# Exponential and Weierstrass Equations

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**Abstract**

We axiomatize the theories of the exponential and the Weierstrass differential equations and show that they can be obtained from amalgamation constructions in the style of Hrushovski.

## Contents

# 1 Introduction

In this paper we axiomatize the theories of the exponential and the Weierstrass differential equations and show that they can be obtained from amalgamation constructions in the style of Hrushovski. More precisely, let $\langle F; +, \cdot, D\rangle$ be a differentially closed field and define a binary predicate $E(x, y)$ by $Dy = yDx$. By the theory of the exponential equation we mean the theory of the reduct $\langle F; +, \cdot, E\rangle$, and similarly for Weierstrass equations.

Hrushovski originally produced his constructions in [6] as counterexamples to Zilber's trichotomy conjecture on strongly minimal sets. The impetus behind the conjecture was the idea that strongly minimal sets might all come from mainstream mathematical objects. The results presented here show that it is possible to view some of these constructions as natural mathematical objects and not just as pathologies. Only the amalgamation part of Hrushovski's construction is involved here, not the collapse to finite Morley Rank.

The amalgamation constructions make use of a predimension notion, and the first indication that they might be related to other parts of mathematics was the observation that this predimension also appears in Schanuel's conjecture of transcendental number theory. If for complex numbers $a_1, \ldots, a_n$ we define

$$\delta(a_1, \ldots, a_n) = \operatorname{td}(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) - \operatorname{ldim}_{\mathbb{Q}}(a_1, \ldots, a_n)$$

then Schanuel's conjecture states precisely that $\delta(a_1, \ldots, a_n) \geqslant 0$.

In [1], James Ax proved a differential field version of Schanuel's conjecture ("Schanuel condition"). This theorem plays an important role in the current work. Brownawell and Kubota extended this work to Weierstrass $\wp$-functions in [2] and, using this result and a theorem of Seidenberg, I proved a version for Weierstrass equations in arbitrary differential fields in [8]. Here I give a different proof of this result, essentially adapting Brownawell and Kubota's proof rather than using Seidenberg's theorem. The use of some model theoretic ideas makes the resulting proof shorter than the original.

Section 2 of this paper gathers together background on differentials and differential forms in differential algebra. Amongst other sources for this are [1], [2], and David Pierce's paper [12]. This section also includes some technical lemmas towards the proofs in the later sections.

The third section gives the universal theory of the reducts, including the Schanuel conditions mentioned above. Section 4 does the amalgamation construction to produce the complete first order theories. The ideas here were refined after discussions with Assaf Hasson, and can be seen as an example of the general procedure described in [5].

The final section proves that the reducts of differentially closed fields satisfy the existentially closed condition which completes the theory of the amalgams. Cecily Crampin proved essentially this for the exponential equation, and it is to appear in her thesis. The current proof is adapted from hers.

The presentation of this paper is not as clear as I would like. However, I will not now improve it substantially until I have written up my DPhil, and after some requests I have decided to make it available as it is.

**Acknowledgements** This is work carried out as part of my DPhil supported by the EPSRC and under the supervision of Boris Zilber. I thank him for many useful discussions. I would also like to thank Rachel for inspiration.

# 2 Differentials

## 2.1 Derivations and Kähler differentials

Let $R$ be a ring and $M$ an $R$-module. A *derivation* of $R$ into $M$ is a function $R \xrightarrow{D} M$ such that for every $a, b \in R$, $D(a+b) = Da + Db$ and $D(ab) = aDb + bDa$ (the Leibniz rule).

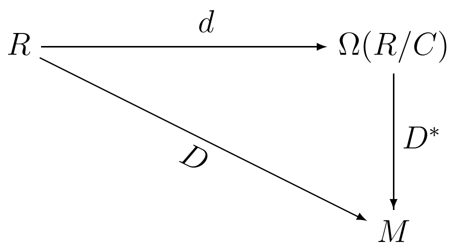
An element $a \in R$ is called a *constant* of $D$ iff $Da = 0$, and the set of all constants is denoted $C$, or $C_D$ if the derivation has to be specified. A well known fact (see for example [4, section 16], which is a good reference for this section) is that if $R$ is a field of characteristic 0 then $C$ is a subfield of $R$ and is algebraically closed in $R$.

From now on we assume all rings are entire (have no zero divisors) and are of characteristic zero. We restrict attention to derivations which are constant on a subfield $C$ which by the above we may assume to be relatively algebraically closed.

Given a ring $R$ and a subfield $C$ there is a *universal derivation* from $R$ which is constant on $C$ constructed as follows. Let $\Omega(R/C)$ be the $R$-module which is generated by the set of symbols $\{da \mid a \in R\}$ subject to the relations that $d : R \to \Omega(R/C)$ is a derivation (is additive and satisfies the Leibniz rule) and that $da = 0$ for each $a \in C$. So $\Omega(R/C)$ is given by the set of terms of the form $\sum_{i=1}^{n} a_i db_i$ for $n \in \mathbb{N}$ and $a_i, b_i \in R$, quotiented out by the equivalence relation generated by the relations described. The $R$-module structure and the map $d$ are defined in the obvious way.

The universal property satisfied by $d$ is the following. If $R \xrightarrow{D} M$ is any derivation of $R$ constant on $C$ then there is a unique $R$-linear map

$\Omega(R/C) \xrightarrow{D^*} M$ such that the following diagram commutes.



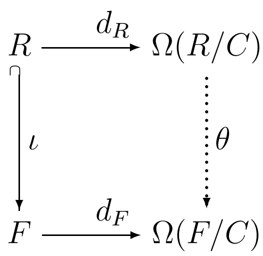

The map $D^*$ is given by setting $D^*(da) = Da$ and extending $R$-linearly, and the universality follows immediately from the construction.

The elements of $\Omega(R/C)$ are called the *Kähler differentials* of $R/C$ or just *differentials*.

**Lemma 2.1.** *Suppose* $b_1, \ldots, b_n \in R$, $f$ *is a polynomial over* $C$ *and* $f(\bar{b}) = 0$. *Then* $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} db_i = 0$.

*Proof.* By induction on $f$ using the Leibniz and addition rules. □

Suppose that $C \subseteq R \subseteq F$ with $F$ and $C$ fields (of characteristic 0) and $R$ a ring and consider the following diagram.




The maps $d_R$ and $d_F$ are the universal derivations of $R/C$ and $F/C$ respectively. The $F$-vector space $\Omega(F/C)$ is also an $R$-module and the map $d_F \circ \iota$ is a derivation. The map $\theta$ is that arising from the universal property of $d_R$, and is given by $\theta(d_R a) = d_F a$ and $R$-linearity.

**Proposition 2.2.** *The kernel of* $\theta$ *consists of the torsion elements of* $\Omega(R/C)$.

*Proof.* Since $F$ is a field, $\Omega(F/C)$ has no torsion, and so every torsion element of $\Omega(R/C)$ must be in the kernel.

Now suppose that $\omega = \sum_{i=1}^{n} a_i d_R b_i \in \Omega(R/C)$ and that $\theta(\omega) = \sum_{i=1}^{n} a_i d_F b_i = 0$.

By reordering, we may assume that $b_1, \ldots, b_m$ form a transcendence base for $C(b_1, \ldots, b_n)/C$, for some $m \leqslant n$. For $i = m+1, \ldots, n$ let $f_i \in C[x_1, \ldots, x_m, y]$ be a polynomial giving the algebraic dependence of $b_i$ on

$b_1, \ldots, b_m$. That is, $f_i(b_1, \ldots, b_m, b_i) = 0$ and $\frac{\partial f_i}{\partial y} \neq 0$. These are polynomial equations in $R$, so from lemma 2.1 we have $\frac{\partial f_i}{\partial x_i} d_R b_i = -\sum_{j=1}^m \frac{\partial f_i}{\partial x_j} d_R b_j$ for each $i$. Thus

$$\left(\prod_{i=m+1}^{n} \frac{\partial f_i}{\partial x_i}\right) \omega = \sum_{j=1}^{m} \left( a_i \left(\prod_{i=m+1}^{n} \frac{\partial f_i}{\partial x_i}\right) + \sum_{k=m+1}^{n} a_k \frac{\partial f_k}{\partial x_j} \prod_{i=m+1, i\neq k}^{n} \frac{\partial f_i}{\partial x_i} \right) d_R b_j$$

and I'll abbreviate the right hand side as $\sum_{j=1}^m e_j d_R b_j$. Note that each $e_j \in R$. Applying $\theta$, we have

$$\left(\prod_{i=m+1}^{n} \frac{\partial f_i}{\partial x_i}\right) \theta(\omega) = \sum_{j=1}^{m} e_i d_F b_j$$

but $\theta(\omega) = 0$ and thus $\sum_{j=1}^m e_j d_F b_j = 0$. Since $b_1, \ldots, b_m$ are algebraically independent over $C$, it follows that $d_F b_1, \ldots, d_F b_m$ are linearly independent over $F$. Thus each $e_j = 0$ and so $\sum_{j=1}^m \omega = 0$ in $\Omega(R/C)$. Thus $\omega$ is a torsion element of $\Omega(R/C)$. □

In general $\Omega(R/C)$ may have torsion, but if $R$ is a field then it is a vector space so this can't happen. Indeed lemma 1 of [2] shows that if $R$ is a valuation ring of $F$ containing $C$, or more generally is a local ring, then $\Omega(R/C)$ is torsion-free. Thus if $R$ is a subfield of $F$ or a valuation ring of $F$, the natural map $\Omega(R/C) \longrightarrow \Omega(F/C)$ is an embedding of $R$-modules.

