} = 0$$
$$p_0 = a_0 \quad q_0 = 1,$$

where a_0 is defined to be $\lfloor x \rfloor$. Because all the a_i are positive for $i > 0$, if p_m (q_m) and p_{m+1} (q_{m+1}) are the same parity for some m, all subsequent terms will be of that parity. From this it follows that q_n is always positive. Now take the case $\theta > 1$. Then it is clear that all the terms p_n^+ are positive. We have $-\theta < -1$, and hence $p_0^- \leq -2$. The following inequality holds

$$p_1 = a_1 p_1 + a + 0 = -2a_1 + 1 \leq -1,$$

therefore each p_n^- is negative.

Using the recurrence relations, since all the p_n are of the same parity:

$$\begin{aligned} |p_n| &= |a_n p_{n-1} + p_{n-2}| \\ &\geq a_n |p_{n-1}| . \end{aligned}$$

Suppose we have equality, then $a_n = 1$ and $p_n = p_{n-1}$. But then

$$|p_{n+1}| = |a_{n+1} p_n + n_{n-1}| = (a_{n+1} + 1) |p_n| > |p_n| .$$

Hence $|p_n| \to \infty$. Exactly the same argument works for q_n, and the case for $d_n^{\pm}$ follows from the algebra of limits.

To consider the limit of the sequence e_n^- we note that the fractions p_n^-/q_n^- tend to $-\theta < -1$. Hence there exists N such that for all $n > N$ we have $p_n^- < -q_n^-$, and hence $e_n^- < 0$. Since all subsequent values of e_n^- are of the same parity, the above argument implies that $|e_n^-| \to \infty$.

$\square$

Using this result we are able to prove the following:

Proposition 3.3.7. $C(Pic^0(Z_L)) \subseteq Hom(L, U(1))$.

Proof. Let $A_l(v) \in \mathrm{Pic}^0(Z_L)$. Then Proposition 3.3.5 implies that there exists $\phi \in \mathrm{Hom}(L, U(1))$ such that for all $l \in L$ we have $\phi(l) = C(A)_l(v)$ for some $v \in \mathbb{C}$. Since this represents the trivial element of $H^1(L, \mathcal{H})$ there exists a vanishing holomorphic function h such that

$$h(v + l) = \phi(l)h(v). \tag{3.15}$$

We will show that $|\phi(l)| = 1$ for all $l \in L$. If $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ then note that this is equivalent to showing that $\phi(\omega_1), \phi(\omega_2) \in U(1)$. Without loss of generality we may assume that $\omega_1 < \omega_2$ since if $\{\omega_1, \omega_2\}$ generate L then so do $\{\omega_1, \omega_1 + \omega_2\}$.

With the notation of Lemma 3.3.6 we let p_n^+, q_n^+ denote the convergents in the continued fraction expansion for $\theta = \omega_2/\omega_1$, and p_n^-, q_n^- the corresponding integers for $-\theta$. Using the Corollary to the proof of Theorem 1.4.7 of [7] we have

$$\left|p_n^+\omega_1 - q_n^+\omega_2\right| = \left|\omega_1 q_n^+\right| \left|\frac{p_n^+}{q_n^+} - \theta\right| \leq \frac{|\omega_1|}{q_n^+}.$$

Hence by Lemma 3.3.6 the sequence $p_n^+\omega_1 - q_n^+\omega_2$ tends to 0 as $n \to \infty$. In a similar way one can show that $p_n^-\omega_1 + q_n^-\omega_2$ tends to 0 as $n \to \infty$.

We will show that $\phi(\omega_1), \phi(\omega_2) \in U(1)$ by showing that every other possibility cannot occur. In each case we assume the existence of a nonvanishing holomorphic function h satisfying (3.15) and deduce a contradiction.

1. Suppose $|\phi(\omega_1)| > |\phi(\omega_2)| > 1$. Let $a_n = p_n^+\omega_1 - q_n^+\omega_2$. Then

$$\begin{aligned}
|h(0)| &= \lim_{n\to\infty} |h(a_n)| \\
&= \lim_{n\to\infty} |\phi(\omega_1)|^{p_n^+} |\phi(\omega_2)|^{-q_n^+} |h(0)| \\
&\geq \lim_{n\to\infty} |\phi(\omega_2)|^{p_n^+} |\phi(\omega_2)|^{-q_n^+} |h(0)| \\
&= \lim_{n\to\infty} |\phi(\omega_2)|^{d_n^+} |h(0)|
\end{aligned}$$

   Since $d_n^+ \to \infty$,and $|\phi(\omega_2)| > 1$ the right hand side tends to infinity contradicting that h is holomorphic.

2. Suppose $|\phi(\omega_1)| < |\phi(\omega_2)| < 1$. Then let $\psi := \phi^{-1}$, and $f := h^{-1}$. Then $f(v+l) = \psi(l)f(v)$, and ψ satisfies $|\psi(\omega_1)| > |\psi(\omega_2)| > 1$. By step 1 above no such f (and hence h) can exist.

3. Suppose $|\phi(\omega_1)| > 1 > |\phi(\omega_2)|$. Define a sequence a_n by $a_n = p_n^-\omega_1 + q_n^-\omega_2$. Then $|\phi(\omega_1)|, |\phi(-\omega_2)| > \eta > 1$ for some η, and we have $|\phi(\omega_1)| = \xi\eta$ and $|\phi(-\omega_2)| = \mu\eta$ for some $\xi, \mu > 1$. Then

$$\begin{aligned}
|h(0)| &= \lim_{n\to\infty} |h(a_n)| \\
&= \lim_{n\to\infty} |\phi(\omega_1)|^{p_n^-} |\phi(\omega_2)|^{q_n^-} \\
&= \lim_{n\to\infty} |\phi(\omega_1)|^{p_n^-} |\phi(-\omega_2)|^{-q_n^-} \\
&= \lim_{n\to\infty} (\xi\eta)^{p_n^-} (\mu\eta)^{-q_n^-} \\
&= \lim_{n\to\infty} \eta^{d_n^-} \xi^{p_n^-} \mu^{-q_n^-}. \\
&= \lim_{n\to\infty} \eta^{d_n^-} \xi^{p_n^- + q_n^-} (\xi\mu)^{-q_n^-} \\
&= \lim_{n\to\infty} \eta^{d_n^-} \xi^{e_n^-} (\xi\mu)^{-q_n^-}.
\end{aligned}$$

   Since all the terms η, ξ and $\xi\mu$ are greater than 1, and all the exponents tend to $-\infty$, this limit is equal to 0, contradicting the assumption that h is nonvanishing.

4. Suppose $|\phi(\omega_1)| < 1 < |\phi(\omega_2)|$. Then let $\psi := \phi^{-1}$, and $f := h^{-1}$. Then

$f(v+l) = \psi(l)f(v)$, and ψ satisfies $|\phi(\omega_1)| > 1 > |\phi(\omega_2)|$. By step 3 above no such f (and hence h) can exist.

5. Suppose $1 > |\phi(\omega_1)| > |\phi(\omega_2)|$. Then there exists $b \in \mathbb{N}$ such that $|\phi(\omega_1)|^b < |\phi(\omega_2)|$. Let $a_n = bp_n^+\omega_1 - bq_n^+\omega_2$. Then

$$\begin{aligned}
|h(0)| &= \lim_{n\to\infty} |h(a_n)| \\
&= \lim_{n\to\infty} |\phi(\omega_1)|^{bp_n^+} |\phi(\omega_2)|^{-bq_n^+} |h(0)| \\
&\leq \lim_{n\to\infty} |\phi(\omega_2)|^{p_n^+} |\phi(\omega_2)|^{-bq_n^+} |h(0)| \\
&\leq \lim_{n\to\infty} |\phi(\omega_2)|^{p_n^+} |\phi(\omega_2)|^{-q_n^+} |h(0)| \\
&= \lim_{n\to\infty} |\phi(\omega_2)|^{d_n^+} |h(0)| .
\end{aligned}$$

Since $d_n^+ \to \infty$ and $|\phi(\omega_2)| < 1$, the right hand side tends to zero. This contradicts that fact that h is non vanishing.

6. Suppose $1 < |\phi(\omega_1)| < |\phi(\omega_2)|$. Then let $\psi := \phi^{-1}$, and $f := h^{-1}$. Then $f(v+l) = \psi(l)f(v)$, and ψ satisfies $1 > |\phi(\omega_1)| > |\phi(\omega_2)|$. By step 5 above no such f (and hence h) can exist.

7. Suppose $\phi(\omega_1) = 1$. Then let $a_n = p_n^+\omega_1 - q_n^+\omega_2$. Then

$$\begin{aligned}
|h(0)| &= \lim_{n\to\infty} |h(a_n)| \\
&= \lim_{n\to\infty} |\phi(\omega_1)|^{p_n^+} |\phi(\omega_2)|^{-q_n^+} |h(0)| \\
&= \lim_{n\to\infty} |\phi(\omega_2)|^{-q_n^+} |h(0)| .
\end{aligned}$$

If $|\phi(\omega_2)| > 1$, then the right hand side tends to 0 as $n \to \infty$. If $|\phi(\omega_2)| < 1$, the right hand side tends to infinity. In both cases we reach a contradiction.

8. Suppose $\phi(\omega_2) = 1$. Then let $a_n = p_n^+\omega_1 - q_n^+\omega_2$. Then

$$\begin{aligned} |h(0)| &= \lim_{n\to\infty} |h(a_n)| \\ &= \lim_{n\to\infty} |\phi(\omega_1)|^{p_n^+} |\phi(\omega_2)|^{-q_n^+} |h(0)| \\ &= \lim_{n\to\infty} |\phi(\omega_1)|^{p_n^+} |h(0)| . \end{aligned}$$

   If $|\phi(\omega_1)| > 1$, then the right hand side tends to infinity as $n \to \infty$. If $|\phi(\omega_1)| < 1$, the right hand side tends to 0. In both cases we reach a contradiction.

The only possibility left is that $|\phi(\omega_1)| = |\phi(\omega_2)| = 1$. □

Hence $C(\mathrm{Pic}^0(Z_L)) \subseteq \mathrm{Hom}(L, U(1))$. Our aim is to show that this is an isomorphism. Observe that we have a homomorphism

$$\begin{aligned} D : \mathrm{Hom}(L, U(1)) &\to \mathrm{Pic}^0(Z_L) \\ \phi &\mapsto [D(\phi)_l(v)] \end{aligned}$$

where $D(\phi)_l(v)$ is defined to by the constant cocycle $D(\phi)_l(v) = \phi(l)$.

Lemma 3.3.8. *$C \circ D = 1$.*

Proof. Let $\phi \in \mathrm{Hom}(L, U(1))$. Then $D(\phi)_l(v) = \phi(l) = e^{2\pi i p(l,v)}$ where p is constant in v. By the proof of Lemma 3.3.5,

$$C \circ D(\phi)(l) = h(v + l)h(v)^{-1}\phi(l),$$

where $h(v) = e^{2\pi i p(0,v)}$. But since p is independent of v, $h(v + l)h(v)^{-1} = 1$. Hence $C \circ D(\phi) = \phi$. □

Lemma 3.3.9. *$D \circ C = 1$.*

Proof. Let $A_l(v) = e^{2\pi i a(l,v)} \in \mathrm{Pic}^0(Z_L)$. Then by the proof of Proposition 3.3.5

$$C(A)(l) = e^{2\pi i[a(l,v)+a(0,v+l)-a(0,v)]}$$

some some $v \in \mathbb{C}$. Hence

$$D \circ C(F) = \left[e^{2\pi i[a(l,v)+a(0,v+l)-a(0,v)]}\right] = \left[e^{2\pi i a(l,v)}\right] = [A_l(v)].$$

$\square$

Proposition 3.3.10. *Let Z_L be a Quantum Torus corresponding to a pseudolattice L. Then $Pic^0(Z_L) \cong Hom(L, U(1))$.*

Proof. Lemmas 3.3.8 and 3.3.9 assert that the homomorphism $C : \mathrm{Pic}^0(Z_L) \to \mathrm{Hom}(L, U(1))$ is a bijection. $\square$

Corollary 3.3.11. *We have a split short exact sequence*

$$0 \longrightarrow Hom(L, U(1)) \longrightarrow H^1(L, \mathcal{H}^*) \longrightarrow Alt^2(L, \mathbb{Z}) \longrightarrow 0. \tag{3.16}$$

Proof. The exactness follows from Propositions 3.3.3 and 3.3.10. That the sequence is split follows from the existence and properties of the map $\sigma : \mathrm{Alt}^2(L, \mathbb{Z}) \to H^1(L, \mathcal{H}^*)$ in the statement of Proposition 3.3.3. $\square$

Theorem 3.3.12. *With the notation of Corollary 3.3.11 and its proof we have*

$$H^1(L, \mathcal{H}^*) \simeq Hom(L, U(1)) \oplus Alt^2(L, \mathbb{Z}).$$

Specifically, every element of $H^1(L, \mathcal{H}^*)$ has a unique representative in $Z^1(L, \mathcal{H}^*)$ of the form

$$\mu(l)\hat{\sigma}(\eta)_l(v)$$

for some $\mu \in Hom(L, U(1))$ and $\eta \in Alt^2(L, \mathbb{Z})$, where $\hat{\sigma}$ is as defined in (3.11).

Proof. This follows from Corollary 3.3.11 and the theory of split exact sequences. See Chapter III §3 Proposition 3.2 of [27]. $\square$

3.3.4 The Appel-Humbert Theorem for Quantum Tori

Theorem 3.3.12 serves to give a description of $H^1(L,\mathcal{H}^*)$ in terms of the groups $\mathrm{Hom}(L,U(1))$ and $\mathrm{Alt}^2(L,\mathbb{Z})$. The Appel-Humbert Theorem for Complex Tori classifies isomorphism classes of line bundles with respect to different data, characterising them in terms of a hermitian form and a type of character. The aim of this section is to prove a similar result for line bundles over Quantum Tori.

Definition 3.3.13. Given a pseudolattice L, let $P(L)$ denote the set of pairs (E,χ) such that

- $E \in \mathrm{Alt}^2(L,\mathbb{Z})$;
- $\chi : L \to U(1)$ such that for $l_1, l_2 \in L$ we have

$$\chi(l_1 + l_2) = \chi(l_1)\chi(l_2)e^{\pi i E(l_1,l_2)}.$$

We say that χ is a *semi-character* for E.

$P(L)$ becomes a group with the law of composition

$$(E_1,\chi_1)(E_2,\chi_2) = (E_1 + E_2, \chi_1\chi_2).$$

Note that $P(L)$ forms part of a short exact sequence

$$0 \longrightarrow \mathrm{Hom}(L,U(1)) \xrightarrow{\alpha} P(L) \xrightarrow{\beta} \mathrm{Alt}^2(L,\mathbb{Z}) \longrightarrow 0$$

where $\alpha(\mu) = (0,\mu)$ and $\beta(E,\chi) = E$.

Proposition 3.3.14. *There exists a homomorphism*

$$\phi : H^1(L,\mathcal{H}^*) \to P(L).$$

Proof. By Theorem 3.3.12 we can represent each line bundle $\mathcal{L}$ uniquely as a representative of the form

$$\mu(l)\hat{\sigma}(\eta)_l(v)$$

for some $\mu \in \mathrm{Hom}(L, U(1))$ and $\eta \in \mathrm{Alt}^2(L, \mathbb{Z})$. Given a line bundle $\mathcal{L}$ represented by such a cocycle we know that if $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ then

$$Ch(\mathcal{L})(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2) = s_\eta(ad - bc)$$

where $s_\eta \in \mathbb{Z}$ is as defined in §3.3.2. Define

$$\begin{array}{rcl} \phi : H^1(L, \mathcal{H}^*) & \to & P(L) \\ \mathcal{L} = [\mu(l)\hat{\sigma}(\eta)_l(v)] & \mapsto & (Ch(\mathcal{L}), \mu\chi_\eta) \end{array}$$

where $\chi_\eta(a\omega_1 + b\omega_2) := e^{\pi i s_\eta ab}$. This is well defined since if $l_1 = a\omega_1 + b\omega_2$ and $l_2 = c\omega_1 + d\omega_2$ then

$$\begin{array}{rcl} \chi_\eta(l_1 + l_2) & = & e^{\pi i s_\eta (a+c)(b+d)} \\ & = & e^{\pi i s_\eta (ab+cd+ad+bc)} \\ & = & \chi_\eta(l_1)\chi_\eta(l_2)e^{\pi i s_\eta (ad-bd)} \\ & = & \chi_\eta(l_1)\chi_\eta(l_2)e^{\pi i Ch(\mathcal{L})(l_1,l_2)}. \end{array}$$

The penultimate line follows since $bc \in \mathbb{Z}$, and so $e^{\pi i bc} = e^{-\pi i bc}$.

The property that ϕ is a homomorphism follows immediately from the observation that if $\mathcal{L}$ and $\mathcal{M}$ are line bundles whose isomorphism classes are represented by the cocycles $\mu_{\mathcal{L}}(l)\hat{\sigma}(\eta_{\mathcal{L}})_l(v)$ and $\mu_{\mathcal{M}}(l)\hat{\sigma}(\eta_{\mathcal{M}})_l(v)$ respectively, then since $\hat{\sigma}$ is a homomorphism the class $[\mathcal{L}] \otimes [\mathcal{M}]$ is represented by the cocycle

$$\mu_{\mathcal{L}}(l)\hat{\sigma}(\eta_{\mathcal{L}})_l(v)\mu_{\mathcal{M}}(l)\hat{\sigma}(\eta_{\mathcal{M}})_l(v) = (\mu_{\mathcal{L}}\mu_{\mathcal{M}})(l)\hat{\sigma}(\eta_{\mathcal{L}} + \eta_{\mathcal{M}})_l(v).$$

Hence

$$\begin{aligned}\phi([\mathcal{L}]\otimes[\mathcal{M}]) &= (\mu_{\mathcal{L}}\mu_{\mathcal{M}}\chi_{\eta_{\mathcal{L}}}\chi_{\eta_{\mathcal{M}}}, \eta_{\mathcal{L}}+\eta_{\mathcal{M}}) \\ &= (\mu_{\mathcal{L}}\chi_{\eta_{\mathcal{L}}}, \eta_{\mathcal{L}})(\mu_{\mathcal{M}}\chi_{\eta_{\mathcal{M}}}, \eta_{\mathcal{L}}) \\ &= \phi([\mathcal{L}])\phi([\mathcal{M}]).\end{aligned}$$

□

Hence we have the following diagram,which at present we do not know is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(L,U(1)) \arrow[r] \arrow[d, equal] & H^1(L,\mathcal{H}^*) \arrow[r, "Ch"] \arrow[d, "\phi"] & \mathrm{Alt}^2(L,\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(L,U(1)) \arrow[r, "\alpha"] & P(La) \arrow[r, "\beta"] & \mathrm{Alt}^2(L,\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

If we can prove this is a commutative diagram we can apply the snake lemma to show that ϕ is an isomorphism.

Lemma 3.3.15. *We have a commutative triangle:*

$$\begin{tikzcd} Hom(L,U(1)) \arrow[r] \arrow[dr, "\alpha"] & H^1(L,\mathcal{H}^*) \arrow[d, "\phi"] \\ & P(L) \end{tikzcd}$$

Proof. Let $\mu \in \mathrm{Hom}(L,U(1))$. The image $A_l(v)$ of μ in $H^1(L,\mathcal{H}^*)$ is independent of v and has $Ch(A_l(v)) = 0$. Hence $\sigma \circ Ch(A_l(v)) = [\hat{\sigma}(\eta_0)] = [1]$, where η_0 denotes the element of $\mathrm{Alt}^2(L,\mathbb{Z})$ which maps every element to 0. Hence $\phi(\mu) = (0,\mu) = \alpha(\mu)$.

