GENERALISED HASSE VARIETIES AND THEIR JET SPACES

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ABSTRACT. Building on the abstract notion of prolongation developed in [7], the theory of *iterative Hasse rings* and *schemes* is introduced, simultaneously generalising difference and (Hasse-)differential rings and schemes. This work provides a unified formalism for studying difference and differential algebraic geometry, as well as other related geometries. As an application, Hasse jet spaces are constructed generally, allowing the development of the theory for arbitrary systems of algebraic partial difference/differential equations, where constructions by earlier authors applied only to the finite dimensional case. In particular, it is shown that under appropriate separability assumptions a Hasse variety is determined by its jet spaces at a point.

1. INTRODUCTION

The algebraic theories of ordinary and partial differential equations, difference equations, Hasse-differential equations, and mixed difference-differential equations bear many formal analogies and some of the theory may be developed uniformly under the rubric of equations over rings with fixed additional operators. In this paper, a continuation of [7], we propose a unified theory of rings with stacks of compatible operators, what we call iterative Hasse rings, and then undertake a detailed study of the infinitesimal structure of Hasse varieties showing how to define jet spaces for these Hasse varieties and that the jet spaces determine the varieties under a separability hypothesis.

Before we consider Hasse rings in full generality, let us consider the special case of ordinary differential rings. Here we have a commutative ring R given together with a derivation $\partial : R \to R$. At one level, to say that ∂ is a derivation is simply to say that ∂ is additive and satisfies the Leibniz rule. On the other hand, we could say that the exponential map $R \to R[\epsilon]/(\epsilon^2)$ given by $x \mapsto x + \partial(x)\epsilon$ is a ring homomorphismm. When R is $\mathbb{Q}$-algebra, this truncated exponential map lifts to a ring homomorphism $R \to R[[\epsilon]]$ given by $x \mapsto \frac{1}{n!}\partial^n(x)\epsilon^n$. If we define $\partial_n(x) := \frac{1}{n!}\partial^n(x)$, then the exponential map takes the form $x \mapsto \sum \partial_n(x)\epsilon^n$. Let us note that we have a formula relating composites of the ∂_n operators with single applications. Indeed, $\binom{n+m}{n}\partial_{n+m} = \frac{(n+m)!}{n!m!}\frac{1}{(n+m)!}\partial^{n+m} = \frac{1}{n!}\partial^n \circ \frac{1}{m!}\partial^m = \partial_n \circ \partial_m$.

From the defining equation for ∂_n, it is clear that it gives no more information than is already given by the first derivative ∂. However, we could consider the general category of Hasse-differential rings which are rings R given together with

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a sequence of additive operators $\partial_n : R \to R$ for which the function $R \to R[[\epsilon]]$ given by $a \mapsto \sum_{n=0}^{\infty} \partial_n(a)\epsilon^n$ is a ring homomorphism, and the operators satisfy the rule that $\partial_0 = \text{id}$ and $\binom{n+m}{n}\partial_{n+m} = \partial_n \circ \partial_m$. Dropping the hypothesis that R is a $\mathbb{Q}$-algebra, one finds Hasse-differential rings for which the higher operators are genuinely independent of the first derivative. Indeed, the language of Hasse-differential rings is the appropriate framework for studying differential equations in positive characteristic.

As explained already by Matsumara (Section 27 of [5]), the iteration rule, $\binom{n+m}{n}\partial_{n+m} = \partial_n \circ \partial_m$, may be expressed as a commuting diagram. Let $(R, \langle \partial_i : i \in \mathbb{N} \rangle)$ be a Hasse-differential ring. That is to say, the map $E_\epsilon : R \to R[[\epsilon]]$ given by $x \mapsto \sum \partial_n(x)\epsilon^n$ is a ring homomorphism. Extending each ∂_n continuously to $R[[\epsilon]]$ by defining $\partial_0(\epsilon) := \epsilon$ and $\partial_n(\epsilon) := 0$ for $n > 0$, we obtain a second exponential homomorphism $E_\eta : R[[\epsilon]] \to R[[\epsilon]][[\eta]]$. On the other hand, there is a natural continuous homomorphism $\Delta : R[[\zeta]] \to R[[\epsilon]][[\eta]]$ given by $\zeta \mapsto (\epsilon + \eta)$. Expanding the powers of $(\epsilon + \eta)$, one sees easily that the iteration rule holds if and only if $\Delta \circ E_\zeta = E_\eta \circ E_\epsilon$. That is, the following diagram is commutative.

$$\begin{array}{ccc} R & \xrightarrow{E_\epsilon} & R[[\epsilon]] \\ {\scriptstyle E_\zeta}\downarrow & & \downarrow{\scriptstyle E_\eta} \\ R[[\zeta]] & \xrightarrow{\zeta \mapsto (\epsilon+\eta)} & R[[\epsilon]][[\eta]] \end{array}$$

We generalise this ring theoretic treatment of iterative Hasse-differential rings to produce a theory of generalised Hasse rings by encoding the generalised Leibniz rules via exponential maps and the iteration rules via a commutative diagram analogous to the one describing the iteration rule for Hasse-derivations. To present a notion of an iterative Hasse ring we need two kinds of data. First, we need a projective system of finite free ring schemes $\underline{\mathcal{D}} := \langle \pi_{i,j} : \mathcal{D}_i \to \mathcal{D}_j \rangle_{0 \le j \le i < \omega}$. That is, we ask that each $\mathcal{D}_i$ is, as an additive group scheme, simply some finite Cartesian power of the usual additive group scheme while multiplication is given by some regular functions. A $\underline{\mathcal{D}}$-ring structure on R is then given by a sequence of ring homomorphisms $E_i : R \to \mathcal{D}_i(R)$ which are compatible with the projective system. Fixing the identifications of each $\mathcal{D}_i$ with a power of the additive group, the map E_i may be presented as $x \mapsto (\partial_0^{(i)}(x), \ldots, \partial_{m_i}^{(i)}(x))$ where each $\partial_k^{(i)} : R \to R$ is an additive operator. To say that these operators give R a $\underline{\mathcal{D}}$-ring structure is equivalent to imposing certain generalised Leibniz rules and identities relating the components of E_i to those of E_j. The second kind of data we require is a collection of morphisms of ring schemes $\Delta_{i,j} : \mathcal{D}_{i+j} \to \mathcal{D}_i \circ \mathcal{D}_j$. For iterativity, we require the following diagrams to commute.

$$\begin{array}{ccc} R & \xrightarrow{E_j} & \mathcal{D}_j(R) \\ {\scriptstyle E_{i+j}}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_j(E_i)} \\ \mathcal{D}_{i+j}(R) & \xrightarrow{\Delta_{i,j}} & \mathcal{D}_i(\mathcal{D}_j(R)) \end{array}$$

We were led to this notion of iteration by considering Matsumura's presentation of the theory for Hasse-derivations.

This theory of iterative Hasse rings is developed in Section 2. In the appendix we discuss several other examples showing that this formalism captures many of the interesting cases of rings with distinguished operators.

Our main goal is to understand algebraic equations involving Hasse operators and these equations are naturally encoded by Hasse schemes, or really, Hasse subschemes of algebraic schemes. To make the issues more concrete, a $\underline{\mathcal{D}}$-equation in some $\underline{\mathcal{D}}$-ring R is simply an algebraic equation on the variables and several of the operators $\partial_k^{(i)}$ applied to the variables. As such, the set of solutions naturally forms a subset of the R-points of some algebraic scheme X and the equations themselves are encoded by projective systems of subschemes of prolongation spaces of X. We shall refer to these projective systems as $\underline{\mathcal{D}}$-schemes. They are studied in some detail in Section 3.

If X is an algebraic variety over a field k, then by the nth jet space of X at a point $p \in X(k)$ we mean the space $\operatorname{Hom}_k\left(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^{n+1}, k\right)$. In Section 4 we define jet spaces for $\underline{\mathcal{D}}$-varieties and show that they have enough points to distinguish between different $\underline{\mathcal{D}}$-subvarieties, at least under an appropriate separability hypothesis. We have already encountered the main difficulty in [7]; that the prolongation space and jet space functors do not commute. The interpolation map comparing the jet space of a prolongation with the prolongation of a jet space, introduced in [7], is the main technical ingredient in our construction of jet spaces for $\underline{\mathcal{D}}$-varieties.

To close this introduction, let us be clear about our aims in the present paper. We develop the geometry of algebraic equations involving additional operators. While our setting may be regarded as a generalisation of difference, differential, and Hasse-differential algebra, our main goal is to unify these subjects rather than to generalise them (though our formalism does allow for such a generalisation). This unification manifests itself not only in proofs and constructions which apply equal well to each of the principal examples, but in a precise formalism for studying confluence between Hasse-differential and difference algebraic geometry. In terms of the geometry, our primary goal is to make sense of the linearisation of general $\underline{\mathcal{D}}$-equations through a jet space construction and then to show that these linear spaces determine the $\underline{\mathcal{D}}$-varieties, at least under suitable separability hypotheses. Using noetherianity, this last point is a tautology for algebraic varieties, but it is far from obvious even when one specialises to a well-known theory of fields with operators such as partial difference or differential algebra. For *finite dimensional* difference/differential varieties, jet spaces were constructed by Pillay and Ziegler [8]. Our theory extends theirs to the infinite dimensional setting.

In the present paper, we do not develop the model theory of general $\underline{\mathcal{D}}$-fields and leave such questions as the existence of model companions, simplicity, the behaviour of ranks, *et cetera* to a later work. Jet spaces were the key technical devices of the Pillay-Ziegler geometric proofs of the dichotomy theorem for minimal types in differentially closed fields of characteristic zero. In [6], arc spaces substituted for jet spaces to extend the dichotomy theorem to *regular* types. While arc spaces did the job in the differential case, jet spaces are preferable because they give a direct linearisation of the equations. Provided that the foundational model theoretic issues are resolved, our theorem on $\underline{\mathcal{D}}$-jet spaces determining $\underline{\mathcal{D}}$-varieties should give information about canonical bases of (quantifier-free) types in the corresponding theory of $\underline{\mathcal{D}}$-fields.

Likewise, there are some closely allied algebraic issues we do not pursue here. For example, jet spaces are clearly connected to a general theory of $\underline{\mathcal{D}}$-modules.

Moreover, we have not fleshed out the theory of specialisations of $\underline{\mathcal{D}}$-rings nor in its local form a theory of valued $\underline{\mathcal{D}}$-fields. Each of these further developments motivates our research into jet spaces for Hasse varieties and will be taken up in a sequel.

2. Generalised Hasse rings

Let us recall the following conventions and definitions from [7]. In this paper, all our rings are commutative and unitary and all our ring homorphisms preserve the identity. All schemes are separated. A *variety* is a reduced scheme of finite-type over a field, but is not necessarily irreducible.

The *standard ring scheme* $\mathbb{S}$ over A is the scheme Spec $\big(A[x]\big)$ endowed with the usual ring scheme structure. So for all A-algebras R, $\mathbb{S}(R) = (R, +, \times, 0, 1)$. An $\mathbb{S}$-*algebra scheme* $\mathcal{E}$ over A is a ring scheme together with a ring scheme morphism $s_{\mathcal{E}} : \mathbb{S} \to \mathcal{E}$ over A. We view $\mathbb{S}$ as an $\mathbb{S}$-algebra via the identity $\mathrm{id} : \mathbb{S} \to \mathbb{S}$. A *morphism* of $\mathbb{S}$-algebra schemes is then a morphism of ring schemes respecting the $\mathbb{S}$-algebra structure. Similarly one can define $\mathbb{S}$-*module* schemes and morphisms. By a *finite free* $\mathbb{S}$-*algebra scheme* we mean an $\mathbb{S}$-algebra scheme $\mathcal{E}$ together with an isomorphism of $\mathbb{S}$-module schemes $\psi_{\mathcal{E}} : \mathcal{E} \to \mathbb{S}^{\ell}$, for some $\ell \in \mathbb{N}$. Fixing $\psi_{\mathcal{E}}$ means that we have a canonical choice of basis $\{1, e_1, \dots, e_{\ell-1}\}$ for $\mathcal{E}(A)$ over A. Replacing the e_i by corresponding indeterminates X_i, we can write $\mathcal{E}(A)$ as the A-algebra $A[X_1, \dots, X_{\ell-1}]/I$ where I is generated by expressions that explain how the monomials in $\{e_1, \dots, e_{\ell-1}\}$ are written as A-linear combinations of this basis. In fact, this will induce canonical identifications of $\mathcal{E}(R)$ with $R[X_1, \dots, X_{\ell-1}]/I_R$ for all A-algebras R, where I_R is the ideal generated by I. In particular, for all A-algebras R, we can identify $\mathcal{E}(R)$ with $R \otimes_A \mathcal{E}(A)$, both as A-algebras and $\mathcal{E}(A)$-algebras.

Given a finite free $\mathbb{S}$-algebra scheme $\mathcal{E}$ over A, an $\mathcal{E}$*-ring* is an A-algebra k together with an A-algebra homomorphism $e : k \to \mathcal{E}(k)$. A detailed study of $\mathcal{E}$-rings was carried out in [7], and we will assume the results of that paper in what follows.

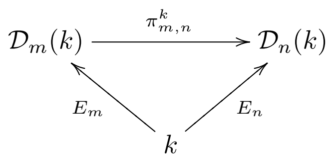
We are interested in rings equipped with an entire directed system of $\mathcal{E}$-ring structures for various $\mathcal{E}$.

Definition 2.1 (Hasse system)**.** Suppose A is a fixed ring. A *generalised Hasse system* over A is an inverse system of finite free $\mathbb{S}$-algebra schemes over A, $\underline{\mathcal{D}} = \{\mathcal{D}_n \mid n \in \mathbb{N}\}$, such that $\mathcal{D}_0 = \mathbb{S}$, and the transition maps $\pi_{m,n} : \mathcal{D}_m \to \mathcal{D}_n$, for $m \geq n$, are surjective ring scheme morphisms over A. We denote by $s_n : \mathbb{S} \to \mathcal{D}_n$ the $\mathbb{S}$-algebra structure on $\mathcal{D}_n$ and by $\psi_n : \mathcal{D}_n \to \mathbb{S}^{\ell_n}$ the $\mathbb{S}$-module isomorphisms witnessing that $\mathcal{D}_n$ is finite and free.

Definition 2.2 (Hasse ring)**.** Suppose $\underline{\mathcal{D}}$ is a Hasse system over A. A *generalised Hasse ring* (or $\underline{\mathcal{D}}$*-ring*) *over* A is an A-algebra equipped with a system of $\mathcal{D}_n$-ring structures that are compatible with π. That is, a $\underline{\mathcal{D}}$-ring is a pair (k, E) where k is an A-algebra and $E = \{E_n : k \to \mathcal{D}_n(k) \mid n \in \mathbb{N}\}$ is a sequence of A-algebra homomorphisms such that

(i) $E_0 = \mathrm{id}$,

(ii) the following diagram commutes for all $m \geq n$



Remark 2.3. One may equally well describe a $\mathcal{D}$-ring by giving a collection of maps $\{\partial_{i,n} : k \to k\}_{n\in\mathbb{N}, i\leq \ell_n}$ via the correspondence $\psi_n \circ E_n = (\partial_{1,n}, \dots, \partial_{\ell_n,n})$. That the collection $\{\partial_{i,n}\}$ so defines a $\mathcal{D}$-ring structure on k is equivalent to the satisfaction of a certain system of functional equations.

Our choice of a natural-number-indexing for Hasse systems is convenient but not absolutely necessary. Indeed, some contexts may be more naturally dealt with by considering Hasse systems that look like $\{\mathcal{D}_{\overline{n}} \mid \overline{n} \in \mathbb{N}^r\}$ or $\{\mathcal{D}_{\overline{n}} \mid \overline{n} \in \mathbb{Z}^r\}$ with a correspondingly adjusted definition for the system of surjective morphisms π. However, indexing by $\mathbb{N}$ does simplify the exposition somewhat, and all our examples can be made to fit into this setting.

The first example of a Hasse system is where each $\mathcal{D}_n = \mathbb{S}$ and $\pi_{m,n} = \psi_n = \mathrm{id}$. Then for any A-algebra k, the only $\underline{\mathcal{D}}$-ring structure on k is the trivial one with $E_n = \mathrm{id}$. This example captures the context of rings without any additional structure. Our main example, that of Hasse-differential rings, is discussed below. See the Appendix for a discussion of several other examples including difference rings, an analogue of q-iterative difference rings, and difference-differential rings.

Example 2.4 (Hasse-Differential rings)**.** Consider the Hasse system $\mathrm{HD}_e = \{\mathcal{D}_n : n \in \mathbb{N}\}$ where for any ring R

- $\mathcal{D}_n(R) = R[\eta_1, \dots, \eta_e]/(\eta_1, \dots, \eta_e)^{n+1}$, where $\eta_1, \dots, \eta_e$ are indeterminates;
- $s^R_n : R \to \mathcal{D}_n(R)$ is the natural inclusion;
- $\psi^R_n : \mathcal{D}_n(R) \to R^{\ell_n}$ is an identification via a fixed ordering of the monomial basis of $R[\eta_1, \dots, \eta_e]/(\eta_1, \dots, \eta_e)^{n+1}$ over R; and,
- for $m \geq n$, $\pi^R_{m,n} : \mathcal{D}_m(R) \to \mathcal{D}_n(R)$ is the quotient map.

We have described this Hasse system by describing what happens at R-points, but it is not hard to check that $\mathcal{D}_n$ is indeed a ring scheme and that s_n, ψ_n, and $\pi_{m,n}$ are all ring scheme morphisms. Writing $E_n(x) = \sum_{\alpha\in\mathbb{N}^e, |\alpha|\leq n} \partial_\alpha(x)\eta^\alpha$ we see that an HD_e-ring in this case is a ring k together with a sequence of additive maps $\{\partial_\alpha : k \to k\}_{\alpha\in\mathbb{N}^e}$ satisfying $\partial_\alpha(xy) = \sum_{\beta+\gamma=\alpha} \partial_\beta(x)\partial_\gamma(y)$, $\partial_{\overline{0}} = \mathrm{id}$, and $\partial_\alpha(1) = 0$ for $|\alpha| > 0$.

A ring equipped with e Hasse-derivations can be viewes as an HD_e-ring. Recall that a *Hasse-derivation* on a ring k is a sequence of additive maps from k to k, $\mathbf{D} = (D_0, D_1 \dots)$, such that

- $D_0 = \mathrm{id}$ and
- $D_n(xy) = \sum_{a+b=n} D_a(x)D_b(y)$.

(cf. Section 27 of [5], for example.) Suppose $\mathbf{D}_1, \dots, \mathbf{D}_e$ is a *sequence* of e Hasse derivation on k and set $E(x) = \sum_{\alpha\in\mathbb{N}^e} D_{1,\alpha_1} D_{2,\alpha_2} \cdots D_{e,\alpha_e}(x)\eta^\alpha$. Then $E : k \to$

$k[[\eta_1,\dots,\eta_e]]$ is a ring homomorphism and we can view it as a system $(E_n)_{n\in\omega}$ where E_n is the composition of E with the quotient $k[[\eta_1,\dots,\eta_e]] \to k[\eta_1,\dots,\eta_e]/(\eta_1,\dots,\eta_e)^{n+1}$. Then (k,E) is an HD_e-ring.

This example specialises further to the case of *partial differential fields in characteristic zero.* Suppose k a field of characteristic zero and $\partial_1,\dots,\partial_e$ are derivations on k. Then $D_{i,n} := \frac{\partial_i^n}{n!}$, for $1 \le i \le e$ and $n \ge 0$, defines a sequence of Hasse-derivations on k. The HD_e-ring structure on k is given in multi-index notation by $E_n(x) := \sum_{\alpha\in\mathbb{N}^e,|\alpha|\le n} \frac{1}{\alpha!}\partial^\alpha(x)\eta^\alpha$ where $\partial := (\partial_1,\dots,\partial_e)$.

On the other hand we can specialise in a different direction to deal with *fields of finite imperfection degree.* The following example is informed by [11]: suppose k is a field of characteristic $p>0$ with imperfection degree e. Let $t_1,\dots,t_e$ be a p-basis for k. Then $t_1,\dots,t_e$ are algebraically independent over $\mathbb{F}_p$. Consider $\mathbb{F}_p[t_1,\dots,t_e]$ and for $1\le i\le e$ and $n\in\mathbb{N}$, define

$$\mathbf{D}_{i,n}(t_1^{\alpha_1}\cdots t_e^{\alpha_e}) := \begin{pmatrix}\alpha_i\\ n\end{pmatrix} t_1^{\alpha_1}\cdots t_i^{\alpha_i-n}\cdots t_e^{\alpha_e}.$$

and extend by linearity to $\mathbb{F}_p[t_1,\dots,t_e]$. Then $(\mathbf{D}_1,\dots,\mathbf{D}_e)$ is a sequence of Hasse-derivations on $\mathbb{F}_p[t_1,\dots,t_e]$. Moreover, they extend uniquely to Hasse derivations on k (see Lemma 2.3 of [11]). This gives rise to an HD_e-ring structure on k.