It is easy to see that the set of derivations from $R$ to $R$ over a subfield $C$ forms an $R$-module, which we write $\mathrm{Der}(R/C)$. The universal property of $d$ gives a natural isomorphism between $\mathrm{Der}(R/C)$ and the $R$-module of $R$-linear maps $\Omega(R/C) \longrightarrow R$. When $R$ is a field, $F$, this module is the dual vector space $\Omega(F/C)^*$.

Any vector space over a field embeds naturally in its double dual, so we may consider $\Omega(F/C)$ to be embedded in $\mathrm{Der}(F/C)^*$. From this point of view, the differential $dx$ is the map which sends a derivation $D$ to its value at $x$, and other differentials are $F$-linear combinations of such maps. When the transcendence degree of $F/C$ is finite, the embedding is an isomorphism and every linear map in $\mathrm{Der}(F/C)^*$ is of this form, but when the transcendence degree is infinite this is not true.

One advantage of thinking about differentials as elements of $\mathrm{Der}(F/C)^*$ is that it gives a "coordinate free" representation of them. If we define something for $\mathrm{Der}(F/C)^*$ then we don't have to check that it is well-defined under a change of representation as $\sum a_i db_i$.

The Schanuel conditions of this paper are relationships between transcendence degree (algebraic dimension) and linear dimension. Differentials can be used as a tool for proving these because of the following well-known fact, which can be found for example as Theorem 16.14 of [4] (although since the map $d$ is not injective, the precise statement there is incorrect).

**Theorem 2.3.** *If* $\mathrm{td}(F/C)$ *is finite and* $B$ *is a transcendence base of* $F$ *over* $C$ *then* $\{db \mid b \in B\}$ *is a basis for* $\Omega(F/C)$ *as an* $F$*-vector space. In particular* $\dim \Omega(F/C) = \mathrm{td}(F/C)$.

## 2.2 Extensions of differential fields

In order to prove existentially closed properties of solutions to differential equations, we need to see what extensions a given differential field can have. For this we need a more general notion of differential. This is basically the idea described in [12] and independently of this was also used by Cecily Crampin and Alex Wilkie.

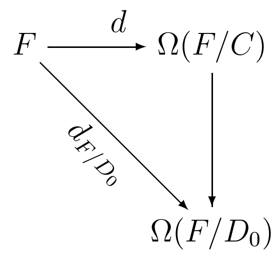
Let $\langle F_0, D_0 \rangle$ be a differential field of characteristic zero, which we assume to be algebraically closed and with constant field $C$, and let $F$ be a field extension of $F_0$. We want to consider the set of extensions of $D_0$ to $F$, but this isn't a vector space so instead we consider a slightly larger set.

Define $\mathrm{Der}(F/D_0) = \{D \in \mathrm{Der}(F/C) \mid \exists \lambda \in F(D \restriction_{F_0} = \lambda D_0)\}$. This is an $F$-vector subspace of $\mathrm{Der}(F/C)$ and the set of actual extensions of $D_0$ is a codimension 1 affine subspace (assuming that $F_0 \neq C$).

If we choose one extension $D_1$ of $D_0$ to $F$, this gives a bijection between the set of all extensions of $D_0$ and $\mathrm{Der}(F/F_0)$ given by $D \mapsto D - D_1$. From this we see that $\dim \mathrm{Der}(F/D_0) = \mathrm{td}(F/F_0) + 1$.

**Lemma 2.4.** *The space* $\mathrm{Der}(F/D_0)$ *is a Lie algebra over* $C$ *(so with the* $F$*-vector space structure is a Lie ring).* $\square$

We assume for convenience that $\mathrm{td}(F/F_0)$ is finite. Define $\Omega(F/D_0)$ to be the dual space of $\mathrm{Der}(F/D_0)$. We can consider this as a quotient of $\Omega(F/C)$ by the restriction map $\omega \mapsto \omega \restriction_{\mathrm{Der}(F/D_0)}$. From another point of view, this is the map arising from the universal property of $\Omega(F/C)$. There is a derivation $F \xrightarrow{d_{F/D_0}} \Omega(F/D_0)$ and the diagram



commutes. Where no ambiguity is likely to arise I will write the restriction of $\omega$ just as $\omega$, and $d_{F/D_0}$ as $d$.

Suppose that $b_1, \ldots, b_n$ is a transcendence base for $F/F_0$, and let $b_0 \in F_0 \setminus C$. Then $db_0, \ldots, db_n$ is a basis for $\Omega(F/D_0)$. (If $F_0 = C$ then $\Omega(F/D_0) = \Omega(F/C)$ and $db_1, \ldots, db_n$ is a basis.)

**Lemma 2.5.** *$\Omega(F/D_0) \cong \Omega(F/F_0) \oplus \langle dx \rangle$ as $F$-vector spaces, where $x \in F_0 \setminus C$.*

*Proof.* There is a natural restriction map from $\Omega(F/D_0)$ to $\Omega(F/F_0)$ which is surjective and has kernel $\langle dx \rangle$. □

## 2.3 Kinds of differentials

The cohomological notions of exact and closed differential are standard, but we give the definitions for completeness.

**Definition 2.6.** A differential $\omega \in \Omega(F/C)$ is *exact* iff there is $x \in F$ such that $\omega = dx$.

There is a natural map $\Omega(F/C) \longrightarrow \Omega^{(2)}(F/C)$, also denoted $d$, where $\Omega^{(2)}(F/C)$ is the space of alternating bilinear forms on $\mathrm{Der}(F/C)$. A differential $\omega$ is *closed* iff $d\omega = 0$.

The ideas surrounding valuation rings and their importance in this context are sketched in detail, but with some proofs missing, in [7], which is motivated in large part by [9] where the remaining proofs can be found.

**Definition 2.7.** Let $F/C$ be a field extension and $R$ a valuation ring of $F/C$. A differential $\omega \in \Omega(F/C)$ is said to be *regular at $R$* iff it is in $\Omega(R/C)$, considered as a sub $R$-module of $\Omega(F/C)$ via the canonical embedding. Otherwise $\omega$ is said to be *singular at $R$*.

**Definition 2.8.** A differential $\omega$ is *of the first kind* (a *dfk*) iff it is regular at every valuation ring of $F/C$. It is *of the second kind* (a *dsk*) iff for every valuation ring $R$ of $F/C$ there is $x \in F$ such that $\omega - dx \in \Omega(R/C)$.

An exact differential is of the second kind but is not of the first kind unless it is the zero differential.

Until now we have been looking at the algebraic objects differentials. We also need to consider the geometric notion of *differential forms* which, roughly speaking, are the assignment of a differential to each point on an algebraic variety. More precisely, the algebraic notion of derivation corresponds to the geometric notion of a tangent vector at a point of a variety. The dual notion of a differential corresponds to a cotangent vector. Thus a differential form on a variety $V$ is simply an element of the contangent bundle $T^*(V)$. For a more detailed explanation, see [13, p195].

The simplest differential form is the exact form $dx$ on the variety $F$. This has a pole at $\infty$ and is regular elsewhere. Another important differential form is the logarithmic form $\lambda$ on $F$ or $F^*$ which sends $x$ to the logarithmic differential $\frac{dx}{x}$. This has poles at 0 and $\infty$, and indeed has residues there so it is not of the second kind unless it is zero.

Let $\mathcal{E}$ be an elliptic curve given by

$$\mathcal{E} = \left\{(y, z) \in F^2 \;\middle|\; z^2 = f(y)\right\}$$

for a given cubic $f \in C[t]$. The Weierstrass differential form associated with $\mathcal{E}$ is the form $w(y, z) = \frac{dy}{z}$. It is of the first kind, that is it has no poles on $\mathcal{E}$, and it is closed. In fact $w$ spans the $C$-space of differential forms of the first kind on $\mathcal{E}$. See [10, p163] for details.

It is easy to see that the logarithmic differential form $\lambda$ is a group homomorphism from the multiplicative group $F^*$ to the additive group $\Omega(F/C)$. Correspondingly, Proposition 2.1 of [8] shows that the Weierstrass differential form $w$ is a group homomorphism from the algebraic group $\mathcal{E}$ to the additive group $\Omega(F/C)$. Furthermore, if $\mathcal{E}$ has complex multiplication by $\tau$ and is thus a $\mathbb{Z}[\tau]$-module then $w$ is a homomorphism of $\mathbb{Z}[\tau]$-modules.

The following relationship between differential forms and differentials will be important.