□

Lemma 3.3.16. *We have a commutative triangle*

$$\begin{tikzcd} Hom(L,U(1)) \arrow[r, "Ch"] \arrow[d, "\phi"] & Alt^2(L,\mathbb{Z}) \\ P(L) \arrow[ur, "\beta"] & \end{tikzcd}$$

Proof. By Theorem 3.3.12, we may represent the isomorphism class of a line bundle $\mathcal{L}$ uniquely by a cocycle of the form

$$\mu(l)\hat{\sigma}(\eta)_l(v).$$

Then $\phi([\mathcal{L}]) = (Ch(\mathcal{L}), \mu\chi_\eta)$, and hence $\beta \circ \phi(\mathcal{L}) = Ch(\mathcal{L})$. □

Corollary 3.3.17. *We have a commutative diagram with exact rows:*

$$\begin{tikzcd} 0 \arrow[r] & Hom(L,U(1)) \arrow[r, "\alpha"] \arrow[d, equal] & H^1(L,\mathcal{H}^*)) \arrow[r, "\beta"] \arrow[d, "\phi"] & Alt^2(L,\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Hom(L,U(1)) \arrow[r] & P(L) \arrow[r, "Ch"] & Alt^2(L,\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

Theorem 3.3.18 (Appel-Humbert Theorem for Quantum Tori). *Let L be a pseudolattice. Then $\phi : H^1(L,\mathcal{H}^*) \to P(L)$ is an isomorphism.*

Proof. This follows from the application of the snake lemma [27] to the commutative diagram in Corollary 3.3.17 □

3.4 Geometric Line Bundles and the Heisenberg Group

So far in this chapter we have viewed line bundles $\mathcal{L}$ over a Quantum Torus Z_L as elements of a certain group of cocycles. We have shown that this yields a nontrivial notion of line bundles where the classical one of Definition 3.2.1 fails due to the non-Hausdorff nature of Quantum Tori. Whereas this is a perfectly satisfactory definition, it is difficult to reconcile this abstract definition with the idea that line bundles are *topological* objects. The subject of this section is to show that we can view $\mathcal{L}$ as a topological space. We see that this leads to the study of objects which we may not have considered had we thought of line bundles solely as cocycles.

3.4.1 Line Bundles and Pull Backs

Before we consider the problem of defining a line bundle over a Quantum Torus as a topological space, we note a few facts concerning line bundles over Hausdorff spaces.

Given a morphism $f : X \to Y$ of topological spaces, and a line bundle $\pi_{\mathcal{L}} : \mathcal{L} \to Y$ we can define the pullback of $\mathcal{L}$ by f to obtain a line bundle $f^*(\mathcal{L})$ over X. In the context of category theory the object $f^*(\mathcal{L})$ is the pullback of the following diagram

$$\begin{tikzcd} & \mathcal{L} \arrow[d, "\pi_{\mathcal{L}}"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

However, an explicit definition for $f^*(\mathcal{L})$ is given by [21]:

$$f^*(\mathcal{L}) := \{(x, l) \in X \times \mathcal{L} : f(x) = \pi_{\mathcal{L}}(l)\} \tag{3.17}$$

The projection $\pi_{f^*(\mathcal{L})} : f^*(\mathcal{L}) \to X$ is given by $\pi_{f^*(\mathcal{L})}(x, l) = x$.

The pullback has the following universal property:

> Let $\varphi : \mathcal{M} \to \mathcal{L}$ be any morphism of line bundles over $f : X \to Y$. Then there exists a unique morphism of line bundles $\tilde{\varphi} : \mathcal{M} \to f^*(\mathcal{L})$.

The morphism $\tilde{\varphi}$ can be computed explicitly when we use the description of $f^*(\mathcal{L})$

in (3.17) as

$$\tilde{\varphi}(m) = (\pi_{\mathcal{M}}(m), \varphi(m)).$$

Recall that given a Quantum Torus Z_L we have a natural projection

$$\begin{aligned} \pi : \mathbb{R} &\to Z_L \\ x &\mapsto x + L. \end{aligned}$$

The underlying philosophy in our approach is to define a topological space $\mathcal{L}$ together with a projection $\pi_{\mathcal{L}}$ such that the pullback of the following diagram gives line bundle (which is necessarily trivial) on $\mathbb{R}$:

$$\begin{tikzcd} & \mathcal{L} \arrow[d, "\pi_{\mathcal{L}}"] \\ \pi : \mathbb{R} \arrow[r] & Z_L. \end{tikzcd}$$

It is not immediately clear how we should do this. In the next section we look for another characterisation of this property in terms of the *Heisenberg Group*.

3.4.2 The Heisenberg Group

Suppose X is a topological space endowed with a group law $+$. Given $x \in X$ we have a natural "translation by x" map

$$\begin{aligned} T_x : X &\to X \\ y &\mapsto y + x \end{aligned}$$

Definition 3.4.1. Let X be a topological group, and $\pi_{\mathcal{L}} : \mathcal{L} \to X$ a line bundle over X. Define

$$K(\mathcal{L}) := \{x \in X : T_x^*(\mathcal{L}) \cong \mathcal{L}\}.$$

The group $K(\mathcal{L})$ is fundamental in defining the Heisenberg Group associated to a line bundle $\mathcal{L}$ over a topological space X.

Definition 3.4.2 (Heisenberg Group). Let $\pi_{\mathcal{L}} : \mathcal{L} \to X$ be a line bundle over a topological group X. The Heisenberg Group $H(\mathcal{L})$ of $\mathcal{L}$, is defined to be the set of pairs (x, ϕ) such that

- $x \in K(\mathcal{L})$;
- $\phi : \mathcal{L} \to T_x^*(\mathcal{L})$ is an isomorphism.

The group law on $H(\mathcal{L})$ is given by

$$(x_1, \phi_1).(x_2, \phi_2) = (x_1 + x_2, T_{x_1}^*(\phi_2) \circ \phi_1).$$

Given a line bundle $\mathcal{L}$ over X, an alternative representation of the group $H(\mathcal{L})$ is given by:

Proposition 3.4.3. *As a set $H(\mathcal{L})$ is in bijection with the set of those automorphisms of $\mathcal{L}$ lying over T_x for some $x \in K(\mathcal{L})$. In this representation elements of $H(\mathcal{L})$ are given by pairs (x, f) such that $x \in K(\mathcal{L})$ and $f : \mathcal{L} \to \mathcal{L}$ is a bijection such that*

$$\pi_{\mathcal{L}}(f(l)) = \pi_{\mathcal{L}}(l) + x.$$

The law of composition is given by

$$(x_1, f_1).(x_2, f_2) = (x_1 + x_2, f_1 \circ f_2). \tag{3.18}$$

Proof. See Remark 6.1.2 of [5]. $\square$

The Heisenberg group can be viewed as part of a short exact sequence:

Proposition 3.4.4 (Proposition 6.1.1 of [5]). *There is an exact sequence*

$$1 \longrightarrow \mathbb{C}^* \xrightarrow{\iota} H(\mathcal{L}) \xrightarrow{p} K(\mathcal{L}) \longrightarrow 0.$$

The image of $\mathbb{C}^*$ *in* $H(\mathcal{L})$ *lies in the centre of* $H(\mathcal{L})$*.*

Proposition 3.4.4 establishes that $H(\mathcal{L})$ is a *central extension* of $K(\mathcal{L})$. According to the theory of such extensions we have an alternating pairing $K(\mathcal{L})$:

Definition 3.4.5. Define a map $e^{\mathcal{L}} : K(\mathcal{L}) \times K(\mathcal{L}) \to \mathbb{C}^*$ by

$$(x_1, x_2) \mapsto \iota^{-1}(g_1 g_2 g_1^{-1} g_2^{-1}),$$

where $g_i = (x_i, \phi_i)$ for some isomorphism ϕ_i such that $g_i \in H(\mathcal{L})$.

3.4.3 Topological Line Bundles over Quantum Tori

In this section we shall see how for Hausdorff spaces, the Heisenberg group can be used to characterise those line bundles over a space X which arise as pullbacks of line bundles over a space Y. We will use this idea to define line bundles over Quantum Tori as topological spaces.

Our starting point is the following result:

Proposition 3.4.6 (Theorem 8.10 of [35]). *Let* $f : X \to Y$ *be a map of abelian varieties spaces, and let* $\mathcal{L}$ *be a line bundle over* X*. Then there is a bijective correspondence between those line bundles* $\mathcal{M}$ *over* Y *such that* $f^*(\mathcal{M}) \cong \mathcal{L}$*, and those homomorphisms* $\ker(f) \to H(\mathcal{L})$ *lifting* $\ker(f) \hookrightarrow X$*.*

This bijection is realised in the following way. Write $H = \ker(f)$. Then it is easily shown that given an action of H on $\mathcal{L}$ compatible with the natural action of H on X by translation, the quotient $\mathcal{L}/H$ determines a line bundle $\mathcal{M}$ over Y with the required properties. Conversely such an $\mathcal{M}$ defines such an action.

The Quantum Torus is *not* an abelian variety, but we apply the philosophy supplied by Proposition 3.4.6. Our natural response is to consider the map

$$\pi : \mathbb{R} \to Z_L$$

in this context, which would lead to a study of homomorphisms $L \to H(\mathcal{L})$ for line bundles $\mathcal{L}$ over $\mathbb{R}$. However, in light of our previous definition of line bundles using cocycles we will modify this slightly.

The motivation for the approach to line bundles in terms of cocycles came from the theory of line bundles over Complex Tori outlined in §3.2.1. Examining these objects led us to define line bundles in terms of cohomology. However, the cohomological description of line bundles over a complex torus X_Λ characterises a specific class of line bundles. The class of bundles characterised by the group $Z^1(\Lambda, \mathcal{H}^*)$ are said to be *holomorphic*, due to the existence of holomorphic sections of the line bundle.

One approach in defining *holomorphic* line bundles on Quantum Tori as topological spaces, would be to say that their pullback to $\mathbb{R}$ should be isomorphic to a holomorphic line bundle over $\mathbb{R}$. However, given that $\mathbb{R}$ has no complex structure this idea seems nonsensical. However, we avoid this problem by making the following definition:

Definition 3.4.7. A holomorphic line bundle on $\mathbb{R}$ is a line bundle $\pi_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}$ such that there exists a holomorphic line bundle $\mathcal{M}$ on $\mathbb{C}$ such

$$\mathcal{L} = \{m \in \mathcal{M} : \pi_{\mathcal{M}}(m) \in \mathbb{R}\}.$$

Morphisms between holomorphic line bundles on $\mathbb{R}$ are the restrictions of morphisms between such line bundles on $\mathbb{C}$.

By this definition, holomorphic line bundles over $\mathbb{R}$ are essentially the same as holomorphic line bundles over $\mathbb{C}$. However, all such line bundles over $\mathbb{C}$ are trivial - they are isomorphic to $\mathcal{T} := \mathbb{C} \times \mathbb{C}$. The projection $\pi_{\mathcal{T}}$ is the projection on to the first factor.

Lemma 3.4.8. *As a set, $H(\mathcal{T})$ is in bijection with $\mathcal{H}^*$.*

Proof. Since every line bundle over $\mathbb{C}$ is isomorphic to $\mathcal{T}$, $K(\mathcal{T}) = \mathbb{C}$. Using the description of $H(\mathcal{L})$ given by Proposition 3.4.3, if $x \in K(\mathcal{T})$ we would like to establish those automorphisms $\phi : \mathcal{T} \to \mathcal{T}$ which lie over T_x. We represent an element of $\mathcal{T}$ by $(v, z) \in \mathbb{C} \times \mathbb{C}$. Suppose ϕ is such an isomorphism, and that

$$\phi(v, z) = (\phi_1(v, z), \phi_2(v, z))$$

for some functions $\phi_1, \phi_2 : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. The condition that ϕ lies over T_x implies that

$$T_x \circ \pi_{\mathcal{T}}(z, v) = \pi_{\mathcal{T}} \circ \phi(v, z).$$

Hence $T_x(v) = \phi_1(v, z)$. The condition that ϕ is an isomorphism of line bundles implies that $\phi_2(v, z) = A(v)z$ for some $A(v) \in \mathbb{C}^*$. The condition that $\mathcal{T}$ is a holomorphic line bundle implies that the association $v \mapsto A(v)$ is a holomorphic function. $\square$

Definition 3.4.9 (Topological Line Bundle over a Quantum Torus). Let Z_L be a Quantum Torus. A Topological Line Bundle over Z_L is a homomorphism

$q : L \to \mathcal{H}(\mathcal{T})$ such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r] \arrow[dr] & \mathcal{H}(\mathcal{T}) \arrow[d, "p"] \\ & \mathbb{C} = K(\mathcal{T}) \end{tikzcd} \tag{3.19}$$

where p is as in Proposition 3.4.4. By Lemma 3.4.8 every such morphism is of the form $l \mapsto (l, A_l(v))$ for some $A_l(v) \in \mathcal{H}^*$. This defines an action of L on $\mathcal{T}$ by

$$l(v, z) := (v + l, A_l(v)z).$$

As a topological space, we define $\mathcal{L}$ to be the quotient of $\mathcal{T}$ by this action and write

$$\mathcal{L} = \mathcal{T}/_q L.$$

Using this description of $H(\mathcal{T})$ we see that the previous notion of line bundles over Quantum Tori in terms of cohomology, and the topological one of Definition 3.4.9 are the same:

Proposition 3.4.10. *Let Z_L be a Quantum Torus. We have a bijection between line bundles over Z_L and topological line bundles over Z_L.*

Proof. Let $\mathcal{L}$ be a topological line bundle. Then we have a homomorphism $q : L \to H(\mathcal{T})$ such that $\mathcal{L} = \mathcal{T}/_q L$. We have $q(l) = (x_l, A_l(v))$ for some $x_l \in K(\mathcal{T})$, $A_l(v) \in \mathcal{H}^*$. That (3.19) commutes implies that $l = x_l$, and the condition that q is a homomorphism implies that

$$\begin{aligned} A_{l_1+l_2}(v) &= T^*_{l_2}(A_{l_1}(v))A_{l_2}(v) \\ &= A_{l_1}(v + l_2)A_{l_2}(v). \end{aligned}$$

This shows that $A_l \in Z^1(L, \mathcal{H}^*)$, and represents a line bundle over Z_L.

Conversely, if $A_l(v) \in Z^1(L, \mathcal{H}^*)$ then define a homomorphism

$$\begin{aligned} q : L &\rightarrow H(\mathcal{T}) \\ l &\mapsto (l, A_l(v)), \end{aligned}$$

where the isomorphism $A_l(v)$ is multiplication by $A_l(v) \in \mathbb{C}^*$ on each fibre over $v \in \mathbb{C}$. This represents a topological line bundle over Z_L. $\square$

Notation: Given a line bundle $\mathcal{L} = \mathcal{T}/_q L$ we have a natural projection $\pi_{\mathcal{L}} : \mathcal{L} \rightarrow Z_L$. If $[v, z]$ represents the equivalence class of $(v, z) \in \mathcal{T} = \mathbb{C} \times \mathbb{C}$ then this is given by

$$\begin{aligned} \pi_{\mathcal{L}} : \mathcal{L} &\rightarrow Z_L \\ [v, z] &\mapsto v + L. \end{aligned}$$

We defined the notion of isomorphism between line bundles abstractly in terms of their image in the cohomology group $H^1(L, \mathcal{H}^*)$. Removing the local conditions from the classical notion of morphism between line bundles we can attempt to define what an isomorphism between topological line bundles is:

Definition 3.4.11 (Isomorphisms of Topological Line Bundles). Let $\mathcal{L}_1 = \mathcal{T}/_{q_1} L$ and $\mathcal{L}_2 = \mathcal{T}/_{q_2} L$ be line bundles over Z_L. An isomorphism h between $\mathcal{L}_1$ and $\mathcal{L}_2$ is a map $h : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ which is linear on each fibre and such that the following diagram commutes

$$\begin{tikzcd} \mathcal{L}_1 \arrow[r, "h"] \arrow[dr, "\pi_{\mathcal{L}_1}"'] & \mathcal{L}_2 \arrow[d, "\pi_{\mathcal{L}_2}"] \\ & Z_L, \end{tikzcd}$$

and pulls back to an isomorphism of holomorphic line bundles over $\mathbb{C}$.

Proposition 3.4.12. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Line Bundles over Z_L which admit an isomorphism between them, corresponding to factors of automorphy A^1_l and A^2_l. Then*

A_l^1 and A_l^2 represent the same class in $H^1(L, \mathcal{H}^*)$.

Proof. Recall that A_l^i gives an action of L on $\mathbb{C} \times \mathbb{C}$, and that $\mathcal{L}_i$ is the quotient of $\mathbb{C} \times \mathbb{C}$ by this action. Write $[v, z]_{A_i}$ for the class of (v, z) in $\mathcal{L}_i$.

Suppose ϕ is an isomorphism. Then write $\phi([v, z]_{A^1}) = [\phi_1(v, z), \phi_2(v, z)]_{A^2}$. The commutativity condition

$$\pi_{\mathcal{L}_1}(\lambda) = \pi_{\mathcal{L}_2} \circ \phi(\lambda) \ \text{ for all } \lambda \in \mathcal{L}_1$$

implies that we have $\pi_{\mathcal{L}_1}(x, v) = x + L$. The condition that ϕ is a linear isomorphism on fibres implies that $\phi_2(v, z) = \Phi(v)z$ for some function Φ, and condition the isomorphism pulls back to one of holomorphic line bundles over $\mathbb{C}$ implies that $\Phi \in \mathcal{H}^*$. Let $[v, z] \in \mathcal{L}_1$, and pick $l \in L$. Then for ϕ to be well defined we require that

$$\phi([v, z]_{A^1}) = \phi([v + l, A_l^1(v)z]_{A^1}).$$

We have

$$\phi([v, z]_{A^1}) = [v, \Phi(v)z]_{A^2} = [v + l, A_l^2(v)\Phi(v)z]_{A^2},$$

and

$$\phi([v + l, A_l^1(v)z]_{A^1}) = [v + l, \Phi(v + l)A_l^1(v)z]_{A^2}.$$

Therefore

$$A_l^2(v) = \frac{\Phi(v + l)}{\Phi(v)} A_l^1(v).$$

□

3.4.4 The Pull back of 1-morphisms

Given a morphism $f : X \to Y$ of topological spaces, we can define the pullback of a line bundle over Y to obtain one over X. We used this as the motivation for our definition of topological line bundles over Quantum Tori in the case where X was equal to $\mathbb{R}$. In Proposition 2.5.5 of §2.5 in Chapter 2 we classified those continuous maps between Quantum Tori. For a certain class of such morphisms we can define the pullback of a line bundle over a Quantum Torus:

Definition 3.4.13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous 1-morphism between Quantum Tori Z_{L_1} and Z_{L_2}. Let $\mathcal{L}$ be a line bundle over Z_{L_2} corresponding to a factor of automorphy $A_l(v) \in Z_1(L, \mathcal{H}^*)$. Then the pullback $f^*(\mathcal{L})$ of $\mathcal{L}$ with respect to f is defined to be the line bundle corresponding to the cocycle

$$f^*(A)_l(v) := A_l(f(v)).$$

Note that the condition that f is a 1-morphism, and that f is continuous implies that f is necessarily translation by an element in $\mathbb{R}$. Conversely, given any $x \in Z_L$, the translation map $T_x : Z_L \to Z_L$ is a 1-morphism. Hence if f represents a 1-morphism between two Quantum Tori Z_{L_1} and Z_{L_2} then we have $L_1 = L_2$, and if $\mathcal{L}$ is a line bundle over Z_L then $f^*(\mathcal{L}) = T_x^*(\mathcal{L})$ for some $x \in Z_L$.

3.5 The relationship between $K(\mathcal{L}), e^{\mathcal{L}}$ and $Ch(\mathcal{L})$

This chapter has so far been devoted to defining line bundles over Quantum Tori from two different perspectives. The approach first taken was to draw on results concerning line bundles over Complex Tori to derive a nontrivial definition for line bundles over Quantum Tori. Using this definition we were able to prove a structure theorem for isomorphism classes of line bundles over Quantum Tori using the Chern class. In §3.4 we gave a topological definition of a line bundle over a Quantum Torus using the Heisenberg Group. This gave rise to the definitions of the group $K(\mathcal{L})$ and an alternating pairing $e^{\mathcal{L}}$.

The purpose of this section is to reconcile these ideas. Theorem 3.5.7 is the main result of this section, describing the relationship between the objects $Ch(\mathcal{L})$, $K(\mathcal{L})$ and $e^{\mathcal{L}}$ associated to a line bundle $\mathcal{L}$ over a Quantum Torus.

3.5.1 An alternating pairing on $\Lambda(\mathcal{L})$

In this section we go part of the way in describing the link between $Ch(\mathcal{L})$ and $e^{\mathcal{L}}$, describing in Proposition 3.5.6 the pairing $e^{\mathcal{L}}$ in terms of the cohomology theory of $\mathcal{L}$.