It is not the case that every HD_e-ring is a Hasse-differential ring. In section 2.2 below we will introduce the notion of *iterativity* for Hasse systems and rings, and this will allow us to capture exactly the class of Hasse-differential rings.

2.1. **Hasse prolongations.** A generalised Hasse structure on a ring k induces, for every algebraic variety X over k, a sequence of (abstract) prolongations of X in the sense of [7]. We recall the construction here.

First some notation. Suppose $\underline{\mathcal{D}}$ is a generalised Hasse system over A and (k,E) is a $\underline{\mathcal{D}}$-ring. For each n, $E_n : k \to \mathcal{D}_n(k)$ gives $\mathcal{D}_n(k)$ a new k-algebra structure which we will denote by $\mathcal{D}_n^{E_n}(k)$ and refer to as the *exponential* k-algebra. Note that as A-algebras, $\mathcal{D}_n^{E_n}(k)$ and $\mathcal{D}_n(k)$ are identical. More generally, given any A-algebra $a : k \to R$, E_n also induces an *exponential* k-algebra structure on $\mathcal{D}_n(R)$, namely the one given by

$$k \xrightarrow{E_n} \mathcal{D}_n(k) \xrightarrow{\mathcal{D}_n(a)} \mathcal{D}_n(R)$$

which we will denote by $\mathcal{D}_n^{E_n}(R)$.

Definition 2.5 (Prolongations)**.** Suppose $\underline{\mathcal{D}}$ is a Hasse system over A, (k,E) is a $\underline{\mathcal{D}}$-ring, and X is a scheme over k. The *nth prolongation of* X, $\tau(X,\mathcal{D}_n,E_n)$, or just $\tau_n(X)$ for short, is the Weil restriction of $X\times_k \mathcal{D}_n^{E_n}(k)$ from $\mathcal{D}_n(k)$ to k (when it exists). We usually write $\tau(X)$ for $\tau_1(X)$.

The characteristic property of prolongations is that for any k-algebra R, there is a canonical identification

$$\tau_n(X)(R) = X\big(\mathcal{D}_n^{E_n}(R)\big). \qquad (1)$$

Indeed, this is Lemma 4.5 of [7].

Remark 2.6. (a) It is not always the case that the Weil restriction, and hence the prolongation, exists. However, if X is such that every finite set of points in X is contained in an affine open subscheme, then $\tau_n(X)$ does exist. So for example, if we restrict our attention to quasi-projective schemes, then we do not have to worry about existence. For more details on Weil restrictions see Section 2 of [7].

(b) Definition 2.5 is just the definition of an abstract prolongation (Definition 4.1 of [7]), specialised to the finite free $\mathbb{S}$-algebra schemes $\mathcal{D}_n$. It follows from the work in that paper that τ_n is a covariant functor which preserves étale morphisms, smooth embeddings, and closed embeddings (cf. Proposition 4.6 of [7].

For $m \geq n$, the morphisms $\pi_{m,n} : \mathcal{D}_m \to \mathcal{D}_n$ induce morphisms $\hat{\pi}_{m,n} : \tau_m(X) \to \tau_n(X)$. Indeed, since k is a $\underline{\mathcal{D}}$-ring, we have that $\pi^k_{m,n} : \mathcal{D}^{E_m}_m(k) \to \mathcal{D}^{E_n}_n(k)$ is a k-algebra homomorphism, and so, for any fixed k-algebra R, so is the corresponding $\pi^R_{m,n} : \mathcal{D}^{E_m}_m(R) \to \mathcal{D}^{E_n}_n(R)$. Now on R-points, using the identification (1) above, $\hat{\pi}_{m,n}$ is just the map induced by $\pi^R_{m,n}$. See section 4.1 of [7] for more details on the morphism between prolongations induced by a morphism of finite free $\mathbb{S}$-algebra schemes.

Setting $m = 0$ we see that the nth prolongation obtains the structure of a scheme over X; namely, $\hat{\pi}_{n,0} : \tau_n(X) \to X$.

Proposition 2.7. *Suppose X is a variety (so reduced and of finite-type) over a $\underline{\mathcal{D}}$-field k. For all $m \geq n$, $\hat{\pi}_{m,n} : \tau_m(X) \to \tau_n(X)$ is a dominant morphism.*

Proof. Let $K = k^{\mathrm{alg}}$ be the algebraic closure of k. On K-points $\hat{\pi}_{m,n}$ is the map $X\big(\mathcal{D}^{E_m}_m(K)\big) \to X\big(\mathcal{D}^{E_n}_n(K)\big)$ induced by $\pi_{m,n} : \mathcal{D}^{E_m}_m(K) \to \mathcal{D}^{E_n}_n(K)$. Hence the proposition will follow from the following general claim:

Claim 2.8. *If $\rho : R \to S$ is a surjective map of artinian K-algebras and $P \in X(S)$ is a smooth S-point of X, then there is an R-point $Q \in X(R)$ sent to P by the map induced by ρ.*

Proof of Claim 2.8. First of all, we can decompose R and S as products of artinian k-algebras, $R \cong (\prod_{i=1}^n A_i) \times C$ and $S \cong \prod_{i=1}^n B_i$, where the A_is and B_is are local, and there exist local surjective homomorphisms $\rho_i : A_i \to B_i$, such that for all $x = (a_1, \ldots, a_n, c) \in R$, $\rho(x) = \big(\rho_1(a_1), \ldots, \rho_n(a_n)\big)$. Now for each $i \leq n$, let $P_i \in X(B_i)$ be the image of P under the map $X(S) \to X(B_i)$ induced by the projection $S \to B_i$. Since A_i is artinian, $\rho_i : A_i \to B_i$ is local, and P_i is a smooth B_i-point of X, we can lift P_i to a point $Q_i \in X(A_i)$. As K is algebraically closed, we can find $Q_C \in X(C)$. Now, letting $Q \in X(R)$ be the point which projects to $Q_C \in X(C)$ and $Q_i \in X(A_i)$ for $i \leq n$, we get that ρ maps Q to P as desired. □

We complete the proof of Proposition 2.7. Using the functoriality of the prolongations, we may assume that X is irreducible over k. Now, by the claim, every smooth point of $X\big(\mathcal{D}^{E_n}_n(K)\big)$ is in the image of $X\big(\mathcal{D}^{E_m}_m(K)\big) \to X\big(\mathcal{D}^{E_n}_n(K)\big)$. Let Y be the proper k-closed subvariety of singular points of X. Then under the identification $X\big(\mathcal{D}^{E_n}_n(K)\big) = \tau_n(X)(K)$, the set $Y\big(\mathcal{D}^{E_n}_n(K)\big)$ is identified with $\tau_n(Y)(K)$, which is a proper k-closed subset of $\tau_n(X)(K)$. Hence $\hat{\pi}_{m,n} : \tau_m(X) \to \tau_n(X)$ is dominant, as desired. □

Definition 2.9. Let $\nabla_n : X(k) \to \tau_n(X)(k)$ be the map which, under the identification $\tau_n(X)(k) = X\big(\mathcal{D}^{E_n}_n(k)\big)$, is induced by $E_n : k \to \mathcal{D}^{E_n}_n(k)$.

Note that ∇ is only defined on the k-points. It is not an algebraic section, but rather a $\underline{\mathcal{D}}$-section.

Lemma 2.10. *For each $n < \omega$, ∇_n is a section to $\hat{\pi}^k_{n,0} : \tau_n(X)(k) \to X(k)$ and satisfies $\hat{\pi}_{n+1,n} \circ \nabla_{n+1} = \nabla_n$.*

Proof. Immediate from the definitions. □

We record the following fact from [7] for later use:

Fact 2.11 (Proposition 4.7(b) of [7])**.** *Suppose $f : X \to Y$ is a morphism of schemes over k and $a \in Y(k)$. Then $\tau_n(X)_{\nabla_n(a)}$, the fibre of $\tau_n(f) : \tau_n(X) \to \tau_n(Y)$ over $\nabla_n(a)$, is $\tau_n(X_a)$.* □

Example 2.12. In our main examples the prolongation spaces specialise to the expected objects. So for pure rings (when $\mathcal{D}_n = \mathbb{S}$) we get $\tau_n(X) = X$. For rings equipped with endomorphisms $\sigma_1, \sigma_2, \ldots$ (this is Example 5.1 of the Appendix, when $\underline{\mathcal{D}} = \text{End}$) $\tau_n X = X \times X^{\sigma_1} \times \cdots \times X^{\sigma_n}$, and $\nabla_n(x) = \big(x, \sigma_1(x), \ldots, \sigma_n(x)\big)$. In the Hasse-differential case of Example 2.4, the $\tau_n(X)$ and ∇_n are the usual differential prolongations with their differential sections. For example, if (k, δ) is an ordinary differential field of characteristic zero, then $\nabla_n(x) = \big(x, \delta(x), \ldots, \frac{\delta^n(x)}{n!}\big)$. See Example 4.2 of [7] for more details on these particular cases. It is also worth pointing out that if $\mathcal{D}_n = k[\epsilon]/(\epsilon)^{n+1}$ and $E_n = s_n : k \to \mathcal{D}_n(k)$ is the usual inclusion, then $\tau_n(X)$ is the nth arc space of X, $\text{Arc}_n(X)$, and ∇_n is the zero section. (The arc spaces are the higher tangent bundles; $\text{Arc}_1(X)$ is the tangent bundle of X.)

2.2. **Iterativity.** As explained in Section 4.2 of [7] we can compose finite free $\mathbb{S}$-algebra schemes. Specialising to Hasse systems, for all $m, n \in \mathbb{N}$ we get finite free $\mathbb{S}$-algebra schemes $\mathcal{D}_{(m,n)} := \mathcal{D}_m\mathcal{D}_n$. So for any A-algebra R, $\mathcal{D}_{(m,n)}(R) = \mathcal{D}_m\big(\mathcal{D}_n(R)\big)$ where the R-algebra structure is given by

$$R \xrightarrow{s_n^R} \mathcal{D}_n(R) \xrightarrow{s_m^{\mathcal{D}_n(R)}} \mathcal{D}_m\big(\mathcal{D}_n(R)\big).$$

There are also the $\mathcal{D}_{(m,n)}$-ring structures on k, $E_{(m,n)} := E_mE_n$, given by

$$k \xrightarrow{E_m} \mathcal{D}_m(k) \xrightarrow{\mathcal{D}_m(E_n)} \mathcal{D}_m\big(\mathcal{D}_n(k)\big).$$

What Proposition 4.12 of [7] tells us is that $\tau_n\big(\tau_m(X)\big) = \tau(X, \mathcal{D}_{(m,n)}, E_{(m,n)})$ and $\nabla_n \circ \nabla_m = \nabla_{\mathcal{D}_{(m,n)}, E_{(m,n)}}$. Note that in this context, for $m' \leq m$ and $n' \leq n$, we have the ring scheme morphisms $\pi_{(m,n),(m',n')} : \mathcal{D}_{(m,n)} \to \mathcal{D}_{(m',n')}$ given by the composition

$$\mathcal{D}_m\big(\mathcal{D}_n(R)\big) \xrightarrow{\mathcal{D}_m(\pi^R_{n,n'})} \mathcal{D}_m\big(\mathcal{D}_{n'}(R)\big) \xrightarrow{\pi^{\mathcal{D}_{n'}(R)}_{m,m'}} \mathcal{D}'_m\big(\mathcal{D}_{n'}(R)\big).$$

It is a matter of fact that all the examples of Hasse rings corresponding to the various Hasse systems that we are particularly interested in satisfy some further relations not implied by the definition of being a Hasse ring. These further relations can be viewed as certain iterativity conditions relating $E_{(m,n)}$ with E_{m+n}. We formalise this as follows.

Definition 2.13. An *iterative* Hasse system is a Hasse system $\underline{\mathcal{D}}$ over a ring A together with a sequence of closed embeddings of ring schemes over A

$$\Delta = \big(\Delta_{(m,n)} : \mathcal{D}_{m+n} \to \mathcal{D}_{(m,n)}\big)_{m,n\in\mathbb{N}}$$

such that:

(a) Δ is compatible with π. That is, for all $m' \leq m$ and $n' \leq n$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}_{m+n} & \xrightarrow{\Delta_{(m,n)}} & \mathcal{D}_{(m,n)} \\ {\scriptstyle \pi_{m+n,m'+n'}}\downarrow & & \downarrow{\scriptstyle \pi_{(m,n),(m',n')}} \\ \mathcal{D}_{m'+n'} & \xrightarrow{\Delta_{(m',n')}} & \mathcal{D}_{(m',n')} \end{array}$$

(b) Δ is associative in the sense that for all ℓ, m, n, and any A-algebra R,

$$\begin{array}{ccc} \mathcal{D}_\ell\big(\mathcal{D}_{m+n}(R)\big) & \xrightarrow{\mathcal{D}_\ell(\Delta_{(m,n)})} & \mathcal{D}_\ell\big(\mathcal{D}_m(\mathcal{D}_n(R))\big) \\ {\scriptstyle \Delta_{(\ell,m+n)}}\uparrow & & \uparrow{\scriptstyle \Delta^{\mathcal{D}_n(R)}_{(\ell,m)}} \\ \mathcal{D}_{\ell+m+n}(R) & \xrightarrow{\Delta_{(\ell+m,n)}} & \mathcal{D}_{\ell+m}\big(\mathcal{D}_n(R)\big) \end{array}$$

commutes.

(c) $\Delta_{(m,0)} = \Delta_{(0,n)} = \mathrm{id}$ for all $m, n \geq 0$.

We say that (k, E) is an *iterative Hasse ring* (or more accurately Δ-iterative) if it is a $\underline{\mathcal{D}}$-ring and

$$\begin{array}{ccccc} \mathcal{D}_{m+n}(k) & & \xrightarrow{\Delta^k_{(m,n)}} & & \mathcal{D}_m\left(\mathcal{D}_n(k)\right) \\ & {\scriptstyle E_{m+n}}\nwarrow & & \nearrow{\scriptstyle E_{(m,n)}} & \\ & & k & & \end{array}$$

commutes for all $m, n \in \mathbb{N}$. That is, $\Delta^k_{(m,n)} : \mathcal{D}^{E_{m+n}}_{m+n}(k) \to \mathcal{D}^{E_{(m,n)}}_{(m,n)}(k)$ is a k-algebra map for all $m, n \in \mathbb{N}$.

Remark 2.14. The iteration maps induce morphisms $\hat{\Delta}_{(m,n)} : \tau_{m+n}(X) \to \tau_n\big(\tau_m(X)\big)$ such that the following diagram commutes:

$$\begin{array}{ccccc} \tau_{m+n}(X) & & \xrightarrow{\hat{\Delta}_{(m,n)}} & & \tau_n\big(\tau_m(X)\big) \\ & {\scriptstyle \nabla_{m+n}}\nwarrow & & \nearrow{\scriptstyle \nabla_n\circ\nabla_m} & \\ & & X & & \end{array}$$

(cf. Propositions 4.8(a) and 4.12 of [7]). Moreover, since the iteration maps are closed embeddings, these induced morphisms are also closed embeddings (cf. Proposition 4.8(c) of [7]).

We will need the following lemma later:

Lemma 2.15. *Suppose $(\underline{\mathcal{D}}, \Delta)$ is an iterative Hasse system. Then for all $m, n \in \mathbb{N}$, and all A-algebras R, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{D}_m\big(\mathcal{D}_{n+1}(R)\big) & \xleftarrow{\Delta_{(m,n+1)}} & \mathcal{D}_{m+n+1}(R) \\ {\scriptstyle \mathcal{D}_m(\pi^R_{n+1,n})}\downarrow & & \downarrow{\scriptstyle \Delta_{(m+1,n)}} \\ \mathcal{D}_m\big(\mathcal{D}_n(R)\big) & \xleftarrow{\pi_m^{\mathcal{D}_n(R)}} & \mathcal{D}_{m+1}\big(\mathcal{D}_n(R)\big) \end{array}$$

Proof. This is a combination of the associativity of Δ together with its compatibility with π. We will prove that the desired diagram commutes by proving that three other diagrams commute. First of all,

(2)
$$\begin{array}{ccc} \mathcal{D}_m\big(\mathcal{D}_{n+1}(R)\big) & \xleftarrow{\Delta_{(m,n+1)}} & \mathcal{D}_{m+n+1}(R) \\ {\scriptstyle \mathcal{D}_m(\Delta_{(1,n)})}\downarrow & & \downarrow{\scriptstyle \Delta_{(m+1,n)}} \\ \mathcal{D}_m\big(\mathcal{D}_1\big(\mathcal{D}_n(R)\big)\big) & \xleftarrow{\Delta_{(m,1)}^{\mathcal{D}_n(R)}} & \mathcal{D}_{m+1}\big(\mathcal{D}_n(R)\big) \end{array}$$

commutes as it is an instance of Definition 2.13(b) (associativity). Next, note that the following diagram is an instance of Definition 2.13(a) with $(1, n)$ and $(0, n)$ (the compatibility of Δ with π), and hence commutes:

$$\begin{array}{ccc} \mathcal{D}_{n+1}(R) & \xrightarrow{\pi^R_{n+1,n}} & \mathcal{D}_n(R) \\ {\scriptstyle \Delta_{(1,n)}}\downarrow & \nearrow{\scriptstyle \pi_0^{\mathcal{D}_n(R)}} & \\ \mathcal{D}_1\big(\mathcal{D}_n(R)\big) & & \end{array}$$

Applying the functor $\mathcal{D}_m$ we get that

(3)
$$\begin{array}{ccc} \mathcal{D}_m\big(\mathcal{D}_{n+1}(R)\big) & \xrightarrow{\mathcal{D}_m(\pi^R_{n+1,n})} & \mathcal{D}_m\big(\mathcal{D}_n(R)\big) \\ {\scriptstyle \mathcal{D}_m(\Delta_{(1,n)})}\downarrow & \nearrow{\scriptstyle \mathcal{D}_m\left(\pi_0^{\mathcal{D}_n(R)}\right)} & \\ \mathcal{D}_m\big(\mathcal{D}_1\big(\mathcal{D}_n(R)\big)\big) & & \end{array}$$

commutes. Finally, the following is also an instance of Definition 2.13(a) with $(m, 1)$ and $(m, 0)$, applied to the ring $\mathcal{D}_n(R)$

(4)
$$\begin{array}{ccc} & & \mathcal{D}_m\big(\mathcal{D}_n(R)\big) \\ & \nearrow{\scriptstyle \mathcal{D}_m\left(\pi_0^{\mathcal{D}_n(R)}\right)} & \uparrow{\scriptstyle \pi_m^{\mathcal{D}_n(R)}} \\ \mathcal{D}_m\big(\mathcal{D}_1\big(\mathcal{D}_n(R)\big)\big) & \xleftarrow{\Delta_{(m,1)}^{\mathcal{D}_n(R)}} & \mathcal{D}_{m+1}\big(\mathcal{D}_n(R)\big) \end{array}$$

Putting the commuting diagrams (2), (3), and (4) together proves the lemma. □

We now point out that the Hasse system coming from our main example admits a natural iteration such that the corresponding iterative Hasse rings form exactly the intended class: rings equipped with commuting iterative Hasse-derivations. See the appendix for a discussion of iterativity for other examples.

Consider the Hasse system HD_e from Example 2.4. So, for R any ring, $\mathcal{D}_n(R) = R[\eta_1,\ldots,\eta_e]/(\eta_1,,\ldots,\eta_e)^{n+1}$. We define Δ so that for all R, $\Delta^R_{(m,n)}$ from

$$\mathcal{D}_{m+n}(R) = R[\eta_1,\ldots,\eta_e]/(\eta_1,,\ldots,\eta_e)^{m+n+1}$$

to

$$\mathcal{D}_m\big(\mathcal{D}_n(R)\big) = R[\zeta_1,\ldots,\zeta_e,\epsilon_1,\ldots,\epsilon_e]/(\zeta_1,\ldots,\zeta_e)^{n+1}(\epsilon_1,\ldots,\epsilon_e)^{m+1}$$

is given by $\eta_i \mapsto (\zeta_i + \epsilon_i)$.