**Lemma 2.9.** *Suppose $\omega$ is a differential form of the first kind defined on a variety $V$, that $y$ is a generic point of $V$ and that $\omega(y) = 0$. Then $\omega$ is zero on all of $V$.*

*Proof.* Let $z \in V$. Then there is a specialization (field morphism) $F \xrightarrow{\pi} F$ fixing $C$ such that $\pi(y) = z$. Let $R$ be the associated valuation ring. $\omega(y) \in \Omega(C(y)/C)$, and is a differential of the first kind with respect to this field extension, since being a dfk is independent of the choice of field extension. Suppose $\omega$ is given by $x \mapsto \sum_{i=1}^n f_i(x)dx_i$. Since $\omega(y)$ is a dfk it is regular at $R' = R \cap C(y)$, that is it has a representation as $\omega(y) = \sum_{i=1}^m a_i db_i$ with each $a_i, b_i \in R'$. We can deduce that $\sum_{i=1}^n f_i(y)dy_i = \sum_{i=1}^m a_i db_i$ from the algebraic relations on $y$ over $C$, so it follows that $\sum_{i=1}^n \pi(f_i(y))d\pi(y_i) = \sum_{i=1}^m \pi(a_i)d\pi(b_i)$. Since the $f_i$ are rational functions over $C$ we have that $\pi(f_i(y)) = f_i(\pi(y)) = f_i(z)$, and thus $\omega(z) = \sum_{i=1}^m \pi(a_i)d\pi(b_i)$. In particular, it is a well defined differential in $\Omega(F/C)$. Now $(\pi(a), \pi(b))$ satisfies all the equations over $C$ that $(a, b)$ does, so the same proof that $\sum_{i=1}^m a_i db_i = 0$ shows that $\omega(x) = \sum_{i=1}^m \pi(a_i)d\pi(b_i) = 0$. $\square$

## 2.4 Lie Derivatives

If $\Delta \subseteq \operatorname{Der}(F/C)$ is a sub Lie ring (in particular, when $\Delta = \operatorname{Der}(F/D_0)$) then there is a Lie derivative

$$\Delta \times \Delta^* \xrightarrow{L} \Delta^*$$

given for $D, D_1 \in \Delta$ and $\omega \in \Delta^*$ by

$$L_D\omega(D_1) = D(\omega D_1) - \omega[D, D_1]$$

This can be thought of as a map $\Delta^* \xrightarrow{L_D} \Delta^*$ for each $D \in \Delta$. Recall that when $\Delta = \operatorname{Der}(F/D_0)$, $\Delta^* = \Omega(F/D_0)$.

When $\Delta$ is finite dimensional, the image of the universal derivation $F \xrightarrow{d} \Delta^*$ spans $\Delta^*$, and every differential can be written in the form $\sum_{i=1}^n a_i db_i$ for some $a_i, b_i \in F$. The form so represented will be the image under the quotient map $\Omega(F/C) \longrightarrow \Omega(F/C)/\operatorname{Ann}(\Delta)$ of the Kähler differential with the same representation. In these terms the Lie derivative is given by

$$L_D \sum_{i=1}^n a_i db_i = \sum_{i=1}^n Da_i db_i + \sum_{i=1}^n a_i d(Db_i)$$

**Lemma 2.10.** *Let $\Delta$ be a sub Lie ring of $\operatorname{Der}(F/C)$, $D \in \Delta$, $\omega \in \Delta^*$ and $a \in F$. The Lie derivative has the following properties.*

- *$L_D$ is an additive function*
- *$L_D(a\omega) = aL_D\omega + (Da)\omega$*
- *$L_D$ is $C_D$-linear*
- *If $\omega$ is closed then $L_D\omega = d(\omega D)$*
- *$L_D(da) = d(Da)$*

*Proof.* These are straightforward calculations from the definitions, and are done for example in [14]. □

## 2.5 Linear dependence results

If $\Delta \subseteq \operatorname{Der}(F/C)$, write $C_\Delta = \{x \in F \mid (\forall D \in \Delta)Dx = 0\}$, and write $\operatorname{Ann}(\Delta) = \{\omega \in \Omega(F/C) \mid (\forall D \in \Delta)\omega D = 0\}$.

**Lemma 2.11.** *If $\Delta \subseteq \operatorname{Der}(F/C)$ and $x \in F$ then $x \in C_\Delta$ iff $dx \in \operatorname{Ann}(\Delta)$.*

*Proof.* $x \in C_\Delta$ iff for all $D \in \Delta$, $Dx = 0$ iff for all $D \in \Delta$, $(dx)D = 0$ iff $dx \in \operatorname{Ann}(\Delta)$. □

If we take $\Delta = \operatorname{Der}(F/D_0)$ for a field extension $F$ of a differential field $\langle F_0, D_0\rangle$ with constants $C$ then $C_\Delta$ is the algebraic closure of $C$ in $F$. Also $\Omega(F/D_0) \cong \Omega(F/C)/\operatorname{Ann}(\operatorname{Der}(F/D_0))$.

We use the Lie derivative for the following result.

**Lemma 2.12.** *Suppose $\Delta_0$ is a sub Lie ring of $\operatorname{Der}(F/C)$, that $\Delta \subseteq \Delta_0$, and that $\eta_1, \ldots, \eta_m \in \Omega(F/C)$ are closed and lie in $\operatorname{Ann}(\Delta)$. Then the $\eta_i$ are linearly independent over $F$ iff they are linearly independent over $C_\Delta$.*

*Proof.* One direction is immediate. For the other, suppose there are $\alpha_i \in F$ such that $\sum_{i=1}^n \alpha_i\omega_i = 0$ and some $\alpha_i \neq 0$. We may choose the $\alpha_i$ such that the least possible number $s$ of them is non-zero, and that some $\alpha_i = 1$. For each $D \in \Delta$ we get

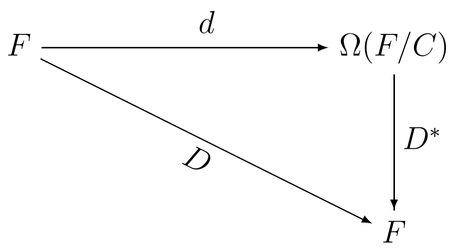
$$\begin{aligned} 0 = L_D \sum_{i=1}^n \alpha_i\omega_i &= \sum_{i=1}^n (D\alpha_i)\omega_i + \sum_{i=1}^n \alpha_i L_D \omega_i \\ &= \sum_{i=1}^n (D\alpha_i)\omega_i + \sum_{i=1}^n \alpha_i d(\omega_i D) \\ &= \sum_{i=1}^n (D\alpha_i)\omega_i \end{aligned}$$

since $\omega_i D = 0$ for each $i$, and using the prevous lemma on properties of the Lie derivative. Some $D\alpha_i = 0$ but then, by our minimal choice of $\alpha_i$, we have that $D\alpha_i = 0$ for every $i$, that is that every $\alpha_i \in C_D$. This holds for every $D \in \Delta$, so every $\alpha_i \in \bigcap_{D\in\Delta} C_D = C_\Delta$. So the $\omega_i$ are $C_\Delta$-linearly dependent. □

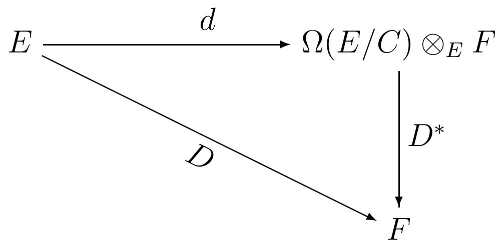
In particular, if $\dim_F(\eta_1, \ldots, \eta_n) = n - 1$, and for some $c_i \in F$ we have $\sum c_i\eta_i = 0$ with one of the $c_i$ equal to 1 and all nonzero, then all of the $c_i$ lie in $C_\Delta$.

**Lemma 2.13.** *Let $\langle F, D\rangle$ be a differential field with constants $C$, and let $E$ be a subfield of $F$ properly containing $C$. ($E$ is not necessarily a differential subfield.) Then $\operatorname{Ann}(D) \cap \Omega(E/C)$ is an $E$-subspace of $\Omega(E/C)$ of codimension 1.*

*Proof.* It is easy to see that it is a subspace. For the codimension, we have the diagram




from the universality of $d$, and $\mathrm{Ann}(D)$ is the kernel of the linear map $D^*$. The diagram restricts to




where again $D^*$ is $F$-linear, with kernel

$$\mathrm{Ann}(D) \cap (\Omega(F/C) \otimes_E F) = (\mathrm{Ann}(D) \cap \Omega(E/C)) \otimes_E F$$

Since $E \neq C$ there is $x \in E \smallsetminus C$. Then $D^*(dx) = Dx \neq 0$, so $D^*$ is not the zero map and thus has rank 1, and so its kernel has codimension 1. But

$$\dim_E(\mathrm{Ann}(D) \cap \Omega(E/C)) = \dim_F(\mathrm{Ann}(D) \cap \Omega(E/C)) \otimes_E F$$

and so $\mathrm{Ann}(D) \cap \Omega(E/C)$ has codimension 1 in $\Omega(E/C)$, as claimed. □

It is possible to consider fields with a family of $r$ independent derivations. In this case the result (and proof) generalize immediately to get that the common annihilator of the derivations has codimension $r$.

Write $K(\mathcal{E})$ for the field of fractions of the endomorphism ring of the elliptic curve $\mathcal{E}$. So $K(\mathcal{E})$ is $\mathbb{Q}$ if $\mathcal{E}$ has no complex multiplication or $\mathbb{Q}(\tau)$ for some imaginary quadratic $\tau$ if $\mathcal{E}$ has complex multiplication by $\tau$.

**Lemma 2.14.** *The algebraic subgroups of $\mathcal{E}^n$ are given by equations of the form $\bigoplus_{i=1}^n m_i X_i = O$ for $m_i \in K(\mathcal{E})$.*

*Proof.* Algebraic subgroups are closed in the Zariski topology, so their complex realizations must be closed in the complex topology. Thinking of $\mathcal{E}$ as the quotient of the complex plane by a lattice, the result follows. □

We make use of the following general theorem which was the inspiration for Zilber's better known model theoretic version.