We begin by giving a description of $K(\mathcal{L})$ in the language of cocycles:

Lemma 3.5.1. *Let $\mathcal{L}$ be a line bundle over Z_L. Then $K(\mathcal{L})$ is isomorphic to the set with addition*

$$\left\{x \in Z_L : (\exists \tilde{x} \in \pi^{-1}(x)) \wedge \left(\frac{A_l(v+\tilde{x})}{A_l(v)} \in B^1(L, \mathcal{H}^*)\right)\right\}$$

Proof. Let $\mathcal{L}$ be a line bundle with factor of automorphy $A_l(v)$. By Proposition 3.4.12, $x \in K(\mathcal{L})$ if and only if there exists $\tilde{x} \in \pi^{-1}(x)$ such that

$$\frac{A_l(v+\tilde{x})}{A_l(v)} \in B^1(L, \mathcal{H}^*).$$

This set is independent of the choice for $\tilde{x}$. If $\tilde{x}'$ is another element of $\mathbb{C}$ such that $\pi(\tilde{x}') = x$ then $\tilde{x}' - \tilde{x} = l' \in L$, and we have

$$\frac{A_l(v+\tilde{x}')}{A_l(v)} = \frac{A_{\omega+l'}(v+\tilde{x})}{A_{l'}(l'+v+\tilde{x})A_l(v)} = \frac{A_{l'}(v+\tilde{x}+l)}{A_{l'}(v+\tilde{x})}\frac{A_l(v+\tilde{x})}{A_l(v)} \in B^1(L, \mathcal{H}^*).$$

$\square$

This motivates the following definition:

Definition 3.5.2. Let $\mathcal{L}$ be a line bundle over a Quantum Torus Z_L. Then define $\Lambda(\mathcal{L}) = \pi^{-1}(K(\mathcal{L}))$. If $\mathcal{L}$ is represented by $A_l(v) \in Z^1(L, \mathcal{H}^*)$ then by Lemma 3.5.1 we have

$$\Lambda(\mathcal{L}) = \left\{ \tilde{x} \in \mathbb{C} : \frac{A_l(v+\tilde{x})}{A_l(v)} \in B^1(L, \mathcal{H}^*) \right\}.$$

So given $\tilde{x} \in \Lambda(\mathcal{L})$, we obtain a coboundary representing an isomorphism $\mathcal{L} \cong T_{\pi(\tilde{x})}(\mathcal{L})$. In order to show that this is unique we will need the following lemma:

Lemma 3.5.3. *Suppose g and h are holomorphic functions such that*

$$\frac{h(v+l)}{h(v)} = \frac{g(v+l)}{g(v)}$$

for all $v \in \mathbb{C}, l \in L$. Then $h(v)g(v)^{-1}$ is constant in v.

Proof. If the above relation holds we have

$$\frac{h(v+l)}{g(v+l)} = \frac{h(v)}{g(v)}.$$

The right hand side is independent of l so the left hand side is. Since L is dense in $\mathbb{R}$, the function $f(v) := g(v)h(v)^{-1}$ is therefore constant on the real axis, and since it is holomorphic therefore constant on $\mathbb{C}$. □

Corollary 3.5.4. *If $\tilde{x} \in \Lambda(\mathcal{L})$ there exists a unique nonvanishing holomorphic function $h_{\tilde{x}}$ such that $h_{\tilde{x}}(0) = 1$ and*

$$\frac{A_l(v+\tilde{x})}{A_l(v)} = \frac{h_{\tilde{x}}(v+l)}{h_{\tilde{x}}(v)}.$$

Definition 3.5.5. Fix $v \in \mathbb{C}$ and a line bundle $\mathcal{L}$ over a Quantum Torus Z_L. Define a pairing on $\Lambda(\mathcal{L})$ by

$$\begin{array}{rcl} H_v(\ ,\ ) : \Lambda(\mathcal{L}) \times \Lambda(\mathcal{L}) & \to & \mathbb{C}^* \\ (\tilde{x}_1, \tilde{x}_2) & \mapsto & \frac{h_{\tilde{x}_2}(v+\tilde{x}_1)}{h_{\tilde{x}_2}(v)} \end{array}$$

By Corollary 3.5.4 each element $\tilde{x}$ of $\Lambda(\mathcal{L})$ defines a unique coboundary $h_{\tilde{x}}$ such that $h_{\tilde{x}}(0) = 1$ exhibiting an isomorphism $\mathcal{L} \cong T^*_{\pi(\tilde{x})}(\mathcal{L})$.

Let

$$\hat{H}(\mathcal{L}) = \{(\tilde{x}, h_{\tilde{x}}(v)) : \tilde{x} \in \Lambda(\mathcal{L})\}.$$

Define an action of L on $\hat{H}(\mathcal{L})$ by

$$l(\tilde{x}, h_{\tilde{x}}(v)) = \left(\tilde{x} + l, \frac{A_l(v+\tilde{x})}{A_l(\tilde{x})} h_{\tilde{x}}(v)\right).$$

It is easily seen that this construction yields an isomorphism

$$H(\mathcal{L}) \cong \hat{H}(\mathcal{L})/L. \tag{3.20}$$

The group law in this representation is given by

$$(\tilde{x}_1, h_{\tilde{x}_1}(v)).(\tilde{x}_2, h_{\tilde{x}_2}(v)) = (\tilde{x}_1 + \tilde{x}_2, h_{\tilde{x}_2}(v + \tilde{x}_1) h_{\tilde{x}_1}(v)).$$

Using this description of $H(\mathcal{L})$ we can describe the relationship between $e^{\mathcal{L}}$ and the pairing on $\Lambda(\mathcal{L})$ of Definition 3.5.5:

Proposition 3.5.6. *For $\tilde{x}_1, \tilde{x}_2 \in \Lambda(\mathcal{L})$, and any $v \in \mathbb{C}$ we have*

$$e^{\mathcal{L}}(x_1, x_2) = H_v(\tilde{x}_1, \tilde{x}_2) H_v(\tilde{x}_2, \tilde{x}_1)^{-1}$$

where $x_i = \pi(\tilde{x}_i)$ for $i = 1, 2$.

Proof. Firstly note that by the definition of the group law we have

$$(\tilde{x}, h_{\tilde{x}}(v))^{-1} = (-\tilde{x}, h_{\tilde{x}}(v - \tilde{x})^{-1}).$$

Then if $g_i = (\tilde{x}_i, h_{\tilde{x}_i}(v))$ for $i = 1, 2$

$$\begin{aligned}
[g_1, g_2] &= g_1 g_2 g_1^{-1} g_2^{-1} \\
&= (\tilde{x}_1, h_{\tilde{x}_1}(v))(\tilde{x}_2, h_{\tilde{x}_2}(v))(-\tilde{x}_1, h_{\tilde{x}_1}(v - \tilde{x}_1)^{-1})(-\tilde{x}_2, h_{\tilde{x}_2}(v - \tilde{x}_2)^{-1}) \\
&= (\tilde{x}_1 + \tilde{x}_2, h_{\tilde{x}_1}(v) h_{\tilde{x}_2}(v + \tilde{x}_1))(-\tilde{x}_1 - \tilde{x}_2, h_{\tilde{x}_1}(v - \tilde{x}_1)^{-1} h_{\tilde{x}_2}(v - \tilde{x}_1 - \tilde{x}_2)^{-1}) \\
&= (0, h_{\tilde{x}_1}(v) h_{\tilde{x}_1}(v + \tilde{x}_2)^{-1} h_{\tilde{x}_2}(v + \tilde{x}_1) h_{\tilde{x}_2}(v)^{-1}) \\
&= (0, H_v(\tilde{x}_1, \tilde{x}_2) H_v(\tilde{x}_2, \tilde{x}_1)^{-1}).
\end{aligned}$$

Hence $e^{\mathcal{L}}(x_1, x_2) := \iota^{-1}([g_1, g_2]) = H_v(\tilde{x}_1, \tilde{x}_2) H_v(\tilde{x}_2, \tilde{x}_1)^{-1}$. Note that since the left hand side is independent of v, the right hand side is. □

3.5.2 The relationship between $K(\mathcal{L})$ and $Ch(\mathcal{L})$

In the theory of Complex Tori, given a line bundle $\mathcal{M}$ over a torus X_Λ it is shown that a certain group $K(\mathcal{M})$ (analogous to the group we have defined for Quantum Tori) is either finite, or the whole of $\mathcal{M}$. The proof of this result relies on the fact that torus X_Λ can be viewed as a complete projective variety, and that the corresponding pairing $e^{\mathcal{M}}$ is a morphism of projective varieties.

The aim of this section is to prove the following:

Theorem 3.5.7. *Let $\mathcal{L}$ be a line bundle over a Quantum Torus Z_L. Then there are two possibilities:*

1. *If $Ch(\mathcal{L})$ is nontrivial, then $K(\mathcal{L})$ is finite. In this case we have*

$$Ch(\mathcal{L})(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2) = s_\eta(ad - bc)$$

for some $s_\eta \in \mathbb{Z}$. We have an isomorphism

$$K(\mathcal{L}) \cong (\mathbb{Z}/s_\eta\mathbb{Z}) \times (\mathbb{Z}/s_\eta\mathbb{Z}).$$

2. *The following statements are equivalent:*

   (a) $Ch(\mathcal{L}) = 0$;

   (b) $K(\mathcal{L}) = Z_L$;

   (c) $e^{\mathcal{L}} \equiv 1$.

Proof of part 1: Suppose that $Ch(\mathcal{L}) = \eta$ is nontrivial.

I first claim that it suffices to only consider those line bundles represented by the cocycles $\hat{\sigma}(\eta)_l(v)$ defined in (3.11). Clearly if $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic line bundles then $K(\mathcal{L}_1) = K(\mathcal{L}_2)$. By Theorem 3.3.12 it suffices to only consider those line bundles represented by cocycles of the form

$$\mu(l)\hat{\sigma}_l(v)$$

for some $\mu \in \mathrm{Hom}(L, U(1))$ and $\eta \in \mathrm{Alt}^2(L, \mathbb{Z})$. Now let $\mathcal{L}_\mu$ denote the line bundle represented by the above cocycle, and $\mathcal{L}_1$ the line bundle represented by the cocycle $\hat{\sigma}_l(v)$. Then $x = \pi(\tilde{x}) \in K(\mathcal{L}_\mu)$ if and only if

$$\frac{\mu(l)\hat{\sigma}_l(v + \tilde{x})}{\mu(l)\hat{\sigma}_l(v)} \in B^1(L, \mathcal{H}^*)$$

for some $\tilde{x} \in \mathbb{C}$ such that $\pi(\tilde{x}) = x$. But this occurs if and only if

$$\frac{\hat{\sigma}_l(v + \tilde{x})}{\hat{\sigma}_l(v)} \in B^1(L, \mathcal{H}^*),$$

which is precisely the condition that $x \in K(\mathcal{L}_1)$.

We now show that $K(\mathcal{L})$ is isomorphic to $(\mathbb{Z}/s_\eta\mathbb{Z}) \times (\mathbb{Z}/s_\eta\mathbb{Z})$. By the previous discussion we assume that $\mathcal{L}$ is represented by the cocyle $\hat{\sigma}_l(v)$. If $x = \pi(\tilde{x}) \in K(\mathcal{L})$ then there exists a unique $h_{\tilde{x}} \in \mathcal{H}^*$ such that $h_{\tilde{x}}(0) = 1$ and

$$\frac{\hat{\sigma}_l(v+\tilde{x})}{\hat{\sigma}_l(v)} = \frac{h_{\tilde{x}}(v+l)}{h_{\tilde{x}}(v)}.$$

However, explicit calculation shows that

$$\frac{\hat{\sigma}_l(v+\tilde{x})}{\hat{\sigma}_l(v)} = e^{2s_\eta \frac{\pi i}{\omega_1} b\tilde{x}} \tag{3.21}$$

where $l = a\omega_1 + b\omega_2 \in L$. This latter expression is independent of v.

There exists a holomorphic function $H_{\tilde{x}}$ such that $h_{\tilde{x}}(v) = e^{2\pi i H_{\tilde{x}}(v)}$. By (3.21), for all $l \in L$, $H_{\tilde{x}}(v+l) - H_{\tilde{x}}(v)$ is a holomorphic function independent of v, and by continuity is constant on lines of constant imaginary part. Differentiating once with respect to v we see that $H'_{\tilde{x}}(v+l) = H'_{\tilde{x}}(v)$ for all $l \in L$, $v \in \mathbb{C}$. Hence $H'_{\tilde{x}}$ is a holomorphic function which is constant on lines of constant imaginary part, and therefore constant everywhere. Hence there exist constants $k(\tilde{x})$ and $c(\tilde{x})$ such that

$$H_{\tilde{x}}(v) = \frac{k(\tilde{x})}{\omega_1} v + c(\tilde{x}).$$

Since we are only concerned with the quotient $h_{\tilde{x}}(v+l)h_{\tilde{x}}(v)^{-1}$ we assume without loss of generality that $c(\tilde{x}) = 0$, and hence

$$h_{\tilde{x}}(v) = e^{\frac{2\pi i}{\omega_1} k(\tilde{x}) v}.$$

Now we compute

$$\frac{h_{\tilde{x}}(v+l)}{h_{\tilde{x}}(v)} = e^{\frac{2\pi i}{\omega_1} k(\tilde{x})(a\omega_1 + b\omega_2)}.$$

Equating this last expression with that of (3.21) we obtain

$$e^{2\frac{\pi i}{\omega_1}s_\eta b\tilde{x}} = e^{2\frac{\pi i}{\omega_1}k(\tilde{x})(a\omega_1+b\omega_2)}.$$

Note that the right hand side is dependent on a, whereas the left hand side is not. Since this equality holds for all $a \in \mathbb{Z}$ we therefore have $k(\tilde{x}) \in \mathbb{Z}$. We deduce that

$$s_\eta b\tilde{x} \in \mathbb{Z}(a\omega_1 + b\omega_2) + \mathbb{Z}\omega_1.$$

This holds for all $a, b \in \mathbb{Z}$, and hence $s_\eta \tilde{x} \in L$.

Conversely, if $s_\eta \tilde{x} \in L$, then we have

$$\tilde{x} = \frac{\alpha}{s_\eta}\omega_1 + \frac{\beta}{s_\eta}\omega_2$$

for some $\alpha, \beta \in \mathbb{Z}$. Define

$$h_{\tilde{x}}(v) = e^{\frac{2\pi i}{\omega_1}\beta v}.$$

Then

$$\frac{\hat{\sigma}_l(v+\tilde{x})}{\hat{\sigma}_l(v)} = \frac{h_{\tilde{x}}(v+\tilde{x})}{h_{\tilde{x}}(v)}.$$

Hence $\Lambda(\mathcal{L}) \cong \frac{1}{s_\eta}L$, and the result follows.

The explicit formula for $Ch(\mathcal{L})$ follows from Proposition 3.5.6.

Proof of part 2: Now consider the case when $Ch(\mathcal{L})$ is trivial, and is represented by a cocycle $A_l(v)$.

$2a \Rightarrow 2b$: Suppose $Ch(\mathcal{L}) = 0$. By Proposition 3.3.5 $A_l(v)$ is cohomologous to a constant cocycle $K_l(v)$. By Lemma 3.5.1 we can use this representative of the cohomology class to determine $K(\mathcal{L})$:

$$K(\mathcal{L}) = \left\{ x \in Z_L : (\exists \tilde{x} \in \pi^{-1}(x)) \wedge \left(\frac{K_l(v+\tilde{x})}{K_l(v)} \in H^1(L, \mathcal{H}^*) \right) \right\}.$$

But since $K_l(v)$ is constant in v, for all $\tilde{x} \in \mathbb{C}$ we have

$$\frac{K_l(v+\tilde{x})}{K_l(v)} = 1,$$

and therefore $K(\mathcal{L}) = Z_L$.

$2b \Rightarrow 2c$: Suppose $K(\mathcal{L}) = Z_L$. By the first part of the theorem, if $\eta = Ch(\mathcal{L})$ is non zero then $\Lambda(\mathcal{L}) = \frac{1}{s_\eta} L$. Since $K(\mathcal{L}) = Z_L$ we have $\Lambda(\mathcal{L}) = \mathbb{R}$, so we must have $\eta = 0$. Hence $\mathcal{L}$ is represented by a cocycle of the form

$$A_l(v) = \mu(l) \frac{h(v+l)}{h(v)}$$

for some $\mu \in \mathrm{Hom}(L, U(1))$ and $h \in \mathcal{H}^*$. We see that for $\tilde{x} \in \mathbb{R}$

$$\frac{A_l(v+\tilde{x})}{A_l(v)} = \frac{h(v+\tilde{x}+l)}{h(v+l)} \Bigg/ \frac{h(v+\tilde{x})}{h(v)}.$$

From this it follows that

$$h_{\tilde{x}}(v) = \frac{h(v+\tilde{x})}{h(v)} \frac{h(0)}{h(\tilde{x})}.$$

Using the explicit formula for $H_v(\tilde{x}_1, \tilde{x}_2)$ in Definition 3.5.5 we have

$$H_v(\tilde{x}_1, \tilde{x}_2) = \frac{h(v+\tilde{x}_1+\tilde{x}_2)}{h(v+\tilde{x}_1)h(v+\tilde{x}_2)h(v)}.$$

Observe that this is symmetric in $\tilde{x}_1$, and $\tilde{x}_2$, and hence by Proposition 3.5.6 we find that $e^{\mathcal{L}}(\tilde{x}_1, \tilde{x}_2) = 1$.

$2c \Rightarrow 2a$: Suppose $e^{\mathcal{L}} \equiv 1$, and assume that $Ch(\mathcal{L}) \neq 0$. Then $s_\eta \neq 0$, and by the results of part 1 we have $Ch(\mathcal{L})(\omega_1/s_\eta, \omega_2/s_\eta) = 1/s_\eta$, and hence $e^{\mathcal{L}}(\omega_1/s_\eta, \omega_2/s_\eta) = e^{-2\pi i/s_\eta} \neq 1$. This is a contradiction, and hence we must have $Ch(\mathcal{L}) = 0$.

□

As an immediate corollary we have

Corollary 3.5.8. *Let $\mathcal{L}$ be a line bundle over Z_L with Chern class η. Extend $Ch(\mathcal{L})$ to a pairing on $\frac{1}{s_\eta}L$ by linearity, and let $\tilde{x}_1, \tilde{x}_2 \in \Lambda(\mathcal{L})$. Then*

$$e^{\mathcal{L}}(\tilde{x}_1, \tilde{x}_2) = e^{2\pi i Ch(\mathcal{L})(\tilde{x}_1, \tilde{x}_2)}.$$

Proof. By Proposition 3.5.6 we have

$$e^{\mathcal{L}}(\tilde{x}_1, \tilde{x}_2) = \frac{h_{\tilde{x}_2}(v + \tilde{x}_1)}{h_{\tilde{x}_2}(v)} \frac{h_{\tilde{x}_1}(v)}{h_{\tilde{x}_1}(v + \tilde{x}_2)}.$$

Let $s_\eta \tilde{x}_1 = a\omega_1 + b\omega_2$ and $s_\eta \tilde{x}_2 = c\omega_1 + d\omega_2$. Then

$$\begin{aligned} e^{\mathcal{L}}(\tilde{x}_1, \tilde{x}_2) &= e^{2\frac{\pi i}{s_\eta \omega_1} d(a\omega_1 + b\omega_2)} e^{-2\frac{\pi i}{s_\eta \omega_1} b(c\omega_1 + d\omega_2)} \\ &= e^{2\pi i \frac{ad - bc}{s_\eta}} \\ &= e^{2\pi i Ch(\mathcal{L})(\tilde{x}_1, \tilde{x}_2)}. \end{aligned}$$

□

3.6 Other approaches to Quantum Tori

So far in this thesis we have studied Quantum Tori as topological spaces with the potential to solve a specific problem in number theory. This idea was originally formulated by Manin in [32], who drew on successes in in the field of Noncommutative Geometry to provide a basis for his proposed theory of *Real Multiplication*. Manin's paper has sparked considerable research in to Noncommutative Tori, and it is the purpose of this section introduce two aspects of this research and indicate how they relate to our studies.

Our discussion of line bundles over Quantum Tori has not been the first proposal of a notion of vector bundles over such Noncommutative spaces. As we remarked in Chapter 1, a motivation for the philosophy behind Noncommutative Geometry was the classification of isomorphism classes of vector bundles over a compact Hausdorff space X as finitely generated projective right $C(X)$-modules. These can be viewed as elements of the K-group of the C*-algebra $C(X)$, a concept which one can associate to both commutative and noncommutative C*-algebras. Hence through Noncommutative Geometry, we define vector bundles over the Noncommutative Torus A_θ to be finitely generated projective right A_θ-modules.