Proposition 2.16. *The system $\Delta = (\Delta_{(m,n)} : m,n \in \mathbb{N})$, above, makes HD_e into an iterative Hasse system. The Δ-iterative HD_e-rings in this case are exactly the rings equipped with e commuting Hasse derivations satisfying the additional identities*

$$D_a D_b = \left(\begin{array}{c} a+b \\ b \end{array}\right) D_{a+b}$$

for all $a, b, \in \mathbb{N}$. (Hasse derivations satusfying these identities are called iterative *Hasse derivations.)*

Proof. We first observe that (HD_e,Δ) is an iterative system. Indeed, $\Delta_{m,n}$ is a closed embedding of ring schemes, it is compatible with π, and it is associative (the latter is just the associativity of $+$), and $\Delta_{(m,0)} = \Delta_{(0,n)} = \mathrm{id}$.

Now suppose (k,E) is an HD_e-ring. For each n, write $E_n(x) = \sum_{\alpha\in\mathbb{N}^e, |\alpha|\le n} \partial_\alpha(x)\eta^\alpha$. Let $D_{i,n} := \partial_{(0,\ldots,n,\ldots,0)}$, where here the multi-index has n in the ith co-ordinate and 0 everywhere else. So $\mathbf{D}_1 := (D_{1,0}, D_{1,1},\ldots),\ldots,\mathbf{D}_e := (D_{e,0}, D_{e,1},\ldots)$ form a sequence of e Hasse derivations. Now, writing out $\Delta_{(m,n)} \circ E_{m+n}$ using the binomial coefficients, we see that Δ-iterativity in this case is equivalent to

$$\partial_\alpha\partial_\beta \quad = \quad \left(\begin{array}{c} \alpha+\beta \\ \beta \end{array}\right)\partial_{\alpha+\beta} \tag{5}$$

for all multi-indices α and β. In particular it implies that each $\mathbf{D}_i$ is an iterative Hasse derivation and that they all commute (indeed all the ∂_α commute). Conversely, suppose $\partial_\alpha = D_{1,\alpha_1}\cdots D_{e,\alpha_e}$ for each α, and $\mathbf{D}_1,\ldots,\mathbf{D}_e$ form a sequence of e iterative commuting Hasse derivations. Then it is not hard to see that (5) holds and so (k,E) is Δ-iterative. □

2.3. **Jets and interpolation for Hasse prolongations.** For a scheme X over a ring k, by the *nth jet space of* X, denoted by $\mathrm{Jet}^n(X) \to X$, we mean the linear space associated to the (coherent) sheaf of $\mathcal{O}_X$-modules $\mathcal{I}/\mathcal{I}^{n+1}$, where $\mathcal{I}$ is the kernel of the map $\mathcal{O}_X \otimes_k \mathcal{O}_X \to \mathcal{O}_X$ given on sections by $f\otimes g \mapsto fg$. Moreover, Jet^n is a covariant functor, its action on morphisms $f : X \to Y$ being the natural one induced by $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. More concretely, if k is a field and $p \in X(k)$ then $\mathrm{Jet}^n(X)_p(k) = \mathrm{Hom}_k\big(\mathfrak{m}_{X,p}/\mathfrak{m}^{n+1}_{X,p}, k\big)$, and $\mathrm{Jet}^n(f)_p : \mathrm{Jet}^n(X)_p \to \mathrm{Jet}^n(Y)_{f(p)}$ is given by precomposing with $f^\sharp_p : \mathfrak{m}_{Y,f(p)}/\mathfrak{m}^{n+1}_{Y,f(p)} \to \mathfrak{m}_{X,p}/\mathfrak{m}^{n+1}_{X,p}$. For details we refer the reader to section 5 of [7], which is dedicated to a review of the relevant properties of this functor.

One of the main purposes of [7] was the introduction of a certain *interpolating map* between the jet space of an abstract prolongation and the prolongation of the jet space. In this section we consider that map, specialised to our setting of Hasse prolongation. Fix an iterative Hasse system $\underline{\mathcal{D}}$, an iterative $\underline{\mathcal{D}}$-ring

(k, E), and a scheme X over k. For each $m, n \in \mathbb{N}$ we have an interpolating map $\phi^X_{m,n} : \operatorname{Jet}^m \tau_n(X) \to \tau_n \operatorname{Jet}^m(X)$, which is a morphism of linear spaces over $\tau_n(X)$, satisfying the following properties (these are parts of Proposition 6.4 of [7], specialised to this setting):

Proposition 2.17. (a) *The interpolating map is compatible with π. That is, for all $n \geq n'$, the following diagram commutes:*

$$\begin{array}{ccc}
\operatorname{Jet}^m \tau_n(X) & \xrightarrow{\operatorname{Jet}^m(\hat{\pi}_{n,n'})} & \operatorname{Jet}^m \tau_{n'}(X) \\
\Big\downarrow \phi_{m,n} & & \Big\downarrow \phi_{m,n'} \\
\tau_n \operatorname{Jet}^m(X) & \xrightarrow[\hat{\pi}^{\operatorname{Jet}^m(X)}_{n,n'}]{} & \tau_{n'} \operatorname{Jet}^m(X)
\end{array}$$

(b) *The interpolating map is compatible with Δ in the sense that for all n, n' the following diagram commutes:*

$$\begin{array}{ccc}
\operatorname{Jet}^m \tau_{n+n'}(X) & \xrightarrow{\operatorname{Jet}^m(\hat{\Delta}_{(n,n')})} & \operatorname{Jet}^m \tau_{n'}\tau_n(X) \\
\Big\downarrow \phi_{m,n+n'} & & \Big\downarrow \phi_{m,\mathcal{D}_{(n,n')},E_{(n,n')}} \\
\tau_{n+n'} \operatorname{Jet}^m(X) & \xrightarrow[\hat{\Delta}^{\operatorname{Jet}^m(X)}_{(n,n')}]{} & \tau_{n'}\tau_n \operatorname{Jet}^m(X)
\end{array}$$

(c) *The following diagram commutes for all n, n'*

$$\begin{array}{ccc}
\operatorname{Jet}^m \tau_{n'}\tau_n(X) & & \\
\Big\downarrow \phi_{m,\mathcal{D}_{(n,n')},E_{(n,n')}} & \searrow^{\phi^{\tau_n(X)}_{m,n'}} & \\
 & & \tau_{n'} \operatorname{Jet}^m \tau_n(X) \\
 & \swarrow_{\tau_{n'}(\phi^X_{m,n})} & \\
\tau_{n'}\tau_n \operatorname{Jet}^m(X) & &
\end{array}$$

Proof. Part (a) is Proposition 6.4(c) of [7], applied to $\alpha = \pi_{n,n'}$. Part (b) is Proposition 6.4(c) of [7], applied to $\alpha = \Delta_{n,n'}$. Part (c) is Proposition 6.4(b) of [7], applied to $(\mathcal{E}, e) = (\mathcal{D}_n, E_n)$ and $(\mathcal{F}, f) = (\mathcal{D}_{n'}, E_{n'})$. □

Besides the above foundational properties of the interpolating map, the main observation from [7] is the following fact:

Fact 2.18 (Corollary 6.8 of [7])**.** *Suppose k is a field and X is of finite-type. If $p \in X(k)$ is smooth, then for all $m, n \in \mathbb{N}$, $\phi^X_{m,n}$ restricts to a surjective linear map between the fibres of $\operatorname{Jet}^m \tau_n(X)$ and $\tau_n \operatorname{Jet}^m(X)$ over $\nabla_n(p) \in \tau_n(X)(k)$.* □

3. Hasse subschemes

Fix an iterative Hasse system $(\underline{\mathcal{D}}, \Delta)$ and an iterative $\underline{\mathcal{D}}$-ring (k, E).

It is possible to develop a theory of $\underline{\mathcal{D}}$-schemes in analogy with algebraic schemes starting with a theory of sheaves of $\underline{\mathcal{D}}$-rings. This would generalise, for example, the approach taken by Kovacic in [4] and Benoist in [2] in differential-algebraic geometry. As their work exhibits, there are a number of subtle and interesting foundational problems that arise in doing so. Moreover, for the $\underline{\mathcal{D}}$-jet space theory we wish to develop here, it seems essential that our $\underline{\mathcal{D}}$-schemes come equipped with a fixed embedding in an algebraic scheme. So we will restrict ourselves to the following approach: we fix an algebraic scheme X over k and introduce only a theory of Hasse subschemes of X.

Definition 3.1. Suppose X is a scheme over k. A *Hasse* (or $\underline{\mathcal{D}}$-) *subscheme of* X *over* k is a collection of closed subschemes over k, $\underline{Z} = \big(Z_n \subseteq \tau_n(X) : n \in \mathbb{N}\big)$, such that:

(1) For all $n \in \mathbb{N}$, the structure morphism $\hat{\pi}_n : \tau_{n+1}(X) \to \tau_n(X)$ restricts to a morphism from Z_{n+1} to Z_n.
(2) For all $m \in \mathbb{N}$, the morphism $\hat{\Delta}_{(m,1)} : \tau_{m+1}(X) \to \tau\big(\tau_m(X)\big)$ induced by iterativity, restricts to a morphism from Z_{m+1} to $\tau(Z_m)$.

By the *k-rational points of $\underline{Z}$* we mean the subset of $X(k)$ given by

$$\underline{Z}(k) := \big\{p \in X(k) : \nabla_n(p) \in Z_n(k), \text{ for all } n \in \mathbb{N}\big\}$$

We will also utilise the following terminology:

- $\underline{Z}$ is *dominant* if each projection $Z_{n+1} \to Z_n$ is dominant.
- $\underline{Z}$ is *separable* if each projection $Z_{n+1} \to Z_n$ is separable.
- $\underline{Z}$ is (absolutely) *irreducible* if each Z_n is (absolutely) irreducible.
- If X is in fact a variety (so reduced and of finite-type over the *field* k), then we say that $\underline{Z}$ is a Hasse *subvariety* if each Z_n is a subvariety of $(\tau_n X)_{\mathrm{red}}$.

Note that if X is a variety over a field k, then every closed subvariety, Y, can be viewed as a dominant Hasse subscheme by considering $\underline{Y} := \big(\tau_n(Y) : n \in \mathbb{N}\big)$; so that $\underline{Y}(k) = Y(k)$. (Domination is by Proposition 2.7.) Moreover, we can consider the dominant Hasse subvariety $\underline{Y}_{\mathrm{red}} := \big(\tau_n(Y)_{\mathrm{red}} : n \in \mathbb{N}\big)$, which also has the property that $\underline{Y}_{\mathrm{red}}(k) = Y(k)$.

3.1. **Some preliminary observations.**

We establish a few lemmas which clarify the definitions a little.

Lemma 3.2. *Suppose $\underline{Z}$ is a $\underline{\mathcal{D}}$-subscheme of a scheme X over k. For all $m, n \in \mathbb{N}$, the morphism $\hat{\Delta}_{(m,n)} : \tau_{m+n}(X) \to \tau_n\big(\tau_m(X)\big)$ induced by iterativity, restricts to a morphism from Z_{m+n} to $\tau_n(Z_m)$. In particular, $Z_n \subseteq \tau_n(Z_0)$ for each $n \in \mathbb{N}$.*

Proof. Note that part (2) of Definition 3.1 is just the $n = 1$ case of this lemma. The 'in particular' clause follows from the $m = 0$ case of the main clause using the fact that $\Delta_{(0,n)} = \mathrm{id}$.

We prove the lemma by induction on n, the case of $n = 0$ being trivially true as $\hat{\Delta}_{(m,0)} = \mathrm{id}$. Now, for any n, consider the following diagram which commutes by the associativity of Δ (part (b) of Definition 2.13):

$$\begin{array}{ccc} \tau_{m+n+1}X & \xrightarrow{\hat{\Delta}^X_{m,n+1}} & \tau_{n+1}\tau_m X \\ \Big\downarrow \hat{\Delta}^X_{m+n,1} & & \Big\downarrow \hat{\Delta}^{\tau_m X}_{n,1} \\ \tau\tau_{m+n}X & \xrightarrow{\tau(\hat{\Delta}^X_{m,n})} & \tau\tau_n\tau_m X \end{array}$$

Let us chase Z_{m+n+1} from the top left to the bottom right, counter-clockwise. By part (2) of Definition 3.1, $\hat{\Delta}_{m+n,1}$ takes Z_{m+n+1} to $\tau(Z_{m+n})$. By the induction hypothesis, $\hat{\Delta}_{(m,n)}$ takes Z_{m+n} to $\tau_n(Z_m)$. Applying the functor τ we get that $\tau(\hat{\Delta}_{m,n})$ takes $\tau(Z_{m+n})$ to $\tau\tau_n Z_m$. So, counter-clockwise, Z_{m+n+1} gets sent to $\tau\tau_n Z_m$. So, from the above diagram, we get that $\hat{\Delta}_{m,n+1}$ restricts to a morphism from Z_{m+n+1} to $\big(\hat{\Delta}_{n,1}^{\tau_m X}\big)^{-1}(\tau\tau_n Z_m)$. Now, as $\hat{\Delta}_{n,1}^{\tau_m X}$ is a closed embedding (Remark 2.14), and $\hat{\Delta}_{n,1}^{\tau_m X} \upharpoonright \tau_{n+1} Z_m = \hat{\Delta}_{n,1}^{Z_m} : \tau_{n+1} Z_m \to \tau\tau_n Z_m$ (this is the functoriality of $\hat{\Delta}$, cf. Proposition 4.8(b) of [7]), we get that $\big(\hat{\Delta}_{n,1}^{\tau_m X}\big)^{-1}(\tau\tau_n Z_m) = \tau_{n+1} Z_m$. So $\hat{\Delta}_{m,n+1}$ restricts to a morphism from Z_{m+n+1} to $\tau_{n+1} Z_m$, as desired. □

Dominant Hasse subschemes, as we have defined them, are given by a *compatible* sequence of algebraic conditions on the prolongation spaces. It might be more natural to consider arbitrary algebraic conditions. The following lemma holds the germ of an abstract version of the Frobenius integrability condition. That is, we present an effective procedure by which one may check whether or not a system of $\underline{\mathcal{D}}$-equations is consistent, here, in the sense of generating a nontrivial iterative $\underline{\mathcal{D}}$-variety.

Lemma 3.3. *Suppose X is a scheme over k and $Y_n \subseteq \tau_n(X)$ is a sequence of closed subschemes. Then there exists a dominant Hasse subscheme $\underline{Z} = (Z_n)$ such that for any $\underline{\mathcal{D}}$-ring k' extending k, $\underline{Z}(k') = \{p \in X(k') : \nabla_n(p) \in Y_n(k'), n < \omega\}$.*

Proof. Let C be the set of all sequences of closed subschemes $W_n \subseteq \tau_n(X)$ such that for all $\underline{\mathcal{D}}$-ring extensions k' of k,

$$\{p \in X(k') : \nabla_n(p) \in W_n(k'), n < \omega\} = \{p \in X(k') : \nabla_n(p) \in Y_n(k'), n < \omega\}.$$

Order C by $(W_n) \subseteq (W'_n)$ if $W_n \subseteq W'_n$ for each $n < \omega$. Note that the intersection of any decreasing chain in C is again in C. So by Zorn's Lemma we have a minimal element (Z_n) of C. We claim that $\underline{Z} := (Z_n)$ is a dominant Hasse subscheme of X.

Fixing m we show that $\hat{\pi}_{m+1,m}$ restricts to a map from Z_{m+1} to Z_m. Indeed let (W_n) be defined by $W_n := Z_n$ for $n \neq m+1$ and $W_{m+1} := \hat{\pi}_{m+1,m}^{-1}(Z_m) \cap Z_{m+1}$. Then, since by Lemma 2.10 $\hat{\pi}\big(\nabla_{m+1}(p)\big) = \nabla_m(p)$, (W_n) is again in C and $(W_n) \subseteq (Z_n)$. By minimality we have $W_m = Z_m$. So $\hat{\pi}_{m+1,m}$ restricts to a map from Z_{m+1} to Z_m, as desired.

Next, we show that $\hat{\pi}_{m+1,m}$ restricts to a dominant map from Z_{m+1} to Z_m. Let $W_m := \overline{\hat{\pi}_{m+1,m}(Z_{m+1})}$ and $W_n := Z_n$ for all $n \neq m$. Again Lemma 2.10 implies that (W_n) is in C and so by minimality $W_m = Z_m$, as desired.

Finally, fixing m we need to show that $\hat{\Delta}_{(m,1)}$ restricts to a morphism from Z_{m+1} to $\tau(Z_m)$. Define (W_n) so that $W_n = Z_n$ for each $n \neq m+1$ and $W_{m+1} := Z_{m+1} \cap \hat{\Delta}_{(m,1)}^{-1}\big(\tau(Z_m)\big)$. Since $\hat{\Delta}_{(m,1)}\big(\nabla_{m+1}(p)\big) = \nabla\big(\nabla_m(p)\big)$ by Remark 2.14, (W_n) is in C. By minimality we get $W_{m+1} = Z_{m+1}$, which means that $\hat{\Delta}_{(m,1)}$ restricts to a morphism from Z_{m+1} to $\tau(Z_m)$, as desired. □

Example 3.4. Consider the iterative Hasse system HD_e of Example 2.4 and Proposition 2.16. If $(k, \partial_1, \ldots, \partial_e)$ is a (partial) differential field of characteristic zero (viewed in the natural way as an iterative HD_e-field) then every system of differential-polynomial equations over k, in say ℓ differential variables, gives rise to a dominant Hasse subscheme of $\mathbb{A}^\ell$. Indeed, such differential-polynomial equations correspond to algebraic condition on the prolongation spaces – and thus give rise to a system of closed subschemes $Y_n \subseteq \tau_n(\mathbb{A}^\ell)$. Now apply Lemma 3.3.

Before moving on, let us briefly discuss the issue of irreducibility for Hasse subvarieties. The definition we have given, namely that each Z_n is irreducible, is rather strong. For example, one cannot expect that every Hasse subvariety can be written as a finite union of irreducible Hasse subvarieties. However, we do have the following:

Lemma 3.5. *Suppose X is a variety (so reduced and of finite type over the $\underline{\mathcal{D}}$-field k) and $\underline{Z}$ is a dominant Hasse subvariety of X over k. Then for each $N < \omega$ there exists finitely many dominant Hasse subvarieties $\underline{Y}^1, \dots, \underline{Y}^\ell \subseteq \underline{Z}$ such that*

- *for any $\underline{\mathcal{D}}$-ring k' extending k, $\underline{Z}(k') = \underline{Y}^1(k') \cup \cdots \cup \underline{Y}^\ell(k')$, and*
- *for all $m \leq N$, Y^i_m is k-irreducible for $i = 1, \dots, \ell$.*

Proof. The proof is by Noetherian induction on Z_N. If Z_N is k-irreducible then so are all the Z_m for $m \leq N$ by dominance – and hence we are done. Suppose we can decompose Z_N as a union of two proper k-closed sets, say U and V. Replacing Z_N by U and then by V in the sequence (Z_n), and then applying Lemma 3.3 to the these two sequences, we get dominant Hasse subvarieties $\underline{Z}^U$ and $\underline{Z}^V$ over k such that $\underline{Z}(k') = \underline{Z}^U(k') \cup \underline{Z}^V(k')$ for any $\underline{\mathcal{D}}$-ring extension k' of k. Now $Z^U_N \subseteq U \subsetneq Z_N$ and $Z^V_N \subseteq V \subsetneq Z_N$. By induction there exists $\underline{Y}^1, \dots, \underline{Y}^\ell$ satisfying the lemma for $\underline{Z}^U$, and $\underline{W}^1, \dots, \underline{W}^s$ satisfying the lemma for $\underline{Z}^V$. But then $\{\underline{Y}^i, \underline{W}^j : i = 1, \dots, \ell, j = 1, \dots, s\}$ works for $\underline{Z}$. □

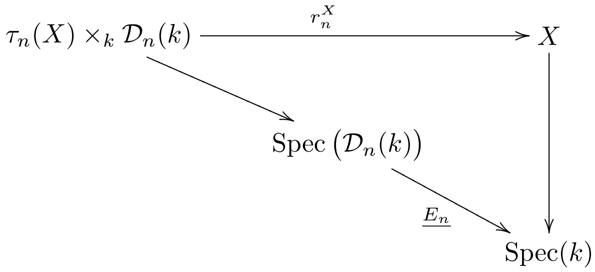
Iterating the above construction we see that every dominant Hasse subvariety can be written as a union of $2^{\aleph_0}$-many k-irreducible Hasse subvarieties.

3.2. **Some $\underline{\mathcal{D}}$-algebra.** While it is our intention to avoid developing the algebraic side of this theory in detail, the definition of a $\underline{\mathcal{D}}$-subscheme given above is best motivated by thinking about $\underline{\mathcal{D}}$-polynomial rings and ideals. For the sake of simplicity (and by working locally) we fix an affine scheme $X = \operatorname{Spec}\big(k[X]\big)$ over an iterative $\underline{\mathcal{D}}$-ring k. Note that for each n, $\tau_n(X)$ will then also be an affine scheme over k.

