ORDERED O-STABLE GROUPS

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Abstract. An ordered structure M is called o-λ-stable if for any subset A with $|A| \leq \lambda$ and for any cut in M there are at most λ 1-types over A which are consistent with the cut. It is proved in the paper that an ordered o-stable group is abelian. Also there were investigated definable subsets and unary functions of o-stable groups.

1. Introduction, notations and preliminaries

Since the notion of an o-minimal structure appeared in [9] various generalizations were introduced. Among them are weakly o-minimal structures [6, 8] and quasi-o-minimal theories [3, 4]. It is easy to see that any cut in o-minimal structure has only one completion up to a complete type over the model. For weakly o-minimal structures a similar result has been proved by Kulpeshov [7]: Let M be a totally ordered structure. Then M is weakly o-minimal iff the following holds on M: any cut $\langle C, D\rangle$ in M has at most two complete 1-types over M extending $\langle C, D\rangle$, and if a cut $\langle C, D\rangle$ in M has two complete 1-types over M extending $\langle C, D\rangle$, then the set of all realizations of each of these types is convex in any elementary extension of M. It immediately follows from the notion of quasi-o-minimality that each cut has at most continuum extensions up to complete types over a model.

What is common for all of these notions. That each cut has a few number of extensions. Recall that stable theories have a few types. So we can combine these things and introduce the notion of an o-stable theory: each cut in each model of this theory has a few complete types which extend it.

Let $\mathcal{M} = (M, <, \dots)$ be a totally ordered structure, a is an element of M and A, B subsets. As usually we write $a < A$ if $a < b$ for any $b \in A$, and $A < B$ if $a < b$ for any $a \in A$ and $b \in B$. A partition $\langle C, D\rangle$ of M is called a *cut* if $C < D$. Given a cut $\langle C, D\rangle$ we construct a partial type $\{c < x < d : c \in C,\ d \in D\}$, which we also call a cut and use the same notation $\langle C, D\rangle$. If the set C is definable, then the cut is called *quasirational*, if in addition $\sup C \in M$ then the cut $\langle C, D\rangle$ is called *rational*, a non-definable cut is called *irrational*. If $C = (-\infty, c)$ we denote this cut by $(c - 0, c)$, and if $C = (-\infty, c]$ we denote it by $(c, c + 0)$. If $C = M$ we denote this cut $+\infty$ and $\sup A$ stands for $\langle C, D\rangle$, where $C = \{c \in M : c < \sup A\}$. If the set C is definable we will sometimes distinguish cuts defined by $\sup C$ and $\inf D$ as: $\sup C$ stands for $\langle C, D\rangle \cup \{C(x)\}$ and $\inf D$ stands for $\langle C, D\rangle \cup \{\neg C(x)\}$. A cut $\langle C, D\rangle$ in an ordered group is called *non-valuational* [8, 17] if $d - c$ converges to 0 whenever c and d converge to $\sup C$ and $\inf D$ accordingly. A cut, which is not non-valuational, is called *valuational*. Observe that for a valuational cut $\langle C, D\rangle$ there is a convex subgroup H such that $\sup C = \sup(a+H)$ for some a, and this cut is definable iff H is definable. An ordered group G is said to be of *non-valuational*

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type, if any quasirational cut is non-valuational. Note that G is of non-valuational type iff there is no definable non-trivial convex subgroup in G.

A cut $\langle C, D\rangle$ in $(M, <)$ is called definable in $\mathcal{M}$ iff the sets C, D are definable in $\mathcal{M}$. The set of all cuts $\langle C, D\rangle$ definable in $\mathcal{M}$ and such that D has no lowest element will be denoted by $\overline{M}$. The set M can be regarded as a subset of $\overline{M}$ by identifying an element $a \in M$ with the cut $\langle(-\infty, a], (a, +\infty)\rangle$. After such an identification, $\overline{M}$ is naturally equipped with a linear ordering extending $(M, <)$: $\langle C_1, D_1\rangle \le \langle C_2, D_2\rangle$ iff $C_1 \subseteq C_2$. Clearly, $(M, <)$ is a dense substructure of $(\overline{M}, <)$.