This is the stance taken by Schwarz in [50]. For a Noncommutative Torus A_θ, he defines a finitely generated projective right A_θ-module $\mathcal{E}$ in terms of the Schwartz functions on $\mathbb{R}$. For each choice of $\tau \in \mathfrak{H}$, there is a choice of holomorphic structure on $\mathcal{E}$, and a unique holomorphic element of $\mathcal{E}$ corresponding to this structure. This unique element is called a *holomorphic theta vector*.

Schwarz's construction can be applied to Complex Tori too, and in this case he exhibits the relationship between the holomorphic theta vectors, and the standard theta functions over such Tori. In [8], Chang-Young and Kim describe the relationship between theta vectors of Noncommutative Tori, and a quantum theta function discussed by Manin in [32]. These quantum theta functions are related to the zeta functions studied by H.Stark in [59] when he formulated a series of conjectures concerning Hilbert's twelfth problem. We will discuss how theta functions arise from our development of line bundles, and Stark's conjectures in Chapter 4.

In Chapter 1 we remarked on a recent trend, using techniques in Model Theory to solve problems in Number Theory. Quantum Tori have been studied from a Model Theoretic viewpoint by Zilber in a series of lectures and studies [64–66]. His work has shown that Quantum Tori can be defined in a class of structures known as *Analytic-Zariski* structures, which represent a variation to structures arising from the Zariski topology on an algebraic curve. In §3.6.1 we examine the possibility that the objects we have studied in the previous sections are definable in such a structure.

Several results in this chapter hint at a deep relationship between Quantum Tori and elliptic curves. This is not an original observation, and the notion of a duality between Complex and Noncommutative Tori has been studied by both Manin and Nikolaev. In §3.6.2 we look at how our results indicate the existence of such a relationship, and refer to the work of Nikolaev who has studied this relationship.

3.6.1 Line Bundles in Model Theory

Throughout our development of line bundles over Quantum Tori, the theory of Complex Tori has been a guiding star. It was a description of line bundles over such tori in terms of cohomology which formed the basis of our description for line bundles over Quantum Tori. By the Uniformization Theorem, Complex Tori can be viewed as algebraic curves, and the theory of line bundles over these objects is closely linked to this fact.

If X_Λ is a complex torus, it is the existence of an integral valued alternating form on the lattice Λ that ensures the existence of "very ample" line bundles over X_Λ. Fundamentally, it is the existence of these line bundles that imply that X_Λ can be viewed as an algebraic variety.

Given a Quantum Torus Z_L it is possible to define an integral valued alternating form on the pseudolattice L given by the Chern class. We may postulate that this assures us of the existence of "very ample line bundles over Z_L". It is this fact that has allowed us to define many objects previously in this chapter associated to Quantum Tori, which we would normally associated to abelian varieties. Although Quantum Tori are not algebraic varieties, through the work of Zilber they can be defined in a class of structures called *Analytic-Zariski structures.*

In Chapter 2 §2.2 we defined a structure in the context of mathematical logic. By adding further axioms to the ones we described, we obtain more specialised structures. A *Zariski structure* is such a specialised structure, introduced and studied by Zilber and Hrushovski. They introduce additional set theoretic axioms, which aim to characterise the Zariski topology on an algebraic curve and are satisfied by the usual Zariski structure on an algebraic variety. Adding a further condition to the Zariski axioms they showed in [20] that such a Zariski structure is indeed isomorphic to the Zariski structure of some curve over an algebraically closed field.

We may hope that the Quantum Torus Z_L lies in a Zariski structure, and hence we can be able to realise it as an algebraic object. However the Quantum Torus fails to satisfy some of the appropriate axioms. In [65], Zilber introduces the notion of an *Analytic-Zariski* structure, in which some of the axioms for Zariski structures are modified, and some new axioms are present. These structures, despite not being isomorphic to structures over algebraic curves may still have properties we commonly associate to algebraic varieties. For example we can talk of compact, complete and irreducible Analytic-Zariski structures. The Quantum Torus (or a group which is isomorphic to it) is definable in an Analytic-Zariski structure, and is compact and complete.

An important concept in model theory is that of stability:

Definition 3.6.1 (Stable Theory). Let T be a complete theory in a countable language, and let κ be an infinite cardinal. We say that T is κ-stable if whenever $\mathcal{M} \models T$, $A \subseteq M$ and $|A| = \kappa$ then $\left|S_n^{\mathcal{M}}\right| = \kappa$.

Given a structure $\mathcal{M}$, we say that $\mathcal{M}$ is κ-stable if its theory is. The following result attributed to Shelah shows that we have a trichotomy:

Proposition 3.6.2 (Theorem 4.5.48 of [34]). *If T is a complete theory in a countable language, then one of the following holds:*

1. *there are no cardinals κ such that T is κ-stable;*

2. *T is stable for all $\kappa \geq 2^{\aleph_0}$;*

3. *T is κ-stable iff $\kappa^{\aleph_0} = \kappa$.*

In the first case we say that T is unstable*, otherwise we say it is* stable*. If T satisfies condition 2 we say that T is* superstable.

Every $\aleph_0$-stable theory is superstable, but there exists superstable theories which are not $\aleph_0$-stable, and stable theories which are not superstable. In this context, superstability can be viewed as a weakening of the property of $\aleph_0$-stability.

This model theoretic concept associated to a structure is conjectured to have strong links to the topological nature of the structure. For example, it is conjectured [29] that all simple $\aleph_0$-stable groups arise from the Zariski-structure on an algebraic variety. In [66] Zilber conjectures that a structure associated to Z_L is *superstable*, which is a weakening of the notion of $\aleph_0$-stability which algebraic varieties are known to satisfy.

The structure considered by Zilber is the two sorted structure

$$\mathcal{T}_\theta := ((\mathbb{C}, +, \mathcal{A}_\theta), \exp, \mathbb{C}^*) \tag{3.22}$$

where

$$\mathcal{A}_\theta = (i + \theta)\mathbb{R} + 2\pi\mathbb{Z} + 2\pi i\mathbb{Z}.$$

It is shown in [66] that that the quotient $\mathbb{C}/\mathcal{A}_\theta$ is isomorphic to the Kronecker foliation of the torus $\mathbb{T}^2_\theta$. Provided a conjecture in transcendence theory (known as Schanuel's conjecture) holds, the theory of this structure is superstable. Although we know that this structure is not isomorphic to a Zariski structure over an algebraic curve, its stability theory suggests that the theory of this structure may contain some of the characteristics we associate to algebraic varieties. Indeed, Zilber has proved that this structure is both compact and complete.

The approach supplied by Model Theory provides a philosophy that may be invaluable when tackling the problem of Real Multiplication. In Complex Multiplication we use torsion points on an algebraic variety (an elliptic curve) to generate abelian extensions of imaginary quadratic fields. Although it is not possible to achieve this for Real Multiplication, Zilber's approach may provide a category (of Analytic-Zariski structures) in which to look for the objects which could potentially provide solutions to Hilbert's Problem.

Definability of e^s

For a line bundle over a Complex Torus $X_\Lambda = \mathbb{C}/\Lambda$, we have an analogous theory to that developed for Quantum Tori. For each line bundle $\mathcal{M}$ over X_Λ we obtain a subgroup $K(\mathcal{M})$ of X_Λ and an alternating pairing

$$e^{\mathcal{M}} : K(\mathcal{M}) \times K(\mathcal{M}) \to \mathbb{C}^*.$$

The group $K(\mathcal{M})$ is an algebraic subvariety of X_Λ, and the pairing $e^{\mathcal{M}}$ is a morphism of algebraic varieties.

As mentioned above, Zilber has identified a category which may serve as an analogy for that of algebraic varieties for the purpose of Real Multiplication - that of Analytic-Zariski structures. We would like to discuss whether the objects $K(\mathcal{L})$ and $e^{\mathcal{L}}$ associated to line bundles over Quantum Tori are definable in such a structure.

We can view a Quantum Torus with parameter θ as a definable subgroup of the structure defined in (3.22) via the map

$$\begin{array}{rcl} E : \mathbb{R}/L_\theta & \to & \mathbf{T}_\theta := \mathbb{C}^*/\mathcal{G}_0 \\ x + L_\theta & \mapsto & e^{2\pi x}.\mathcal{G}_\theta \end{array}$$

where $\mathcal{G}_\theta := \exp(\mathcal{A}_\theta)$. Throughout this rest of this section, we identify $K(\mathcal{L})$ with its image under E. As a consequence of Theorem 3.5.7 we obtain the following result, which is a promising start to viewing the pairing $e^{\mathcal{L}}$ as a definable function in an Analytic-Zariski structure:

Lemma 3.6.3. *If Shanuel's conjecture holds, then $K(\mathcal{L})$ is an Analytic-Zariski set.*

Proof. We consider the cases $s \neq 0$ and $s = 0$ separately. In the case when $s \neq 0$, by part 1 of Theorem 3.5.7 we have

$$\Lambda(\mathcal{L}) \cong \{x \in \mathbb{C}^* : x^s \in \mathcal{G}_\theta\}.$$

$K(\mathcal{L})$ is the quotient of this by $\mathcal{G}_\theta$, and hence a definable subset of the structure $(\mathbb{C}^*, \mathcal{G}_\theta, .)$.

If $s = 0$, by part 2 of Theorem 3.5.7 we have $K(\mathcal{L}) \cong \mathbf{T}_\theta = \mathbb{C}^*/\mathcal{G}_\theta$, which is an Analytic-Zariski structure by Zilber's study (modulo Schanuel's conjecture). $\square$

Unfortunately, this is the limit to the extent we can achieve our goal of defining the pairing $e^{\mathcal{L}}$ in an Analytic-Zariski structure at present. The graph of $e^{\mathcal{L}}$ is not definable in any of the structures that Zilber considers in [64], [65] and [66]. In order to acquire a structure in which the graph of $e^{\mathcal{L}}$ is definable, it would be desirable to have a log-function between $\mathbb{C}^*$ and $\mathbb{C}$. It is unknown whether the addition of this function to any of Zilber's structures would alter the stability of such a structure. We do mention that in [66], Zilber defines a "random logarithm" from $\mathbb{C}^*$ to $e^{2\pi\theta\mathbb{Z}}$ where the resulting structure is unstable.

3.6.2 A Duality between Elliptic Curves and Quantum Tori

In this section we describe how the results of this chapter suggest the existence of a relationship between Quantum and Complex Tori. The first of these is Theorem 3.3.18, which provides an analogue of the Appel-Humbert Theorem for Quantum Tori. We consider the corresponding result for Complex Tori:

Theorem 3.6.4 (Appel-Humbert Theorem, Theorem 1.5 of [23]). *Let $X_\Lambda := \mathbb{C}/\Lambda$ be a Complex Torus. The Line Bundles over X_Λ are characterised up to isomorphism by the following data:*

1. *A hermitian form H on $\mathbb{C}$;*

2. *A semi-character $\chi : \Lambda \to U(1)$ such that*

$$\chi(\lambda_1 + \lambda_2) = \chi(\lambda_1)\chi(\lambda_2)e^{\pi i E(\lambda_1, \lambda_2)} \tag{3.23}$$

   where $E = \Im H$ is an $\mathbb{R}$-bilinear alternating form on $\mathbb{C}$.[4]

The set $P(\Lambda)$ of pairs (H, χ) of such data form a group under the law of composition

$$(H_1, \chi_1)(H_2, \chi_2) = (H_1 + H_2, \chi_1\chi_2).$$

However, we note that a Hermitian form H is determined by the alternating form $E := \Im(H)$:

$$H(z, w) = E(iz, w) + iE(z, w).$$

So by the above result we may characterise isomorphism classes of line bundles over complex tori by pairs (E, χ) where E is an alternating form on Λ and χ satisfies (3.23).

Let L and Λ be a pseudolattice and a complex lattice respectively, and suppose ψ is an isomorphism of additive groups $\Lambda \cong L$. Let $\chi : L \to U(1)$ be a semi-character for E. This corresponds canonically to a semi-character for E with domain Λ given by

$$\psi^*(\chi) = \chi \circ \psi : \Lambda \to U(1).$$

Proposition 3.6.5. *We have an isomorphism $H^1(\Lambda, \mathcal{H}^*) \cong H^1(L, \mathcal{H}^*)$.*

[4] Here $\Im(H)$ denotes the imaginary part of H.

Proof. If (E, χ) is the Appel-Humbert data for an element of $H^1(L, \mathcal{H}^*)$, map this to $(E, \psi^*(\chi))$. Since ψ is an isomorphism this is an isomorphism. □

We conclude that we have a bijection between isomorphism classes of line bundles over Quantum and Complex Tori. This is a surprising result, since as we remarked at the end of §3.2, the analytic techniques used when considering elements of $Z^1(L, \mathcal{H}^*)$ are different from those used for line bundles over Complex Tori.

If we look closer at the proof of Theorem 3.5.7, we see that we have an even stronger result.

Proposition 3.6.6. *We have an isomorphism* $Z^1(\Lambda, \mathcal{H}^*) \cong Z^1(L, \mathcal{H}^*)$.

Proof. The proof shows that every element of $Z^1(L, \mathcal{H}^*)$ is equal to

$$\chi(l)\hat{\sigma}(\eta)_l(v)\frac{h(v+l)}{h(v)}$$

for some $\chi \in \mathrm{Hom}(L, U(1))$, $\eta \in \mathrm{Alt}^2(L, \mathbb{Z})$ and $h \in \mathcal{H}^*$. Fix an isomorphism $\psi : \Lambda \cong L$. Then $\psi^*(\chi) \in \mathrm{Hom}(\Lambda, U(1))$, $\psi^*(\eta) \in \mathrm{Alt}^2(\Lambda, \mathbb{Z})$. Define

$$\begin{array}{rcl} \Phi : Z^1(L, \mathcal{H}^*) & \to & Z^1(\Lambda, \mathcal{H}^*) \\ \chi(l)\hat{\sigma}(\eta)_l(v)\frac{h(v+l)}{h(v)} & \mapsto & \psi^*(\chi)(\lambda)\hat{\sigma}(\psi^*(\eta))_\lambda(v)\frac{h(v+\lambda)}{h(v)} \end{array}$$

Since ψ is an isomorphism, it follows that Φ is. □

Hence we have a bijection, not just between isomorphism classes of line bundles of Quantum and Complex Tori, but between line bundles themselves. If we let $\Lambda_n = \omega_1\mathbb{Z} + \omega_2 z_n\mathbb{Z}$ be a sequence of lattices with $z \in \mathfrak{H}$ tending to 1 as $n \to \infty$, then $\Lambda_n \to L$ as $n \to \infty$. We can view line bundles over Quantum Tori as occurring as the *limit* of a sequence line bundles over the Complex Tori determined by the lattices Λ_n. The idea that Quantum Tori can be viewed as limits of Complex Tori is an idea noted by Manin in [32]. We note that in [33], Manin generalises the notion of the Heisenberg group to define line bundles over Noncommutative Tori. It would be interesting to investigate the relationship between this notion of line bundles and the one we have developed.

The connection between Quantum and Complex Tori has been investigated by Nikolaev in [40]. In his paper he describes a bijection between elliptic curves E_τ associated to a lattice $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau \in \mathfrak{H}$, and pairs (A_θ, ω) of Noncommutative Tori with parameter θ and positive functionals $A_\theta \to \mathbb{C}$ of norm ω. This not only gives a relationship between Complex and Noncommutative tori, but also suggests that the arithmetic of certain complex and noncommutative tori may be linked. Indeed, he claims that this bijection yields Noncommutative Tori with Real Multiplication from elliptic curves with Complex Multiplication. In a later preprint [39], he makes some precise conjectures concerning generators of the Hilbert class field of a real quadratic field. The determination of such generators would provide valuable insight in to the Real Multiplication analogue of the modular j-function.

Zilber's representation of the Quantum Torus also lends its self to the suggestion of a duality between Quantum and Complex Tori. Recall we have defined the group

$$\mathcal{A}_\theta = \mathbb{R}(\theta + i) + 2\pi\mathbb{Z} + 2\pi i\mathbb{Z}.$$

As remarked in §3.6, the quotient $\mathbb{C}/\mathcal{A}_\theta$ is isomorphic to the Kronecker foliation of the torus with parameter θ, and hence to Z_θ.

Alternatively, we note that

$$\mathcal{A}_\theta = \mathbb{R}(\theta + i) + 2\pi\mathbb{Z} + 2\pi\theta\mathbb{Z}.$$

Hence the quotient $\mathbb{C}/\mathcal{A}_\theta$ can be viewed as a "foliation of the Quantum Torus $\mathbb{C}/2\pi L_\theta$" by the subgroup $\mathbb{R}(\theta + i) + 2\pi L_\theta$.

Chapter 4

Theta Functions over Quantum Tori

4.1 Introduction

In the previous section we defined the notion of a line bundle over a Quantum Torus Z_L as an element of the group of cocycles $Z^1(L, \mathcal{H}^*)$. We now discuss the existence of "sections" of these line bundles.

The motivation for the work of Chapter 3 was the fact that line bundles over a Complex Torus X_Λ are in bijection with the group of cocycles $Z^1(\Lambda, \mathcal{H}^*)$. Suppose $\pi_{\mathcal{L}} : \mathcal{L} \to X_\Lambda$ is a line bundle over such a Complex Torus X_Λ, and corresponds to a cocycle $A_\lambda(v) \in Z^1(\Lambda, \mathcal{H}^*)$. The topological space $\mathcal{L}$ is viewed as the quotient of $\mathbb{C} \times \mathbb{C}$ by the action of Λ given by

$$\lambda(z, v) = (z + \lambda, A_\lambda(v)z).$$

A section of $\mathcal{L}$ is a map $\sigma : X \to \mathcal{L}$, such that $\pi_{\mathcal{L}} \circ \sigma = 1_{X_\Lambda}$. If p and $\tilde{p}$ denote the natural projections

$$\mathbb{C} \longrightarrow X_\Lambda \qquad \text{and}$$

$$\mathbb{C} \times \mathbb{C} \longrightarrow \mathcal{L}$$

respectively, then we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C} \times \mathbb{C} \arrow[r, "\tilde{p}"] \arrow[d, "\pi"] & \mathcal{L} \arrow[d, "\pi_{\mathcal{L}}"'] \\
\mathbb{C} \arrow[r, "p"] & X_\Lambda. \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

where π is the projection on to the first coordinate.

The natural projection $\tilde{p} : \mathbb{C} \times \mathbb{C} \to \mathcal{L}$ is a covering map, so the section σ lifts to a section $\tilde{\sigma} : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ of the trivial bundle on $\mathbb{C}$ satisfying

$$\pi \circ \sigma = 1_{\mathbb{C}}. \tag{4.1}$$

By (4.1) we have

$$\tilde{\sigma}(z) = (z, \theta(z))$$

for some $\theta \in \mathcal{H}^*$. Since $\tilde{\sigma}$ is a lift of σ, for all $\lambda \in \Lambda$ we have

$$\tilde{p} \circ \tilde{\sigma}(z) = \tilde{p} \circ \tilde{\sigma}(z + \lambda),$$

which implies that

$$\theta(z + \lambda) = A_\lambda(v)\theta(z). \tag{4.2}$$

Conversely, if $\theta \in \mathcal{H}^*$ satisfies the periodicity condition of (4.2) with respect to the lattice Λ, the map

$$\sigma : z + \Lambda \mapsto \tilde{p}((z, \theta(z)))$$

defines a section of $\mathcal{L}$. This prompts the following definition:

Definition 4.1.1 (Theta function). A holomorphic theta function for a group $G \subseteq \mathbb{C}$ is a holomorphic map $\theta : \mathbb{C} \to \mathbb{C}$ such that for all $v \in \mathbb{C}$

$$\theta(v+g) = A_g(v)\theta(v) \tag{4.3}$$

for some $A_g \in Z^1(G, \mathcal{H}^*)$.

Hence theta functions for a complex lattice Λ correspond to sections of holomorphic line bundles over the Complex Torus X_Λ. To determine the existence of sections of holomorphic line bundles over Quantum Tori Z_L, we need to determine whether there are any holomorphic theta functions for the pseudolattice L.

This chapter is split in to two main parts, the first consisting of §4.2 and §4.3. In the first of these we show that unlike the case for complex lattices Λ, there are no nontrivial holomorphic theta functions corresponding to a pseudolattice L. In §4.3, we weaken the condition of holomorphicity to allow our theta function to have poles. We show that the double sine function studied by Shintani [54, 55] and Kurokawa [24, 25] can be interpreted to be a *meromorphic* theta function for a pseudolattice.