Remark 3.6. In this section we will repeatedly, and implicitly, use the fact that for any k-algebra R, $\mathcal{D}_n(R)$ is canonically isomorphic to $R \otimes_k \mathcal{D}_n(k)$ both as a k-algebra and as a $\mathcal{D}_n(k)$-algebra (see the discussion at the beginning of section 2). Moreover, under this identification, $\mathcal{D}_n^{E_n}(R)$ is identified with the ring $R \otimes_k \mathcal{D}_n(k)$ together with the k-algebra structure given by $a \mapsto 1 \otimes_k E_n(a)$. The canonicity in the above identifications stem from the fact that we have a fixed isomorphism $\psi_n : \mathcal{D}_n \to \mathbb{S}^{\ell_n}$.

Let $k\langle X\rangle$ denote the direct limit of the co-ordinate rings $k[\tau_m X]$ under the homomorphisms induced by $\hat{\pi}_{m+1,m} : \tau_{m+1}X \to \tau_m X$. There is a natural $\underline{\mathcal{D}}$-ring structure on $k\langle X\rangle$, which we now describe. Recall that coming from the definition of prolongations via Weil restrictions we have an nth *canonical morphism* $r^X_n : \tau_n(X) \times_k \mathcal{D}_n(k) \to X$ for each $n \in \mathbb{N}$. It is this morphism that gives us the identification $\tau_n(X)(R) = X\big(\mathcal{D}_n^{E_n}(R)\big)$. Indeed, that identification is by $p \mapsto$

$r_n^X \circ \big(p \times_k \mathcal{D}_n(k)\big)$. See Definition 4.3 of [7] for details. One thing to remark is that r_n^X is not over k in the usual manner, rather we have the commuting diagram



where $\underline{E_n}$ is the morphism of schemes induced by $E_n : k \to \mathcal{D}_n(k)$. Put another way, working at the level of co-ordinate rings, r_n^X induces a k-algebra morphism from $k[X]$ to $\mathcal{D}_n^{E_n}\big(k[X]\big)$.

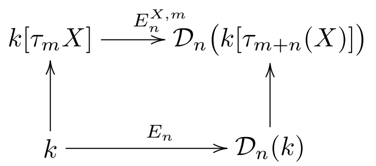
Fixing $n, m \in \mathbb{N}$ we can consider the nth canonical morphism applied to $\tau_m X$, namely $r_n^{\tau_m X} : \tau_n(\tau_m X) \times_k \mathcal{D}_n(k) \to \tau_m X$. Pre-composing with $\hat{\Delta}_{m,n} \times_k \mathcal{D}_n(k)$, we obtain

$$\tau_{m+n}(X) \times_k \mathcal{D}_n(k) \to \tau_m X.$$

On co-ordinate rings this induces

$$E_n^{X,m} : k[\tau_m X] \to \mathcal{D}_n\big(k[\tau_{m+n}X]\big).$$

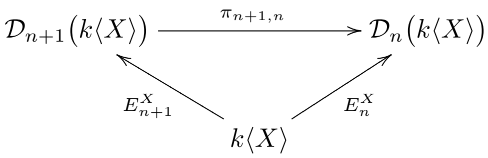
These maps lift E_n. That is,



commutes; or, if you like, $E_n^{X,m} : k[\tau_m X] \to \mathcal{D}_n^{E_n}\big(k[\tau_{m+n}X]\big)$ is a k-homomorphism. We view these maps as approximating a $\underline{\mathcal{D}}$-ring structure on $k[\tau_m X]$, extending the $\underline{\mathcal{D}}$-ring structure on k. Since $\mathcal{D}_n\big(k[\tau_{m+n}X]\big) = k[\tau_{m+n}X] \otimes_k \mathcal{D}_n(k)$, and tensor products commute with direct limits, taking direct limits of $E_n^{X,m}$ give us $E_n^X : k\langle X\rangle \to \mathcal{D}_n\big(k\langle X\rangle\big)$ which lift E_n. We set $E^X := (E_n^X : n \in \mathbb{N})$ and show that it endows $k\langle X\rangle$ with the structure of an iterative $\underline{\mathcal{D}}$-ring.

Proposition 3.7. *$\big(k\langle X\rangle, E^X\big)$ is an iterative $\underline{\mathcal{D}}$-ring.*

Proof. Since $r_0^{\tau_m(X)} = \mathrm{id}$ for all m, $E_0^X = \mathrm{id}$. Next we show that E^X is compatible with π; that



commutes for all $n \in \mathbb{N}$. Passing to the finite stages of the direct limits involved, this reduces to showing that for every $m \in \mathbb{N}$,

$$\begin{array}{ccc}
\mathcal{D}_{n+1}\big(k[\tau_{m+n+1}X]\big) & \xrightarrow{\pi_{n+1,n}} & \mathcal{D}_n\big(k[\tau_{m+n+1}X]\big) \\
{\scriptstyle E^{X,m}_{n+1}}\uparrow & & \uparrow{\scriptstyle E^{X,m+1}_{n}} \\
k[\tau_m X] & \xrightarrow{(\hat{\pi}^X_m)^*} & k[\tau_{m+1}X]
\end{array}$$

commutes. Looking at the definition of E^X, we see that the commuting of the above diagram is equivalent to the commuting of the following diagram:

$$\begin{array}{ccc}
\tau_{m+n+1}X \times_k \mathcal{D}_{n+1}(k) & \xleftarrow{\mathrm{id}\times_k \pi_{n+1,n}} & \tau_{m+n+1}X \times_k \mathcal{D}_n(k) \\
{\scriptstyle \hat{\Delta}_{(m,n+1)}\times_k \mathcal{D}_{n+1}(k)}\downarrow & & \downarrow{\scriptstyle \hat{\Delta}_{(m+1,n)}\times_k \mathcal{D}_{n}(k)} \\
\tau_{n+1}(\tau_m X) \times_k \mathcal{D}_{n+1}(k) & & \tau_n(\tau_{m+1}X) \times_k \mathcal{D}_n(k) \\
{\scriptstyle r^{\tau_m X}_{n+1}}\downarrow & & \downarrow{\scriptstyle r^{\tau_{m+1} X}_{n}} \\
\tau_m X & \xleftarrow{\hat{\pi}^X_m} & \tau_{m+1}X
\end{array} \tag{6}$$

To prove that diagram (6) commutes we will introduce four auxiliary diagrams and prove that they commute. First of all,

$$\begin{array}{ccc}
\tau_{m+n+1}X \times_k \mathcal{D}_{n+1}(k) & \xleftarrow{\mathrm{id}\times_k \pi_{n+1,n}} & \tau_{m+n+1}X \times_k \mathcal{D}_n(k) \\
{\scriptstyle \hat{\Delta}_{(m,n+1)}\times_k \mathcal{D}_{n+1}(k)}\downarrow & & \downarrow{\scriptstyle \hat{\Delta}_{(m,n+1)}\times_k \mathcal{D}_{n}(k)} \\
\tau_{n+1}(\tau_m X) \times_k \mathcal{D}_{n+1}(k) & \xleftarrow{\mathrm{id}\times \pi_{n+1,n}} & \tau_{n+1}(\tau_m X) \times_k \mathcal{D}_n(k)
\end{array} \tag{7}$$

clearly commutes. Next, the commuting of

$$\begin{array}{ccc}
\tau_{n+1}(\tau_m X) \times_k \mathcal{D}_{n+1}(k) & \xleftarrow{\mathrm{id}\times_k \pi_{n+1,n}} & \tau_{n+1}(\tau_m X) \times_k \mathcal{D}_n(k) \\
{\scriptstyle r^{\tau_m X}_{n+1}}\downarrow & & \downarrow{\scriptstyle \hat{\pi}^{\tau_m(X)}_{n+1,n}\times_k \mathcal{D}_{n}(k)} \\
\tau_m X & \xleftarrow{r^{\tau_m X}_{n}} & \tau_n(\tau_m X) \times_k \mathcal{D}_n(k)
\end{array} \tag{8}$$

is the compatibility of the canonical morphism r_n with $\pi_{n+1,n}$, which is Lemma 4.9 of [7]. Now, Lemma 2.15 tells us that

$$\begin{CD}
\tau_{n+1}(\tau_m X) @<{\hat{\Delta}_{(m,n+1)}}<< \tau_{m+n+1}X \\
@V{\hat{\pi}^{\tau_m X}_{n+1,n}}VV @VV{\hat{\Delta}_{(m+1,n)}}V \\
\tau_n(\tau_m X) @<{\tau_n(\hat{\pi}^X_m)}<< \tau_n(\tau_{m+1}X)
\end{CD}$$

commutes. Tensoring this with $\mathcal{D}_n(k)$, gives us that

$$\begin{CD}
\tau_{n+1}(\tau_m X)\times_k \mathcal{D}_n(k) @<{\hat{\Delta}_{(m,n+1)}\times_k\mathcal{D}_n(k)}<< \tau_{m+n+1}X\times_k \mathcal{D}_n(k) \\
@V{\hat{\pi}^{\tau_m X}_{n+1,n}\times_k\mathcal{D}_n(k)}VV @VV{\hat{\Delta}_{(m+1,n)}\times_k\mathcal{D}_n(k)}V \\
\tau_n(\tau_m X)\times_k \mathcal{D}_n(k) @<{\tau_n(\hat{\pi}^X_m)\times_k\mathcal{D}_n(k)}<< \tau_n(\tau_{m+1}X)\times_k \mathcal{D}_n(k)
\end{CD} \tag{9}$$

commutes. Finally, the functoriality of the Weil restriction, and the associated canonical morphism, means that

$$\begin{CD}
\tau_n(\tau_m X)\times_k \mathcal{D}_n(k) @<{\tau_n(\hat{\pi}^X_m)\times_k\mathcal{D}_n(k)}<< \tau_n(\tau_{m+1}X)\times_k \mathcal{D}_n(k) \\
@V{r_n^{\tau_m X}}VV @VV{r_n^{\tau_{m+1}X}}V \\
\tau_m X @<{\hat{\pi}^X_m}<< \tau_{m+1}X
\end{CD} \tag{10}$$

commutes. We leave it to the reader to paste together the commuting diagrams (7), (8), (9), and (10), to obtain the desired diagram (6). We have shown that $\big(k\langle X\rangle, E^X\big)$ is a $\underline{\mathcal{D}}$-ring.

It remains to show that E^X is iterative. We need to show that

$$\begin{array}{ccc}
\mathcal{D}_{r+n}\big(k\langle X\rangle\big) & \xrightarrow{\Delta_{(r,n)}} & \mathcal{D}_r\big(\mathcal{D}_n(k\langle X\rangle)\big) \\
{\scriptstyle E^X_{r+n}}\nwarrow & & \nearrow{\scriptstyle E^X_{(r,n)}} \\
& k\langle X\rangle &
\end{array}$$

commutes for all $r,n\in\mathbb{N}$. Passing to the finite stages of the direct limit this reduces to showing that for all $m\in\mathbb{N}$,

$$\begin{CD}
\mathcal{D}_{r+n}\big(k[\tau_{m+r+n}X]\big) @>{\Delta_{(r,n)}}>> \mathcal{D}_r\big(\mathcal{D}_n(k[\tau_{m+r+n}X])\big) \\
@A{E^{X,m}_{r+n}}AA @AA{\mathcal{D}_r\left(E^{X,m+r}_n\right)}A \\
k[\tau_m X] @>{E^{X,m}_r}>> \mathcal{D}_r\big(k[\tau_{m+r}X]\big)
\end{CD}$$

commutes. Writing out the definitions of the maps, we have to show that

(11)

$$\begin{array}{ccccc}
\tau_{m+r+n}X \times_k \mathcal{D}_{r+n}(k) & & \xleftarrow{\mathrm{id} \times_k \underline{\Delta_{(r,n)}}} & & \tau_{m+r+n}X \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
\Big\downarrow{\scriptstyle \hat{\Delta}_{(m,r+n)} \times_k \mathcal{D}_{r+n}(k)} & & & & \Big\downarrow{\scriptstyle \hat{\Delta}_{(m+r,n)} \times_k \mathcal{D}_r\left(\mathcal{D}_n(k)\right)} \\
\tau_{r+n}(\tau_m X) \times_k \mathcal{D}_{r+n}(k) & & & & \tau_n(\tau_{m+r}X) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
\Big\downarrow{\scriptstyle r_{r+n}^{\tau_m X}} & & & & \Big\downarrow{\scriptstyle r_n^{\tau_{m+r}X} \times_k \mathcal{D}_r(k)} \\
\tau_m X & \xleftarrow[r_r^{\tau_m X}]{} & \tau_r(\tau_m X) \times_k \mathcal{D}_r(k) & \xleftarrow[\hat{\Delta}_{(m,r)} \times_k \mathcal{D}_r(k)]{} & \tau_{m+r}X \times_k \mathcal{D}_r(k)
\end{array}$$

commmutes. Here, $r_n^{\tau_{m+r}X} \times_k \mathcal{D}_r(k)$ requires some explanation as $r_n^{\tau_{m+r}X}$ is not over k in the usual sense: Let $a : k[\tau_{m+r}X] \to \mathcal{D}_n^{E_n}\big(k[\tau_{m+r}X]\big)$ be the k-homomorphism induced by $r_n^{\tau_{m+r}X}$. What we mean by

$$r_n^{\tau_{m+r}X} \times_k \mathcal{D}_r(k) : \tau_n(\tau_{m+r}X) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \to \tau_{m+r}X \times_k \mathcal{D}_r(k)$$

is really the morphism of schemes induced by the k-algebra homomorphism

$$a \otimes_k \mathcal{D}_r(k) : k[\tau_{m+r}X] \otimes_k \mathcal{D}_r(k) \to \mathcal{D}_n^{E_n}\big(k[\tau_{m+r}X]\big) \otimes_k \mathcal{D}_r(k)$$

composed with the (not over k) isomorphism between $\mathcal{D}_n^{E_n}\big(k[\tau_{m+r}X]\big) \otimes_k \mathcal{D}_r(k)$ and $k[\tau_{m+r}X] \otimes_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big)$.

Toward the proof that diagram (11) commutes, suppose Y is a scheme over k. Recall that $\tau_n(\tau_r Y)$ is the prolonagtion of Y with respect to $\mathcal{D}_{(r,n)}$ and $E_{(r,n)}$ (cf. Proposition 4.12 of [7]). A such, we have the canonical morphism

$$r_{(r,n)}^Y := r_{\mathcal{D}_{(r,n)}, E_{(r,n)}}^Y : \tau_n(\tau_r Y) \times_l \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \to Y.$$

Now, by Lemma 4.9 of [7], the following diagram commutes:

$$\begin{array}{ccc}
\tau_{r+n}(Y) \times_k \mathcal{D}_{r+n}(k) & \xleftarrow{\mathrm{id} \times_k \underline{\Delta_{(r,n)}}} & \tau_{r+n}(Y) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
\Big\downarrow{\scriptstyle r_{r+n}^Y} & & \Big\downarrow{\scriptstyle \hat{\Delta}_{(r,n)}^Y \times_k \mathcal{D}_r\left(\mathcal{D}_n(k)\right)} \\
Y & \xleftarrow{r_{(r,n)}^Y} & \tau_n(\tau_r Y) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big)
\end{array}$$

But we also have, be Lemma 4.4 of [7], that

$$\begin{array}{ccc}
Y & \xleftarrow{r_{(r,n)}^Y} & \tau_n(\tau_r Y) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
{\scriptstyle r_r^Y} \nwarrow & & \swarrow {\scriptstyle r_n^{\tau_r Y} \times_k \mathcal{D}_r(k)} \\
& \tau_r(Y) \times_k \mathcal{D}_r(k) &
\end{array}$$

commutes. Putting these two diagrams together we get

$$\begin{array}{ccccc}
\tau_{r+n}(Y)\times_k \mathcal{D}_{r+n}(k) & & \xleftarrow{\ \mathrm{id}\times_k \underline{\Delta_{(r,n)}}\ } & & \tau_{r+n}(Y)\times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
\Big\downarrow {\scriptstyle r^Y_{r+n}} & & & & \Big\downarrow {\scriptstyle \hat{\Delta}^Y_{(r,n)}\times_k \mathcal{D}_r(\mathcal{D}_n(k))} \\
Y & \xleftarrow{\ r^Y_r\ } & \tau_r(Y)\times_k \mathcal{D}_r(k) & \xleftarrow{\ r_n^{\tau_r Y}\times_k \mathcal{D}_r(k)\ } & \tau_n(\tau_r Y)\times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big)
\end{array}$$

commutes. Now apply the above commuting diagram to $Y=\tau_m X$. This yields the commuting diagram

$$\begin{array}{ccccc}
\tau_{r+n}(\tau_m X)\times_k \mathcal{D}_{r+n}(k) & & \xleftarrow{\ \mathrm{id}\times_k \underline{\Delta_{(r,n)}}\ } & & \tau_{r+n}(\tau_m X)\times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
\Big\downarrow {\scriptstyle r^{\tau_m X}_{r+n}} & & & & \Big\downarrow {\scriptstyle \hat{\Delta}^{\tau_m X}_{(r,n)}\times_k \mathcal{D}_r(\mathcal{D}_n(k))} \\
\tau_m X & \xleftarrow{\ r^{\tau_m X}_r\ } & \tau_r(\tau_m X)\times_k \mathcal{D}_r(k) & \xleftarrow{\ r_n^{\tau_r\tau_m X}\times_k \mathcal{D}_r(k)\ } & \tau_n(\tau_r\tau_m X)\times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big)
\end{array}$$

We can lift the top line of this diagram to get the commuting diagram

(12)

$$\begin{array}{ccccc}
\tau_{m+r+n}(X)\times_k \mathcal{D}_{r+n}(k) & & \xleftarrow{\ \mathrm{id}\times_k \underline{\Delta_{(r,n)}}\ } & & \tau_{m+r+n}(X)\times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
\Big\downarrow {\scriptstyle \hat{\Delta}^X_{(m,r+n)}\times_k \mathcal{D}_{r+n}(k)} & & & & \Big\downarrow {\scriptstyle \hat{\Delta}^X_{(m,r+n)}\times_k \mathcal{D}_r(\mathcal{D}_n(k))} \\
\tau_{r+n}(\tau_m X)\times_k \mathcal{D}_{r+n}(k) & & & & \tau_{r+n}(\tau_m X)\times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
\Big\downarrow {\scriptstyle r^{\tau_m X}_{r+n}} & & & & \Big\downarrow {\scriptstyle \hat{\Delta}^{\tau_m X}_{(r,n)}\times_k \mathcal{D}_r(\mathcal{D}_n(k))} \\
\tau_m X & \xleftarrow{\ r^{\tau_m X}_r\ } & \tau_r(\tau_m X)\times_k \mathcal{D}_r(k) & \xleftarrow{\ r_n^{\tau_r\tau_m X}\times_k \mathcal{D}_r(k)\ } & \tau_n(\tau_r\tau_m X)\times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big)
\end{array}$$

The desired diagram (11) is obtained from diagram (12) above by pasting onto its right side the following two commuting diagrams:

$$\begin{array}{ccc}
\tau_{r+n}(\tau_m X) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) & \xleftarrow{\hat{\Delta}^X_{(m,r+n)} \times_k \mathcal{D}_r(\mathcal{D}_n(k))} & \tau_{m+r+n}(X) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
\Big\downarrow {\scriptstyle \hat{\Delta}^{\tau_m X}_{(r,n)} \times_k \mathcal{D}_r(\mathcal{D}_n(k))} & & \Big\downarrow {\scriptstyle \hat{\Delta}_{(m+r,n)} \times_k \mathcal{D}_r(\mathcal{D}_n(k))} \\
\tau_n(\tau_r \tau_m X) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) & \xleftarrow[\tau_n(\hat{\Delta}_{(m,r)}) \times_k \mathcal{D}_r(\mathcal{D}_n(k))]{} & \tau_n(\tau_{m+r} X) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big)
\end{array}$$

which is by the associativity of Δ, and

$$\begin{array}{ccc}
\tau_n(\tau_r \tau_m X) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) & \xleftarrow[\tau_n(\hat{\Delta}_{(m,r)}) \times_k \mathcal{D}_r(\mathcal{D}_n(k))]{} & \tau_n(\tau_{m+r} X) \times_k \mathcal{D}_r\big(\mathcal{D}_n(k)\big) \\
\Big\downarrow {\scriptstyle r_n^{\tau_r \tau_m X} \times_k \mathcal{D}_r(\mathcal{D}_n(k))} & & \Big\downarrow {\scriptstyle r_n^{\tau_{m+r} X} \times_k \mathcal{D}_r(\mathcal{D}_n(k))} \\
\tau_r(\tau_m X) \times_k \mathcal{D}_r(k) & \xleftarrow{\hat{\Delta}_{(m,r)} \times_k \mathcal{D}_r(k)} & \tau_{m+r} X \times_k \mathcal{D}_r(k)
\end{array}$$

which is by the functoriality of the prolongation and the associated canonical morphism. This proves that diagram (11) commutes, and completes the proof of Proposition 3.7 □