A subset A of M is called *convex* if for any $a, b \in A$ the interval $[a, b] \subseteq A$. A *convex component* of a set A is a maximal convex subset of A. The *convex hull* $\overline{A}$ of A is defined as $\overline{A} = \{b \in M : \exists a_1, a_2 \in A(a_1 \le b \le a_2)\}$. An ordered structure is called *weakly o-minimal* if any definable subset consists of finitely many convex components [6, 8].

Let P be some property. We say that P holds *eventually in* A if there is an element $a \in M$ such that $a < \sup A$ and P holds on $(a, \infty) \cap A$. If $A = M$, we omit it. If sets B and C are eventually equal in A we denote this by $B \overset{\infty}{=}_A C$. Let $B \subseteq A \subseteq M$. The set B is said to be *dense in* A if for any $a_1 < a_2$ form A there is $b \in B$ with $a_1 < b < a_2$. If $A = M$ we omit it. A *dense component* of B in A is a maximal subset of B which is dense in A.

Let T be an $\mathcal{L}$-theory and $\phi(\bar{x}, \bar{y})$ a formula. We say that ϕ has the *independence property* (relatively T) if for all $n < \omega$ there is a model $\mathcal{M} \models T$ and two sequences $(\bar{a}_i : i < n)$ and $(\bar{b}_J : J \subseteq n)$ in M such that $\mathcal{M} \models \phi(\bar{a}_i, \bar{b}_J)$ if and only if $i \in J$. We say that T has the independence property if some formula has the independence property.

Notation 1.1. Let s be a partial n-type, A a set. Then

$$S^n_s(A) \triangleq \{p \in S^n(A) : p \cup s \text{ is consistent}\}$$

Note, s need not to be a type over A.

Definition 1.2.
(1) An ordered structure $\mathcal{M}$ is *o-stable in* λ if for any $A \subseteq M$ with $|A| \le \lambda$ and for any cut $\langle C, D\rangle$ in $\mathcal{M}$ there are at most λ one-types over A which are consistent with the cut $\langle C, D\rangle$, i.e. $|S^1_{\langle C,D\rangle}(A)| \le \lambda$.
(2) A theory T is *o-stable in* λ if every model of T is. Sometimes we write T is λ-o-stable or o-λ-stable
(3) T is *o-stable* if there exists a λ in which T is o-stable.
(4) T is *o-superstable* if there exists a λ such that T is o-stable in all $\mu \ge \lambda$.
(5) T is *strongly o-stable* if in addition any definable cut in a model $\mathcal{M}$ of T is definable in the language of pure ordering, or, that is the same, if $\sup A \in M$ for any definable subset A of $\mathcal{M}$.

In the following lemma we prove that an o-stable theory does not have the strict order property inside a cut.

Lemma 1.3 (Strict order property inside a cut)**.** *Let $\mathcal{M}$ be a model of an o-stable theory, and $\mathcal{N}$ a sufficiently saturated elementary extension of $\mathcal{M}$. Then for any formula $\phi(x, \bar{y})$ there is a bound $n = n_\phi$ for the following chain in an arbitrary cut $\langle C, D\rangle$ in $\mathcal{M}$*

$$\phi(\mathcal{N}, \bar{a}_1) \cap \langle C, D\rangle(\mathcal{N}) \subset \phi(\mathcal{N}, \bar{a}_2) \cap \langle C, D\rangle(\mathcal{N}) \subset \cdots \subset \phi(\mathcal{N}, \bar{a}_k) \cap \langle C, D\rangle(\mathcal{N})$$

Proof. Assume the contrary, that there is a formula $\phi(x, \bar{y})$ such that for any natural number n there is a cut $\langle C_n, D_n\rangle$ in $\mathcal{M}$ and there is a sequence of $len(\bar{y})$-tuples $\langle \bar{a}_i^n : i < n\rangle$ such that

$$\phi(\mathcal{N}, \bar{a}_0^n) \cap \langle C_n, D_n\rangle(\mathcal{N}) \subset \phi(\mathcal{N}, \bar{a}_1^n) \cap \langle C_n, D_n\rangle(\mathcal{N}) \subset \cdots \subset \\ \subset \phi(\mathcal{N}, \bar{a}_{n-1}^n) \cap \langle C_n, D_n\rangle(\mathcal{N})$$