In the 1970's Stark made a series of conjectures [57–60] regarding the values of L-functions associated to number fields at $s = 0$. The second half of this chapter concerns the application of the functions we discuss in the first part to these conjectures. In §4.4 we give an introduction to Stark's ideas, and how they are related to our goal of understanding an explicit class field theory for real quadratic fields. The remainder of §4.4 is devoted to an account of the work of Shintani. In [54] Shintani described the values of an L-function associated to a real quadratic field in terms of specific values of the double sine function, and in a later paper [55] used these values to generate abelian extensions of specific real quadratic fields. In the context of this thesis, this is an important result, stating that in specific cases special values of *meromorphic* theta functions associated to Quantum Tori can generate abelian extensions of certain real quadratic fields.

We see that Shinatani's result can be interpreted as a solution in a special case to the Rank One Abelian Stark conjecture [46, 62], which concerns the case when the L-function has a simple zero at $s = 0$. There exist higher order Stark conjectures [49, 63] which concern the cases when the L-function has zeros of higher order at $s = 0$. Motivated by these conjectures and Shintani's result we investigate whether it is possible to write higher derivatives of L-functions associated to real quadratic fields in terms of meromorphic theta functions for a pseudolattice. Our main result is Theorem 4.6.1, which writes the m^{th} derivative of an L-function as an element of a certain field, whose generators contain the special values of various functions defined in §4.5 which are shown to be theta functions for pseudolattices.

We use Shintani's work of [54] to reduce the proof of Theorem 4.6.1 to a result concerning a type of zeta function. This result is proved in §4.7 using a blend of induction (of which Shintani's result is the starting case), and the calculation of various contour integrals. In §4.8 we discuss the possible implications this has to Real Multiplication, and where this result could be improved.

4.2 Holomorphic Theta functions for L

We begin by using the results of Chapter 4 to show that there are no nontrivial[1] holomorphic theta functions for a pseudolattice.

[1] A trivial theta function is a nonzero multiple of the exponential function.

Proposition 4.2.1. *Let L be a pseudolattice. There are no nontrivial holomorphic theta functions for L.*

Proof. Note that if Θ is a theta function for $A_l \in Z^1(L,\mathcal{H}^*)$ and

$$B_l(v) = A_l(v)\frac{h(v+l)}{h(v)}$$

for some non-vanishing holomorphic function h, then $\Theta(v)h(v)$ is a theta function for B_l. Hence it suffices to show that there are no nonconstant holomorphic theta functions satisfying (4.3) for a representative of each cohomology class in $Z^1(L,\mathcal{H}^*)$.

Suppose Θ is a holomorphic theta function for a line bundle $\mathcal{L}$. By Theorem 3.3.12, the isomorphism class of $\mathcal{L}$ in $H^1(L,\mathcal{H}^*)$ has a unique representative

$$\mu(l)\hat{\sigma}(\eta)_l(v)$$

where $\mu \in \mathrm{Hom}(L, U(1))$ and $\hat{\sigma}(\eta)_l(v)$ is as defined in §3.3.2. For $v \in \mathbb{R}$ we have $|\mu(l)\hat{\sigma}(\eta)_l(v)| = 1$. Hence for all $v \in \mathbb{R}$ we have

$$|\Theta(v+l)| = |\mu(l)\hat{\sigma}(\eta)_l(v)\Theta(v)| = |\Theta(v)|\,. \tag{4.4}$$

First note that if $\Theta(v)$ has a zero, then it is identically zero, for the above relation implies that it has an accumulation point of zeros. Therefore we may assume that $\Theta(v)$ is nonvanishing.

Fix $r \in \mathbb{R}$. Since Θ is nonvanishing there exists a function $x_r(v)$ holomorphic in v such that

$$\frac{\Theta(v+r)}{\Theta(v)} = e^{2\pi i x_r(v)}. \tag{4.5}$$

Without loss of generality we assume that $x_0(v) = 0$. Equation (4.4) implies that

$x_r(v) \in \mathbb{R}$ for all $v \in \mathbb{C}$. Since $x_r(v)$ is a holomorphic function in v this implies that it is constant.

Now fix $v \in \mathbb{C}$, and let $r, s \in \mathbb{R}$. Then

$$\frac{\Theta(v+r+s)}{\Theta(v)} = \frac{\Theta(v+r+s)}{\Theta(v+r)} \frac{\Theta(v+r)}{\Theta(v)}.$$

Hence there exists $n(v) \in \mathbb{Z}$ such that

$$\begin{aligned} x_{r+s}(v) &= x_s(v+r) + x_r(v) + 2\pi i n(v) \\ &= x_s(v) + x_r(v) + 2\pi i n(v). \end{aligned}$$

Since $x_0(v) = 0$ we see that $n(v) = 0$, and as a function of $r \in \mathbb{R}$, $x_r(v)$ is a homomorphism. Hence for all $r \in \mathbb{R}$, $x_r(v) = \alpha r$ for some $\alpha \in \mathbb{R}$.

Now consider the left hand side of (4.5). As r varies over $\mathbb{C}$ this is a holomorphic function. Hence for fixed v, there exists a function $x_v(w)$ holomorphic in w such that for all $v \in \mathbb{C}$

$$\frac{\Theta(v+w)}{\Theta(v)} = e^{2\pi i x_v(w)}.$$

Again we assume without loss of generality that $x_v(0) = 0$, and therefore $x_z(v) = x_v(z)$ for all $v, w \in \mathbb{C}$. On $\mathbb{R}$ we therefore have $x_v(w) = \alpha w$, and hence by holomorphicity this holds on the whole plane.

Since Θ is holomorphic we may compute its derivative along any path. Let $z \in \mathbb{C}$, and let $\gamma_z(t)$ be the path $z + t$. Then

$$\Theta'(z) = \lim_{t \to 0} \frac{\Theta(\gamma_z(t)) - \Theta(\gamma_z(0))}{t} = \lim_{t \to 0} \frac{\Theta(z+t) - \Theta(z)}{t}$$

$$= \lim_{t \to 0} \frac{e^{2\pi i \alpha t} - 1}{t} \Theta(z) = 2\pi i \alpha \Theta(z).$$

Hence

$$\frac{d}{dz} \log(\Theta(z)) = 2\pi i \alpha$$

and hence $\Theta(z) = Ae^{2\pi i \alpha z}$ for some $A \in \mathbb{C}^*$.

□

This could be viewed as a set back in defining Real Multiplication analogues to functions which form the foundation of Complex Multiplication, such as the Weierstrass $\wp$-function and modular discriminant Δ. When X_τ is the Complex Torus corresponding to the lattice $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ there are four holomorphic Jacobi theta functions denoted by $\theta_i(z, \tau)$ for $i = 1, 2, 3, 4$. These are related to the $\wp$ and Δ-functions via the following expressions:

$$\wp(z; \tau) = -\log(-\theta_1(z; \tau))'' + c \text{ for some constant } c;$$

$$\theta_3(0; \tau)^{24} = \frac{\Delta^2(\frac{\tau+1}{2})}{\Delta(\tau + 1)} \text{ where } \Delta = \eta^{24}.$$

We may have hoped that the existence of nontrivial holomorphic theta functions associated to Quantum Tori would have enabled us to define similar functions for a real irrational parameter θ in place of the complex modulus τ.

The nonexistence of nontrivial holomorphic theta functions for pseudolattices leads us to consider the existence of theta functions which are meromorphic. For Complex Tori, elliptic functions and meromorphic theta functions can be constructed out of quotients of holomorphic theta functions [12]. For Quantum Tori this technique fails due to Proposition 4.2.1. In the next section we examine how it is possible to define meromorphic theta functions for L.

4.3 Meromorphic Theta Functions for L

Let $\mathcal{H}$ denote the ring of holomorphic functions on $\mathbb{C}$, and denote by $\mathcal{K}$ the field of fractions of $\mathcal{H}$. Then $\mathcal{K}^*$ is the multiplicative group of those meromorphic function which are not identically zero. Consider the group of 1-cocycles $Z^1(L, \mathcal{K}^*)$. These can be viewed as cocycles corresponding to *meromorphic theta functions* for the pseudolattice L. We saw in the previous section that any holomorphic theta function for L is constant. This motivates the following question:

Question 4.3.1 (Existence of meromorphic theta functions for L). Does there exist $A_l(v) \in Z^1(L, \mathcal{K}^*)$, and a nonconstant meromorphic function F on $\mathbb{C}$ such that for any $l \in L$, $v \in \mathbb{C}$ we have

$$F(v + l) = A_l(v)F(v)?$$

Definition 4.3.2. Let $\omega = (\omega_1, \omega_2)$ be a 2-tuple of elements $\omega_1, \omega_2 \in \mathbb{R}_{>0}$. The *double sine function* with parameter ω is the unique meromorphic function $S_2^\omega(z)$ on $\mathbb{C}$ such that:

$$S_2^\omega(z, \omega) = 2 \sin\left(\frac{\pi z}{\omega_2}\right) S_2^\omega(z + \omega_1, \omega) \tag{4.6}$$

$$S_2^\omega(z, \omega) = 2 \sin\left(\frac{\pi z}{\omega_1}\right) S_2^\omega(z + \omega_2, \omega) \tag{4.7}$$

$$S_2^\omega\left(\frac{\omega_1 + \omega_2}{2}, \omega_2\right) = 1. \tag{4.8}$$

This existence of such a function can be deduced from the properties of the double gamma function. The development of the double gamma function by Barnes in [4] in 1901 was motivated by Lerch's formula

$$\log \Gamma(x) = \zeta'(0, x) + \frac{1}{2} \log(2\pi), \tag{4.9}$$

where $\zeta(s,x)$ is the Riemann-Hurwitz zeta function. For $x, \omega_1, \omega_2 \in \mathbb{R}_{>0}$, and $s \in \mathbb{C}$ with $\Re(x) > 0$, the double Riemann-Hurwitz zeta function is defined to be

$$\zeta_2(s, x, (\omega_1, \omega_2)) = \sum_{n_1, n_2 \in \mathbb{N}} \frac{1}{(n_1\omega_1 + n_2\omega_2 + x)^s}.$$

This series converges absolutely for $\Re(s) > 1$ and has an analytic continuation to the complex plane. The relationship between the gamma function and Riemann-Hurwitz zeta function in (4.9) motivates the double gamma function $\Gamma_2(x, \omega)$ to be defined by the following relation:

$$\log\left(\Gamma_2(x, \omega)\right) = \frac{\partial}{\partial s}\zeta_2(s, x, \omega)\bigg|_{s=0} + A$$

where A is some normalising constant. We can now define the double sine function by the following formula:

$$S_2^{\omega}(z) := \frac{\Gamma_2(\omega_1 + \omega_2 - z, \omega)}{\Gamma_2(z, \omega)}.$$

Proposition 4.3.3. *Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a pseudolattice. Then $S_2^{\omega}(z)$ is a meromorphic theta function for L. More generally, suppose that G is a meromorphic function such that there exist meromorphic functions $f(v)$ and $g(v)$ such that for all $v \in \mathbb{C}$*

$$G(v + \omega_1) = f(v)G(v) \tag{4.10}$$

$$G(v + \omega_2) = g(v)G(v). \tag{4.11}$$

Then $G(v)$ is a meromorphic theta function for L. If $l = n\omega_1 + m\omega_2 \in L$ then we have

$$G(v + l) = A_l(v)G(v) \tag{4.12}$$

where

$$A_l(v) := \prod_{r=0}^{n-1} \prod_{s=0}^{m-1} f(z + r\omega_1) g(z + s\omega_2).$$

Proof. It suffices to prove the general case. It is an immediate consequence of the periodicity relations of (4.10) and (4.11) to show that (4.12) is satisfied. We need to show that $A_l(v) \in Z^1(L, \mathcal{H}^*)$. Let $l_1 = n_1\omega_1 + m_1\omega_2$ and $l_2 = n_2\omega_1 + m_2\omega_2 \in L$, and let $l = l_1 + l_2 = n\omega_1 + m\omega_2$. Then

$$\begin{aligned}
A_{l_1+l_2}(v) &= \prod_{r=0}^{n-1} \prod_{s=0}^{m-1} f(v + r\omega_1) g(v + s\omega_2) \\
&= \prod_{r=n_2}^{n-1} \prod_{s=m_2}^{m-1} f(v + r\omega_1) g(v + s\omega_2) \\
&\qquad \times \prod_{r=0}^{n_2-1} \prod_{s=0}^{m_2-1} f(v + r\omega_1) g(v + s\omega_2) \\
&= \prod_{r=0}^{n_1-1} \prod_{s=0}^{m_1-1} f(v + (r + n_2)\omega_1) g(v + (s + m_2)\omega_2) \\
&\qquad \times A_{l_2}(v) \\
&= A_{l_1}(v + l_2) A_{l_2}(v)
\end{aligned}$$

□

Having exhibited the existence of meromorphic theta functions for pseudolattices, the remainder of this chapter concerns their possible application to Real Multiplication. In the next section we examine the work of Stark and Shintani to Hilbert's twelfth problem, and observe that meromorphic theta functions for pseudolattices have an important role to play in this area.

4.4 Stark's Conjecture and Hilbert's Twelfth Problem

In this section we give a brief overview of a series of conjectures made by Stark concerning the values of L-functions associated to number fields at $s = 0$. This leads on to give an account of the work of Shintani, who proved a version of Stark's conjecture in special cases when the ground field was a real quadratic field. We aim to stress the importance of the double sine function in Shintani's method, and its application in his approach to a solution of Hilbert's twelfth problem for certain real quadratic fields. Motivated by so called "higher order" Stark conjectures, and Shintani's results we will study generalisations of the double sine function in §4.5.

4.4.1 L-functions and Stark's conjecture

Let K be a number field, and suppose that M is an abelian extension of K with Galois group G. Class field theory supplies a homomorphism

$$\tilde{\psi}_{M/K} : I_K \to G$$

where I_K denotes the group of fractional ideals of K. Let V be a representation of G with character χ. Then define

$$L(\chi, s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\chi, s)$$

where $\mathfrak{p}$ runs over the prime ideals in $\mathcal{O}_K$ and

$$L_{\mathfrak{p}}(\chi, s) = \left(1 - \chi(\tilde{\psi}_{M/K}(\mathfrak{p}))N_{K/\mathbb{Q}}\mathfrak{p}^{-s}\right)^{-1}.$$

Let S be a finite set of places of K which is non-empty and contains all the infinite places of K. We define the L-function associated to S by

$$L_S(\chi, s) := \prod_{\mathfrak{p} \notin S} L_{\mathfrak{p}}(\chi, s).$$

These functions are known as L-functions, and when χ is a nonprincipal character they have analytic continuations to the entire complex plane. There exists a functional equation for these functions relating their values at s to their values at $1 - s$.

We can write

$$L_S(\chi, s) = \sum_{g \in G} \chi(g) L_S(s, g)$$

where

$$L_S(s, g) = \sum_{\substack{\mathfrak{a} : (\mathfrak{a}, S) = 1 \\ \tilde{\psi}_{M/K}(\mathfrak{a}) = g}} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s}.$$

When χ is nonprincipal the functional equation implies (see [63]) that the order of vanishing of $L_S(\chi, s)$ is equal to

$$r(\chi) = |\{v \in S : v \text{ splits completely in } L\}|. \tag{4.13}$$

Suppose L is ramified at precisely one of the infinite primes, and that S contains precisely the ramified finite primes and the infinite ones. Then $r(\chi) = 1$.

In a series of four papers [57–60] between 1971 and 1980, Stark studied the values of the L-functions attached to such Galois extensions of number fields at the value $s = 1$, which are related via the functional equation to the values at $s = 0$. If as above, the L-function has a first order zero at $s = 0$, the simple pole of the gamma factor of the functional equation picks out the derivative of $L_S(\chi, s)$ at $s = 0$.

Under these conditions studying the value of the L-function at $s = 1$ is equivalent to studying the value of the derivative at $s = 0$.

Conjecture 4.4.1 (Rank One Abelian Stark Conjecture [46],[62]). *Let M/K be an abelian extension, and S a finite set of places of K containing the infinite ones, one of which splits completely in M. Let m be the number of roots of unity contained in K. There exists an S-unit (not necessarily unique) $\varepsilon \in M$ such that for every character χ of G we have*

$$\frac{d}{ds} L_S(\chi, s)\bigg|_{s=0} = -\frac{1}{m} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon^\sigma|_w \, .$$

Variations on this conjecture exist for when both infinite primes ramify (known as the Brumer-Stark conjecture), and when K is totally real. This last case was studied by Tangedal in [62].

In the last of Stark's papers he proves a version of Conjecture 4.4.1 for the cases case $k = \mathbb{Q}$, and when k is an imaginary quadratic field. The latter result uses the the work of Ramachandra in [43], which also was a driving force behind the work of Shintani, whose work we study in the next section.

4.4.2 Real Quadratic Fields and the work of Shintani

In 1976 Shintani [53] introduced a generalisation of the Riemann-Hurwitz zeta function and proved its analytic continuation to the complex plane. Shintani used this function to reprove the result of Siegel and Klingen [17, 55]:

> Suppose k is a totally real field, and let χ a character of the ray class group of F modulo an integral ideal $\mathfrak{f}$. Let S be the finite set of those primes dividing $\mathfrak{f}$. Then for each $n \in \mathbb{N}$ we have $L_S(1-n, \chi) \in \mathbb{Q}$.

Shintani showed that for a totally real field k, it is possible to express L-functions associated to k as linear combinations of these "Shintani L-functions", reducing the study of the value of $L_S(s,\chi)$ to that of Shintani's L-functions. In a subsequent paper [54], Shintani proved a formula relating the value of his L-functions at $s=1$ to the double gamma function studied by Barnes in [4], analogous to the Kronecker limit formula for imaginary quadratic fields. Using these ideas he went on to prove a refined Stark conjecture for real quadratic fields in [55].

Shintani's Limit Formula

In this section we give an account of Shintani's Kronecker limit formula for real quadratic fields.

Let F be a real quadratic field such that $\mathrm{Gal}(F/\mathbb{Q})$ is generated by σ. Given an integral ideal $\mathfrak{g}$ of $\mathcal{O}_F$ we let $F^+_{1,\mathfrak{g}}$ denote the group of principal fractional ideals of F generated by those elements α such that

1. α is totally positive. i.e. $\alpha > 0$ and $\alpha^\sigma > 0$;
2. $\mathrm{ord}_\mathfrak{p}(\alpha - 1) > 0$ for all $\mathfrak{p}|\mathfrak{g}$.

The group $I_F^\mathfrak{g}/F^+_{1,\mathfrak{g}}$ is denoted by $G^+_\mathfrak{g}(F)$, and is called the *narrow class group of F modulo* $\mathfrak{g}$, where $I_F^\mathfrak{g}$ denotes the group of fractional ideals coprime to $\mathfrak{g}$. When $\mathfrak{g} = \mathcal{O}_F$ we denote this group by $G^+(F)$, and its order by h^+. Given a fractional ideal $\mathfrak{a}$ we let $[\mathfrak{a}]^+$ denote the class it represents in $G^+(F)$.

Now fix an integral ideal $\mathfrak{f}$ of F, and put

$$S(\mathfrak{f}) := \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } \mathcal{O}_F \text{ dividing } \mathfrak{f}\} \cup \{1, \sigma\}.$$

Let χ be a character of $G_{\mathfrak{f}}^{+}(F)$, and suppose ε is a fundamental totally positive unit of F.

Define a simplicial cone in $\mathbb{R}^2$ by

$$C := \{x(1,1) + y(\varepsilon, \varepsilon^{\sigma}) : x > 0, y > 0\}.$$

We choose and fix a set of representatives $\{\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{h^+}\}$ of the narrow class group $G^+(F)$ of F. For each $g \in G_{\mathfrak{f}}^{+}(F)$ there exists a unique i such that $g = [\mathfrak{f}\mathfrak{a}_i]^+$ in $G^+(F)$.

With this notation, for $g \in G_{\mathfrak{f}}^{+}(F)$ we define the finite set

$$R(g) = \{z = x(1,1) + y(\varepsilon, \varepsilon^{\sigma}) \in C \cap (\mathfrak{f}\mathfrak{a}_i)^{-1} : x\mathfrak{f}\mathfrak{a}_i \in g, 0 < x \leq 1, 0 \leq y < 1\}.$$

Shintani showed that

$$L_S(s,g) = N(\mathfrak{f}\mathfrak{a}_i)^{-s} \sum_{z = x_1 + \varepsilon x_2 \in R(g)} \zeta(s, (\varepsilon, \varepsilon^{\sigma}), (x_1, x_2)) \tag{4.14}$$

where $\zeta(s, (\varepsilon, \varepsilon^{\sigma}), (x_1, x_2))$ is a special case of a family of zeta function we will call "Shintani L-functions". Higher dimensional versions of this function were studied in [53], which he used to evaluate the zeta functions associated to totally real algebraic number fields at negative integers, obtaining the result of Siegel and Klingen stated previously.