Now let $\underline{Z}$ be a $\underline{\mathcal{D}}$-subscheme of X. Since the maps $\tau_{m+1} X \to \tau_m X$ restrict to $Z_{m+1} \to Z_m$, on co-ordinate rings we have that $k[\tau_m X] \to k[\tau_{m+1} X]$ induces maps on the quotients $k[Z_m] \to k[Z_{m+1}]$. Letting $k\langle \underline{Z} \rangle$ be the direct limit of the $k[Z_m]$, we have that $k\langle \underline{Z} \rangle$ is a quotient of $k\langle X \rangle$ (by the direct limit of the ideals that define the Z_m). We call $k\langle \underline{Z} \rangle$ the *$\underline{\mathcal{D}}$-co-ordinate ring* of $\underline{Z}$. The $\underline{\mathcal{D}}$-ring structure on $k\langle X \rangle$ induces one on $k\langle \underline{Z} \rangle$. Indeed, for each m and n, the map $\tau_{m+n}(X) \times_k \mathcal{D}_n(k) \to \tau_m X$ which is obtained by pre-composing the canonical map $r_n^{\tau_m X} : \tau_n(\tau_m X) \times_k \mathcal{D}_n(k) \to \tau_m X$ with $\hat{\Delta}_{m,n} \times_k \mathcal{D}_n(k)$, restricts to a map $Z_{m+n} \times_k \mathcal{D}_n(k) \to Z_m$. Hence $E_n^{X,m}$ induces

$$E_n^{\underline{Z},m} : k[Z_m] \to \mathcal{D}_n\big(k[Z_{m+n}]\big).$$

Taking direct limits we get $E_n^{\underline{Z}} : k\langle \underline{Z} \rangle \to \mathcal{D}_n\big(k\langle \underline{Z} \rangle\big)$, which is induced by E_n^X. Setting $E^{\underline{Z}} := (E_n^{\underline{Z}} : n \in \mathbb{N})$ we have shown:

Corollary 3.8. *If $\underline{Z}$ is a $\underline{\mathcal{D}}$-subscheme of X then $\big(k\langle \underline{Z} \rangle, E^{\underline{Z}}\big)$ is an iterative $\underline{\mathcal{D}}$-ring and the quotient map $\rho : k\langle X \rangle \to k\langle \underline{Z} \rangle$ is a $\underline{\mathcal{D}}$-*homomorphism*; that is,*

$$\begin{array}{ccc}
k\langle X \rangle & \xrightarrow{E_n^X} & \mathcal{D}_n\big(k\langle X \rangle\big) \\
\Big\downarrow {\scriptstyle \rho} & & \Big\downarrow {\scriptstyle \mathcal{D}_n(\rho)} \\
k\langle \underline{Z} \rangle & \xrightarrow{E_n^{\underline{Z}}} & \mathcal{D}_n\big(k\langle \underline{Z} \rangle\big)
\end{array}$$

commutes for each n. □

3.3. **Generic points in fields.** The geometric theory of $\underline{\mathcal{D}}$-subvarieties over $\underline{\mathcal{D}}$-fields goes much more smoothly if we can be guaranteed that in some $\underline{\mathcal{D}}$-field extension our $\underline{\mathcal{D}}$-subvariety has a generic point. This will not always be the case. In this section we introduce a condition on the Hasse system which will guarantee us the existence of generic points in $\underline{\mathcal{D}}$-field extensions.

Definition 3.9. Suppose (k, E) is an iterative $\underline{\mathcal{D}}$-ring and $S \subseteq k \setminus \{0\}$ is a multiplicatively closed set. We say that E *localises to* $S^{-1}k$ if for every $n \in \mathbb{N}$, $E_n : k \to \mathcal{D}_n(k)$ extends to a ring homomorphism $\tilde{E}_n$ such that

$$\begin{array}{ccc} k & \xrightarrow{E_n} & \mathcal{D}_n(k) \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_n(\iota)} \\ S^{-1}k & \xrightarrow{\tilde{E}_n} & \mathcal{D}_n(S^{-1}k) \end{array}$$

commutes. We say that the Hasse system $\underline{\mathcal{D}}$ *extends to fields* if whenever (R, E) is an iterative $\underline{\mathcal{D}}$-integral domain, and K is the fraction field, then E localises to K.

Remark 3.10. *Suppose (k, E) is an iterative $\underline{\mathcal{D}}$-ring and $S \subseteq k \setminus \{0\}$ is a multiplicatively closed set such that E localises to $S^{-1}k$. Then each $\tilde{E}_n$ is unique and $\big(S^{-1}k, \tilde{E} := (\tilde{E}_n : n \in \mathbb{N})\big)$ is an iterative $\mathcal{D}$-ring.* Indeed, if $\frac{\iota a}{\iota b} \in S^{-1}k$ then as $\tilde{E}_n$ is a ring homomorphism, $\tilde{E}_n(\iota b)$ is a unit in $\mathcal{D}_n(S^{-1}k)$ and $\tilde{E}_n(\frac{\iota a}{\iota b}) = \frac{E_n(\iota a)}{E_n(\iota b)} = \frac{\mathcal{D}_n(\iota)\big(E_n(a)\big)}{\mathcal{D}_n(\iota)\big(E_n(b)\big)}$. This gives uniqueness. Two straighforward diagram chases now show that $\tilde{E} := (\tilde{E}_n : n \in \mathbb{N})$ is compatible with π and Δ (since E is), hence making $S^{-1}k$ into an iterative $\underline{\mathcal{D}}$-ring.

Proposition 3.11. *Suppose $\underline{\mathcal{D}}$ is an iterative Hasse system over A such that for all n, the kernel of $\pi_{n,0} : \mathcal{D}_n(A) \to A$ is a nilpotenty ideal, I_n. Then for any iterative $\underline{\mathcal{D}}$-ring (k, E) over A, and any multiplicatively closed set $S \subseteq k \setminus \{0\}$, E localises to $S^{-1}k$. In particular, $\underline{\mathcal{D}}$ extends to fields.*

Proof. Fix $n \in \mathbb{N}$. By the universal property of localisations, the existence of such $\tilde{E}_n$ will follow once we show that $\mathcal{D}_n(\iota)\big(E_n(s)\big)$ is a unit in $\mathcal{D}_n(S^{-1}k)$, for each $s \in S$. Consider the commuting square

$$\begin{array}{ccc} k & \xleftarrow{\pi^k_{n,0}} & \mathcal{D}_n(k) \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_n(\iota)} \\ S^{-1}k & \xleftarrow{\pi^{S^{-1}k}_{n,0}} & \mathcal{D}_n(S^{-1}k) \end{array}$$

Now, the kernel of the surjective homomorphism $\pi_{n,0} : \mathcal{D}_n(S^{-1}k) \to S^{-1}k$ is the nilpotent ideal $S^{-1}k \otimes_A I_n$, and hence the units of $\mathcal{D}_n(S^{-1}k)$ are just the pullbacks of the units of $S^{-1}k$. In particular, as $\pi_{n,0}\big(\mathcal{D}_n(\iota)(E_n(s))\big) = \iota(s)$ is a unit in $S^{-1}k$, we get that $\mathcal{D}_n(\iota)\big(E_n(s)\big)$ is a unit in $\mathcal{D}_n(S^{-1}k)$, as desired. So the required extensions $\tilde{E}_n : S^{-1}k \to \mathcal{D}_n(S^{-1}k)$ exist. □

Corollary 3.12. *The iterative Hasse system* HD_e *used to study Hasse-differential rings in Example 2.4 extends to fields.* □

Here is why this property of extending to fields is useful: under this assumption our Hasse varieties will always have many rational points in field extensions. More precisely,

Proposition 3.13. *Suppose $\underline{\mathcal{D}}$ extends to fields, (k, E) is an iterative $\underline{\mathcal{D}}$-field, X is an algebraic variety over k, and $\underline{Z}$ is a dominant irreducible Hasse subvariety of X over k. Then there exists an iterative $\underline{\mathcal{D}}$-field extension (K, E) of (k, E) and a point $b \in \underline{Z}(K)$ such that $\nabla_n(b)$ is k-generic in Z_n for all $n \in \mathbb{N}$. We say that b is* k-generic *in $\underline{Z}$.*

Proof. Suppose $\underline{Z}$ is a dominant irreducible Hasse subvariety of X over k. We construct a $\underline{\mathcal{D}}$-extension K of k such that $\underline{Z}(K)$ contains a "k-generic" point. Let $k\langle\underline{Z}\rangle$, which is the direct limit of the $k[Z_n]$, be the Hasse co-ordinate ring. By irrreducibility each $k[Z_n]$, and hence $k\langle\underline{Z}\rangle$, are integral domains. Let K be the fraction field of $k\langle\underline{Z}\rangle$, the *Hasse rational function field* of $\underline{Z}$. Since $\underline{\mathcal{D}}$ extends to fields, the iterative $\underline{\mathcal{D}}$-ring structure on $k\langle\underline{Z}\rangle$ extends to an iterative $\underline{\mathcal{D}}$-field structure on K.

Let $a : k\langle\underline{Z}\rangle \to K$ be the inclusion of the integral domain in its fraction field. For each $n \in \mathbb{N}$, let $a_n : k[Z_n] \to K$ be the homomorphism obtained from a by precomposing with the direct limit map from $k[Z_n]$ to $k\langle\underline{Z}\rangle$. The dominance of the maps $Z_{m+1} \to Z_m$ imply that a_n factors through $k(Z_n)$, the rational function field of Z_n. That is, $a_n \in Z_n(K)$ is k-generic in Z_n.

Next we prove that for each $n \in \mathbb{N}$, $\nabla_n(a_0) = a_n$. First note that the following diagram commutes since a is the inclusion in the fraction field and the $\underline{\mathcal{D}}$-structure on K extends that on $k\langle\underline{Z}\rangle$:

$$\begin{array}{ccc} K & \xrightarrow{\quad E_n \quad} & \mathcal{D}_n(K) = K \otimes_k \mathcal{D}_n(k) \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle a\otimes_k \mathcal{D}_n(k)} \\ k\langle\underline{Z}\rangle & \xrightarrow{\quad E_n \quad} & k\langle\underline{Z}\rangle \otimes_k \mathcal{D}_n(k) \end{array}$$

On the other hand, since E_n on $k\langle\underline{Z}\rangle$ comes as the direct limit of the maps $E_n^{\underline{Z},m}$ (see Section 3.2), we have

$$\begin{array}{ccc} k\langle\underline{Z}\rangle & \xrightarrow{\quad E_n \quad} & k\langle\underline{Z}\rangle \otimes_k \mathcal{D}_n(k) \\ \uparrow & & \uparrow \\ k[Z_0] & \xrightarrow{\quad E_n^{\underline{Z},0} \quad} & k[Z_n] \otimes_k \mathcal{D}_n(k) \end{array}$$

Putting the two diagrams together, and remembering how $E_n^{\underline{Z},0}$ is defined and that $\nabla_n(a_0)$ is the point in $\tau_n Z_0(K)$ induced by $E_n \circ a_0$, we get

$$\begin{array}{ccccc} K & & \xrightarrow{\qquad\qquad E_n \qquad\qquad} & & \mathcal{D}_n(K) \\ {\scriptstyle a_0}\uparrow & & \nearrow{\scriptstyle \nabla_n(a_0)\otimes_k \mathcal{D}_n(k)} & & \uparrow{\scriptstyle a_n\otimes_k \mathcal{D}_n(k)} \\ k[Z_0] & \xrightarrow{r_n^{Z_0}} & k[\tau_n Z_0] \otimes_k \mathcal{D}_n(k) & \longrightarrow & k[Z_n] \otimes_k \mathcal{D}_n(k) \end{array}$$

Hence, as K-points of $\tau_n Z_0$, $\nabla_n(a_0) = a_n$.

We have shown that $b := a_0 \in \underline{Z}(K)$ and that b is k-generic in $\underline{Z}$. $\square$

Definition 3.14. An iterative $\underline{\mathcal{D}}$-field (k, E) will be called *rich* if whenever X is an algebraic variety over k and $\underline{Z}$ is a dominant irreducible Hasse subvariety of X over k, then $\nabla_n\big(\underline{Z}(k)\big)$ is Zariski-dense in Z_n for all $n \in \mathbb{N}$.

Corollary 3.15. *Suppose $\underline{\mathcal{D}}$ extends to fields. Then every iterative $\underline{\mathcal{D}}$-field extends to a rich iterative $\underline{\mathcal{D}}$-field.*

Proof. Suppose (k, E) is an iterative $\underline{\mathcal{D}}$-field. We build a rich $\underline{\mathcal{D}}$-field, L, as a direct limit of an ω_1-chain of $\underline{\mathcal{D}}$-field extensions of k. Start with $L_0 = k$. Given L_m, list all of the dominant irreducible $\underline{\mathcal{D}}$-varieties over L_m, $(\underline{Z}_\alpha : \alpha < \kappa)$. We build $L_{m+1,\beta}$ by transfinite recursion on $\beta < \kappa$. At stage β, if $\beta > 0$ then let $M = M_{m,\beta}$ be the union of $L_{m+1,\gamma}$ for $\gamma < \beta$. If $\beta = 0$ then let $M = L_m$. Let $\underline{Z}$ be an M-irreducible component of Z_β. Let $L_{m+1,\beta} \supseteq M$ and $a = a_{m,\beta} \in \underline{Z}_\beta(L_{m+1,\beta})$ be an M-generic point of $\underline{Z}$, as given by Proposition 3.13. We then let L_{m+1} be the union of the $L_{m+1,\alpha}$, $\alpha < \kappa$. At limit stages we take unions, and we set $L := L_{\omega_1}$.

Suppose now that $\underline{Z} = (Z_n)$ is a dominant irreducible Hasse variety over L and W a proper subvariety of some Z_n. Then Z and W are defined over some countable subfield of L and as such are defined over (and irreducible over) some L_m. So, for some β, $\underline{Z} = \underline{Z}_\beta$ for the listing of the dominant irreducible Hasse varieties over L_m. We have $L_m \subseteq M_{m,\beta} \subseteq L_{m+1,\beta} \subseteq L_{m+1} \subseteq L$. As $\underline{Z}$ is irreducible over L, it was already irreducible over $M_{m,\beta}$. Thus, $a_{m,\beta} \in \underline{Z}(L_{m+1,\beta})$ is $M_{m,\beta}$-generic, and hence L_m-generic. In particular, $\nabla_n(a_{m,\beta})$ is not an element of W. Thus, $\nabla_n(\underline{Z}(L))$ is not contained in $W(L)$. We have shown that $\nabla_n\big(\underline{Z}(L)\big)$ is Zariski-dense in Z_n, for all $n \in \mathbb{N}$. $\square$

We make immediate use of the existence of sufficiently many rational points in the following proposition, which we will need later, and which says that applying ∇ to the rational points of a Hasse subvariety produces the rational points of another Hasse subvariety.

Proposition 3.16. *Suppose (k, E) is a rich iterative $\underline{\mathcal{D}}$-field, X is a variety over k, and $\underline{Z}$ is a dominant irreducible Hasse subvariety of X over k. Then for each $m \in \mathbb{N}$ there exists a dominant Hasse subvariety $\underline{Y}$ of Z_m with $\nabla_m\big(\underline{Z}(k)\big) = \underline{Y}(k)$. We denote this Hasse subvariety by $\nabla_m\underline{Z}$.*

Proof. There is an obvious candidate for $\underline{Y}$: set $\underline{Y} = (Y_n)$ where Y_n is the image of Z_{m+n} in $\tau_n(Z_m)$ under $\hat{\Delta}_{(m,n)}$. Since $\hat{\Delta}_{(m,n)}$ is a closed embedding Y_n is a closed subvariety of $\tau_n(Z_m)$.

We first show that $\underline{Y}$ is a dominant Hasse subvariety. For the first condition we need to check that $\tau_{n+1}(Z_m) \to \tau_n(Z_m)$ induces a dominant map from Y_{n+1} to Y_n. But this follows from the fact that $\tau_{m+n+1}(X) \to \tau_{m+n}(X)$ induces a dominant map from Z_{m+n+1} to Z_{m+n}, and from the compatibility of Δ with π (cf. the commuting diagram in Definition 2.13(a)). The second condition requires us to confirm that $\hat{\Delta}^{Z_m}_{(n,1)} : \tau_{n+1}(Z_m) \to \tau\big(\tau_n(Z_m)\big)$ induces a map from Y_{n+1} to $\tau(Y_n)$. Unravelling definitions we see that it suffices to show that the following diagram commutes:

$$\begin{array}{ccc}
\tau_{m+n+1}(X) & \xrightarrow{\hat{\Delta}_{(m+n,1)}} & \tau\big(\tau_{m+n}(X)\big) \\
\Big\downarrow{\scriptstyle \hat{\Delta}^X_{(m,n+1)}} & & \Big\downarrow{\scriptstyle \tau\left(\hat{\Delta}^X_{(m,n)}\right)} \\
\tau_{n+1}\big(\tau_m(X)\big) & \xrightarrow{\hat{\Delta}^{\tau_m(X)}_{(n,1)}} & \tau\big(\tau_n(\tau_m(X))\big)
\end{array}$$

Reading the above diagram at the level of rings we see that it is a case of the associativity of Δ (cf. the commuting diagram in part (b) of Definition 2.13). Therefore, $\underline{Y}$ so defined is a dominant Hasse subvariety of Z_m.

Next we need to show that $\nabla_m\big(\underline{Z}(k)\big) = \underline{Y}(k)$. First fix $p \in \underline{Z}(k)$. Then $\nabla_m(p) \in Z_m \subseteq \tau_m(X)$ and so $\nabla_n\big(\nabla_m(p)\big) \in \tau_n\big(\tau_m(X)\big)$ for all n. But $\nabla_n\big(\nabla_m(p)\big) = \hat{\Delta}_{(m,n)}\big(\nabla_{m+n}(p)\big)$. Since $\nabla_{m+n}(p) \in Z_{m+n}$, $\nabla_n\big(\nabla_m(p)\big) \in Y_n$ for all n. Hence $\nabla_m(p) \in \underline{Y}(k)$. We have shown that $\nabla_m\big(\underline{Z}(k)\big) \subseteq \underline{Y}(k)$.

So far we have not used the assumption that $\underline{Z}$ has many k-rational points. One consequence of this assumption is that Y_n is the Zariski closure of $\nabla_n\big(\nabla_m \underline{Z}(k)\big)$, for all n. Indeed, Y_n is the image of Z_{m+n} under the closed embedding $\hat{\Delta}_{(m,n)}$, $\nabla_n\big(\nabla_m(\underline{Z}(k))\big)$ is the image of $\nabla_{m+n}\big(\underline{Z}(k)\big)$ under the same map, and $\nabla_{m+n}\big(\underline{Z}(k)\big)$ is Zariski dense in Z_{m+n} by assumption.