Observe that for any triple (i, j, n) with $i < j < n$ there are $c \in C$ and $d \in D$ such that either $\phi(\mathcal{M}, \bar{a}_i^n) \cap (c, \sup C) \subset \phi(\mathcal{M}, \bar{a}_j^n) \cap (c, \sup C)$, or $\phi(\mathcal{M}, \bar{a}_i^n) \cap (\sup C, d) \subset \phi(\mathcal{M}, \bar{a}_j^n) \cap (\sup C, d)$ if the set C is definable, and $\phi(\mathcal{M}, \bar{a}_i^n) \cap (c, d) \subset \phi(\mathcal{M}, \bar{a}_j^n) \cap (c, d)$ if C is not definable.

We add two new $(1 + len(\bar{y}))$-ary predicates $P(x, \bar{y})$ and $R(x, \bar{y})$ naming the following sets

$$\bigcup_{n<\omega} C_n \times \{a_i^n : i < n\}, \quad \bigcup_{n<\omega} D_n \times \{a_i^n : i < n\}$$

correspondingly. Obviously, at least one of the following two properties holds for infinitely many natural numbers:

$$\phi(\mathcal{N}, \bar{a}_1^n) \cap P(\mathcal{N}, \bar{a}_1^n) \cap \langle C_n, D_n\rangle(\mathcal{N}) \subset \phi(\mathcal{N}, \bar{a}_2^n) \cap P(\mathcal{N}, \bar{a}_1^n) \cap \langle C_n, D_n\rangle(\mathcal{N}) \subset \\ \subset \cdots \subset \phi(\mathcal{N}, \bar{a}_{n-1}^n) \cap P(\mathcal{N}, \bar{a}_1^n) \cap \langle C_n, D_n\rangle(\mathcal{N})$$

$$\phi(\mathcal{N}, \bar{a}_1^n) \cap R(\mathcal{N}, \bar{a}_1^n) \cap \langle C_n, D_n\rangle(\mathcal{N}) \subset \phi(\mathcal{N}, \bar{a}_2^n) \cap R(\mathcal{N}, \bar{a}_1^n) \cap (\langle C_n, D_n\rangle(\mathcal{N}) \subset \\ \subset \cdots \subset \phi(\mathcal{N}, \bar{a}_{n-1}^n) \cap R(\mathcal{N}, \bar{a}_1^n) \cap \langle C_n, D_n\rangle(\mathcal{N})$$

Say, the first one holds. Let I be an ordered set of indices. Consider the following type $p(\bar{x}_i : i \in I)$ which consists of formulae of the following form:

$$\forall v(P(v, x_1) \to \phi(\mathcal{N}, \bar{x}_i) \cap (v, \sup P(\mathcal{N}, \bar{x}_1)) \subset \phi(\mathcal{N}, \bar{x}_j) \cap (v, \sup P(\mathcal{N}, \bar{x}_1)))$$

for all $i < j \in I$. Clearly, p is finitely consistent. Let $\mathcal{N}_1^+ = (\mathcal{N}_1, P)$ be an elementary extension of $(\mathcal{M}, P)$ realizing the type p by a sequences $(\bar{b}_i : i \in I)$. Let $C = P(\mathcal{N}_1^+)$ and D be the compliment of C. Let also $\langle \mathcal{C}, \mathcal{D}\rangle$ be a cut in I. Then the following type

$$r_{\langle \mathcal{C}, \mathcal{D}\rangle}(x) = \{\neg\phi(x, \bar{b}_i) : i \in \mathcal{C}\} \cup \{\phi(x, \bar{b}_i) : i \in \mathcal{D}\}$$

is consistent with the cut $\langle C, D\rangle$. Thus the cut $\langle C, D\rangle$ may have $2^{|I|}$ extensions and the elementary theory T of $\mathcal{M}$ is not o-$|I|$-stable. Since I is arbitrary, T is not o-stable. □

Another property of o-stable theories is the following:

Fact 1.4. [15] *An o-stable theory does not have the independence property.*

Lemma 1.3 and Fact 1.4 implies the following criterion of o-stability.