**Theorem 2.15 (Indecomposability Theorem).** *Let $G$ be an algebraic group and $V$ an irreducible subvariety of $G$, containing the identity. Then the subgroup generated by $V$ is algebraic.*

*Proof.* See for example [11, p261]. $\square$

James Ax proved the following by an argument using Puiseux series.

**Proposition 2.16.** *Let $y_1, \ldots, y_n \in F^*$ and suppose that $\lambda(y_1), \ldots, \lambda(y_n)$ are linearly dependent over $C$. Then they are linearly dependent over $\mathbb{Q}$.* $\square$

In analogy, we prove the following.

**Proposition 2.17.** *Let $Y_1, \ldots, Y_n \in \mathcal{E}$ and suppose that $w(Y_1), \ldots, w(Y_n)$ are linearly dependent over $C$. Then they are linearly dependent over $K(\mathcal{E})$.*

*Proof.* Suppose $Y_1, \ldots, Y_n \in \mathcal{E}$ are such that $\sum_{i=1}^n c_i w(Y_i) = 0$ for some $c_i \in C$, not all zero. Consider the differential form

$$\eta : \mathcal{E}^n \to \Omega(F/C)$$

given by $\eta(X_1, \ldots, X_n) = \sum_{i=1}^n c_i w(X_i)$. Then the assumption is that $\bar{Y}$ lies in the kernel of $\eta$, which is a proper subgroup of $\mathcal{E}^n$ since $\eta$ is a non-zero group homomorphism.

Let $V$ be the irreducible subvariety of $\mathcal{E}^n$ over $C$ which has $\bar{Y}$ as a generic point. Then $V$ contains some element $\bar{P} \in {\mathcal{E}_C}^n$, which is contained in the kernel of $\eta$. Let $\bar{Y}' = \bar{Y} \ominus \bar{P}$ and $V'$ be the irreducible subvariety of $\mathcal{E}^n$ over $C$ which has $\bar{Y}'$ as a generic point. Then $\bar{Y}' \in \ker \eta$ and the identity $O$ of $\mathcal{E}^n$ lies in $V'$. Also $w(Y_i') = w(Y_i)$ for each $i$.

The form $\eta$ is a $C$-linear combination of Weierstrass differentials. These are of the first kind and the set of all differentials of the first kind is a $C$-linear subspace of $\Omega(F/C)$, so $\eta$ is a differential form of the first kind. Thus its restriction to $V'$ is also of the first kind. By assumption, $\eta$ is zero at a generic point of $V'$, so by lemma 2.9, $V'$ lies in the kernel of $\eta$.

Now $V'$ is an algebraic subvariety of $\mathcal{E}^n$ containing the identity so, by the Indecomposability Theorem, the subgroup $H$ generated by $V'$ is an algebraic subgroup. Since $V'$ is contained in the proper subgroup $\ker \eta$ of $\mathcal{E}^n$ it follows that $H$ is also contained in $\ker \eta$, and so $H$ is a proper algebraic subgroup of $\mathcal{E}^n$. From the description of the algebraic subgroups of $E^n$ given in lemma 2.14 we see that there are $m_1, \ldots, m_n \in K$ such that $\bigoplus_{i=1}^n m_i Y_i = O$. Since $\eta$ is a homomorphism this gives a $K$-linear dependence between the $w(Y_i)$. $\square$

# 3 The universal theory

## 3.1 Group structures

In this section as before, $\langle F, D\rangle$ is a differential field of characteristic 0, and $C$ is the constant subfield.

Consider the algebraic groups $\langle F; +\rangle$, $\langle F^*; \cdot\rangle$ and elliptic curves $\langle \mathcal{E}; \oplus\rangle$ where the affine part of $\mathcal{E}$ is given by $\mathcal{E} = \{(y,z) \in F^2 \mid z^2 = f(y)\}$ for a cubic $f$ with coefficients in $C$ and distinct roots (in a splitting field, although usually $C$ and $F$ will be algebraically closed).

We consider the exponential differential equation

$$\frac{Dy}{y} = Dx$$

for $x \in F$ and $y \in F^*$, and the Weierstrass differential equation

$$\frac{Dy}{z} = Dx$$

for $x \in F$ and $(y,z) \in \mathcal{E}$, for different elliptic curves $\mathcal{E}$. This equation can also be written just in terms of $x$ and $y$ as

$$\frac{(Dy)^2}{f(y)} = (Dx)^2.$$

Let $E \subseteq F \times F^*$ be the solution set to the exponential equation and $W \subseteq F \times \mathcal{E}$ be the solution set to the Weierstrass equation for a given elliptic curve $\mathcal{E}$. The main aim of this project is to study the reducts of differentially closed fields with the field structure and these relations but without the derivation.

**Theorem 3.1.** *$E$ is a subgroup of $F \times F^*$ and $W$ is a subgroup of $F \times \mathcal{E}$. Where $\mathcal{E}$ has complex multiplication by $\tau$, $W$ is a sub $\mathbb{Z}[\tau]$-module. The fibres of the projections of $E$ and $W$ are cosets of the constant subgroups $C$ of $F$, $C^*$ of $F^*$ and $\mathcal{E}_C$ of $\mathcal{E}$.*

*Proof.* By the comments in section 2.3, the differential forms

$$\begin{array}{rcl} F \times F^* & \xrightarrow{\omega} & \Omega(F/C) \\ (x,y) & \longmapsto & dx - \dfrac{dy}{y} \end{array}$$

and

$$\begin{array}{rcl} F \times \mathcal{E} & \xrightarrow{\omega} & \Omega(F/C) \\ (x,(y,z)) & \longmapsto & dx - \dfrac{dy}{z} \end{array}$$

are group homomorphisms. In the case of an elliptic curve with complex multiplication the differential form is also a $\mathbb{Z}[\tau]$-module homomorphism. The sets $E$ and $W$ are the kernels of these hence are subgroups. The kernels of the exact, logarithmic and Weierstrass differential forms are the constant subgroups, hence the fibres of points are the cosets of these. □

## 3.2 Schanuel conditions

The main result of James Ax's paper [1] is the following.

**Theorem 3.2 (Ax).** *Suppose $n \geqslant 1$ and $x_i, y_i \in F$ such that $y_i \neq 0$ and $(x_i, y_i) \in E$ for $i = 1, \dots, n$. Then*

$$\mathrm{td}_C(x_1, \dots, x_n, y_1, \dots, y_n) - \dim_{\mathbb{Q}}(Dx_1, \dots, Dx_n) \geqslant 1$$

We prove the corresponding Schanuel condition for Weierstrass equations.

**Theorem 3.3.** *Let $\mathcal{E}$ be an elliptic curve with associated cubic $f$, and let $n \geqslant 1$. Suppose that $x_1, y_1, \dots, x_n, y_n \in F$ such that $f(y_i) \neq 0$ and $(x_i, y_i) \in W$ for each $i = 1, \dots, n$. Then*

$$\mathrm{td}_C(x_1, \dots, x_n, y_1, \dots, y_n) - \dim_{K(\mathcal{E})}(Dx_1, \dots, Dx_n) \geqslant 1$$

*Proof.* Let $L$ be the subfield (not necessarily differential subfield) $C(x_1, y_1, \dots, x_n, y_n)$ of $F$. It is enough to assume that $\mathrm{td}(L/C) < n + 1$ and show that the $Dx_i$ are linearly dependent over $K(\mathcal{E})$.

For each $i$, there are two $Y_i \in \mathcal{E}$ such that $Y_i = (y_i, z_i)$ for some $z_i$. Choose the one such that $(w(Y_i) - dx_i)D = 0$, and let $\eta_i = w(Y_i) - dx_i$. Then for each $i$ we have $\eta_i \in \mathrm{Ann}(D) \cap \Omega(L/C)$. Suppose $\mathrm{td}(L/C) < n + 1$. Then by lemma 2.13, $\dim_L(\mathrm{Ann}(D) \cap \Omega(L/C)) < n$. So $\eta_1, \dots, \eta_n$ must be $L$-linearly dependent and thus $F$-linearly dependent. They are closed differentials and so by lemma 2.12 they are $C$-linearly dependent. Say

$$\sum_{i=1}^{n} c_i \eta_i = 0$$

with the $c_i \in C$, not all zero. Then

$$\sum_{i=1}^{n} c_i w(Y_i) = \sum_{i=1}^{n} c_i dx_i = d \sum_{i=1}^{n} c_i x_i$$

Each $w(Y_i)$ is of the first kind and differentials of the first kind form a $C$-subspace, so the left hand side here is a differential of the first kind. The right hand side is exact, and an exact differential can only be of the first kind if it is zero, so both sides are zero. So $\sum_{i=1}^n c_i w_i = 0$, that is the Weierstrass differentials $w(Y_i)$ are linearly dependent over $C$. By proposition 2.17 they are linearly dependent over $K(\mathcal{E})$, so we may assume the $c_i \in K(\mathcal{E})$. Then $\sum_{i=1}^n c_i Dx_i = (\sum_{i=1}^n c_i dx_i)D = 0$, so the $Dx_i$ are $K(\mathcal{E})$-linearly dependent as required. $\square$

Essentially the same proof also gives the following generalization.