In [54], Shintani is able to give an explicit formula for the value of the derivative his L-function at $s = 0$ in terms of Barnes' double gamma function. Shintani's result can be expressed ass

$$\frac{d}{ds}\zeta(s,(\varepsilon,\varepsilon^{\sigma}),(x_1,x_2))\bigg|_{s=0} = \log\left(T(x_1+\varepsilon x_2,(\varepsilon,\varepsilon^{\sigma}))\right) \tag{4.15}$$

where

$$T(x_1+\varepsilon x_2,(\varepsilon,\varepsilon^{\sigma})) = \left\{\frac{\Gamma_2(x_1+x_2\varepsilon,(1,\varepsilon))\Gamma_2(x+y\varepsilon^{\sigma},(1,\varepsilon^{\sigma}))}{\rho((1,\varepsilon))\rho((1,\varepsilon^{\sigma}))}\right\} e^{\frac{\varepsilon-\varepsilon^{\sigma}}{4}\log\left(\frac{\varepsilon^{\sigma}}{\varepsilon}\right)\left(x_1^2+x_1-\frac{1}{6}\right)}.$$

The numbers $\rho((a_1,a_2))$ are normalising constants which occur in the theory of the double gamma function [4]. The main result of [54] is deduced from (4.14) and (4.15):

Theorem 4.4.2 (Shintani, [54]). *Let F be a real quadratic field, and $\mathfrak{f}$ an integral ideal of $\mathcal{O}_F$. Let $S = S(\mathfrak{f})$ and suppose $g \in G_{\mathfrak{f}}^{+}(F)$. Then*

$$\frac{d}{ds}L_S(s,g)\bigg|_{s=0} = \log T(g)$$

where

$$T(g) = \prod_{z=x_1+\varepsilon x_2\in R(g)} T((z,(\varepsilon,\varepsilon^{\sigma})).$$

Hence if χ is a character of $G_{\mathfrak{f}}^{+}(F)$

$$\frac{d}{ds}L_S(\chi,0)\bigg|_{s=0} = \sum_{g\in G_{\mathfrak{f}}^{+}(F)} \chi(g)\log T(g).$$

This final expression is reminiscent of the one in the Rank One Abelian Stark conjecture (Conjecture 4.4.1). With this comparison, Stark's conjecture suggests that the class invariants $T(g)$ are units in some ray class field over F.

4.4.3 Shintani's Class Invariants

In 1978 Shintani produced a paper proving a modified version of Stark's conjecture for real quadratic fields, subject to various conditions. Astonishingly, he was unaware of Stark's conjecture when he formulated his results.

As before, let F be a real quadratic field, and $\mathfrak{f}$ an integral ideal of F. Fix a totally positive integer ν such that $\nu + 1 \in \mathfrak{f}$, and let $[\nu]_{\mathfrak{f}}^{+}$ denote the class it represents in $G_{\mathfrak{f}}^{+}(F)$. By the Existence Theorem of class field theory (Theorem 1.1.1), there exists an abelian extension $M_{\mathfrak{f}}$ of F such that the reciprocity map induces an isomorphism

$$G_{\mathfrak{f}}^{+}(F) \cong \mathrm{Gal}(M_{\mathfrak{f}}/F).$$

We shall abuse the notation and shall identify $[\nu]_{\mathfrak{f}}^{+}$ with its image under the reciprocity map as an element of this Galois group. For $g \in \mathrm{Gal}(M_{\mathfrak{f}}/F)$, Shintani studies the value of $L_S(s, g) - L(s, [\nu]_{\mathfrak{f}}^{+} g)$ using Theorem 4.4.2. The properties of ν imply this has a particularly nice form:

$$L_S(s, g) - L_S(s, [\nu]_{\mathfrak{f}}^{+} g) = \sum_{z \in R(g)} \log \left\{ F(z, (1, \varepsilon)) F(z^{\sigma}, (1, \varepsilon^{\sigma})) \right\} \tag{4.16}$$

where the function $F(z, (1, \varepsilon))$ is related to the double sine function introduced in Definition 4.3.2 by

$$F(z, (1, \varepsilon)) = S_2^{(1,\varepsilon)}(z)^{-1}.$$

Based on this result Shintani defines the natural class invariant

$$X_{\mathfrak{f}}(g) = \prod_{z \in R(g)} F(z, (1, \varepsilon)) F(z^{\sigma}, (1, \varepsilon^{\sigma})).$$

With the notation of Theorem 4.4.2 we have

$$X_{\mathfrak{f}}(g) = T(g)T([\nu]_{\mathfrak{f}}^{+}g)^{-1}.$$

Theorem 4.4.2 implies that if Stark's conjecture is true, the invariants $X_{\mathfrak{f}}(g)$ should be units.

For a subgroup G of $G_{\mathfrak{f}}^{+}(F)$, given $c \in G_{\mathfrak{f}}^{+}(F)/G$ define

$$X_{\mathfrak{f}}(c, G) = \prod_{g \in G} X_{\mathfrak{f}}(cg),$$

and let $M_{\mathfrak{f}}(G)$ denote the subfield of $M_{\mathfrak{f}}$ fixed by the elements of G. Using these invariants Shintani proves the following subject to some conditions on G and further rather restrictive hypotheses on the ideal $\mathfrak{f}$.

Theorem 4.4.3. *There exists a positive rational number m such that*

1. *The invariant $X_{\mathfrak{f}}(c, G)^m$ is a unit in the field $M_{\mathfrak{f}}(G)$. Moreover for every $g \in G_{\mathfrak{f}}^{+}(F)$ we have*

$$\{X_{\mathfrak{f}}(c, G)^m\}^{\psi_{M_{\mathfrak{f}}/F}(g)} = X_{\mathfrak{f}}(cg, G)^m$$

2. *Consider the system of invariants*

$$\bigcup_{\mathfrak{f}' \| \mathfrak{f}} \left\{ X_{\mathfrak{f}_0}(c, \tilde{G})^m : c \in G_{\mathfrak{f}_0}^{+}(F)/\tilde{G} \right\}.$$

The union is taken over all divisors $\mathfrak{f}_0$ of $\mathfrak{f}$ which satisfy the same conditions that $\mathfrak{f}$ does, and $\tilde{G}$ is the image of G under the natural homomorphism

$$G_{\mathfrak{f}}^{+}(F) \longrightarrow G_{\mathfrak{f}_0}^{+}(F).$$

Then this system generates $M_{\mathfrak{f}}(G)$ over F.

The conditions on G imply that precisely one of the infinite primes of F splits in $M_{\mathfrak{f}}(G)$, so we are in the case considered by the Rank One Abelian Stark conjecture. Theorem 4.4.3 not only serves to give a special case of Stark's conjecture, but also has the two ingredients listed in the introduction which are required as a solution to Hilbert's twelfth problem: A system of generators with an explicit action of the Galois group.

4.5 Generalisations of the Double Gamma Function

Shintani's results imply that the double sine function will play an important role in any solution to the Rank One Abelian Stark conjecture for real quadratic fields. The description of this function as a meromorphic theta function for a pseudolattice, and hence a section of a line bundle over a Quantum Torus leads us to question the existence of other such functions. In this section we generalise the notion of the double gamma function originally defined by Barnes, with a view to investigating its relationship to the values of L-series attached to real quadratic fields.

Let $\omega_1, \omega_2 \in \mathbb{R}$ be such that the quotient ω_2/ω_1 is not negative. In [4], the double gamma function was defined by the integral equation

$$\Gamma_2(z,\omega) := \exp\left\{ \frac{1}{2\pi i} \oint_{I(\lambda,\infty)} e^{-zt} \frac{1}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})} \frac{\log(-t)+\gamma}{t} dt \right\}. \quad (4.17)$$

In this representation and in what follows, for $r \in \mathbb{R}_{>0} \cup \{\infty\}$, $I(\lambda, r)$ is the contour from r towards zero along the positive real axis to λ, around zero anticlockwise by a circle of radius λ and then out along the real axis to r.

We aim to generalise this integral definition to define a family $\Gamma_2^r(z,\omega)$ of functions which satisfy periodicity conditions with respect to the group $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, for which we have $\Gamma_2^1(z,\omega) = \Gamma_2(z,\omega)$.

Barnes supplies the following defining relation for the double sine function

$$\log\left(\Gamma_2(z,\omega)\right) = \frac{d}{ds}\zeta_2(s,z,\omega)\Big|_{s=0}$$

where

$$\zeta_2(s,z,\omega) = \sum_{n,m=0}^{\infty} \frac{1}{(z+m\omega_1+n\omega_2)^s}$$

for $\Re(s) > 1$ and $\Re(z) > 0$. The function $\zeta_2(s,z,\omega)$ has meromorphic continuation to the whole plane as a function of s and z.

Definition 4.5.1. Let $r \in \mathbb{N}$, and suppose $\omega = (\omega_1,\omega_2) \in \mathbb{R}^2$ is such that the quotient ω_2/ω_1 is not negative. For $z \in \mathbb{C}$ define

$$\log(\Gamma_2^r(z,\omega)) := \left(\frac{d}{ds}\right)^r \zeta_2(s,z,\omega)\Big|_{s=0}. \tag{4.18}$$

We have an integral formula for $\zeta_2(s,z,\omega)$ given by

$$\zeta_2(s,z,\omega) = \frac{\Gamma(1-s)}{2\pi i}\oint_{I(\lambda,\infty)} e^{-zt}\frac{(-t)^{s-1}}{(1-e^{\omega_1 t})(1-e^{\omega_2 t})}dt.$$

Integrating this r times we obtain an integral expression for $\Gamma_2^r(z,\omega)$:

$$\log\left(\Gamma_2^r(z,\omega)\right) = \frac{1}{2\pi i}\sum_{m=0}^{r}(-1)^m\binom{r}{m}\Gamma^{(m)}(1)\oint_{I(\lambda,\infty)}\frac{e^{-zt}}{(1-e^{\omega_1 t})(1-e^{\omega_2 t})}\frac{\log(-t)^{r-m}}{t}dt. \tag{4.19}$$

Definition 4.5.2. For $\omega = (\omega_1,\omega_2) \in \mathbb{R}^2$ such that the quotient ω_2/ω_1 is not negative, $z \in \mathbb{C}$ with $\Re(z) > 0$ define

$$G_2^r(z,\omega) = \exp\left(\frac{1}{2\pi i}\oint_{I(\lambda,\infty)} \frac{e^{-zt}}{(1-e^{\omega_1 t})(1-e^{\omega_2 t})}\frac{\log(-t)^r}{t}dt\right)$$

Lemma 4.5.3. *Fix $z \in \mathbb{C}$ and $\omega \in \mathbb{R}^2_{>0}$. Let W denote the field generated over $\mathbb{Q}$ by the values $\Gamma^{(i)}(1)$ for $i = 0, \dots, r$. Let V vector space over W generated by the values $\log\left(\Gamma_2^j(z,\omega)\right)$ for $j = 0, \dots, r$. Then V is equal to the vector space over W generated by $\log\left(G_2^j(z,\omega)\right)$ for $i, j = 0, \dots, r$.*

Proof. Define a matrix A with coefficients

$$A_{ij} = (-1)^j \binom{i}{j} \Gamma^{(j)}(1) \in W.$$

By (4.19) we have

$$\log\left(\Gamma_2^r(z,\omega)\right) = \sum_{j=0}^{r} A_{rj} \log\left(G_2^{r-j}(z,\omega)\right).$$

The matrix A_{ij} is upper triangular, with nonzero diagonal entries, and therefore invertible. $\square$

Corollary 4.5.4. *$G_2^r(z,\omega)$ is a meromorphic theta function on $\mathbb{C}$ for the pseudolattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.*

Proof. The meromorphicity follows from the meromorphicity of the $\zeta_2(z,s,\omega)$, (4.18) and Lemma 4.5.3. Observe that

$$\zeta_2(s, z+\omega_1, \omega) = \frac{\Gamma(1-s)}{2\pi i}\oint_{I(\lambda,\infty)} e^{-zt}[1 + (1-e^{\omega_1 t})]\frac{(-t)^{s-1}}{(1-e^{\omega_1 t})(1-e^{\omega_2 t})}dt.$$

$$= \zeta_2(s,z,\omega) + \frac{\Gamma(1-s)}{2\pi i}\oint_{I(\lambda,\infty)} e^{-zt}\frac{(-t)^{s-1}}{(1-e^{\omega_2 t})}dt.$$

The second term is equal to a zeta function $\zeta_1(s, z, \omega_2)$ which has meromorphic continuation to the whole plane [3]. Let $\Gamma_1^r(z,\omega_2) = \exp\left(\zeta_1^{(r)}(0,z,\omega_2)\right)$, and hence

$$\Gamma_2^r(z+\omega_1,\omega) = \Gamma_1^r(z,\omega_1)\Gamma_2^r(z,\omega).$$

A similar expression holds for $\Gamma_2(z+\omega_2,\omega)$.

Hence the functions $\Gamma_2^r(z,\omega)$ are meromorphic theta functions for L. By Lemma 4.5.3, the functions $G_2^r(z,\omega)$ are. □

The aim of this chapter is to write the higher derivatives of L-functions associated to real quadratic fields in terms of meromorphic theta functions for a pseudolattice. In order to achieve this we will need to introduce another function, which does not seem to have any analogy in Shintani's work.

For $t,u,z,v \in \mathbb{C}$ and $\omega,\lambda \in \mathbb{R}^2$ define

$$\mathfrak{g}(t,u,z,v,\omega,\lambda) = \frac{e^{zt}e^{(|\lambda|-v)tu}}{(1-e^{t(\omega_1+u\lambda_1)})(1-e^{t(\omega_2+u\lambda_2)})} - \frac{e^{zt}}{(1-e^{t\omega_1})(1-e^{t\omega_2})} \tag{4.20}$$

where $|\lambda| = \lambda_1+\lambda_2$. This is a holomorphic function in u with a zero at $u=0$. We define a family of functions $C_N(t,v,\omega,\lambda)$ indexed by $N \in \mathbb{N}$ by

$$\mathfrak{g}(t,u,z,v,\omega,\lambda) + \frac{e^{zt}}{(1-e^{t\omega_1})(1-e^{t\omega_2})} = \sum_{N=0}^{\infty} e^{tz} C_N(t,v,\omega,\lambda)u^N. \tag{4.21}$$

We note that

$$C_0(t,v,\omega,\lambda) = \frac{1}{(1-e^{t\omega_1})(1-e^{t\omega_2})},$$

and hence

$$\log(G_2^r(z,\omega)) = \frac{1}{2\pi i}\oint_{I(\lambda,\infty)} \frac{\log(-t)^r}{t} e^{(|\omega|-z)t} C_0(t,v,\omega,\lambda)dt$$

for any $\lambda \in \mathbb{R}^2$, $v \in \mathbb{C}$ where $|\omega| = \omega_1 + \omega_2$.

Suppose h is a function in a real variable vanishing at 0. Let J be the operator defined on such a function by

$$J(h)(u) := -\frac{1}{2\pi i}\int_0^u \frac{1}{t}h(t)dt.$$

Let $\mathfrak{g}(u)$ be the function defined in (4.20), considered as a function of u. Then we have

$$J^k(\mathfrak{g}(u))(1) = (-1)^k \sum_{N=1}^{\infty} \frac{C_N(t, v, \omega, \lambda)}{N^k},$$

where the functions $C_N(t, v, \omega, \lambda)$ are as defined in (4.21). Note that this can be viewed as a variety of zeta function.

Definition 4.5.5. Suppose $\omega, \lambda \in \mathbb{R}^2$ are such that neither of the quotients ω_2/ω_1 or λ_2/λ_1 are negative. For $z, v \in \mathbb{C}$ and $q, k \in \mathbb{N}$ define

$$H^{q,k}(z, v, \omega, \lambda) := \frac{1}{2\pi i}\oint_{I(\lambda,\infty)} e^{(|\omega|-z)t} J^k(\mathfrak{g}(u))(1)\frac{\log(-t)^q}{t}dt. \tag{4.22}$$

Proposition 4.5.6. *For all $k, q, \in \mathbb{N}$, the integral of* (4.22) *converges for $\Re(z) > S$ for some S depending on v, ω and λ. In this region the integral defines an analytic function $H^{k,q}(z, v, \omega, \lambda)$, which is a theta function in z for the pseudolattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.*

Proof. The proof of this result is the subject of §A.1 of the Appendix. □

This last result follows as a result of some crude estimates using Cauchy's integral formula for the derivative of a holomorphic function. We conjecture that this may be strengthened:

Conjecture 4.5.7. *For all $k, q \in \mathbb{N}$, $v \in \mathbb{C}$ and $\omega, \lambda \in \mathbb{R}^2_{>0}$ the integral in* (4.22) *defines a meromorphic function $H^{q,k}(z,v,\omega,\lambda)$, which as a function in z is a theta function for $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$.*

We will implicitly assume Conjecture 4.5.7 for the remainder of this thesis.

4.6 The derivative of L-functions of real quadratic fields

Let F be a real quadratic field and suppose $\mathfrak{f}$ is an integral ideal of F. Let S be a finite set of primes of F containing those primes dividing $\mathfrak{f}$. Let χ be a character of the group $G^+_{\mathfrak{f}}(F)$, and let $L_S(\chi,s)$ denote the corresponding L-function. In §4.4.2, we saw from the results of Shintani that under certain circumstances we can write the value of $L'_S(\chi,0)$ as a linear combination of special values of meromorphic theta functions for pseudolattices lying in F. In this section will prove the following

Theorem 4.6.1. *Let F be a real quadratic field, $\mathfrak{f}$ an integral ideal of F, and χ a character of $I^{\mathfrak{f}}_F$. Let $m\in\mathbb{N}$, and let $L^{(m)}(\chi,s)$ denote the m^{th} derivative of the L-function with respect to s. We may write $L^{(m)}_S(\chi,0)$ as an element of the field $K^m_{\mathfrak{f}}(F)$ generated over F by*

1. *$2\pi i$, the values $\Gamma^{(j)}(1)$ for $j=0,\ldots,m$. The maximal power of $2\pi i$ which occurs is $m+1$;*

2. *the roots of unity of order p, where p is the maximal order of an element of $G^+_{\mathfrak{f}}(F)$;*

3. *the logarithms of a finite number of elements $N_i\in F$ (which are specified in the statement of Lemma 4.6.3);*

4. *the values $Li_n(-\varepsilon^\sigma/\varepsilon)$, $Li_n(-\varepsilon/\varepsilon^\sigma)$ and $Li_n(-1)$ for $n=1\ldots m+1$, where Li_n denotes the n^{th} polylogarithm function, and ε is a generator for the group of totally positive units of F;*

5. *the special values*

$$\log(G_2^r(x_1^i + \varepsilon x_2^i, (1, \varepsilon)))$$
$$\log(G_2^r(x_1^i + \varepsilon^\sigma x_2^i, (1, \varepsilon^\sigma)))$$
$$\log(H^{r,k}(x_1^i + \varepsilon x_2^i, x_1^i + \varepsilon^\sigma x_2^i, (1, \varepsilon), (1, \varepsilon^\sigma)))$$
$$\log(H^{r,k}(x_1^i + \varepsilon^\sigma x_2^i, x_1^i + \varepsilon x_2^i, (1, \varepsilon^\sigma), (1, \varepsilon)))$$

where $\underline{x}^i$ *is one of a finite set of pairs of element of* F *determined by* F *and the choice of* ε*. The highest value of* r *and* k *which occurs is* m.

.

Remark. Throughout the proof I will refer to fields generated over $\mathbb{Q}$ or F by some combinations of these generators. For example, if I wish to refer to the field generated over $\mathbb{Q}$ by those elements in statements 2, 4 and 5 in the statement of Theorem 4.6.1, I shall denote this field by $\mathbb{Q}([2], [4], [5])$.

We will break the proof up in to several stages. The first stage is to recall that we can write the L-function of F as a finite sum of "Shintani L-functions".