Theorem 1.5. *Let a language L contain binary symbol '$<$' and a theory T of L include axioms saying that '$<$' is a total order. The theory T is o-stable iff it has no the independence property and the strict order property inside a cut.*

Proof. The direction 'only if' follows from Lemma 1.3 and Fact 1.4. So we assume that a theory T has no both the independence property and the strict order property inside a cut. If T is not o-stable, similar to stability there is a model $\mathcal{M} \models T$, a cut $\langle C, D\rangle$ in $\mathcal{M}$ and a formula $\varphi(x; \bar{y})$ such that there are $2^{|M|}$ φ-types over M

which are consistent with the cut $\langle C, D\rangle$. We add a new unary relational symbol P naming the set C. Let $(\mathcal{N}, P) \succ (\mathcal{M}, P)$ be sufficiently saturated, and both α and β be such that they realize the cut $\langle C, D\rangle$ and $\alpha < \sup P < \beta$. Claim that each φ-type p which is consistent with the cut $\langle C, D\rangle$ is also consistent with the cut $\sup(P(\mathcal{N}))$ in the model $\mathcal{N}$.

Let $\theta(x; \bar{y}, \alpha, \beta) \triangleq \varphi(x; \bar{y}) \wedge \alpha < x < \beta$, then there are $2^{|M|}$ θ-types over M. The formula θ has no the independence property, then by Shelah's results θ has the strict order property: there is a sequence $(\bar{b}_i : i < \omega)$ of elements of M such that $\theta(\mathcal{N}; \bar{b}_i, \alpha, \beta) \subset \theta(\mathcal{N}; \bar{b}_j, \alpha, \beta)$ iff $i < j$. Thus to obtain a contradiction it is sufficient to show that $\neg\theta(x; \bar{b}_i, \alpha, \beta) \wedge \theta(x; \bar{b}_j, \alpha, \beta)$ is consistent with the cut $\sup(P(\mathcal{N}))$ for any $i < j$. Indeed, then $\neg\varphi(x; \bar{b}_i) \wedge \varphi(x; \bar{b}_j)$ is consistent with the cut $\sup(P(\mathcal{N}))$ if $i < j$, which contradicts to o-stability.

Let $i < j$. If $\neg\theta(x; \bar{b}_i, \alpha, \beta) \wedge \theta(x; \bar{b}_j, \alpha, \beta)$ is not consistent with the cut $\sup(P(\mathcal{N}))$ then there is $\gamma \in P(\mathcal{N})$ such that

$$\theta(\mathcal{N}; \bar{b}_i, \alpha, \beta) \cap (\gamma, \sup(P(\mathcal{N}))) = \theta(\mathcal{N}; \bar{b}_j, \alpha, \beta) \cap (\gamma, \sup(P(\mathcal{N})))$$

or that is the same, then there is $\gamma \in P(\mathcal{N})$ such that

$$\varphi(\mathcal{N}; \bar{b}_i) \cap (\gamma, \sup(P(\mathcal{N}))) = \varphi(\mathcal{N}; \bar{b}_j) \cap (\gamma, \sup(P(\mathcal{N})))$$

Then there is $c \in C$ such that

$$\varphi(\mathcal{M}; \bar{b}_i) \cap (c, \sup(C)) = \varphi(\mathcal{M}; \bar{b}_j) \cap (c, \sup(C))$$

Similarly it can be shown that there is $d \in D$ such that

$$\varphi(\mathcal{M}; \bar{b}_i) \cap (\sup(C), d) = \varphi(\mathcal{M}; \bar{b}_j) \cap (\sup(C), d)$$

Thus $\varphi(\mathcal{M}; \bar{b}_i) \cap (c, d) = \varphi(\mathcal{M}; \bar{b}_j) \cap (c, d)$. This contradicts to existence of α and β in $\mathcal{N}$. □