**Theorem 3.4.** *Let $f_1, \ldots, f_m \in C[t]$ be cubics with distinct roots such that the corresponding elliptic curves are non-isogenous. For $i = 1, \ldots, m$, let $K_i$ be $\mathbb{Q}(\tau_i)$ if the elliptic curve corresponding to $f_i$ has complex multiplication by $\tau_i$ or $\mathbb{Q}$ otherwise, and let $K_0 = \mathbb{Q}$. For each $i = 0, \ldots, m$, let $n_i \in \mathbb{N}$, not all zero, and $x_{i1}, y_{i1}, \ldots, x_{in_i}, y_{in_i} \in F$ such that $Dx_{i1}, \ldots, Dx_{in_i}$ are $K_i$-linearly independent. Suppose for each $i > 0$ and each $k = 1, \ldots, n_i$ that $f_i(y_{ik}) \neq 0$ and*

$$\frac{(Dy_{ik})^2}{f_i(y_{ik})} = (Dx_{ik})^2$$

*and that for each $k = 1, \ldots, n_0$ that $y_{0k} \neq 0$ and that*

$$\frac{Dy_{0k}}{y_{0k}} = Dx_{0k}.$$

*Let $L$ be the field generated over $C$ by all the $x_{ik}$ and $y_{ik}$.*

*Then $\operatorname{td}_C L \geqslant \sum_{i=0}^m n_i + 1$.* $\square$

## 3.3 Uniformity and definability

It has been known and indeed used by Poizat, Wilkie and Zilber that the statement of Ax's theorem is first order in the language of differential fields, that is it can be expressed as a collection of first order sentences in that language. I show below that essentially the same argument gives this for the language $\langle +, \cdot, E \rangle$ of the reduct. This is due to a certain uniformity in parameters property that strengthens the statement of the Schanuel condition. I give the proof for the Weierstrass case, and the same proof works for the exponential case.

The usual language of fields is equivalent to a relational language with one symbol for each irreducible variety $V$ defined over the prime subfield, $\mathbb{Q}$. A variety defined over the constant field $C$ is just a fibre of a variety over $\mathbb{Q}$. Note that the constant field is definable in $\langle F; +, \cdot, W \rangle$, for example by $x \in C \iff W(x, O)$. The field $K(\mathcal{E})$ has an embedding into $\mathbb{C}$ (unique up to complex conjugation) which gives rise to a norm on its cartesian powers.

**Theorem 3.5 (Uniform Schanuel condition).** *Let $V(x, y, z) \subseteq F^n \times \mathcal{E}^n \times F^k$ be an irreducible variety over $\mathbb{Q}$, and $p$ the projection onto the $z$ coordinates.*

*Then there is a finite set $\mathcal{H}_V \subseteq R^n \smallsetminus \{\bar{0}\}$ where $R$ is the endomorphism ring of $\mathcal{E}$ such that for all $(a, b, c) \in V$ with $(a, b) \in W^n$, $c \in C^k$, and $\dim p^{-1}(c) \leqslant n$ there is $m \in \mathcal{H}_V$ such that $\sum_{i=1}^n m_i a_i \in C$ and $\bigoplus_{i=1}^n m_i \cdot b_i \in \mathcal{E}_C$.*

*Proof.* We use a simple compactness argument. Let

$$A = \left\{(a, b, c) \in V \;\middle|\; (a, b) \in W^n, c \in C^k, \text{ and } \dim p^{-1}(c) \leqslant n \right\}.$$

Then $A$ is definable, the dimension part by the theorem on dimension of fibres (see [13, p76]). For $N \in \mathbb{N}$, let

$$\varphi_N(x, y, z) \equiv [(x, y, z) \in A \wedge \neg \bigvee_{\substack{m \in R^n \\ 0 < \|m\| \leqslant N}} \sum_{i-1}^{n} m_i x_i \in C.$$

This is a first order formula since the disjunction is finite. Suppose there is no $\mathcal{H}_V$ with the properties required for the theorem. Then the type $\{\varphi_N \mid N \in \mathbb{N}\}$ is finitely satisfiable and hence, by compactness, is satisfied in some differential field. But this contradicts the Schanuel condition. □

For definiteness we take $\mathcal{H}_V$ to be the set of those vectors from $R^n$ which actually arise. We don't extract any information about the size or nature of $\mathcal{H}_V$ from the proof, although some effectivity is probably possible.

Now for each $V$ the sentence

$$\theta_V \equiv \forall xyz[(x, y, z) \in A \to \bigvee_{m \in \mathcal{H}_V} \sum_{i=1}^{n} m_i a_i \in C$$

is first order in the language $\langle +, \cdot, W \rangle$ and the set of $\theta_V$ over varieties $V$ in the relational language expresses the uniform Schanuel condition.

# 4 Amalgamation and axiomatization

We produce a structure by amalgamation intended to have the same theory as the reduct of a differentially closed field, and give a complete axiomatization of its theory.

Take a language $\mathcal{L} = \langle +, \cdot, (-)^{-1}, 0, 1, g_2, g_3, W, C \rangle$ where $(-)^{-1}$ is a function giving field inverse (with the convention that $0^{-1} = 0$), $C$ is a unary relation symbol and $W$ is a ternary relation symbol, a subset of $F \times \mathcal{E}$ where $\mathcal{E}$ is the elliptic curve related to $g_2$ and $g_3$. Having the inverse means that structures will be fields, not rings. This is convenient but not important.

Let $T_0$ be the $\mathcal{L}$-theory given for a structure $F$ as follows.

(Field) $F$ is a field of characteristic zero.

(C) $C$ is a relatively algebraically closed subfield.

($\mathcal{E}$) We fix constants $g_2$ and $g_3$ in $C$.

(W)
- $W$ is a subgroup (or sub $\mathbb{Z}[\tau]$-module) of $F \times \mathcal{E}$.
- If $W(x_0, Y_0)$ then $W(x, Y_0) \iff x - x_0 \in C$.
- If $W(x_0, Y_0)$ then $W(x_0, Y) \iff Y \ominus Y_0 \in \mathcal{E}_C$.

(SC) For each variety $V$ over $\mathbb{Q}$, the sentence $\theta_V$ expressing the uniform Schanuel condition for $V$ in the language $\mathcal{L}$.

If $\mathcal{E}$ has complex multiplication by $\tau$ then we add $\tau$ to the language as a constant, as it is convenient to have the complex multiplication field $K = K(\mathcal{E})$ in each model of $T_0$.

For a model $F$ of $T_0$, let $\hat{F}$ be the projection of $W$ onto the first coordinate (the additive group). In other words, $\hat{F} = \{x \in F \mid \exists Y \in \mathcal{E}_F[W(x, Y)]\}$. $\hat{F}$ is always a $K$-vector space containing $C$. Let the group rank of $W$ in the model $F$, $\operatorname{grk}_C F$ be defined as the $K$-linear dimension of $\hat{F}$ modulo $C$, that is the size of a minimal subset $B$ of $\hat{F}$ such that every element of $\hat{F}$ can be written as a finite sum of the form $\sum m_i b_i + c$ with $m_i \in K$, $b_i \in B$ and $c \in C$.

If $F$ is a model of $T_0$ such that $\operatorname{td}(F/C)$ is finite then we can define a predimension function

$$\delta(F) = \operatorname{td}_C F - \operatorname{grk}_C F.$$

This is finite and non-negative by the Schanuel condition, and is zero precisely when $F = C$.

We define an $\mathcal{L}$-embedding $F_1 \hookrightarrow F_2$ to be *strong* iff for every finitely generated substructure $X$ of $F_2$ we have $\delta(X \cap F_1) \leqslant \delta(X)$. Write $F_1 \lhd F_2$ for a strong embedding. Let $\mathcal{K}_0$ be the category of finitely generated models of $T_0$ and strong embeddings.

The category $\mathcal{K}_0$ has the hereditary property (if $F_2 \in \mathcal{K}_0$ and $F_1$ is a strong substructure of $F_2$ then the embedding $F_1 \lhd F_2$ is in $\mathcal{K}_0$). We want to show that $\mathcal{K}_0$ also has the joint embedding property (JEP) and the amalgamation property (AP). The field $K(g_2, g_3)$ is a model of $T_0$ and strongly embeds into every model of $T_0$, so the JEP is a special case of the AP.

We will show below that the category $\mathcal{K}_0$ has the amalgamation property. Assuming this for now, by the Fraissé amalgamation theorem (see for example [3]) there is a unique countable $\mathcal{L}$-structure, $M$, such that the strong substructures are exactly the countable models of $T_0$ and whenever $F_1, F_2 \lhd M$ are isomorphic and finitely generated then the isomorphism extends to an automorphism of $M$. This says that $M$ is $\aleph_0$-homogeneous for strong substructures, or that it is $\mathcal{K}_0$-saturated or $\mathcal{K}_0$-existentially closed. ("All possible finite strong extensions happen inside $M$.")

The theory $T_0$ is universal so every substructure of a model of $T_0$ is also a model of $T_0$, in particular every substructure (not necessarily strong) of $M$ is a model of $T_0$.

## 4.1 Axiomatization

Our aim now is to find axioms which, together with $T_0$, give the complete theory of $M$. We would like an axiom scheme expressing the $\mathcal{K}_0$-existential closedness.

- If $F_1 \lhd_{f.g.} M$ and $F_2$ is a finitely generated model of $T_0$, with $F_1 \lhd F_2$ in $\mathcal{K}_0$ then there is a copy of $F_2$ in $M$ such that $F_1 \lhd F_2$ inside $M$.