It remains to show that if $q \in \underline{Y}(k)$ then $q \in \nabla_m\big(\underline{Z}(k)\big)$. First note that it suffices to show that $q \in \nabla_m\big(X(k)\big)$. Indeed, if $q = \nabla_m(p)$ then $\hat{\Delta}_{(m,n)}\big(\nabla_{m+n}(p)\big) = \nabla_n(q) \in Y_n$ for all n, so that $\nabla_{m+n}(p) \in Z_{m+n}$ for all n, which implies that $p \in \underline{Z}(k)$. So we need to find $p \in X(k)$ such that $\nabla_m(p) = q$. This will follow from the following claim

Claim 3.17. *If $q \in \tau_m(X)(k)$ has the property that $\nabla_m(q)$ is contained in the Zariski closure of $\nabla_m\big(\nabla_m(X(k))\big)$, then $q = \nabla_m(p)$ for some $p \in X(k)$.*

Proof. Consider the commuting diagram

$$\begin{array}{ccc}
\tau_m\big(\tau_m(X)\big) & \xrightarrow{\tau_m(\hat{\pi}^X_{m,0})} & \tau_m(X) \\
\Big\downarrow{\scriptstyle \hat{\pi}^{\tau_m(X)}_{m,0}} & & \Big\downarrow{\scriptstyle \hat{\pi}^X_{m,0}} \\
\tau_m(X) & \xrightarrow{\hat{\pi}^X_{m,0}} & X
\end{array}$$

By the functoriality of ∇ (cf. Proposition 4.7(a) of [7]) we have that $\tau_m(\hat{\pi}^X_{m,0})\big(\nabla_m(q)\big) = \nabla_m\big(\hat{\pi}^X_{m,0}(q)\big)$. So $q \in \nabla_m\big(X(k)\big)$ if and only if $\tau_m(\hat{\pi}^X_{m,0})\big(\nabla_m(q)\big) = q$. But the latter identity says that $\nabla_m(q)$ satisfies a certain Zariski closed condition on $\tau_m\big(\tau_m(X)\big)$, namely the condition

$$\tau_m(\hat{\pi}^X_{m,0})(u) = \hat{\pi}^{\tau_m(X)}_{m,0}(u).$$

Since this condition is satisfied by all $u \in \nabla_m\big(\nabla_m(X(k))\big)$, and since $\nabla_m(q)$ is in the Zariski closure of $\nabla_m\big(\nabla_m(X(k))\big)$, we get that $q \in \nabla_m\big(X(k)\big)$, as desired. $\square$

Now fix $q \in \underline{Y}(k)$. So $\nabla_m(q) \in Y_m$, and the latter is in the Zariski closure of $\nabla_m\big(\nabla_m(X(k))\big)$ – as it is the Zariski closure of $\nabla_m\big(\nabla_m(\underline{Z}(k))\big)$. So by the claim, $\nabla_m(p) = q$ for some $p \in X(k)$, as desired. $\square$

4. Hasse jet spaces

We intend to define a Hasse jet space associated to a point in a Hasse subvariety; it will be a linear Hasse subvariety of the jet space of the ambient algebraic variety at that point. In the differential case, for finite dimensional subvarieties, this was done by Pillay and Ziegler [8], but their construction does not extend to infinite dimensional differential varieties. Staying with the differential setting for the moment, one might consider imitating the algebraic construction by defining the nth differential jet space at p as the "differential dual" to the maximal differential ideal at p modulo the $(n+1)$st power of that ideal. This approach fails however, first because such a space is too large to fit naturally into a definable context, but also because such spaces are in another sense too small: they will not determine the differential variety. This latter difficulty stems from non-noetherianity, or more specifically from the fact that there exist differential varieties with points that have the the property that the intersection of all the powers of the maximal ideal at the point is not trivial; an example[1] in one derivation is given by the equation $x\delta^2 x - \delta x = 0$. So neither the algebraic construction, nor the finite dimensional differential construction of Pillay-Ziegler suggest extensions. Our approach is to take the algebraic jet spaces of the sequence of algebraic varieties that define the Hasse subvariety, and then use this sequence to define a Hasse jet space. In order to do so we make essential use of the interpolating map, which we discussed in Section 2.3, and which was introduced and developed in [7].

Fix an iterative Hasse system $\underline{\mathcal{D}}$, an iterative $\underline{\mathcal{D}}$-field (k, E), a variety X over k, a Hasse subvariety $\underline{Z} = (Z_n)$ of X over k, and a natural number m. For each $n \in \mathbb{N}$, note that $(\operatorname{Jet}^m Z_n)_{\mathrm{red}}$ is a subvariety of $\operatorname{Jet}^m \tau_n(X)$, and hence we can consider its image in $\tau_n\big(\operatorname{Jet}^m(X)\big)$ under the interpolating map. Setting T_n to be the Zariski closure of this image we obtain:

Lemma 4.1. $\underline{T} := \Big(T_n := \overline{\phi^X_{m,n}\big((\operatorname{Jet}^m Z_n)_{\mathrm{red}}\big)} : n \in \mathbb{N}\Big)$ *is a Hasse subvariety of* $\operatorname{Jet}^m(X)$.

Proof. By functoriality, $\operatorname{Jet}^m(\hat{\pi}_{n+1,n}) : \operatorname{Jet}^m\big(\tau_{n+1}(X)\big) \to \operatorname{Jet}^m\big(\tau_n(X)\big)$ restricts to a map $\operatorname{Jet}^m(Z_{n+1})_{\mathrm{red}} \to \operatorname{Jet}^m(Z_n)_{\mathrm{red}}$. Transforming this by the interpolating map (cf. part (a) of Proposition 2.17) yeilds that $\hat{\pi}^{\operatorname{Jet}^m(X)}_{n+1,n} : \tau_{n+1}\operatorname{Jet}^m(X) \to \tau_n \operatorname{Jet}^m(X)$ restricts to a map from $\phi^X_{m,n+1}\big((\operatorname{Jet}^m Z_{n+1})_{\mathrm{red}}\big)$ to $\phi^X_{m,n}\big((\operatorname{Jet}^m Z_n)_{\mathrm{red}}\big)$, and hence from T_{n+1} to T_n. This proves the first condition of being a Hasse subvariety.

It remains to prove that for all $n \in \mathbb{N}$, $\hat{\Delta}^{\operatorname{Jet}^m(X)}_{n,1} : \tau_{n+1}\big(\operatorname{Jet}^m(X)\big) \to \tau\big(\tau_n(\operatorname{Jet}^m(X))\big)$ restricts to a map from T_{n+1} to $\tau(T_n)$. Parts (b) and (c) of Corollary 2.17 together give us the following compatibility of the interpolating map with Δ:

$$\hat{\Delta}^{\operatorname{Jet}^m(X)}_{n,1} \circ \phi^X_{m,n+1} = \tau(\phi^X_{m,n}) \circ \phi^{\tau_n(X)}_{m,1} \circ \operatorname{Jet}^m(\hat{\Delta}^X_{n,1}).$$

Hence, to see where $\hat{\Delta}^{\operatorname{Jet}^m(X)}_{n,1}$ takes $T_{n+1} = \overline{\phi^X_{m,n+1}\big((\operatorname{Jet}^m Z_{n+1})_{\mathrm{red}}\big)}$, we can apply the right-hand-side of the above equality to $(\operatorname{Jet}^m Z_{n+1})_{\mathrm{red}}$. We have

$$\operatorname{Jet}^m(\hat{\Delta}^X_{n,1}) : (\operatorname{Jet}^m Z_{n+1})_{\mathrm{red}} \to (\operatorname{Jet}^m \tau Z_n)_{\mathrm{red}}.$$

By functoriality of the interpolating map (Proposition 6.4(a) of [7]),

$$\phi^{\tau_n(X)}_{m,1} : (\operatorname{Jet}^m \tau Z_n)_{\mathrm{red}} \to (\tau \operatorname{Jet}^m Z_n)_{\mathrm{red}}.$$

[1]This example was communicated to us by Phyllis Cassidy.

Finally, since $\phi^X_{m,n} : (\operatorname{Jet}^m Z_n)_{\mathrm{red}} \to T_n$,

$$\tau(\phi^X_{m,n}) : (\tau \operatorname{Jet}^m Z_n)_{\mathrm{red}} \to \tau(T_n).$$

Hence, $\hat{\Delta}^{\operatorname{Jet}^m(X)}_{n,1} : T_{n+1} \to \tau(T_n)$, as desired. □

This allows us to define the Hasse jet spaces:

Definition 4.2 (Hasse jet space)**.** Suppose $\underline{Z}$ is a Hasse subvariety of X. The *mth Hasse jet space* (or *$\underline{\mathcal{D}}$-jet space*) *of $\underline{Z}$* is the Hasse subvariety of $\operatorname{Jet}^m(X)$ given by Lemma 4.1. That is,

$$\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z}) := \left(\overline{\phi^X_{m,n}\big((\operatorname{Jet}^m Z_n)_{\mathrm{red}}\big)} : n \in \mathbb{N}\right).$$

Given $a \in \underline{Z}(k)$ we set the *mth Hasse jet space of $\underline{Z}$ at a* to be the Hasse subvariety of $\operatorname{Jet}^m(X)_a$ given by

$$\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a := \left(\overline{\phi^X_{m,n}\big((\operatorname{Jet}^m Z_n)_{\mathrm{red}}\big)}_{\nabla_n(a)} : n \in \mathbb{N}\right).$$

We showed in Lemma 4.1 that $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})$ is a Hasse subvariety of $\operatorname{Jet}^m X$. Let us note that for $a \in \underline{Z}(k)$, $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a$ *is indeed a Hasse subvariety of* $\operatorname{Jet}^m X_a$. Let

$$T_n := \overline{\phi^X_{m,n}\big((\operatorname{Jet}^m Z_n)_{\mathrm{red}}\big)}$$

and let $(T_n)_{\nabla_n(a)}$ be the (reduced) fibre of $\tau_n \operatorname{Jet}^m X \to \tau_n X$ over $\nabla_n(a)$. So $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a$ is given by the sequence $\big((T_n)_{\nabla_n(a)} : n \in \mathbb{N}\big)$. First of all,

$$(\tau_n \operatorname{Jet}^m X)_{\nabla_n(a)} = \tau_n(\operatorname{Jet}^m X_a)$$

by Fact 2.11, and so $(T_n)_{\nabla_n(a)}$ is a closed subvariety of the nth prolongation of $\operatorname{Jet}^m X_a$. Moreover, the structure morphism $\tau_{n+1}(\operatorname{Jet}^m X_a) \to \tau_n(\operatorname{Jet}^m X_a)$ is just the restriction of $\hat{\pi}_{n+1,n} : \tau_{n+1} \operatorname{Jet}^m X \to \tau_n \operatorname{Jet}^m X$. Hence, as we have already shown that T_{n+1} is sent to T_n, it follows from the functoriality of $\hat{\pi}$ (this is Proposition 4.8(b) of [7]) that $(T_{n+1})_{\nabla_{n+1}(a)}$ is sent to $(T_n)_{\nabla_n(a)}$. A similar argument shows that $\big((T_n)_{\nabla_n(a)} : n \in \mathbb{N}\big)$ satisfies the second condition of being a Hasse subvariety, namely that $(T_{n+1})_{\nabla_{n+1}(a)}$ is sent to $\tau\big((T_n)_{\nabla_n(a)}\big)$ under the iterativity map $\hat{\Delta}_{(n,1)}$.

Remark 4.3. Note that for $a \in \underline{Z}(k)$,

$$\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a(k) = \{\lambda \in \operatorname{Jet}^m X_a(k) : (a, \lambda) \in \operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})(k)\}.$$

Indeed, $\nabla_n(a, \lambda) \in T_n(k)$ if and only if $\nabla_n(\lambda) \in (T_n)_{\nabla_n(a)}(k)$.

4.1. **Main results.** We now establish the main properties of Hasse jet spaces.

First of all, since every algebraic subvariety can be viewed as a Hasse subvariety, it makes sense to ask what the Hasse jet spaces of algebraic varieties look like. As one might hope, the Hasse jet spaces of an algebraic variety coincides with the algebraic jet spaces, at least for "sufficiently general" points.

Notation 4.4 ("Sufficiently general")**.** We say that a property P holds for *sufficiently general points of $\underline{Z}(k)$* if there exist dense Zariski open subsets of U_n of Z_n, for all n, such that P holds for all members of $\{a \in \underline{Z}(k) : \nabla_n(a) \in U_n(k), n \in \mathbb{N}\}$.

Lemma 4.5. *For sufficiently general $a \in \underline{Z}(k)$, $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a$ is given by the sequence* $\big(\phi^X_{m,n}\big([\operatorname{Jet}^m (Z_n)_{\nabla_n(a)}]_{\mathrm{red}}\big) \ : \ n \in \mathbb{N}\big)$.

Proof. For sufficiently general $a \in \underline{Z}(k)$, Zariski closure will commute with fibres, and so $(T_n)_{\nabla_n(a)}$ will be the Zariski closure of $\phi^X_{m,n}\big([\mathrm{Jet}^m(Z_n)_{\nabla_n(a)}]_{\mathrm{red}}\big)$, for all $n \in \mathbb{N}$. Hence it suffices to show that for sufficiently general a, $\phi^X_{m,n}\big([\mathrm{Jet}^m(Z_n)_{\nabla_n(a)}]_{\mathrm{red}}\big)$ is already Zariski closed in $\tau_n \mathrm{Jet}^m(X)_{\nabla_n(a)}$, for all $n \in \mathbb{N}$. But if we choose a so that $\nabla_n(a)$ is smooth, then by Fact 2.18, $\phi^X_{m,n}$ restricts to a surjective *linear* map from $\mathrm{Jet}^m\big(\tau_n(X)\big)_{\nabla_n(a)}$ to $\tau_n \mathrm{Jet}^m(X)_{\nabla_n(a)}$. As $[\mathrm{Jet}^m(Z_n)_{\nabla_n(a)}]_{\mathrm{red}}$ is a linear subvariety of $\mathrm{Jet}^m\big(\tau_n(X)\big)_{\nabla_n(a)}$, $\phi^X_{m,n}\big([\mathrm{Jet}^m(Z_n)_{\nabla_n(a)}]_{\mathrm{red}}\big)$ is Zariski closed. □

Proposition 4.6. *Suppose the alegbraic variety X is viewed as a Hasse subvariety of itself by considering $\underline{X} = (\tau_n(X)_{\mathrm{red}} : n \in \mathbb{N})$. For $a \in X(k)$ sufficiently general, $\mathrm{Jet}^m_{\mathcal{D}}(\underline{X})_a = \mathrm{Jet}^m X_a$. That is, the Hasse and algebraic jet spaces at a coincide.*

Proof. Indeed, $\mathrm{Jet}^m_{\mathcal{D}}(\underline{X})_a = \big(\phi^X_{m,n}\big([\mathrm{Jet}^m(\tau_n X)_{\nabla_n(a)}]_{\mathrm{red}}\big) \ : \ n \in \mathbb{N}\big)$ by Lemma 4.5. But

$$\phi^X_{m,n}\big(\mathrm{Jet}^m(\tau_n X)_{\nabla_n(a)}\big) = (\tau_n \mathrm{Jet}^m X)_{\nabla_n(a)} = \tau_n(\mathrm{Jet}^m X_a)$$

where the first equality is by Fact 2.18 and the second is by Fact 2.11. □

The next lemma gives us an explicit (infinitary) criterion for when an algebraic jet lives in the Hasse jet space at a sufficiently general point.

Lemma 4.7. *For sufficiently general $a \in \underline{Z}(k)$, and arbitrary $\lambda \in \mathrm{Jet}^m(X)_a(k)$, $\lambda \in \mathrm{Jet}^m_{\mathcal{D}}(\underline{Z})_a(k)$ if and only if for all $n \geq 0$ there exists $\gamma_n \in \mathrm{Jet}^m(Z_n)_{\nabla_n(a)}(k)$ such that $\nabla_n(a, \lambda) = \phi^X_{m,n}\big(\nabla_n(a), \gamma_n\big)$.*

Proof. By Lemma 4.5, $\lambda \in \mathrm{Jet}^m_{\mathcal{D}}(\underline{Z})_a(k)$ if and only if for all $n \geq 0$, $\nabla_n(\lambda) \in \phi^X_{m,n}\big((\mathrm{Jet}^m Z_n)_{\nabla_n(a)}\big)(k)$. But as $\phi^X_{m,n}$ is linear on $\mathrm{Jet}^m\big(\tau_n(Z_0)\big)_{\nabla_n(a)}$ by Fact 2.18, taking k-rational points commutes with taking the image under $\phi^X_{m,n}$. It follows that $\nabla_n(\lambda) \in \phi^X_{m,n}\big((\mathrm{Jet}^m Z_n)_{\nabla_n(a)}\big)(k)$ if and only if $\nabla_n(a, \lambda) = \phi^X_{m,n}\big(\nabla_n(a), \gamma_n\big)$ for some $\gamma_n \in \mathrm{Jet}^m(Z_n)_{\nabla_n(a)}(k)$. □

Notice that so far we have not dealt with the question of when the Hasse jet space is a *dominant* Hasse subvariety. In fact, it is not the case that the Hasse jet spaces of dominant Hasse subvarieties are themselves always dominant.

Example 4.8. We consider (oridinary) iterative Hasse-differential fields. That is, we are working in the Hasse system HD_1 and we have an iterative Hasse-differential field $\big(k, \mathbf{D} = (D_0, D_1, \dots)\big)$. (See Example 2.4 and Proposition 2.16). Suppose $\mathrm{char}(k) = p > 0$ and consider the Hasse subvariety of the affine line defined by $\big(D_1(x)\big)^p = x$. That is, let $\underline{Z}$ be the dominant Hasse subvariety of $\mathbb{A}^1$ obtained by applying Lemma 3.3 to the sequence (Y_n) where Y_1 is given by $y^p = x$ in $\tau_1(\mathbb{A}^1) = \mathrm{Spec}(k[x, y])$, and $Y_i = \tau_i(\mathbb{A}^1)$ for all $i \neq 1$. Then $Z_0 = Y_0 = \mathrm{Spec}(k[x])$ and $Z_1 = Y_1 = \mathrm{Spec}\big(k[x, y]/(y^p - x)\big)$. Note that $Z_1 \to Z_0$ is inseparable. Now, since the algebraic tangent space coincides with the first algebraic jet space, and since the interpolating map from the prolongation of a tangent space to the tangent space of a prolongation is the identity (i.e. prolongations commute with tangent spaces), we get that $\mathrm{Jet}^1_{\mathcal{D}}(\underline{Z})$ is given by the sequence of tangent spaces $(TZ_n : n < \omega)$. A straightforward calculation shows that $TZ_1 \to TZ_0$ is not dominant, and so $\mathrm{Jet}^1_{\mathcal{D}}(\underline{Z})$ is not a dominant Hasse subvariety of $T\mathbb{A}^1$. In fact, for any nonzero $a \in k$, the Hasse jet space at a, $\mathrm{Jet}^1_{\mathcal{D}}(\underline{Z})_a$, is not a dominant Hasse subvariety of $(T\mathbb{A}^1)_a$. This is ultimately due to the inseparability of the morphism $\hat{\pi}_{1,0} : Z_1 \to Z_0$. The following proposition explains that such inseparability is the only obstacle.

Proposition 4.9. *Suppose* $\underline{Z}$ *is a dominant and separable* $\underline{\mathcal{D}}$*-subvariety over* k*. Then* $\mathrm{Jet}^m_{\mathcal{D}}(\underline{Z})$ *is dominant. Moreover, for sufficiently general* $a \in \underline{Z}(k)$*,* $\mathrm{Jet}^m_{\mathcal{D}}(\underline{Z})_a$ *is dominant and irreducible.*

Proof. The commuting diagram in part (a) of Corollary 2.17 restricts to

$$\begin{array}{ccc} (\mathrm{Jet}^m Z_{n+1})_{\mathrm{red}} & \xrightarrow{\mathrm{Jet}^m(\hat{\pi}^X_n)} & (\mathrm{Jet}^m Z_n)_{\mathrm{red}} \\ \big\downarrow{\scriptstyle \phi^X_{m,n+1}} & & \big\downarrow{\scriptstyle \phi^X_{m,n}} \\ T_{n+1} & \xrightarrow{\hat{\pi}^{\mathrm{Jet}^m(X)}_{n+1,n}} & T_n \end{array}$$

Now $\hat{\pi}_{n+1,n} : Z_{n+1} \to Z_n$ is dominant and separable by assumption. It follows that $\mathrm{Jet}^m(\hat{\pi}_{n+1,n}) : \mathrm{Jet}^m Z_{n+1} \to \mathrm{Jet}^m Z_n$ is dominant (cf. Lemma 5.9 of [7]). As the two vertical arrows are also dominant so is $T_{n+1} \to T_n$, as desired.

Note that by Lemma 4.5, $(T_n)_{\nabla_n(a)} = \phi^X_{m,n}\big([\mathrm{Jet}^m(Z_n)_{\nabla_n(a)}]_{\mathrm{red}}\big)$ for sufficiently general $a \in \underline{Z}(k)$. Now, the separability and dominance of $\hat{\pi}_{n+1,n} : Z_{n+1} \to Z_n$ imply not only the dominance of $\mathrm{Jet}^m(\hat{\pi}_{n+1,n}) : \mathrm{Jet}^m Z_{n+1} \to \mathrm{Jet}^m Z_n$, but also the surjectivity of that map retricted to sufficiently general fibres (for example, see the proof of Lemma 5.9 of [7]). That is, $\mathrm{Jet}^m(Z_{n+1})_{\nabla_{n+1}(a)} \to \mathrm{Jet}^m(Z_n)_{\nabla_n(a)}$ is surjective for sufficiently general $a \in \underline{Z}(k)$. Hence $(T_{n+1})_{\nabla_{n+1}(a)} \to (T_n)_{\nabla_n(a)}$ is surjective, which implies that $\mathrm{Jet}^m_{\mathcal{D}}(\underline{Z})_a$ is dominant. For irreducibility, note that by choosing a so that $\nabla_n(a)$ is smooth in Z_n, we get that $\mathrm{Jet}^m(Z_n)_{\nabla_n(a)}$ is irreducible, and hence so is $(T_n)_{\nabla_n(a)}$. □

Here is our main theorem.