Definition 4.6.2 (Shintani L-function). Let $\underline{a} = (a_1, a_2), \underline{x} = (x_1, x_2) \in \mathbb{R}^2$. Then we define the Shintani L-function $\zeta(s, \underline{a}, \underline{x})$ for $\Re(s) > 1$ by

$$\zeta(s, \underline{a}, \underline{x}) = \sum_{m,n=0}^{\infty} \frac{1}{(x_1 + m + (x_2 + n)a_1)^s (x_1 + m + (x_2 + n)a_2)^s}. \tag{4.23}$$

Elements of the proof of the following result were discussed in §4.4.2 when we discussed Shintani's Limit Formula:

Lemma 4.6.3 (Shintani, [54]). *Let* $\varepsilon > 1$ *be a generator for the group of totally positive units of* F*, and let* σ *be the non trivial element of* $Gal(F/\mathbb{Q})$*. There exists*

$N \in \mathbb{N}$, 2-tuples $\underline{x}_1, \ldots, \underline{x}_N \in F^2$, elements $N_i \in F$ and $c_i \in \mu_p$ such that

$$L_S(\chi, s) = \sum_{i=1}^{N} c_i N_i^s \zeta(s, (\varepsilon, \varepsilon^\sigma), \underline{x}_i). \tag{4.24}$$

Differentiating the expression for $L_S(\chi, s)$ in (4.24) m times with respect to s, we see that at $s = 0$ the derivative of the L-function is given by

$$L_F^{(m)}(0, \chi) = \sum_{j=0}^{m} \sum_{i=1}^{N} c_i \binom{m}{j} \log(N_i)^{m-j} \zeta^{(j)}(0, (\varepsilon, \varepsilon^\sigma), \underline{x}_i). \tag{4.25}$$

This expression shows the need to adjoin the roots of unity μ_p and the values $\log(N_i)$ which are mentioned in parts 2 and 3 of the statement of Theorem 4.6.1. With this result in mind, Theorem 4.6.1 will follow if we can prove the following:

Proposition 4.6.4. *Let $m \in \mathbb{N}$ and suppose $\underline{x} \in F^2$. Then with the notation of Theorem 4.6.1, $\zeta^{(m)}(0, (\varepsilon, \varepsilon^\sigma), \underline{x}) \in K_{\mathfrak{f}}^m(F)$.*

4.7 Proof of Theorem 4.6.1

We will prove Theorem 4.6.1 by proving Proposition 4.6.4.

An integral formula for $\zeta(s, \underline{a}, \underline{x})$ is given in [54] as

$$4\pi^2 \frac{(1 + e^{2\pi i s})}{\Gamma(1-s)^2} \zeta(s, a, x) = \int_{I(\lambda,\infty)} (-t)^{2s} \frac{dt}{t} \int_{I(\lambda,1)} u^s \frac{du}{u} [g(t, tu) + g(tu, t)] \tag{4.26}$$

where

$$g(t_1, t_2) = \frac{e^{(1-x_1)(t_1+t_2)+(1-x_2)(a_1 t_1 + a_2 t_2)}}{(1 - e^{t_1+t_2})(1 - e^{a_1 t_1 + a_2 t_2})}.$$

We shall proceed by induction.

Proposition 4.7.1. *For $m = 0, 1$, $\zeta^{(m)}(0, (\varepsilon, \varepsilon^\sigma), \underline{x}) \in K_{\mathfrak{f}}^m(F)$.*

Proof. These results follow from the statement and proof of Proposition 3 of [54]. We let B_1 and B_2 denote the first and second Bernoulli polynomials, which have coefficients in $\mathbb{Q}$. The statement of this result implies that

$$\zeta^{(1)}(0,(\varepsilon,\varepsilon^\sigma),\underline{x}) = \log\left(\Gamma_2^1(x_1+x_2\varepsilon, x_1+x_2\varepsilon^\sigma,(1,\varepsilon),(1,\varepsilon^\sigma))\right)$$

$$+\log\left(\Gamma_2^1(x_1+x_2\varepsilon^\sigma, x_1+x_2\varepsilon,(1,\varepsilon^\sigma),(1,\varepsilon))\right) + \frac{\varepsilon^\sigma-\varepsilon}{4\varepsilon\varepsilon^\sigma}\log\left(\frac{\varepsilon^\sigma}{\varepsilon}\right)B_2(x_1).$$

We may rewrite the final term as

$$\frac{\varepsilon^\sigma-\varepsilon}{4\varepsilon\varepsilon^\sigma}\log\left(\frac{\varepsilon^\sigma}{\varepsilon}\right)B_2(x_1) = \frac{\varepsilon^\sigma-\varepsilon}{4\varepsilon\varepsilon^\sigma}\left[Li_1(-\varepsilon^\sigma/\varepsilon) - Li_1(-\varepsilon/\varepsilon^\sigma)\right]B_2(x_1)$$

since $Li_1(x) = -\log(1-x)$.

In the proof of this result, Shintani also shows that

$$\zeta(0,(\varepsilon,\varepsilon^\sigma),\underline{x}) = \frac{1}{4}\left(\frac{1}{\varepsilon}+\frac{1}{\varepsilon^\sigma}\right)B_2(x_1) + B_1(x_1)B_1(x_2) + \frac{1}{4}(\varepsilon+\varepsilon^\sigma)B_2(x_2).$$

Hence we may write the null values of these derivatives of the zeta function in terms of the double gamma function. Since the field W of Lemma 4.5.3 is contained in $\mathbb{Q}([1])$, the result follows. $\square$

Now fix $m \in \mathbb{N}$, and assume the inductive hypothesis holds for all values of b less than m:

If $\underline{x} \in F^2$ then $\zeta^{(b)}(0,(\varepsilon,\varepsilon^\sigma),\underline{x}) \in K_{\mathfrak{f}}^m(F)$ for all $b = 0\ldots m-1$.

We need to show that $\zeta^{(m)}(0,(\varepsilon,\varepsilon^\sigma),\underline{x}) \in K_{\mathfrak{f}}^m(F)$.

Differentiate both sides of (4.26) m times with respect to s. Evaluating at $s=0$

we see that the left hand side of the result is a finite sum of terms of the form

$$T_{a,b}^{m} := 2\pi^2 (2\pi i)^a \zeta^{(b)}(0, \underline{a}, \underline{x}) \left(\frac{d}{ds}\right)^{m-a-b} \Gamma(1-s)\bigg|_{s=0} \tag{4.27}$$

for $a, b \in \mathbb{N}$ such that $a+b \leq m$. By our inductive hypothesis, if $b \neq m$ then $T_{a,b}^{m} \in K_{\mathfrak{f}}^{m}(F)$. Note that it is at this point we are required to adjoin the higher derivatives of the gamma function in the statement of Theorem 4.6.1. To prove that $\zeta^{(m)}(0, (\varepsilon, \varepsilon^{\sigma}), \underline{x}) \in K_{\mathfrak{f}}^{m}(F)$ it is therefore sufficient to show that the m^{th} derivative of the right hand side of (4.26) lies in $K_{\mathfrak{f}}^{m}(F)$. The rest of this section is devoted to proving that this is indeed true.

When we differentiate m times with respect to s, the right hand side of (4.26) becomes

$$I_m(s) := \int_{I(\lambda,\infty)} (-t)^{2s} \frac{dt}{t} \int_{I(\lambda,1)} u^s \frac{du}{u} [g(t,tu) + g(tu,t)][2\log(-t) + \log(u)]^m.$$

Using the binomial theorem we see that at $s = 0$

$$I_m(0) = \sum_{p=0}^{m} \binom{m}{p} 2^p I_{p,m-p} \tag{4.28}$$

where

$$I_{p,q} := \int_{I(\lambda,\infty)} \frac{dt}{t} \int_{I(\lambda,1)} \frac{du}{u} [g(t,tu) + g(tu,t)] \log(-t)^p \log(u)^q.$$

In order to evaluate integrals of this form we will first consider the integrals

$$A_{p,q} := \int_{I(\lambda,\infty)} \frac{dt}{t} \int_{I(\lambda,1)} \frac{du}{u} g(t,tu) \log(-t)^p \log(u)^q$$

where $p = m - q$ and $q = 0, 1, \ldots, m$. Once we have evaluated the integrals $A_{p,q}$, we shall be able to use our result to evaluate the integrals

$$B_{p,q} := \int_{I(\lambda,\infty)} \frac{dt}{t} \int_{I(\lambda,1)} \frac{du}{u} g(tu,t) \log(-t)^p \log(u)^q. \tag{4.29}$$

Having shown that both $A_{p,q}$ and $B_{p,q}$ lie in $K_{\mathfrak{f}}^m(K)$, since $I_{p,q} = A_{p,q} + B_{p,q}$ it shall follow that $I_{p,q} \in K_{\mathfrak{f}}^m(F)$. Hence by (4.28) we will have shown $I_m(0) \in K_{\mathfrak{f}}^m(F)$.

We begin by noting that

$$\begin{aligned} g(t,tu) &= \mathfrak{g}(t,u,(1-x_1)+a_1(1-x_2),(1-x_1)+a_2(1-x_2),(1,a_1),(1,a_2)) \\ &= -\tfrac{e^{zt}}{(1-e^t)(1-e^{a_1 t})} + \textstyle\sum_{N=1}^{\infty} e^{tz} C_N(t,v,(1,a_1),(1,a_2)) u^N. \end{aligned}$$

where $z := (1-x_1)+a_1(1-x_2)$ and $v = (1-x_1)+a_2(1-x_2)$.

Before we proceed we make a remark which will simplify our calculations. Note that the integral expression we have for $\zeta(s,a,\underline{x})$ in (4.26) is independent of λ for sufficiently small λ, and hence I_m is independent of λ. Hence $I_m = \lim_{\lambda\to 0} I_m$. Now suppose we can write I_m as the sum of finitely many integrals:

$$I_m = \sum_{i=1}^{n} \int_{I(\lambda,\infty)} \int_{I(\lambda,1)} f_i(t,u) du\ dt.$$

Then providing each of the limits is finite, we have

$$I_m = \lim_{\lambda\to 0} I_m = \sum_{i=1}^{n} \lim_{\lambda\to 0} \int_{I(\lambda,\infty)} \int_{I(\lambda,1)} f_i(t,u) du\ dt.$$

We will use this idea to calculate the integrals $A_{p,q}$. There are three cases to consider:

- When $q = 0$ we have

$$A_{p,0} = -\int_{I(\lambda,\infty)} \frac{dt}{t} \int_{I(\lambda,1)} \frac{du}{u} e^{zt} C_0(t,v,(1,a_1),(1,a_2)) \log(-t)^p$$

$$+\int_{I(\lambda,\infty)} \frac{dt}{t} \int_{I(\lambda,1)} \frac{du}{u} \sum_{N=1}^{\infty} e^{tz} C_N(t, v, (1, a_1), (1, a_2)) u^N \log(-t)^p.$$

The second integral vanishes since the integrand does not have poles on or within the contour traced as u traces $I(\lambda, 1)$. By our definitions

$$\begin{aligned} A_{p,0} &= 4\pi^2 \log\left(G_2^p(1 + a_1 - z, (1, a_1))\right) \\ &= 4\pi^2 \log\left(G_2^p(x_1 + a_1 x_2, (1, a_1))\right) \end{aligned}.$$

- Now consider the case when neither p or q are zero. Then

$$A_{p,q} = -\int_{I(\lambda,\infty)} \frac{dt}{t} \int_{I(\lambda,1)} \frac{du}{u} \frac{e^{zt}}{(1 - e^t)(1 - e^{a_1 t})} \log(-t)^p \log(u)^q$$

$$+\int_{I(\lambda,\infty)} \frac{dt}{t} \int_{I(\lambda,1)} \frac{du}{u} \sum_{N=1}^{\infty} e^{tz} C_N(t, v, (1, a_1), (1, a_2)) u^N \log(-t)^p \log(u)^q.$$

Using Lemma A.2.1 (found in the Appendix) we find that this is equal to

$$-\frac{(2\pi i)^{q+1}}{q+1} \int_{I(\lambda,\infty)} \frac{dt}{t} \frac{e^{zt}}{(1 - e^t)(1 - e^{a_1 t})} \log(-t)^p$$

$$+\sum_{k=1}^{q-1} (-1)^k \frac{q!}{(q-k)!} (2\pi i)^{q-k} \int_{I(\lambda,\infty)} \frac{dt}{t} \sum_{N=0}^{\infty} e^{tz} \frac{C_N(t, v, (1, a_1), (1, a_2))}{N^{k+1}} \log(-t)^p.$$

Using the definitions of §4.5 we find that

$$A_{p,q} = -\frac{(2\pi i)^{q+2}}{q+1} \log\left(G_2^p(x_1 + a_1 x_2, (1, a_1))\right)$$

$$-(2\pi i)^{q+1} \sum_{k=0}^{q-1} \frac{q!}{(q-k)!} \log\Big(H^{p,k+1}(x_1 + a_1 x_2, x_1 + a_2 x_2, (1, a_1), (1, a_2))\Big).$$

- Finally we consider the case when $p = 0$. In this case

$$A_{0,q} = -\int_{I(\lambda,\infty)} \frac{dt}{t} \int_{I(\lambda,1)} \frac{du}{u} \frac{e^{zt}}{(1 - e^t)(1 - e^{a_1 t})} \log(u)^q$$

$$+\int_{I(\lambda,\infty)}\frac{dt}{t}\int_{I(\lambda,1)}\frac{du}{u}\sum_{N=1}^{\infty}e^{tz}C_N(t,v,(1,a_1),(1,a_2))u^N\log(u)^q.$$

The coefficient of t^{-1} in the integrand of the first integral is calculated to be

$$\frac{1}{2}\frac{1+u}{a_1+a_2u}B_2(x_1)+B_1(x_1)B_1(x_2)+\frac{1}{2}\frac{a_1+a_2u}{1+u}B_2(x_2).$$

We use Lemma A.2.1 again to calculate the integral over $I(\lambda,1)$ to find that

$$A_{0,q}=-\int_{I(\lambda,1)}\left[\frac{1}{2}\frac{1+u}{a_1+a_2u}B_2(x_1)+B_1(x_1)B_1(x_2)+\frac{1}{2}\frac{a_1+a_2u}{1+u}B_2(x_2)\right]\frac{\log(u)^q}{u}$$

$$+\sum_{k=0}^{q-1}(-1)^k\frac{q!}{(q-k)!}(2\pi i)^{q-k}\int_{I(\lambda,\infty)}\frac{dt}{t}\sum_{N=1}^{\infty}e^{tz}\frac{C_N(t,v,(1,a_1),(1,a_2))}{N^{k+1}}.$$

Using the definitions of §4.5 this simplifies to

$$A_{0,q}=-\int_{I(\lambda,1)}\left[\frac{1}{2}\frac{1+u}{a_1+a_2u}B_2(x_1)+B_1(x_1)B_1(x_2)+\frac{1}{2}\frac{a_1+a_2u}{1+u}B_2(x_2)\right]\frac{\log(u)^q}{u}$$

$$-(2\pi i)^{q+1}\sum_{k=0}^{q-1}\frac{q!}{(q-k)!}\log\left(H^{0,k+1}(x_1+a_1x_2,x_1+a_2x_2,(1,a_1),(1,a_2))\right).$$

To prove the result it suffices to show that the first integral lies in $K_{\mathfrak{f}}^m(F)$. Note that

$$\int_{I(\lambda,1)}\frac{\log(u)^q}{u}\frac{1+u}{a_1+a_2u}du$$

$$=\frac{1}{a_1a_2}\int_{I(\lambda,1)}\log(u)^q\left[\frac{1}{u}-\frac{a_2}{a_1+a_2u}\right]\left[(1-a_1)+(a_1+a_2u)\right]du.$$

We are therefore reduced to calculating the following integrals

$$\int_{I(\lambda,1)}\log(u)^q du; \tag{4.30}$$

$$\int_{I(\lambda,1)} \frac{\log(u)^q}{u} du; \tag{4.31}$$

$$\int_{I(\lambda,1)} \frac{\log(u)^q}{a_1 + a_2 u} du. \tag{4.32}$$

Lemma A.2.1 shows that both the integrals (4.30) and (4.31) lie in $K^m_{\mathfrak{f}}(F)$. By the remark made earlier, to determine (4.32) it is sufficient to evaluate

$$\lim_{\lambda \to 0} \int_{I(\lambda,1)} \frac{\log(u)^q}{a_1 + a_2 u} du.$$

It is easy to see that the integral around the circular path is $O(\lambda)$, so tends to 0 as $\lambda \to 0$. This reduces the evaluation of (4.32) to that of

$$\lim_{\lambda \to 0} \left\{ \int_{\lambda}^{1} \frac{(\log(u) + 2\pi i)^q}{a_1 + a_2 u} du - \int_{\lambda}^{1} \frac{\log(u)^q}{a_1 + a_2 u} du \right\}.$$

Expanding this using the binomial theorem we are reduced to showing that the following expression lies in $K^m_{\mathfrak{f}}(F)$:

$$\lim_{\lambda \to 0} \int_{\lambda}^{1} \frac{\log(u)^q}{a_1 + a_2 u} du.$$

This is proved in Lemma A.2.2.

Hence we have shown that for all $p = 0 \dots m$, $A_{p,m-p} \in K_{\mathfrak{f}}(F)$. The details of the proof describes this in more detail:

Lemma 4.7.2. *There exist* $\alpha_l, \beta_{r,k} \in F([1])$ *and polynomials* Q_1, Q_2 *and* $Q_3 \in \mathbb{Q}([1],[4])[X_1, X_2]$ *such that*

$$A_{p,q} = \sum_l \alpha_l \log\Big(G^l_2(x_1 + a_1 x_2, (1, a_1))\Big)$$

$$+ \sum_{r,k} \beta_{r,k} \log\Big(H^{r,k}(x_1 + a_1 x_2, x_1 + a_2 x_2, (1, a_1), (1, a_2))\Big)$$

$$+B_2(x_1)Q_1[a_1,a_2]+B_1(x_1)B_1(x_2)Q_2[a_1,a_2]+B_2(x_2)Q_3[a_1,a_2].$$

In order to show that $I_m(0)\in K_{\mathfrak{f}}^m(F)$ we need to show that $B_{p,m-p}\in K_{\mathfrak{f}}^m(F)$ where $B_{p,q}$ is defined in (4.29). However, note that

$$\begin{aligned} g(tu,t) &= \mathfrak{g}(t,u,(1-x_1)+a_2(1-x_2),(1-x_1)+a_1(1-x_2),(1,a_2),(1,a_1)) \\ &= \mathfrak{g}(t,u,v,z,(1,a_2),(1,a_1)) \end{aligned}$$

If $A_{p,q}$ has an explicit description as in the statement of Lemma 4.7.2, then the above expression implies that we have

$$B_{p,q}=\sum_l \alpha_l \log\Big(G_2^l(x_1+a_2x_2,(1,a_2))\Big)$$

$$+\sum_{r,k}\beta_{r,k}\log\Big(H^{r,k}(x_1+a_2x_2,x_1+a_1x_2,(1,a_2),(1,a_1))\Big)$$

$$+B_2(x_1)Q_1[a_2,a_1]+B_1(x_1)B_1(x_2)Q_2[a_2,a_1]+B_2(x_2)Q_3[a_2,a_1].$$

Hence $B_{p,q}\in K_{\mathfrak{f}}^m(F)$, and therefore $I_{p,q}=A_{p,q}+B_{p,q}$ does. By (4.28) $I_m(0)\in K_{\mathfrak{f}}^m(F)$.

□

4.8 Meromorphic Theta functions and Stark's conjecture

Let F be a real quadratic number field, and let $\mathfrak{f}$ be an integral ideal of F. Let T be a finite set of primes, such that the finite primes in T are precisely those dividing $\mathfrak{f}$, and suppose χ is a character defined on $I_F^{\mathfrak{f}}$. Let S be a set of places of F containing S. Then the relationship between the L-functions corresponding to T and S is given by

$$L_S(\chi,s)=L_T(\chi,s)\prod_{\mathfrak{p}\in T\setminus S}L_{\mathfrak{p}}(\chi,s). \tag{4.33}$$

In §4.4 we examined the Rank One Abelian Stark conjecture, which concerned the case when the L-function had a simple zero at $s = 0$. Higher order conjectures, exist when the L-function has zeros of order greater than one. These conjectures link the values of derivatives of L-functions over number fields to the existence of units in Galois extensions, and have been the subject of study by Tate [63] and Rubin [49]. In the light of these conjectures we suggest that information concerning the derivatives of L-functions associated to real quadratic fields will be relevant to Real Multiplication.

In §4.6 we identified a certain field $K_{\mathfrak{f}}^m(F)$ in which the value $L_S^{(m)}(\chi,0)$ lies. If S contained both real primes of F then we observe from (4.13) that $r(\chi) = 2$, and $L_S(\chi,0)$ has a second order zero at $s = 0$. Although we have not explicitly done so, using our method it is possible to give an exact formula for the value $L^{(2)}(\chi,0)$ in terms of the theta functions G_2^r and $H^{k,r}$, and hence to formulate a conjecture on how these functions may define units in a class field above F.