One way of looking at this is that the Schanuel condition (hereditary positivity of the relative predimension function) gives a sufficient condition (as well as a necessary one) for a system of equations to have solutions.

There are three problems with this. Firstly, it is not possible to tell with a finite number of formulas whether or not a substructure $F \subseteq M$ is strong in $M$. Because of this, our axiom scheme must say something stronger – the assumption that $F_1 \lhd M$ must be relaxed to $F_1 \subseteq M$. Fortunately this stronger condition does hold in $M$ as we will show.

Any strong extension $F_1 \lhd F_2$ can be split into two parts, the first just extending the subfield $C$ and the second keeping $C$ fixed. Existential closedness for extensions of $C$ corresponds to saying that $C$ is algebraically closed and of infinite transcendence degree. The latter part we ignore because it is not first order and because it doesn't hold in all the reducts of a differentially closed field in which we are interested. From the general existential closedness condition it follows that the whole field $M$ will be algebraically closed, and $T_0$ says that $C$ is relatively algebraically closed, so to capture the rest of existential closedness under extensions of $C$ we take the usual axiom scheme expressing the following.

(EC$_1$) $M$ is an algebraically closed field

For extensions which keep $C$ fixed we start by considering the *strong existential closedness* axiom scheme.

(SEC) If $F_1 \subseteq_{f.g.} M$, $F_2$ is a finitely generated model of $T_0$ with $F_1 \lhd F_2$ in $\mathcal{K}_0$ and containing no new constants then there is a copy of $F_2$ in $M$ such that $F_1 \lhd F_2$ inside $M$.

The word "strong" here refers both to the notion of strong embeddings and also to the fact that this axiom is stronger than the existential closedness axiom scheme (EC) which we will give soon.

The second problem we must address is that to express this axiom in a first order way we have to replace finitely generated structures with generating sets for them. We give a *normal form* for a generating set.

Take $F_1 \subseteq_{f.g.} M$ generated by a finite tuple $a$. Some of the coordinates of $a$ are in $C$, some are in a projection of $W \cap {F_1}^3$ and some are in neither, so we may write $a$ as $(a_1, a_2, a_3)$ where $a_1 \in C^t$, $a_2 \in W^n$ and $a_3 \in {F_1}^s$, and $W$ is generated by $a_2$, under the rules specified by the theory $T_0$. Recall that $R$ is the endomorphism ring of $\mathcal{E}$, with field of fractions $K$. The group $W$ is an $R$-module, and the quotient $W/C$ is a $K$-vector space, the dimension of which is $\mathrm{grk}_{\mathrm{C}} F$. By removing some coordinates of $a_2$ we may assume that the remainder are $K$-linearly independent in the group $W$ over $W_C$. The part of $T_0$ which specifies the fibres of $W$ ensures that $F_1$ is still well-defined by this reduced set of generators. A generating set of this form $(a_1, a_2, a_3)$ is said to be in *normal form*.

Suppose $F_1 \lhd F_2$ with no extra constants, that $(a_1, a_2, a_3)$ is a generating set for $F_1$ in normal form, and $b$ is a generating set for $F_2$ over $F_1$. There are no extra constants, so there is no need for a $b_1$ part of the normal form of $b$. Since $W$ projects surjectively onto the field $M$ in all three coordinates, we may extend $b$ so that every coordinate is part of some triple in $W$. Thus we may also do away with the $b_3$ part of $b$. This may have the effect of extending $F_2$, but this is no problem. Thus we say that the normal form of a generating set for an extension $F_2$ over $F_1$ is a tuple $b \in W^m$ which is $K$-linearly independent over $C$ and over $a_2$. This says simply that $(a_1, a_2 b, a_3)$ is a generating set in normal form.

For a tuple $a$ in normal form, let $V = \mathrm{loc}(a)$, the algebraic locus of $a$ over $\mathbb{Q}$, and let $\tilde{V}$ be given as follows.

$$\tilde{V} = \left\{(x_1, x_2, x_3) \in V \;\middle|\; x_1 \in C^t, x_2 \in W^n, \dim V_{x_1 x_3} = \dim V_{a_1 a_3},\ \text{and}\ \mathrm{grk}_{\mathrm{C}}(x_2) = n\right\}$$

Note that this is first-order definable, the dimension of the variety by the fibre condition of algebraic geometry and the linear independence of $x_2$ by the argument giving the uniformity of the Schanuel condition.

Let $U = \mathrm{loc}(ab)$, the algebraic locus of $ab$ over $\mathbb{Q}$, and define

$$\tilde{U} = \left\{(x,y) \in U \;\middle|\; (x) \in \tilde{V}, y \in W^m, \dim U_x = \dim U_a, \text{ and } \mathrm{grk_C}(x_2 y) = n+m\right\}$$

which again is first order definable. If $(x,y) \in \tilde{U}$, the predimension $\delta(x,y) = \mathrm{td}_C(x_2, x_3, y) - (n+m)$. Thus genericity in the sense of the predimension coincides with algebraic genericity. By assumption, $(a,b)$ is generic in $\tilde{U}$, and $b$ is generic in $\tilde{U}_a$ over the field $C$ together with the parameters $a$.

Thus (SEC) can be restated as the following.

- For each strong extension $F_1 \lhd F_2$ in $\mathcal{K}_0$, with generators in normal form and with $\tilde{V}$ and $\tilde{U}$ as defined above,

$$\forall x \exists y \left[x \in \tilde{V} \to (y \text{ is generic in } \tilde{U}_x \text{ over } C(x))\right].$$

The third and final problem with giving an axiomatization is that this is still not a first order statement, because we cannot insist that $y$ is generic in a first order way. Instead we must give all first order approximations to this. If the predimension of $F_2$ is equal to that of $F_1$ this does not matter, because the Schanuel condition ensures that $y$ must be generic. However, the first order theory of $M$ does not imply that the dimension of a model (in the sense of the pregeometry arising from the predimension) is infinite, or indeed that it is greater than 1.

To approximate this in a first order way, we say that for each proper subvariety $U'$ of $U_x$, there is a $y$ satisfying the conditions but not living in $U'$. The existential closedness axiom scheme is as follows.

($\mathrm{EC}_2$) For each strong extension $F_1 \lhd F_2$ in $\mathcal{K}_0$, with generators in normal form and with $\tilde{V}$ and $\tilde{U}$ as defined above, for each proper subvariety $U'$ of $\mathrm{pr}\, U$ definable without parameters,

$$\forall x \exists y \left[x \in \tilde{V} \to (y \in \tilde{U}_x \smallsetminus U')\right].$$

Take (EC) = ($\mathrm{EC}_1$) + ($\mathrm{EC}_2$), and $T_W = T_0$ + (EC). We must show that $M \models T_W$ and that $T_W$ is complete.

It is easy to see that ($\mathrm{EC}_1$) holds in the amalgam $M$. We prove that (SEC) holds in $M$. Suppose that $a$ is a tuple such that $M \models \tilde{V}(a)$, and let $A = \langle a \rangle$. (SEC) says that we can find a tuple $b$ in $M$ which is generic in $\tilde{U}_a$ over $C(a)$. This is equivalent to finding a strong extension $B$ of $A$ inside $M$, corresponding to a generating set $b$, and it does not matter than $a$ may not be a generic point of $\tilde{V}$ since $B$ will still be a strong extension of $A$. If

$A \lhd M$, the existence of the strong extension $B$ follows from the fact that $M$ is $\mathcal{K}_0$-homogeneous and $\mathcal{K}_0$-universal. We have to show that the same holds even if $A$ is not strong in $B$. For this, let $\bar{A}$ be a hull of $A$, that is $A \subseteq \bar{A} \subseteq_{f.g.} M$ with $\delta(\bar{A}) = d(A)$. (We can also ask for $\bar{A}$ to be minimal such, but it is not necessary to prove that such a minimal hull exists so we don't do that here.) In other words, $\bar{A}$ is a finitely generated extension of $A$ in $M$ which is strong in $M$. We now have the situation where $A$ embeds in $\bar{A}$ and $B$, strongly in $B$ but not in $\bar{A}$, and we want to find an amalgamation of this.

**Proposition 4.1.** *The class $\mathcal{K}_0$ of finitely generated models of $T_0$ has the* asymmetric amalgamation property *(AAP). That is, any diagram of the form*

$$\begin{array}{ccc} Y_1 & & \\ \big\uparrow & & \\ X & \lhd & Y_2 \end{array}$$

*can be completed to a commuting diagram*

$$\begin{array}{ccc} Y_1 & \lhd & Z \\ \big\uparrow & & \big\uparrow \\ X & \lhd & Y_2 \end{array}$$

*in the class $\mathcal{K}_0$. In particular, the category $\mathcal{K}_0$ has the AP.*

Assuming this proposition, take $Z$ to be an amalgam of $\bar{A}$ and $B$. Then by the $\mathcal{K}_0$-saturatedness of $M$, there is a copy of $Z$ extending $\bar{A}$ in $M$. This $Z$ contains a copy of $B$ which is a strong extension of $A$ in $M$, and so $M$ satisfies (SEC) as required.