Theorem 4.10. *Hasse subvarieties are determined by their Hasse jets: Suppose* (k, E) *is a rich iterative* $\underline{\mathcal{D}}$*-field,* X *is an algebraic variety over* k*,* $\underline{Z}$ *and* $\underline{Z}'$ *are irreducible, separable and dominant Hasse subvarieties of* X *over* k*, and* $a \in \underline{Z}(k) \cap \underline{Z}'(k)$ *is a sufficiently general point of intersection. If* $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z})_{\nabla_r(a)}(k) = \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z}')_{\nabla_r(a)}(k)$ *for all* $m, r \in \mathbb{N}$*, then* $\underline{Z} = \underline{Z}'$*.*

Note that if $\underline{\mathcal{D}}$ extends to fields, as it does in the differential and difference cases (cf. Corollary 3.12 and Proposition 5.1), then by Corollary 3.15 the hypothesis of richness in the above theorem can always be satisfied by passing to an extension.

Proof of Theorem 4.10. By Proposition 3.16 we have the dominant Hasse subvariety $\nabla_r \underline{Z}$ of $\tau_r X$ whose k-points are $\nabla_r\big(\underline{Z}(k)\big)$, and similarly for $\underline{Z}'$. Fixing r we have by Proposition 4.9 that $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z})_{\nabla_r(a)}$ is a dominant irreducible Hasse subvariety of $\mathrm{Jet}^m(\tau_r X)_{\nabla_r(a)}$ over k. By the $n = 0$ case of Lemma 4.5 we have that $\big[\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z})_{\nabla_r(a)}\big]_0 = \mathrm{Jet}^m\big((\nabla_r \underline{Z})_0\big)_{\nabla_r(a)}$. On the other hand, by the construction of $\nabla_r \underline{Z}$ (see Proposition 3.16), $(\nabla_r \underline{Z})_0 = Z_r$. So $\big[\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z})_{\nabla_r(a)}\big]_0 = \mathrm{Jet}^m(Z_r)_{\nabla_r(a)}$. By richness it follows that $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z})_{\nabla_r(a)}(k)$ is Zariski dense in $\mathrm{Jet}^m(Z_r)_{\nabla_r(a)}$. Similarly for $\underline{Z}'$. So, fixing r, we have that

$$\mathrm{Jet}^m(Z_r)_{\nabla_r(a)} = \mathrm{Jet}^m(Z'_r)_{\nabla_r(a)}$$

for all $m \in \mathbb{N}$. Since the (algebraic) jet spaces of an algebraic subvariety at a point determine that subvariety (cf. Corollary 5.8 of [7]), we have $Z_r = Z'_r$ for all $r \in \mathbb{N}$. Hence $\underline{Z} = \underline{Z}'$. □

4.2. **Hasse jets via $\underline{\mathcal{D}}$-modules.** We aim in this section to give a more concrete algebraic characterisation of when a vector in the algebraic jet spaces lives in the Hasse jet space of a Hasse subvariety. This will require some further familiarity with the construction of the interpolating map in [7]. The use of the term "$\underline{\mathcal{D}}$-modules" in the title of this subsection is meant to be suggestive; we do not formally develop the theory of $\underline{\mathcal{D}}$-modules here.

Let us fix an iterative Hasse system $\underline{\mathcal{D}}$, a $\underline{\mathcal{D}}$-closed field (k, E), a variety X over k, and a Hasse subvariety $\underline{Z}$ of X over k. Fix also a point $a \in \underline{Z}(k)$ and $m \in \mathbb{N}$.

For each $r \geq 0$ the morphism $\hat{\pi}_r : Z_{r+1} \to Z_r$ induces a k-linear map

$$\mathfrak{m}_{Z_r,\nabla_r(a)}/\mathfrak{m}^{m+1}_{Z_r,\nabla_r(a)} \to \mathfrak{m}_{Z_{r+1},\nabla_{r+1}(a)}/\mathfrak{m}^{m+1}_{Z_{r+1},\nabla_{r+1}(a)}.$$

Setting $V_r := \mathfrak{m}_{Z_r,\nabla_r(a)}/\mathfrak{m}^{m+1}_{Z_r,\nabla_r(a)}$ for brevity, we obtain a directed system

$$V_0 \to V_1 \to V_2 \to \cdots$$

Taking k-duals we have a corresponding inverse system of restriction maps

$$V_0^* = \mathrm{Jet}^m(Z_0)_a(k) \leftarrow V_1^* = \mathrm{Jet}^m(Z_1)_{\nabla(a)}(k) \leftarrow V_2^* = \mathrm{Jet}^m(Z_2)_{\nabla_2(a)}(k) \leftarrow \cdots$$

Lemma 4.11. *For each $r \geq 0$, the canonical morphism $r_r^{Z_0} : \tau_r(Z_0) \times_k \mathcal{D}_r(k) \to Z_0$ induces an additive map $e_r : V_0 \to V_r \otimes_k \mathcal{D}_r(k)$.*

Proof. Since $Z_r \subseteq \tau_r(Z_0)$, r^{Z_0} restricts to a morphism $Z_r \times_k \mathcal{D}_r(k) \to Z_0$. On the other hand, we have

$$\begin{array}{ccc}
\mathrm{Spec}(k) & \xrightarrow{\quad a \quad} & Z_0 \\
\uparrow{\scriptstyle \underline{E}_r} & & \uparrow{\scriptstyle r_r^{Z_0}} \\
\mathrm{Spec}\big(\mathcal{D}_r^{E_r}(k)\big) & \xrightarrow{\nabla_r(a)\times_k \mathcal{D}_r(k)} & \tau_r(Z_0) \times_k \mathcal{D}_r(k)
\end{array}$$

Indeed, $\nabla_r(a)$ is by definition the unique morphism that makes the above square commute. So $r_r^{Z_0}$ maps $\nabla_r(a) \times_k \mathcal{D}_r(k)$ to a. Hence it induces $V_0 \to V_r \otimes_k \mathcal{D}_r(k)$. □

The maps $e_r : V_0 \to V_r \otimes_k \mathcal{D}_r(k)$ endow V_0 with something resembling a "Hasse module" structure. For example, while these maps are *not* k-linear they do satisfy

$$e_r(a \cdot \alpha) = E_r(a) \cdot e_r(\alpha)$$

for all $a \in k$ and $\alpha \in V_0$.

Remark 4.12. Note that in the case when X is affine, e_r is just the map induced by the homomorphism $E_r^{Z_0,0} : k[Z_0] \to k[\tau_r(Z_0)] \otimes_k \mathcal{D}_r(k)$ discussed in section 3.2.

Theorem 4.13. *Suppose $\lambda \in \mathrm{Jet}^m(Z_0)_a(k) = V_0^*$ and $\gamma \in \mathrm{Jet}^m(Z_r)_{\nabla_r(a)}(k) = V_r^*$. Then the following are quivalent:*

(i) $\nabla_r(a, \lambda) = \phi^X_{m,r}\big(\nabla_r(a), \gamma\big)$

(ii) *The following diagram commutes*

$$\begin{array}{ccc}
V_0 & \xrightarrow{\quad \lambda \quad} & k \\
\downarrow{\scriptstyle e_r} & & \downarrow{\scriptstyle E_r} \\
V_r \otimes_k \mathcal{D}_r(k) & \xrightarrow{\gamma \otimes_k \mathcal{D}_r(k)} & \mathcal{D}_r(k)
\end{array}$$

Proof. Note that (i) makes sense: $(a,\lambda) \in \mathrm{Jet}^m(Z_0)(k)$ and

$$\big(\nabla_r(a),\gamma\big) \in \mathrm{Jet}^m(Z_r)(k) \subseteq \mathrm{Jet}^m\left(\tau_r(Z_0)\right)(k) \subseteq \mathrm{Jet}^m\,\tau_r X(k)$$

so that both $\nabla_r(a,\lambda)$ and $\phi^X_{m,r}\big(\nabla_r(a),\gamma\big)$ lie in $\tau_r\big(\mathrm{Jet}^m(Z_0)\big)(k)$. In fact, under the usual identifications, they both live in $\mathrm{Jet}^m(Z_0)_{\widehat{\nabla_r(a)}}\big(\mathcal{D}_r^{E_r}(k)\big)$, where $\widehat{\nabla_r(a)}$: $\mathrm{Spec}\left(\mathcal{D}_r^{E_r}(k)\right) \to Z_0$ is the $\mathcal{D}_r^{E_r}(k)$-point of Z_0 associated to $\nabla_r(a) \in \tau_r(Z_0)(k)$.

Claim 4.14. $\mathrm{Jet}^m(Z_0)_{\widehat{\nabla_r(a)}}\big(\mathcal{D}_r^{E_r}(k)\big) = \mathrm{Hom}_{\mathcal{D}_r^{E_r}(k)}\left(V_0 \otimes_k \mathcal{D}_r^{E_r}(k), \mathcal{D}_r^{E_r}(k)\right)$

Proof. We have $\mathrm{Jet}^m(Z_0)_{\widehat{\nabla_r(a)}}\big(\mathcal{D}_r^{E_r}(k)\big) = \mathrm{Hom}_{\mathcal{D}_r^{E_r}(k)}\left(\widehat{\nabla_r(a)}^*(\mathcal{I}/\mathcal{I}^{m+1})\ ,\ \mathcal{D}_r^{E_r}(k)\right)$ where $\mathcal{I}$ is the kernel of the map $\mathcal{O}_{Z_0} \otimes_k \mathcal{O}_{Z_0} \to \mathcal{O}_{Z_0}$ given by $f \otimes g \mapsto fg$ (cf. section 5 of [7]). On the other hand,

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\quad a \quad} & Z_0 \\ & {\scriptstyle \underline{E_r}} \nwarrow \quad \nearrow {\scriptstyle \widehat{\nabla_r(a)}} & \\ & \mathrm{Spec}\left(\mathcal{D}_r^{E_r}(k)\right) & \end{array}$$

commutes. So $\widehat{\nabla_r(a)}^*(\mathcal{I}/\mathcal{I}^{m+1}) = \underline{E_r}^* a^*(\mathcal{I}/\mathcal{I}^{m+1})$. But

$$a^*(\mathcal{I}/\mathcal{I}^{m+1}) = a^{-1}(\mathcal{I}/\mathcal{I}^{m+1}) \otimes_{\mathcal{O}_{Z_0,a}} k = \mathcal{O}_{Z_0,a}/\mathfrak{m}^{m+1}_{Z_0,a} = V_0.$$

Hence, $\widehat{\nabla_r(a)}^*(\mathcal{I}/\mathcal{I}^{m+1})$ is (the sheaf of $\mathcal{D}_r^{E_r}(k)$-modules) $V_0 \otimes_k \mathcal{D}_r^{E_r}(k)$. □

Claim 4.15. *As an element of* $\mathrm{Jet}^m(Z_0)\big(\mathcal{D}_r^{E_r}(k)\big)$, $\phi\big(\nabla_r(a),\gamma\big) = \big(\widehat{\nabla_r(a)},\alpha\big)$ *where* $\alpha : V_0 \otimes_k \mathcal{D}_r^{E_r}(k) \to \mathcal{D}_r^{E_r}(k)$ *is given by*

$$\alpha = \big([\gamma \otimes_k \mathcal{D}_r(k)] \circ e_r\big) \otimes_k \mathrm{id}_{\mathcal{D}_r^{E_r}(k)}\big)$$

Proof. We can view $\gamma \otimes_k \mathcal{D}_r(k)$ as a (*not* k-linear) map from $V_r \times_k \mathcal{D}_r(k)$ to $\mathcal{D}_r^{E_r}(k)$. Precomposing with (the *not* k-linear) $e_r : V_0 \to V_r \times_k \mathcal{D}_r(k)$, we get a map $[\gamma \otimes_k \mathcal{D}_r(k)] \circ e_r : V_0 \to \mathcal{D}_r^{E_r}(k)$. This map *is* k-linear. Indeed, one can check this by tracing through the map (using, for example, (14) below). So the claim makes sense; $\big([\gamma \otimes_k \mathcal{D}_r(k)] \circ e_r\big) \otimes_k \mathrm{id}_{\mathcal{D}_r^{E_r}(k)}\big) : V_0 \otimes_k \mathcal{D}_r^{E_r}(k) \to \mathcal{D}_r^{E_r}(k)$ is a well-defined $\mathcal{D}_r^{E_r}(k)$-linear map.

To prove the claim we first describe $\phi\big(\nabla_r(a),\gamma\big)$ using Claim 4.14 and the construction of the interpolating map in [7]. Applying Jet^m functor to $r_r^{Z_0} \restriction Z_r \otimes_k \mathcal{D}_r(k)$ induces a map

$$v : \mathrm{Jet}^m\left(Z_r \times_k \mathcal{D}_r(k)\right)_{\nabla_r(a)\times_k \mathcal{D}_r(k)}\big(\mathcal{D}_r(k)\big) \to \mathrm{Jet}^m(Z_0)_{\widehat{\nabla_r(a)}}\big(\mathcal{D}_r^{E_r}(k)\big)$$

by Lemma 6.2 of [7]. Since $\mathcal{D}_r^{E_r}(k) = \mathcal{D}_r(k)$ as rings, Claim 4.14 tells us

$$\mathrm{Jet}^m(Z_0)_{\widehat{\nabla_r(a)}}\big(\mathcal{D}_r^{E_r}(k)\big) = \mathrm{Hom}_{\mathcal{D}_r(k)}\left(V_0 \otimes_k \mathcal{D}_r^{E_r}(k), \mathcal{D}_r(k)\right).$$

On the other hand

$$\mathrm{Jet}^m\left(Z_r \times_k \mathcal{D}_r(k)\right)_{\nabla_r(a)\times_k \mathcal{D}_r(k)}\big(\mathcal{D}_r(k)\big) = \mathrm{Hom}_{\mathcal{D}_r(k)}\left(V_r \otimes_k \mathcal{D}_r(k), \mathcal{D}_r(k)\right).$$

Hence v is dual to a $\mathcal{D}_r(k)$-linear map $f : V_0 \otimes_k \mathcal{D}_r^{E_r}(k) \to V_r \otimes_k \mathcal{D}_r(k)$. By definition of the interpolating map in section 6 of [7],

$$\phi\big(\nabla_r(a),\gamma\big) \quad = \quad \big(\widehat{\nabla_r(a)}, [\gamma \otimes_k \mathcal{D}_r(k)] \circ f\big) \tag{13}$$

On the other hand, f is induced by the Weil representing morphism $\tau_r Z_0 \times_k \mathcal{D}_r(k) \to Z_0 \times_k \mathcal{D}_r^{E_r}(k)$. Since $e_r : V_0 \to V_0 \times_k \mathcal{D}_r(k)$ is induced by $r_r^{Z_0}$, which is the above morphism composed with the projection $X \times_k \mathcal{D}_r^{E_r}(k) \to X$, it follows that

$$e_r = f \circ (\mathrm{id}_{V_0}, 1_{\mathcal{D}_r^{E_r}(k)}) \tag{14}$$

where $(\mathrm{id}_{V_0}, 1_{\mathcal{D}_r^{E_r}(k)}) : V_0 \to V_0 \otimes_k \mathcal{D}_r^{E_r}(k)$. It is then not hard to see that

$$\big([\gamma \otimes_k \mathcal{D}_r(k)] \circ e_r\big) \otimes_k \mathrm{id}_{\mathcal{D}_r^{E_r}(k)}\big) = \big(\gamma \otimes_k \mathcal{D}_r(k)\big) \circ f.$$

Claim 4.15 now follows from (13). □

Claim 4.16. *As an element of* $\mathrm{Jet}^m(Z_0)\big(\mathcal{D}_r^{E_r}(k)\big)$,

$$\nabla_r(a, \lambda) = \big(\widehat{\nabla_r(a)}, (E_r \circ \lambda) \otimes_k \mathrm{id}_{\mathcal{D}_r^{E_r}(k)}\big).$$

Proof. We are are viewing $\nabla_r(a, \lambda)$ as a $\mathcal{D}_r^{E_r}(k)$-point of $\mathrm{Jet}^m(Z_0)$. As such we have

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{(a,\lambda)} & \mathrm{Jet}^m(Z_0) \\ & \nwarrow_{\underline{E_r}} \quad\quad {}^{\nabla_r(a,\lambda)}\nearrow & \\ & \mathrm{Spec}\big(\mathcal{D}_r^{E_r}(k)\big) & \end{array}$$

Claim 4.16 follows. □

Finally, we have

$$\begin{aligned} \nabla_r(a, \lambda) = \phi\big(\nabla_r(a), \gamma\big) \quad & \iff \quad (E_r \circ \lambda) \otimes_k \mathrm{id}_{\mathcal{D}_r^{E_r}(k)} = \big([\gamma \otimes_k \mathcal{D}_r(k)] \circ e_r\big) \otimes_k \mathrm{id}_{\mathcal{D}_r^{E_r}(k)} \\ & \iff \quad E_r \circ \lambda = [\gamma \otimes_k \mathcal{D}_r(k)] \circ e_r \end{aligned}$$

where the first equivalence is by Claims 4.15 and 4.16. This completes the proof of Theorem 4.13. □

Corollary 4.17. *An algebraic jet* $\lambda \in \mathrm{Jet}^m(Z_0)_a(k)$ *is in* $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_a(k)$ *if and only if for all* $r \geq 0$ *there exists* $\gamma_r \in \mathrm{Jet}^m(Z_r)_{\nabla_r(a)}(k)$ *extending* λ*, such that* $E_r \circ \lambda = [\gamma_r \otimes_k \mathcal{D}_r(k)] \circ e_r$*.*

Proof. This is just Lemma 4.7 and Theorem 4.13 combined, using also that the interpolating map is over $\mathrm{Jet}^m(X)$ and hence the γ_r's do indeed extend λ. □

5. Appendix: Other Examples

Throughout the main text of the paper we have carried along at least one motivating example, namely that of Hasse-differential rings (cf. 2.4, 2.12, 2.16, 3.4, 3.12 and 4.8). In this appendix we outline several other motivating examples.

5.1. **Rings with endomorphisms.** Consider the Hasse system $\mathrm{End} = \{\mathcal{D}_n \mid n \in \mathbb{N}\}$ where $\mathcal{D}_n$ is $\mathbb{S}^{n+1}$ with the product ring scheme structure, the $\mathbb{S}$-algebra structure given by the diagonal $s_n : \mathbb{S} \to \mathbb{S}^{n+1}$, and $\pi_{m,n}$ the natural co-ordinate projection. Then an End-ring (k, E) is a ring k together with a sequence of endomorphisms $\{\sigma_i : k \to k\}_{i \in \mathbb{Z}_+}$, where $E_n := (\mathrm{id}, \sigma_1, \sigma_2, \dots, \sigma_n)$.

A special case of this is when, for each $n > 0$, $\sigma_{2n} = \tau_1^n$ and $\sigma_{2n+1} = \tau_2^n$, where τ_1 and τ_2 are a pair of endomorphisms of k, possibly commuting, and possibly even satisfying the relation $\tau_2 = \tau_1^{-1}$. In this way one can make any *difference* ring – a ring equipped with a distinguished automorphism – into an End-ring.

A rather more convenient Hasse system for dealing with rings equipped with e commuting automorphisms would be to set $\mathcal{D}_n$ to be $\mathbb{S}^{(2n+1)^e}$ with s_n still the diagonal embedding and $\pi_{n+1,n}$ the natural co-ordinate projection. Then a ring k with commuting automorphisms $\tau_1, \dots, \tau_e$ can be viewed as a End-ring by setting

$$E_n(x) = \big(\tau_1^{\alpha_1}\tau_2^{\alpha_2}\cdots\tau_e^{\alpha_e}(x)\big)_{\{\alpha \in \mathbb{Z}^e:\ \text{each } |\alpha_i| \leq n\}}$$

We can now impose an iterativity condition which will force the iterative End-rings to be rings equipped with e commuting automorphisms. For ease of presentation, let us deviate slightly from standard multi-index notation and write $|\alpha| \leq n$ to mean that $|\alpha_i| \leq n$ for each $i = 1, \dots, e$. Then our iteration map, $\Delta_{(m,n)} : \mathcal{D}_{m+n} \to \mathcal{D}_{(m,n)}$, will be given by by $(x_\alpha)_{|\alpha| \leq n+m} \mapsto \big((x_{\beta+\gamma})_{|\beta| \leq n}\big)_{|\gamma| \leq m}$.