When T is a set of primes containing S, the L-function $L_T(\chi,0)$ may have zeros at $s = 0$ of arbitrary order. Using (4.33) we see that

$$L_T^{(m)}(\chi,0) \in K_{\mathfrak{f}}^m(F)\left(\{\log(N_{F/\mathbb{Q}}(\mathfrak{p})) : \mathfrak{p} \in T \setminus S\}\right).$$

Using analogous techniques to the proof of Theorem 4.6.1 we could obtain an explicit formula for this value in terms of meromorphic theta functions for pseudolattices in F.

Theorem 4.6.1 is a far cry from an immediate application to Hilbert's twelfth problem. It does not give an explicit description for the value of the derivative of the L-function (although this is implicit in the proof), and the field $K_{\mathfrak{f}}^m(F)$ is clearly not a number field. However, motivated by a technique of Shintani's [55] we can write certain L-values purely in terms of theta functions, without the transcendental constant terms:

Theorem 4.8.1. *Let F be a real quadratic field, $\mathfrak{f}$ an integral ideal of F, and χ a character of $G_{\mathfrak{f}}(F)$. Let $m, n \in \mathbb{N}$. Let $\mu \in F$ be a totally positive element of $\mathcal{O}_F$ such that $\nu \equiv 1 \mod \mathfrak{f}$, and suppose $g \in G_{\mathfrak{f}}^{+}(F)$. If $S = S(\mathfrak{f})$ then the value $L_S^{(m)}(0, g) - L_S^{(m)}(0, [\nu]_{\mathfrak{f}}^{+} g)$ is an element of the field $K_{\mathfrak{f}}(F)$ generated over F by a finite number of meromorphic theta functions.*

Proof. By the proof of Theorem 1 of [55], with the notation used previously we have

$$L_S(s, g) = N(\mathfrak{f}\mathfrak{a}_j)^{-s} \sum_{x_1 + \varepsilon x_2 \in R(g)} \zeta(s, (\varepsilon, \varepsilon^{\sigma}), (x_1, x_2)).$$

Differentiating m times with respect to s we get

$$L_S^{(m)}(0, g) = \sum_{x_1 + \varepsilon x_2 \in R(g)} \sum_{k=0}^{m} \binom{m}{k} \log(N(\mathfrak{f}\mathfrak{a}_j))^{m-k} \zeta^{(k)}(0, (\varepsilon, \varepsilon^{\sigma}), (x_1, x_2)).$$

By Lemma 4.7.2, by employing an induction argument it is easy to show that there exist coefficients $a_l, b_{r,k} \in F([1])$ and polynomials R_1, R_2 and $R_3 \in \mathbb{Q}([1, 4])$ such that

$$\begin{aligned}
\zeta^{(m)}(0, (\varepsilon, \varepsilon^{\sigma}), (x_1, x_2)) = &\sum_{l} a_l \log\Big(G_2^l(x_1 + x_2\varepsilon, (1, \varepsilon))\Big) \\
&+ \sum_{l} a_l \log\Big(G_2^l(x_1 + x_2\varepsilon^{\sigma}, (1, \varepsilon^{\sigma}))\Big) \\
&+ \sum_{r,k} b_{r,k} \log\Big(H^{r,k}(x_1 + x_2\varepsilon, x_1 + x_2\varepsilon^{\sigma}, (1, \varepsilon), (1, \varepsilon^{\sigma}))\Big) \\
&+ \sum_{r,k} b_{r,k} \log\Big(H^{r,k}(x_1 + x_2\varepsilon^{\sigma}, x_1 + x_2\varepsilon, (1, \varepsilon^{\sigma}), (1, \varepsilon))\Big) \\
&+ B_2(x_1)R_1[\varepsilon, \varepsilon^{\sigma}] + B_1(x_1)B_1(x_2)R_2[\varepsilon, \varepsilon^{\sigma}] + B_2(x_2)R_3[\varepsilon, \varepsilon^{\sigma}].
\end{aligned}$$

For $z = x_1 + x_2\varepsilon \in R(g)$, by the proof of Theorem 1 of [55] the map

$$z \mapsto \overline{-z} := \begin{cases} 1 - x_1 & \text{if } x_2 = 0,\ 0 < x_1 < 1 \\ 1 - x_1 + (1 - x_2)\varepsilon & \text{if } 0 < x_1, x_2 < 1 \\ 1 + (1 - x_2)\varepsilon & \text{if } x_1 = 1,\ 0 < x_2 < 1 \end{cases}$$

is a bijection between $R(g)$ and $R([\nu]_{\mathfrak{f}}^{+} g)$. If $\overline{-z} = \overline{x}_1 + \overline{x}_2\varepsilon$, then observe that $B_2(x_1) = B_2(\overline{x}_1)$, $B_1(x_1)B_1(x_2) = B_1(\overline{x}_1)B_1(\overline{x}_2)$ and $B_2(x_2) = B_2(\overline{x}_2)$. Hence when we compute $L_S^{(m)}(0, g) - L_S^{(m)}(0, [\nu]_{\mathfrak{f}}^{+} g)$ using the above expression, the presence of the terms with the polynomials R_1, R_2 and R_3 vanish. We define

$$\mathbf{G}^r(z, \varepsilon) := \frac{G_2^r(z, (1, \varepsilon))}{G_2^r(1 + \varepsilon - z, (1, \varepsilon))},$$

$$\mathbf{H}^{r,k}(z, v, \varepsilon, \varepsilon^\sigma) := \frac{H^{r,k}(z, v, (1, \varepsilon), (1, \varepsilon^\sigma))}{H^{r,k}(1 + \varepsilon - z, v, (1, \varepsilon), (1, \varepsilon^\sigma))}.$$

It follows that $L_S^{(m)}(0, g) - L_S^{(m)}(0, [\nu]_{\mathfrak{f}}^{+} g)$ can be written as a finite linear combination with coefficients in F of values of the form

$$\log(N(\mathfrak{f}\mathfrak{a}_j))^i \mathbf{G}^r(x_1 + x_2\varepsilon, \varepsilon)$$
$$\log(N(\mathfrak{f}\mathfrak{a}_j))^i \mathbf{G}^r(x_1 + x_2\varepsilon^\sigma, \varepsilon^\sigma)$$
$$\log(N(\mathfrak{f}\mathfrak{a}_j))^j \mathbf{H}^{r,k}(x_1 + x_2\varepsilon, x_1 + x_2\varepsilon^\sigma, \varepsilon, \varepsilon^\sigma)$$
$$\log(N(\mathfrak{f}\mathfrak{a}_j))^j \mathbf{H}^{r,k}(x_1 + x_2\varepsilon^\sigma, x_1 + x_2\varepsilon, \varepsilon^\sigma, \varepsilon).$$

□

The proof of Theorem 4.6.1 in §4.7 would enable us to give an explicit expression for the value $L_S^{(m)}(\chi, 0)$ as an element of of the field $K^m(F)$, which is transcendental over F. By the proof of Theorem 4.8.1 we could obtain an expression for

$$L_S^{(m)}(0, g) - L_S^{(m)}(0, [\nu]_{\mathfrak{f}}^{+} g)$$

as an element of a field $L_{\mathfrak{f}}(F)$. It would be interesting to investigate whether the field $L_{\mathfrak{f}}(F)$ is algebraic over F, or if not, whether any subfield of it was. Indeed, according to higher order versions of Stark's conjectures certain combinations (defined by the explicit expression for the L-value) of the special values of theta functions are strongly related to units in some algebraic extension of F.

Appendix A

Integral Calculations

A.1 The analyticity of $H^{q,k}$

Lemma A.1.1. *Fix $v \in \mathbb{C}$ and $\omega, \lambda \in \mathbb{R}^2$ such that neither of the quotients ω_2/ω_1 and λ_2/λ_1 are negative, and assume that $|\omega_i| > |\lambda_i|$ for $i = 1, 2$. Then there exists $R, r > 1$ such that for $|t|$ sufficiently large and for all N*

$$|C_N(t, v, \omega, \lambda)| \leq \frac{1}{r^N} \frac{\max\left\{e^{r|t||\lambda_1+\lambda_2-v|}, e^{-r|t||\lambda_1+\lambda_2-v|}\right\}}{R^2}.$$

Proof. By the conditions on ω and λ, there exists $r > 1$ such that the function $\mathfrak{g}(t, u, z, v, \omega, \lambda)$ defined in (4.20) is a meromorphic function in u possessing no poles in the circle $\{|u| < r\}$ other than the one at zero. By the definition of the functions $C_N(t, v, \omega, \lambda)$ in (4.21), by Cauchy's formula we have

$$C_N(t, v, \omega, \lambda) = \frac{1}{2\pi i} \oint_{|u|=r} \frac{1}{u^{N+1}} \frac{e^{(|\lambda|-v)tu}}{(1 - e^{t(\omega_1+\lambda_1 u)})(1 - e^{t(\omega_2+\lambda_2 u)})} du.$$

Therefore we obtain

$$|C_N(t, v, \omega, \lambda)| \leq \frac{1}{2\pi} \frac{2\pi r}{r^{N+1}} \max_{|u|=r} \left\{ \frac{e^{(|\lambda|-v)tu}}{(1 - e^{t(\omega_1+\lambda_1 u)})(1 - e^{t(\omega_2+\lambda_2 u)})} \right\}.$$

Now consider

$$\left|\frac{e^{(|\lambda|-v)tu}}{(1-e^{t(\omega_1+\lambda_1 u)})(1-e^{t(\omega_2+\lambda_2 u)})}\right| = \frac{\left|e^{(|\lambda|-v)tu}\right|}{\left|1-e^{t(\omega_1+\lambda_1 u)}\right|\left|1-e^{t(\omega_2+\lambda_2 u)}\right|}$$

$$\leq \frac{\left|e^{(|\lambda|-v)tu}\right|}{\left|1-\left|e^{t(\omega_1+\lambda_1 u)}\right|\right|\left|1-\left|e^{t(\omega_2+\lambda_2 u)}\right|\right|}$$

$$\leq \frac{\max_{|u|=r}\left|e^{(|\lambda|-v)tu}\right|}{\min_{|u|=r}\left|1-\left|e^{t(\omega_1+\lambda_1 u)}\right|\right|\min_{|u|=r}\left|1-\left|e^{t(\omega_2+\lambda_2 u)}\right|\right|}. \tag{A.1}$$

The remainder of the proof is concerned with obtaining bounds for these maxima and minima. On the circle we have $u = r^{i\theta}$ for $0 \leq \theta \leq 2\pi$. We first consider the denominator of (A.1), and put $t = t_1 + it_2$:

$$\left|1-\left|e^{t(\omega_1+\lambda_1 u)}\right|\right| = \left|1-e^{\Re(t(\omega_1+\lambda_1 u))}\right|$$

$$= \left|1-e^{t_1\omega_1+rt_1\lambda_1\cos(\theta)-rt_2\lambda_1\sin(\theta)}\right|$$

This expression assumes its extremal values when $t_1\cos(\theta) - t_2\sin(\theta)$ does, which are equal to $\pm|t|$. Hence

$$\min_{|u|=r}\left|1-\left|e^{t(\omega_1+\lambda_1 u)}\right|\right| = \min\left\{\left|1-e^{t_1\omega_1+r\lambda_1|t|}\right|, \left|1-e^{t_1\omega_1-r\lambda_1|t|}\right|\right\},$$

and we obtain a similar expression for

$$\min_{|u|=r}\left|1-\left|e^{t(\omega_2+\lambda_2 u)}\right|\right|.$$

Therefore, there exists a $T \in \mathbb{R}$ such that if $|t| > T$ then

$$\min\left\{\min_{|u|=r}\left|1-\left|e^{t(\omega_1+\lambda_1 u)}\right|\right|, \min_{|u|=r}\left|1-\left|e^{t(\omega_2+\lambda_2 u)}\right|\right|\right\} \geq R.$$

Now consider the numerator of (A.1), and put $v = v_1 + iv_2$. We calculate

$$\begin{aligned}(|\lambda| - v)tu &= (\lambda_1 + \lambda_2 - v_1 - iv_2)(t_1 + it_2)u \\ &= ([(\lambda_1 + \lambda_2 - v_1)t_1 + v_2 t_2] + i\,[(\lambda_1 + \lambda_2 - v_1)t_2 - t_1 v_2])\,u\end{aligned}$$

The real part of the above expression is equal to

$$r\,[(\lambda_1 + \lambda_2 - v_1)t_1 + v_2 t_2]\cos(\theta) - r\,[(\lambda_1 + \lambda_2 - v_1)t_2 - t_1 v_2]\sin(\theta).$$

Hence the extremal values of

$$\left|e^{(|\lambda| - v)tu}\right| = \left|e^{(|\lambda| - v)tr(\cos(\theta) + i\sin(\theta))}\right|$$

are $e^{\pm E}$ where

$$\begin{aligned}E^2 &= r^2\,[(\lambda_1 + \lambda_2 - v_1)t_1 + v_2 t_2]^2 + r^2\,[(\lambda_1 + \lambda_2 - v_1)t_2 - t_1 v_2]^2 \\ &= r^2\left((\lambda_1 + \lambda_2 - v_1)^2 t_1^2 + v_2^2 t_2^2 + (\lambda_1 + \lambda_2 - v_1)^2 t_2^2 + t_1^1 v_2^2\right) \\ &= r^2\left((\lambda_1 + \lambda_2 - v_1)^2\,|t|^2 + v_2^2\,|t|^2\right) \\ &= r^2\,|t|^2\,|\lambda_1 + \lambda_2 - v|^2\end{aligned}$$

□

Corollary A.1.2. *Under the same conditions as Lemma A.1.1, the integral*

$$\int_{I(\lambda,\infty)} e^{(|\omega| - z)t} J^k(\mathfrak{g}(u))(1)\frac{\log(-t)}{t}dt$$

converges if $\Re(z) > r\,|\lambda_1 + \lambda_2 - v| + \omega_1 + \omega_2$.

Proof. The integrand is bounded on the circular path around the origin, so it suffices to show that the following integrals converge:

$$\int_{\lambda}^{\infty} e^{(|\omega| - z)t} J^k(\mathfrak{g}(u))(1)\frac{\log(-t)}{t}dt;$$

$$\int_{\lambda}^{\infty} e^{(|\omega|-z)t} J^k(\mathfrak{g}(u))(1) \frac{\log(-t)+2\pi i}{t} dt.$$

Note that since t dominates $\log(t)$ there exists R_1 such that if $t > R_1$.

$$\max\left\{\left|\frac{\log(-t)}{t}\right|, \left|\frac{\log(-t)+2\pi i}{t}\right|\right\} < 1.$$

Once again, on any finite interval (λ, R) the integrands are bounded so it suffices to show that the following integral is convergent for sufficiently large R:

$$\int_{R}^{\infty} \left|e^{(|\omega|-z)t}\right| \left|J^k(\mathfrak{g}(u))(1)\right| dt \tag{A.2}$$

for sufficiently large R.

$$\left|J^k(\mathfrak{g}(u))(1)\right| \leq \left|\sum_{N=1}^{\infty} \frac{C_N(t,v,\omega,\lambda)}{N^k}\right|$$

$$\leq \sum_{N=1}^{\infty} |C_N(t,v,\omega,\lambda)|$$

$$\leq \frac{\max\left\{e^{r|t||\lambda_1+\lambda_2-v|}, e^{-r|t||\lambda_1+\lambda_2-v|}\right\}}{R^2} \sum_{N=1}^{\infty} \frac{1}{r^N}.$$

Then the expression of (A.2) is less than or equal to

$$\frac{1}{1-r}\frac{1}{R^2}\int_{R}^{\infty} \left|e^{(|\omega|-z)t}\right| \max\left\{e^{r|t||\lambda_1+\lambda_2-v|}, e^{-r|t||\lambda_1+\lambda_2-v|}\right\} dt$$

If $\Re(z) > r\,|\lambda_1+\lambda_2-v| + \omega_1+\omega_2$ then this integral converges.

□

Corollary A.1.3. *The integral* (4.22) *defines an analytic theta function for the pseudolattice* $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *in the region* $\Re(z) > r\,|\lambda_1+\lambda_2-v| + \omega_1+\omega_2$.

Proof. Using the integral formula we see that for $i = 1, 2$

$$\log(H^{k,q}(z+\omega_i, v, \omega, \lambda))$$

$$= \frac{1}{2\pi i}\oint_{I(\lambda,\infty)} e^{(|\omega|-z)t}(1-e^{\omega_i t})J^k(\mathfrak{g}(u))(1)\frac{\log(-t)}{t}dt + \log(H^{k,q}(z, v, \omega, \lambda)).$$

In a similar way to the proof of Corollary A.1.2 one can show that the above integral converges for $\Re(z) > r\,|\lambda_1 + \lambda_2 - v| + \omega_1 + \omega_2$. □

A.2 Contour Integrals used in §4.7

In this appendix we calculate two integrals that we use in Chapter 4 §4.7 whilst proving Theorem 4.6.1.

Lemma A.2.1. *Let $m \in \mathbb{Z}$, and $r \in \mathbb{N}$. Then*

$$\int_{I(\lambda,1)} u^m \log^r(u)du = \begin{cases} \sum_{k=0}^{r-1}(-1)^k \frac{r!}{(r-k)!}\frac{(2\pi i)^{r-k}}{(m+1)^{k+1}} & \textit{if } m \neq -1 \\ \frac{(2\pi i)^{r+1}}{r+1} & \textit{if } m = -1. \end{cases}$$

Proof. Since the integrand is holomorphic at all points away from 0, we know the value of the integral is independent of λ. We will show that we can split this integral in to a finite sum of finite integrals, and take the limit as $\lambda \to 0$. We have

$$\int_{I(\lambda,1)} u^m \log^r(u)du = \int_1^\lambda u^m \log^r(u) + \int_\lambda^1 u^m\left[\log(u) + 2\pi i\right]^r$$

$$+ i\lambda^{m+1}\int_0^{2\pi} e^{(m+1)i\theta}\left[\log(\lambda) + 2\pi i\theta\right]^r d\theta.$$

The second integral is $O(\lambda)$. The first integral is equal to

$$\sum_{k=0}^{r-1}\binom{r}{k}(2\pi i)^{r-k}\int_\lambda^1 u^m \log^k(u)du. \tag{A.3}$$

Let

$$I_{m,k} = \int_{\lambda}^{1} u^m \log(u)^k du$$

and note the decomposition $u^m \log(u)^k = u^{-1} \times u^{m+1} \log(u)^k$. Using integration by parts we obtain

$$I_{m,k} = \left[u^{m+1} \log(u)^{k+1} \right]_{\lambda}^{1} - (m+1) I_{m,k+1} - k I_{m,k}.$$

We are only interested in the limit of these integrals as $\lambda \to 0$. Taking this limit we obtain

$$\lim_{\lambda \to 0} I_{m,k} = -\frac{k}{m+1} \lim_{\lambda \to 0} I_{m,k-1}.$$

This yields

$$\lim_{\lambda \to 0} I_{m,k} = (-1)^k \frac{k!}{(m+1)^{k+1}}$$

which when substituted in to (A.3) yields the result. □

Lemma A.2.2. *Let d be a non-negative integer. Then*

$$\lim_{\lambda \to 0} \int_{\lambda}^{1} \frac{\log(u)^r}{a_1 + a_2 u} du = \frac{r!}{a_2} Li_{r+1}\left(-\frac{a_2}{a_1} \right).$$

Proof. We first observe the identity

$$\int_0^t \frac{\log^r(u)}{a_1 + a_2 u} du = \frac{1}{a_2} \sum_{i=0}^{r} (-1)^{i+1} Li_{i+1}(-at/b) \log(t)^{r-i} \frac{r!}{(r-i)!}. \tag{A.4}$$

This is easy to prove by differentiation, and using the identities

$$Li_1(z) = -\log(1-z) \qquad \text{and} \qquad Li_{s+1}(z) = \int_0^z \frac{Li_s(t)}{t} dt.$$

Now split the integral in the statement of the lemma in to two parts:

$$\int_{\lambda}^{1} \frac{\log(u)^d}{a_1 + a_2 u} du = \int_{0}^{1} \frac{\log(u)^d}{a_1 + a_2 u} du - \int_{0}^{\lambda} \frac{\log(u)^d}{a_1 + a_2 u} du.$$

Evaluating the integral of (A.4) at $t = 1$ yields $\frac{r!}{a_2} Li_{r+1}$. The order of vanishing of $Li_j(t)$ is at least 1 for $j \geq 1$. Hence $Li_j(\lambda)\log(\lambda)^n \to 0$ as $\lambda \to 0$ for any $j \geq 1, n \in \mathbb{N}$. □

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