*Proof of Proposition 4.1.* As before, we may assume that the constant field $C$ remains the same in all extensions. Take $X'$ to be the algebraic closure of $X$ as a field, with the group $W$ on $X'$ generated by its restriction to $X$ according to the rules specified in the theory $T_0$. Similarly define $Y_1'$ and $Y_2'$. The embeddings $X \hookrightarrow Y_1$ and $X \lhd Y_2$ extend to $X' \hookrightarrow Y_1'$ and $X' \lhd Y_2'$, uniquely up to isomorphism. As a field, define $Z'$ to be the algebraically closed "free amalgam" of the underlying fields $Y_1'$ and $Y_2'$ over $X'$. That is, if $Y_i$ has transcendence base $B_i$ over $X$ then the transcendence base of $Z'$ over $X'$ is the disjoint union $B_1 \sqcup B_2$, and we choose embeddings of $Y_1'$ and $Y_2'$ into $Z'$ extending the embeddings $X' \hookrightarrow Z'$ and $B_i \hookrightarrow Z'$.

Since we are assuming the constant field $C$ is contained in $X'$, this is well defined in $Z'$. We take $W$ in $Z'$ to be generated by the images of $W$

in $Y_1'$ and $Y_2'$. Finally take $Z$ to be the substructure of $Z'$ which is (finitely) generated by the images of $Y_1$ and $Y_2$. This satisfies all the conditions of being an amalgam except possibly that $Y_1 \lhd Z$. First note that

$$\mathrm{td}_C(Z) = \mathrm{td}(Y_1/X) + \mathrm{td}(Y_2/X) + \mathrm{td}(X/C) = \mathrm{td}(Y_1/C) + \mathrm{td}(Y_2/C) - \mathrm{td}(X/C)$$

and similarly $\mathrm{grk_C}\, Z = \mathrm{grk_C}\, Y_1 + \mathrm{grk_C}\, Y_2 - \mathrm{grk_C}\, X$, so we have

$$\delta(Z) = \delta(Y_1) + \delta(Y_2) - \delta(X)$$

and thus $\delta(Z) - \delta(Z \cap Y_1) = \delta(Z) - \delta(Y_1) = \delta(Y_2) - \delta(X) \geqslant 0$. The same equality holds taking the intersection of the diagram with any substructure of $Z$, so the fact that $X \lhd Y_2$ forces $Y_1 \lhd Z$ as required. □

To show that $T_W$ is complete, we prove that it is *near model complete*, that is, it has quantifier elimination to the level of existential formulas and their negations (which are universal formulas). This is also a useful fact in its own right.

**Proposition 4.2.** *$T_W$ is near model complete and complete.*

*Proof.* Write $\mathrm{etp}(a)$ for the set of existential formulas or their negations satisfied by $a$. Suppose $M$, $M'$ are $\omega$-saturated models of $T_W$, and $a, a'$ are tuples from $M, M'$ respectively such that $\mathrm{etp}(a) = \mathrm{etp}(a')$. Let $b \in M$. We want to find $b' \in M'$ such that $\mathrm{etp}(ab) = \mathrm{etp}(a'b')$. By the standard back and forth argument, this will show that $\mathrm{tp}(a) = \mathrm{tp}(a')$, and so $T_W$ is near model complete. We allow the tuples $a, a'$ to be empty, so this also shows that $\mathrm{tp}(\emptyset)$ is the same in $M$ and $M'$, that is that $\mathrm{Th}(M) = \mathrm{Th}(M')$, and so $T_W$ is complete.

We prove this in three steps.

**Step 1** If $a, a'$ have the same etp then they have isomorphic hulls.

Take $A = \langle a \rangle$. We may assume that $a$ is in the normal form of a generating set. Suppose that $\bar{A}$ is a hull of $A$ in $M$, and $V$ is the locus of a generating set in normal form of $\bar{A}$ over $A$. Then $\mathrm{etp}(a)$ contains the formulas $\exists y \tilde{V}(y)$ and $\neg\exists y(\tilde{V}(y) \wedge V'(y))$ for each proper subvariety $V'$ of $V$, defined over the parameters $a$. Any realisation of $\tilde{V}$ in $M'$ will give an isomorphic copy of $\bar{A}$ containing $a'$ in $M'$.

Take isomorphic hulls $\bar{A}$ of $A = \langle a \rangle$ and $\bar{A}'$ of $A' = \langle a' \rangle$. Note that $\bar{A} \lhd M$ and $\bar{A}' \lhd M'$.

**Step 2** If $A \lhd_{f.g.} M$, $A' \lhd_{f.g.} M'$ and $A \cong A'$ then the isomorphism can be extended to any $b \in M$.

Take $B$ to be a hull of $Ab$ in $M$, and $\bar{b}$ a generating set for $B$ over $A$, in normal form. Since $A \lhd M$ it follows that $A \lhd B$. The structure $M'$ is $\omega$-saturated and satisfies (EC) so it satisfies (SEC), and hence there is $\bar{b}'$ in $M'$ generating $B'$ isomorphic over $A'$ to $B$ over $A$. This isomorphism restricts to $Ab \cong A'b'$ for some $b'$ (which may not be one of the generators in $\bar{b}'$).

**Step 3** If $A \lhd M$, $A' \lhd M'$ and $A \cong A'$ then $\mathrm{etp}(A) = \mathrm{etp}(A')$.

$A \cong A'$ is the same as equality of the quantifier-free types, $\mathrm{qftp}(A) = \mathrm{qftp}(A')$. The etp of $A$ also contains information about what finitely generated extensions of $A$ exist in $M$, and this determines $\mathrm{etp}(A)$. Since $A \lhd M$ the only extensions of $A$ which exist in $M$ are strong extensions. By (SEC), all finitely generated strong extensions do exist. The same holds for $A'$, and so $\mathrm{etp}(A) = \mathrm{etp}(A')$.

Continuing from step 2, we deduce that $\mathrm{etp}(ab) = \mathrm{etp}(a'b')$, and so $T_W$ is near model complete and complete as required. □

We may define $T_E$ and $T_{EW}$ for the exponential equation and for both equations in a precisely analogous way, and the results and proofs go through essentially unchanged.

# 5 The existentially closed condition

We show that the reducts of differentially closed fields in question satisfy the existential closedness condition (EC). The theory $T_0$ was chosen specifically as the universal theory of these reducts, so this will show that the theory of the reducts is $T_W$.

To show (EC), we use the differential nullstellensatz which says that if a finite system of differential equations and inequations over a differentially closed field $F$ has a solution in some extension of $F$ then it has a solution already inside $F$. We also use the characterization of differential field extensions in terms of differential forms.

Any differentially closed field $F$ is algebraically closed, so ($\mathrm{EC}_1$) holds. It remains to show ($\mathrm{EC}_2$). That is, if $F_1 \lhd F_2$ is a strong extension in $\mathcal{K}_0$ with generators in normal form with locuses $V$ and $U$, $U'$ is a proper subvariety of $\mathrm{pr}\, U$, and $a$ is a tuple in $F$ satisfying $\tilde{V}$ then there is a tuple $b$ in $F$ such that $(a, b) \in \tilde{U}$ and $b \notin U'$. By the differential nullstellensatz it is enough to find $b$ in some extension of $F$. $F_1$ can be subsumed into $F$ as a set of parameters. Take $F_2$ to have generators $b$ over $F$, where $b \in W^n$ and is free in this group over $F$. Let $U_a = \mathrm{loc}(b)/F$, the algebraic locus of $b$ over $F$, which has parameters $a$ from the subfield $F_1$ of $F$. Take

$$\tilde{U}_a = \{y \in U_a \mid y \in W^n \text{ and is free over } F\}.$$

This is definable in the language $\langle +, \cdot, W, 0, 1\rangle$ as explained before. We prove that in some differential field extension of $F$ there is $b \in \tilde{U}_a$, generic over $F$.

Note that although this appears to prove (SEC) not just (EC), when we use the differential nullstellensatz to pull this solution back into $F$ we only retain the first order part of the existential closedness condition.

The assumptions on $F_2$ are that it satisfies $T_0$, that it is a strong extension of $F$ and that the chosen generating set is independent in $W$ over $F$. In terms of the variety $\tilde{U}_a$, this means that a point $y$ of $\tilde{U}_a$, generic over $F$, does not satisfy any equation of the form $\bigoplus_{i=1}^n m_i y_i \in F$ with $m_i \in R$ and that for any matrix $M \in \mathrm{Mat}_{r\times n}(R)$,

$$\mathrm{td}_C(My) - \mathrm{grk}_{\mathrm{C}}(My) \geqslant 1.$$

In Boris Zilber's terminology, these two conditions are called respectively freeness over $F$ and normality of the variety $\tilde{U}_a$.

We change notation slightly to match the earlier section on Schanuel conditions. The subvariety $U_a \subseteq (F \times \mathcal{E})^n$ becomes $V$, and a generic point of $V$ is written $(a, b)$ where $a \in F^n$ and $b \in \mathcal{E}^n$. The differentially closed field is now $F_0$, and the extension $F \lhd F_2$ becomes $F_0 \lhd F$. The result we now need, stated in terms of differentials, is as follows.