Proposition 5.1. *The system $\Delta = (\Delta_{(m,n)} : m, n \in \mathbb{N})$, above, makes* End *into an iterative Hasse system. Moreover, the Δ-iterative* End*-rings are exactly the rings equipped with e commuting automorphisms. Finally, the system* End *extends to fields.*

Proof. We leave the straightforward (though somewhat notationally tedious) task of showing that (End, Δ) is an iterative system, to the reader. If (k, E) is an End-ring then by the compatibility of E with π we can write $E_n(x) = \big(\sigma_\alpha(x)\big)_{\{\alpha \in \mathbb{Z}^e : |\alpha| \leq n\}}$ where each σ_α is an endomorphisms of k. Then for (k, E) to be Δ-iterative means exactly that

$$\sigma_\gamma \circ \sigma_\beta \quad = \quad \sigma_{\beta+\gamma} \quad \text{for all } \beta, \gamma \in \mathbb{Z}^e. \tag{15}$$

Clearly, if $\sigma_\alpha = \tau_1^{\alpha_1}\tau_2^{\alpha_2}\cdots\tau_e^{\alpha_e}$ for all $\alpha \in \mathbb{Z}^e$, where $\tau_1, \dots, \tau_e$ are commuting automorphisms of k, then (15) holds. Conversely, for $i = 1, \dots, e$, let $\tau_i := \sigma_{(\dots,0,1,0,\dots)}$ where the 1 is in the ith co-ordinate. Then (15) implies that the $\tau_1, \dots, \tau_e$ commute, are invertible, and $\sigma_\alpha = \tau_1^{\alpha_1}\tau_2^{\alpha_2}\cdots\tau_e^{\alpha_e}$ for all $\alpha \in \mathbb{Z}^e$.

To see that End extends to fields suppose (R, E) is an iterative End-integral domain and K is the fraction field of R. We need to extend each E_n to a ring homomorphism $\tilde{E_n} : K \to \mathcal{D}_n(K)$. It suffices to check that E_n takes nonzero elements in R to units in $\mathcal{D}_n(K)$. But this is the case since the units in $\mathcal{D}_n(K) = K^{(2n+1)^e}$ are just those elements all of whose co-ordinates are nonzero, and $E_n(x) = \big(\tau_1^{\alpha_1}\tau_2^{\alpha_2}\cdots\tau_e^{\alpha_e}(x)\big)_{|\alpha| \leq n}$, where the τ_i are automorphisms of R. $\square$

Note that it is not the case that End-rings always localise, one must require that the multiplicatively closed set by which one is localising is also closed under the operators. Note also that we really needed iterativity here in order to extend to fields: if R is an integral domain and $\sigma : R \to R$ is a nonconstant endomorphism with a nontrivial kernel (*eg* $R = \mathbb{Z}[x]$ and $\sigma(f(x)) := f(0)$), then there is no extension of σ to an endomorphism of the field of fractions of R.

5.2. **Difference-differential rings.** We can combine the above example with the differential example. A Hasse system that is convenient for the study of a ring equipped with one Hasse-derivation together with an endomorphism might be the following: $\mathcal{D}_n(R) = \prod_{i=0}^{n} R[\eta]/(\eta)^{n+1-i}$, $s_n(r) := (r + (\eta)^{n+1-i} : i = 0, \ldots, n)$, $\psi_n : \mathcal{D}_n(R) \to \prod_{i=0}^{n} R^{n+1-i}$ given by the standard monomial basis in each of the $n+1$ factors, and $\pi_{m,n} : \mathcal{D}_m(R) \to \mathcal{D}_n(R)$ given by projecting onto the first n coordinates and then taking the quotient $R[\eta]/(\eta)^{m+n+1-i} \to R[\eta](\eta)^{n+1-i}$ on each of the remaining factors. Given a ring k together with a Hasse-derivation $\mathbf{D}$ and an endomorphism σ, we make k into a $\underline{\mathcal{D}}$-ring by setting $E_n : k \to \mathcal{D}_n(k)$ to be the ring homomorphism

$$E_n(x) = \big(\sum_{j=0}^{n-i} \sigma^i D_j(x)\eta^j \ : i = 0, 1, \ldots, n\big).$$

As before, if one wants to focus on the case of an automorphism a more convenient presentation would be

$$\mathcal{D}_n(R) = \prod_{i=1}^{n} R[\eta]/(\eta)^{n+1-i} \ \times R[\eta]/(\eta)^{n+1} \times \prod_{i=1}^{n} R[\eta]/(\eta)^{n+1-i}$$

and

$$E_n(x) = \big(\sum_{j=0}^{n-i} \sigma^{-i} D_j(x)\eta^j, \sum_{j=0}^{n} D_j(x)\eta^j, \sum_{j=0}^{n-i} \sigma^i D_j(x)\eta^j : i = 1, \ldots, n\big).$$

We can then combine the iterativity maps for HD and End to obtain an iteration map $\Delta_{(m,n)} : \mathcal{D}_{m+n} \to \mathcal{D}_{(m,n)}$ given by

$$f_i(\eta)_{-n+m \leq i \leq n+m} \mapsto \big((f_{\alpha+\beta}(\zeta + \epsilon))_{-n \leq \alpha \leq n}\big)_{-m \leq \beta \leq m}.$$

The corresponding iterative Hasse rings are precisely rings equipped with an iterative Hasse-derivation and an automorphism that commutes with the Hasse-derivation. Moreover, this iterative Hasse system will extend to fields.

5.3. **Higher D-rings.** As a final example we consider a higher order version of the D-rings studied by the second author in [9] and [10], see also Example 3.7 of [7]. As we explain at the end of this section, higher D-rings specialise to both Hasse-differential rings and to difference rings thought of as rings with difference operators.

Let e be a positive integer and let $A := \mathbb{Z}[c_1, \ldots, c_e]$ be the polynomial ring in e indeterminates. We define a Hasse system over A as follows. For each $m \in \mathbb{N}$, let

$$P_m(X, W) := \prod_{i=0}^{m-1} (X - iW) \in \mathbb{Z}[X, W]$$

where for convenience we set $P_0(X, W) := 1$. For $I \in \mathbb{N}^e$ and R an A-algebra define

$$\mathcal{D}_I(R) := R[\epsilon_1, \ldots, \epsilon_e]/\big(P_{I_1+1}(\epsilon_1, c_1), \ldots, P_{I_e+1}(\epsilon_e, c_e)\big).$$

As $P_\ell(X,W)$ divides $P_m(X,W)$ for $\ell \le m$, we have quotient maps $\pi_{I,J} : \mathcal{D}_I(R) \to \mathcal{D}_J(R)$ for $J \le I$. Since, $P_1(X,W) = X$, $\mathcal{D}_{\mathbf{0}}(R) = R$. As $P_m(X, c_\ell)$ is a monic polynomial over k, the rings $\mathcal{D}_I(R)$ are free R-algebras with monomial basis

$$\{\epsilon^J : J \le I\}.$$

So $\underline{\mathcal{D}} = (\mathcal{D}_I : I \in \mathbb{N}^e)$ is a Hasse system over A, albeit indexed by $\mathbb{N}^e$ and thus diverging slightly from our formalism.

Observe that the ring $\mathbb{Z}[W][X,Y]/(P_\ell(X,W), P_m(Y,W))$ is the coordinate ring of the reduced subscheme $X_{\ell,m}$ of $\mathbb{A}^2_{\mathbb{Z}[W]}$ whose underlying space is $\{(iW, jW) : 0 \le i < \ell, 0 \le j < m\}$. Visibly, $P_{\ell+m+1}(X+Y, W)$ is identically zero on $X_{\ell+1,m+1}$. Hence,

$$P_{\ell+m+1}(X+Y,W) \in (P_{\ell+1}(X,W), P_{m+1}(Y,W)).$$

This observation permits a definition of an iteration map. Indeed, changing variables so as to separate out the roles of each of the applications of $\mathcal{D}_I$, for I and J two multi-indices in $\mathbb{N}^e$ and R an A-algebra, let us write $\mathcal{D}_I \circ \mathcal{D}_J(R)$ as

$$R[X_1,\ldots,X_e,Y_1,\ldots,Y_e]/(P_{I_1}(X_1,c_1),\ldots,P_{I_e}(X_e,c_e),P_{I_1}(Y_1,c_1),\ldots,P_{I_e}(Y_e,c_e))$$

and

$$\mathcal{D}_{I+J}(R) := R[Z_1,\ldots,Z_e]/(P_{I_1+J_1}(Z_1,c_1),\ldots,P_{I_n+J_n}(Z_n,c_n)).$$

The iteration map $\Delta_{I,J} : \mathcal{D}_{I+J} \to \mathcal{D}_I \circ \mathcal{D}_J$ is then defined by $Z_i \mapsto X_i + Y_i$ for $1 \le i \le n$. Our observation that $P_{\ell+m+1}(X_i+Y_i, c_i)$ may be expressed as an R-linear combination of $P_{\ell+1}(X_i,c_i)$ and $P_{m+1}(Y_i,c_i)$ shows that $\Delta_{I,J}$ is a homomorphism of R-algebras. Visibly these maps are associative and compatible with the projection maps defining the inverse system.

As usual, a $\underline{\mathcal{D}}$-ring structure on an A-algebra k is given by collection of A-algebra homomorphisms $E_I : k \to \mathcal{D}_I(k)$ compatible with the identification $\mathcal{D}_{\mathbf{0}}(k) = k$ and the maps $\pi_{I,J} : \mathcal{D}_I(k) \to \mathcal{D}_J(k)$ in the inverse system. We may express each such map in terms of the monomial basis as

$$E_I(x) = \sum_{J \le I} \partial_{I,J}(x)\epsilon^J.$$

However, it is not the case in general that for $J \le I$ and $J \le K$ that $\partial_{I,J} = \partial_{K,J}$. For example, taking $e = 1$, we have $\epsilon^2 = 0 \cdot \epsilon^0 + 0 \cdot \epsilon^1 + 1 \cdot \epsilon^2$ in $\mathcal{D}_2(k)$ but $\epsilon^2 = 0 \cdot \epsilon^0 + e \cdot \epsilon^1$ in $\mathcal{D}_1(k)$. If we wish to express the $\underline{\mathcal{D}}$-ring structure on k via a single $\mathbb{N}^e$-indexed sequence of operators $\delta_J : k \to k$, then instead of the monomial basis we should take $\{\beta_J : J \le I\}$ as a basis for $\mathcal{D}_I$, where

$$\beta_J(\epsilon_1,\ldots,\epsilon_e) := \prod_{i=1}^{e} P_{J_i}(\epsilon_i, c_i).$$

Viewing the $\mathcal{D}_I$ as finite free $\mathbb{S}$-algebras with respect to this basis, we have that if (k,E) is a $\underline{\mathcal{D}}$-ring then

$$E_I(x) = \sum_{J \le I} \partial_J(x)\beta_J$$

where $\{\partial_J : J \in \mathbb{N}^e\}$, are A-linear additive endomorphisms of k.

Proposition 5.2. *Suppose k is an A-algebra and $\{\partial_I : I \in \mathbb{N}^e\}$ is a set of A-linear additive endomorphisms of k. For $i \le e$, let $\sigma_i := c_i \cdot \partial_i + \mathrm{id}$ where $\partial_i := \partial_{(0,\ldots,0,1,0,\ldots,0)}$ with the 1 is in the ith co-ordinate. For $K \in \mathbb{N}^e$, set $\sigma^K := \sigma_1^{K_1} \circ$*

$\cdots \circ \sigma_e^{K_e}$. *Then setting* $E_I(x) = \sum_{J \leq I} \partial_J(x)\beta_J$ *for all* $I \in \mathbb{N}^e$, (k, E) *is an iterative* $\underline{\mathcal{D}}$*-ring if and only if the following two rules hold*

- *Product rule:* $\partial_I(xy) = \sum_{J+K=I} \sigma^K\big(\partial_J(x)\big) \cdot \partial_K(y)$,
- *Iteration rule:* $\partial_I \circ \partial_J = \binom{I+J}{I} \partial_{I+J}$.

To carry out this proof we need a few easy combinatorial lemmata.

Let us start with a calculation allowing us to see the iteration rule.

Lemma 5.3. $P_\ell(X+Y, W) = \sum_{m=0}^{\ell} \binom{\ell}{m} P_m(X, W) P_{\ell-m}(Y, W)$

Proof. It suffices to show that the stated equality holds whenever one evaluates at points of the form (aW, bW) where a and b are integers. On the lefthand side, we have $P_\ell(aW + bW, W) = \prod_{i=0}^{\ell-1}((a+b-i)W) = \ell!\binom{a+b}{\ell} W^\ell$. On the righthand side we have

$$\begin{aligned}
\sum_{m=0}^{\ell} \binom{\ell}{m} P_m(aW, W) P_{\ell-m}(bW, W) &= \sum_{m=0}^{\ell} \binom{\ell}{m} m! \binom{a}{m} W^m (\ell-m)! \binom{b}{\ell-m} W^{\ell-m} \\
&= W^\ell \sum_{m=0}^{\ell} \frac{\ell! m! (\ell-m)!}{m!(\ell-m)!} \binom{a}{m} \binom{b}{\ell-m} \\
&= W^\ell \ell! \sum_{m=0}^{\ell} \binom{a}{m} \binom{b}{\ell-m} \\
&= W^\ell \ell! \binom{a+b}{\ell}
\end{aligned}$$

The last equality is obtained by comparing the coefficients of W^ℓ in the expansion of the equality $(1+W)^a(1+W)^b = (1+W)^{a+b}$. □

Now

$$\begin{aligned}
\Delta_{(m,n)} \circ E_{I+J}(x) &= \sum_{K \leq I+J} \partial_K(x) \beta_K(X_1 + Y_1, \ldots, X_e + Y_e) \\
&= \sum_{K \leq I+J} \partial_K(x) \prod_{i=1}^{e} P_{K_i}(X_i + Y_i, c_i).
\end{aligned}$$

Using Lemma 5.3 to expand this, one sees that Δ-iterativity is equivalent to the iteration rule claimed by the proposition.

To see that the claimed Leibniz rule is equivalent to the E_I being homomorphisms, we need to compute the product of two standard basis vectors. First we observe:

Lemma 5.4. $P_n(X, W) P_m(X, W) = \sum_{i=0}^{n} i! \binom{n}{i} \binom{m}{i} W^i P_{m+n-i}(X)$

Proof. The case of $n = 0$ is clear. For the inductive step,

$$\begin{aligned}
P_{n+1}P_m &= \sum_{i=0}^{n} i!\binom{n}{i}\binom{m}{i}W^i(X-(m+n-i)W+(m-i)W)P_{m+n-i} \\
&= \sum_{i=0}^{n} i!\binom{n}{i}\binom{m}{i}W^iP_{m+1+n-i} + i!\binom{n}{i}\binom{m}{i}W^{i+1}(m-i)P_{m+n-i} \\
&= \sum_{i=0}^{n+1}(i!\binom{n}{i}\binom{m}{i} + (i-1)!\binom{n}{i-1}\binom{m}{i-1}(m-i+1))W^iP_{m+n+1-i} \\
&= \sum_{i=0}^{n+1}(\frac{m!}{(m-i)!}\binom{n}{i} + \binom{n}{i-1}\frac{(i-1)!m!(m-i+1)}{(m-i+1)!(i-1)!})W^iP_{m+n+1-i} \\
&= \sum_{i=0}^{n+1} i!\binom{m}{i}\binom{n+1}{i}W^iP_{m+n+1-i}
\end{aligned}$$

□

Lemma 5.4 leads to an expression for the product rule, but not the claimed one. For the sake of definiteness, let us write down the Leibniz rule predicted by Lemma 5.4. Expanding the exponential in two different ways, we have

$$\begin{aligned}
\sum \partial_L(ab)\beta_L &= E(ab) \\
&= E(a)E(b) \\
&= \sum_{I,J}\partial_I(a)\partial_J(b)\beta_I\beta_J \\
&= \sum_{I,J}\sum_{K}\partial_I(a)\partial_J(b)K!\binom{I}{K}\binom{J}{K}c^K\beta_{I+J-K}
\end{aligned}$$

So multiplicativity of E amounts to the product rule:

$$\partial_L(ab) = \sum_{I+J=K+L} K!c^K\binom{I}{K}\binom{J}{K}\partial_I(a)\partial_J(b). \tag{16}$$

To put (16) in the form claimed by Proposition 5.2, we should compute the iterates of σ. Under the hypothesis of iterativity, if $\sigma(x) = c\partial_1(x) + x$, then $\sigma^n(x) = \sum_{i=0}^{n} c^i\frac{n!}{(n-i)!}\partial_i(x)$. Indeed, $\sigma^n(x) = \sum_{i=0}^{n} c^i\binom{n}{i}\partial_1^i(x)$. Via iterativity, we have $i!\partial_i = \partial_1^i$ so that $\binom{n}{i}\partial_1^i = \frac{n!}{(n-i)!}\partial_i$. Putting together this observation with (16), we compute

$$\begin{aligned}
\partial_L(ab) &= \sum_{I+J=K+L} K!c^K\binom{I}{K}\binom{J}{K}\partial_I(a)\partial_J(b) \\
&= \sum_{I'+J=L}\sum_{K=0}^{J} K!c^K\binom{I'+K}{K}\binom{J}{K}\partial_{I'+K}(a)\partial_J(b) \\
&= \sum_{I'+J=L}\sum_{K=0}^{J} K!c^K\binom{J}{K}\partial_K(\partial_{I'}(a))\partial_J(b) \\
&= \sum_{I'+J=L}\sigma^J(\partial_{I'}(a))\partial_J(b)
\end{aligned}$$

The computation is reversible, and so we get that (16) is equivalent to the desired product rule. This completes the proof of Proposition 5.2. □

Let us note some specializations. If $A \to k$ factors through $\mathbb{Z}[c_1, \ldots, c_e] \to \mathbb{Z}[c_1, \ldots, c_e]/(c_1, \ldots, c_e) = \mathbb{Z}$, then an iterative $\underline{\mathcal{D}}$-ring is simply an iterative Hasse-differential ring. If k is a $\mathbb{Q}$-algebra, then it follows from the iteration rule that $\partial_I = \dfrac{1}{I!}\partial_1^{I_1} \circ \cdots \circ \partial_e^{I_e}$ so that the full stack is already determined by the operators $\partial_1, \ldots, \partial_e$. If c_i is a unit in R, then $\partial_i = c_i^{-1}(\sigma_i - \mathrm{id})$. Thus, in the case of $\mathbb{Q}[c_1^{\pm 1}, \ldots, c_e^{\pm 1}]$-algebras, the category of $\underline{\mathcal{D}}$-algebras is equivalent to the that of difference algebras for e commuting endomorphisms. However, in positive characteristic, even when the parameters c_i are units, it is not the case that a $\underline{\mathcal{D}}$-ring is essentially just a difference ring.

Algebras over $\mathbb{Z}[c]$ with additive operators $D : R \to R$ satisfying $D(xy) = xD(y) + yD(x) + cD(x)D(y)$ were considered by the second author in [9] and [10]. André developed a theory of confluence between difference and differential operators in [1] taking both operators $\sigma : R \to R$ and $\delta : R \to R$ as basic where σ is a ring endomorphism and δ is an additive operator satisfying the twisted Leibniz rule $\delta(xy) = \sigma(x)\delta(y) + \delta(x)y$. If there is some $b \in R$ with $\delta(b) \in R^{\times}$, then one may express $\sigma(x) = c\delta(x) + x$ where $c := \frac{\sigma(b)-b}{\delta(b)}$. The operator δ is then a D-operator in the above sense.

Hardouin develops a theory of iterative q-difference operators in [3]. Her axioms are very similar to ours (with $e = 1$). For instance, the Leibniz rules are exactly the same. However, there are some major distinctions. The parameter c is $(q-1)t$ so that the operator $\delta_1(x) = \frac{\sigma_q(x)-x}{(q-1)t}$ where $\sigma_q : \mathbb{C}(t) \to \mathbb{C}(t)$ is the automorphism $f(t) \mapsto f(qt)$ is not $\mathbb{Z}[c]$-linear. Additionally, her iteration rules involve the q-analogues of the binomial coefficients. Most importantly, her exponential maps take values in noncommutative difference algebraic rings. Some aspects of the q-iterative operators may be incorporated into our setting by working with the ring schemes $\mathcal{D}_n(R) := R[\epsilon]/(\prod_{i=0}^{n-1}(\epsilon - q^i))$. We shall address comparisons between these theories and the more general issue of $\underline{\mathcal{D}}$-rings for which we relax the requirement that the operators be linear over the base ring in a companion article.

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