**Proposition 5.1.** *If $V$ is free and normal over $F_0$, $(a, b)$ is generic in $V$ over $F_0$ and $F = F_0(a, b)$ then there is a derivation $D$ on $F$ which extends the original derivation $D_0$ on $F_0$ with no new constants and such that $w(b_i)D = da_i D$ for each $i = 1, \ldots, n$.*

In order to prove this we use the following two lemmas which are essentially the same as lemma 3.1 of [15]. For $x, y \in F^n$, write $x \cdot y$ for the dot product $\sum_{i=1}^n x_i y_i$. For any $y \in F^n$, write $H_y$ for the affine hyperplane $\{x \in F^n \mid x \cdot y = 1\}$. It is easy to see that if every coordinate of $y$ is nonzero and $F^n \xrightarrow{p} F^m$ is any coordinate projection with $m < n$ then the image of $H_y$ under $p$ is all of $F^m$. The following lemma generalises this observation.

**Lemma 5.2.** *Let $F$ be an algebraically closed field of infinite transcendence degree over a set $b$ of parameters, possibly infinite. Let $V \subseteq F^n$ be any irreducible variety defined over $b$ and let $d = \dim V$. Let $c \in F^n$ be algebraically independent over $b$, and let $V'$ be an irreducible component of $V \cap H_c$. Let $a$ be generic in $V'$ over $bc$. Suppose that $e \in F^m$ for some $m$ with $e \in \mathrm{acl}(ba)$ but $a \notin \mathrm{acl}(be)$.*

*Then $\mathrm{td}(e/bc) = \mathrm{td}(e/b)$, that is $e$ is free from $c$ over $b$.*

*Proof.* By the independence of $c$ over $b$, $V$ is not contained in $H_c$. Thus $\dim V' < \dim V$, and so $\dim V' = d - 1$. Thus $\mathrm{td}(a/bc) = d - 1$, and $\mathrm{td}(a/b) = d$ since $a$ is necessarily generic in $V$ over $b$.

Let $U = \mathrm{loc}(c/be)$ be the irreducible variety over $be$ with generic point $c$. Suppose for a contradiction that $\mathrm{td}(e/bc) < \mathrm{td}(e/b)$. Then $e$ is not free from $c$ over $b$, and so $\dim U < |c| = n$. But $\dim U = \mathrm{td}(c/be) \geqslant \mathrm{td}(c/ba) = n - 1$, so $\dim U = n - 1$.

Now $c \in H_a \cap U$, and $\dim H_a = n - 1$, so by the genericity of $c$ in $U$ we see that $U = H_a$. Thus $H_a$ is defined over $be$, and so $(\forall y \in H_a)[y \cdot x = 1]$ is expressible as a formula with parameters $be$. This formula defines $a$ uniquely, and so $a \in \mathrm{acl}(be)$, which contradicts the assumption. □

We apply this to reduce the dimension of a free and normal variety without losing freeness and normality.

**Lemma 5.3.** *If $(\bar{a}, \bar{b})$ is free and normal over $A_1$ and $\mathrm{td}(\bar{a}, \bar{b}/A_1) > n$ then there is $A_2$ extending $A_1$ such that $\mathrm{td}(\bar{a}, \bar{b}/A_2) = n$ and $(\bar{a}, \bar{b})$ is free and normal over $A_2$.*

*Proof.* This is an immediate application of the previous lemma using the definitions of freeness and normality. □

*Proof of Proposition 5.1.* We assume that $\mathrm{td}_{F_0}(a, b) = n$, using lemma 5.3. For $i = 1, \ldots, n$ let $\omega_i = w(b_i) - da_i$. Let $H$ be the subspace $\langle \omega_1, \ldots, \omega_n \rangle$ of $\Omega(F/C)$, and $\Delta = \ker H \subseteq \mathrm{Der}(F/C)$.

**Step 1** Suppose $x \in C_\Delta$. Then $dx \in H$ so there are $c_1, \ldots, c_n \in F$ such that $dx = \sum_{i=1}^n c_i \omega_i$.

Choose a $K$-basis $\gamma_1, \ldots, \gamma_r$ for $c_1, \ldots, c_n$, such that there are $m_{ij} \in R$ with $c_i = \sum_{j=1}^r m_{ij} \gamma_j$. Here $R$ is the ring of endomorphisms of $\mathcal{E}$ and $K$ is its field of fractions. Let

$$\alpha_j = \sum_{i=1}^n m_{ij} a_j, \qquad \beta_j = \bigoplus_{i=1}^n m_{ij} b_i, \qquad \eta_j = w(\beta_j) - d\alpha_j$$

for each $j$. Then $\eta_j = \sum_{i=1}^n m_{ij} \omega_j$ and $dx = \sum_{j=1}^r \gamma_j \eta_j$.

Let $\Delta' = \ker(\eta_1, \ldots, \eta_r)$, a superspace of $\Delta$ in $\mathrm{Der}(F/C)$. If $D \in \Delta'$ then $dx(D) = \sum_{j=1}^r \gamma_j \eta_j D = 0$, so $dx \in \mathrm{Ann}(\Delta')$ and $x \in C_{\Delta'}$. By freeness of $V$ over $F_0$, and nondegeneracy of the matrix $(m_{ij})$, the $\eta_j$ are $K$-linearly independent. Thus by proposition 2.17 they are linearly independent over $C_{\Delta'}$ and by 2.12 they are linearly independent over $F$. Now applying 2.12 to $dx, \eta_1, \ldots, \eta_r$ we see that each $\gamma_i \in C_{\Delta'}$.

Expanding out the $\eta_j$, we get

$$dx + \sum_{j=1}^{r} \gamma_j d\alpha_j = \sum_{j=1}^{r} \gamma_j w(\beta_j)$$

Applying the canonical map $\Omega(F/C) \to \Omega(F/C_{\Delta'})$ we get that

$$d_{F/C_{\Delta'}} x + \sum_{j=1}^{r} \gamma_j d_{F/C_{\Delta'}} \alpha_j = \sum_{j=1}^{r} \gamma_j w_{F/C_{\Delta'}}(\beta_j)$$

and, since the $\gamma_j \in C_{\Delta'}$, the left hand side is exact and the right hand side is a $C_{\Delta'}$-linear combination of Weierstrass differentials, so is of the first kind. Thus both sides are zero. In particular, $\sum_{j=1}^{r} \gamma_j w_{F/C_{\Delta'}}(\beta_j) = 0$.

By renumbering, we may suppose that the $j$ for which $w_{F/C_{\Delta'}}(\beta_j) \neq 0$ are $1, \ldots, s$.

Then $w_{F/C_{\Delta'}}(\beta_1), \ldots, w_{F/C_{\Delta'}}(\beta_s)$ are Weierstrass differentials which are linearly dependent over $C_{\Delta'}$, so by proposition 2.17 we may assume these $\gamma_j \in K$. But the $\gamma_j$ were taken to be linearly independent over $K$, so we must have $s = 0$, that is each $\beta_j \in C_{\Delta'}$. Then also each $\alpha_j \in C_{\Delta'}$ by the definition of $\Delta'$.

Now $\operatorname{td}(C_{\Delta'}/C) = \dim \operatorname{Ann}(\Delta') \leqslant r$, so $\operatorname{td}_C(\bar{\alpha}, \bar{\beta}) \leqslant r$ and thus by the normality of $(\bar{a}, \bar{b})$ we deduce that $r = 0$. Then each $c_i = 0$, so $dx = 0$ and $x \in C$. This shows that $C_\Delta = C$ as required. Since each $c_i = 0$ this also shows that the $\omega_i$ are linearly independent over $F$.

**Step 2** We now show that they are even $F$-linearly independent modulo $\Omega(F_0/C) \otimes_{F_0} F$. Suppose that for some $\gamma_i \in F$ and $\eta \in \Omega(F_0/C)$ we have $\sum_{i=1}^{n} \gamma_i \omega_i = \gamma_0 \eta$. Apply the canonical map $\Omega(F/C) \longrightarrow \Omega(F/F_0)$. The kernel of this map contains $\Omega(F_0/C) \otimes_{F_0} F$, so we get $\sum \gamma_i \bar{\omega_i} = 0$. Now $\bar{\omega_i} = w_{F/F_0} b_i - d_{F/F_0} a_i$, and so the argument of step 1 shows that the $\gamma_i \in K$.

Let $a_0 = \sum \gamma_i a_i$ and $b_0 = \bigoplus \gamma_i \cdot b_i$. Then $\gamma_0 \eta = \sum \gamma_i \omega_i = w b_0 - d a_0$. By freeness, either $\gamma_1, \ldots, \gamma_n$ are all zero or at least one of $a_0, b_0$ does not lie in $F_0$. Then applying the canonical map into $\Omega(F/F_0)$ we have $\bar{\eta} = 0$ and thus $da_0 = wb_0$ which is impossible when neither is in $F_0$. Thus each $\gamma_i$ is zero and $\gamma_0 \eta = 0$ as required.

**Step 3** Consider the canonical map $\Omega(F/C) \longrightarrow \Omega(F/D_0)$. Step 2 says that the images $\bar{\omega_i}$ are $F$-linearly independent in $\Omega(F/D_0)$. Now $\dim \Omega(F/D_0) = \operatorname{td}(F/F_0) + 1 = n + 1$. Also $\dim \Delta = \dim \Omega(F/D_0) - \dim\langle \bar{\omega_1}, \ldots, \bar{\omega_n} \rangle = 1$. Take $D$ to be the unique $D \in \Delta$ such that $D \restriction_{F_0} = D_0$. Then $C_D = C_\Delta$ since $D$ spans $\Delta$, and we have already shown that $C_\Delta = C$. Thus there are no new constants, and this completes the proof. $\square$

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