Fact 1.6. [13] *Let T be a theory of a language $\mathcal{L}$ without the independence property, and $\mathcal{M} \prec \mathcal{N}$ two models of T such that $\mathcal{N}$ is $|M|^+$-saturated. For any formula $\phi(\bar{x})$ with parameters in N we add a new relational symbol $P_\phi(\bar{x})$ interpreted by $P_\phi^{\mathcal{M}} = \phi^{\mathcal{N}} \cap M^k$ in order to form language $\mathcal{L}^*$. Then the expansion $\mathcal{M}^*$ of $\mathcal{M}$ as an $\mathcal{L}^*$-structure admits quantifier elimination. In particular, $\mathcal{M}^*$ has no the independence property.*

Lemma 1.7. *Let the expansion $\mathcal{M}^*$ be defined as in Fact 1.6 and $\mathcal{M}_1^*$ an elementary extension of $\mathcal{M}^*$. Let also $\mathcal{N}_1$ be a sufficiently saturated elementary extension of $\mathcal{M}_1$. Then for any $P_\phi(\bar{x})$ there are parameters in N_1 such that $P_\phi^{\mathcal{M}_1} = \phi^{\mathcal{N}_1} \cap M_1^k$. That is, the property that new relations are externally definable preserves in elementary extensions of $\mathcal{M}^*$.*

Proof. Let $\mathcal{N}$ be an $|M|^+$-saturated elementary extension of $\mathcal{M}$, and $\mathcal{M}_1^*$ an elementary extension of $\mathcal{M}$. By the definition $\psi^*(\mathcal{M}) = \psi(\mathcal{N}, \bar{\alpha}) \cap M$ for some $\bar{\alpha} \in \mathcal{N}$. Consider the formula $\psi(\bar{x}; \bar{y})$. Clearly, the formula $\psi^*(\bar{x})$ is the definition of a $\theta(\bar{y}; \bar{x}) \triangleq \psi(\bar{x}; \bar{y})$-type $p(\bar{y})$ of $\bar{\alpha}$ over M. Let p_1 be a unique θ-type over M_1 extending p with the same definition ψ^*. Let $\mathcal{N}_1$ be an $|M_1|^+$-saturated elementary extension of $\mathcal{M}_1$, and $\bar{\alpha}_1 \in N_1$ realize the type p_1. Then $\psi^*(\mathcal{M}_1^*) = \psi(\mathcal{N}_1, \bar{\alpha}_1) \cap M_1^{n+1}$. □

Theorem 1.8. *Let T be an o-stable theory of a language $\mathcal{L}$, and $\mathcal{M} \prec \mathcal{N}$ two models of T such that $\mathcal{N}$ is $|M|^+$-saturated. For any formula $\phi(\bar{x})$ with parameters in N we add a new relational symbol $P_\phi(\bar{x})$ interpreted by $P_\phi^{\mathcal{M}} = \phi^{\mathcal{N}} \cap M^k$ in order to form a language $\mathcal{L}^*$. Then the elementary theory T^* of the expansion $\mathcal{M}^*$ of $\mathcal{M}$ is o-stable.*

Proof. Since T^* has no the independence property it is sufficient to prove that it has no the strict order property onside a cut. Assume the contrary, that T^* is not o-stable and then has the strict order property inside a cut witnessed by some $\mathcal{L}^*$-formula $\Theta(x, \bar{y})$ and a cut $\langle C, D\rangle$ in some model $\mathcal{M}_0$ of T^*. By Lemma 1.7 without loss of generality we may assume that $\mathcal{M}_0 = \mathcal{M}$. Then there is an infinite sequence $(\bar{b}_i : i \in I)$ such that $\Theta(\mathcal{M}_1^*, \bar{b}_i) \cap \langle C, D\rangle(\mathcal{M}_1^*) \subset \Theta(\mathcal{M}_1^*, \bar{b}_j) \cap \langle C, D\rangle(\mathcal{M}_1^*)$ for all $i < j$ in some elementary extension $\mathcal{M}_1^*$ of $\mathcal{M}^*$. Claim that the convex set C is definable by some $\mathcal{L}^*$-formula. Indeed let $\alpha \in N$ realize the cut $\langle C, D\rangle$. Then $(x < \alpha) \cap M$ defines C. Without lost of generality we may assume that for all $i < j$ there is $c_{i,j}$ such that in the expanded model $\mathcal{M}^*$ the following holds:

$$\Theta(\mathcal{M}^*, \bar{b}_i) \cap (c_{i,j}, \sup C) \subset \Theta(\mathcal{M}^*, \bar{b}_j) \cap (c_{i,j}, \sup C)$$

Recall, that $\Theta(\mathcal{M}^*, \bar{b}_i) = \theta(\mathcal{N}, \bar{b}_i, \alpha) \cap M$ for some $\bar{\alpha} \in N$ and some $\mathcal{L}$-formula θ. Let $\bar{C}$ be the convex hull of C in $\mathcal{N}$, and $\bar{D}$ the compliment of $\bar{C}$ in N. Then for any cut $\langle \mathcal{C}, \mathcal{D}\rangle$ in I the following type

$$\{\neg\theta(x, \bar{b}_i) : i \in \mathcal{C}\} \cup \{\theta(x, \bar{b}_i) : i \in \mathcal{D}\}$$

is consistent with the cut $\langle \bar{C}, \bar{D}\rangle$ in $\mathcal{N}$. This contradicts to o-stability of T. □

There is an analog of Morley's theorem for o-ω-stable theories.

Lemma 1.9. *Let T be o-ω-stable. Then for any $\lambda > \omega$ the theory T is o-λ-stable.*

Proof. The proof is similar to the proof of analogous Morley's theorem. In this proof it is sufficient to replace the set of all one-types with the set of all one-types which are consistent with a given cut. □

Lemma 1.10. *Let T be o-λ-stable, $\mathcal{M} = (M, <, \dots) \models T$, and A a definable subset of M. Then A with the full induced structure is o-λ-stable.*

Proof. Let $\langle C, D\rangle$ be a cut in A. Claim that $\sup C$ defines also a cut in M. The lemma follows. □

Lemma 1.11. *Let T be o-λ-stable, $\mathcal{M} = (M, <, \dots) \models T$, and E is a definable equivalence relation with convex classes. Then M/E with the full induced structure is ordered and o-λ-stable.*

Proof. If a cut in M/E has too many extensions to complete types, then the corresponding cut in M has the same extensions to complete types. □

Lemma 1.12. *Let T be o-λ-stable, $\mathcal{M} = (M, <, \dots) \models T$, and E is a definable equivalence relation with unbounded classes. Then M/E with the full induced structure is λ-stable.*

Proof. Consider a definable set in G/K: it is a union of E-classes. Then this set is consistent with the cut $+\infty$. Since T is o-λ-stable, the elementary theory of M/E is λ-stable. □

Observe that if T is o-ω-stable, then for each cut $\langle C, D\rangle$ we can introduce local Morley rank of a formula as well as of a type.

Definition 1.13. (1) We say that Morley rank of a formula $\phi(x)$ inside a cut $\langle C, D\rangle$ is equal to or greater than 1 and write $RM_{\langle C,D\rangle}(\phi) \geq 1$ for this, if $\{\phi(x)\} \cup \langle C, D\rangle$ is consistent.

(2) $RM_{\langle C,D\rangle}(\phi) \geq \alpha + 1$ if there are infinitely many pairwise inconsistent formulae $\psi_i(x)$ such that $RM_{\langle C,D\rangle}(\phi(x) \wedge \psi_i(x)) \geq \alpha$.

(3) If α is a limit ordinal, then $RM_{\langle C,D\rangle}(\phi) \geq \alpha$ if $RM_{\langle C,D\rangle}(\phi) \geq \beta$ for all $\beta < \alpha$.

(4) $RM_{\langle C,D\rangle}(\phi) = \alpha$ if $RM_{\langle C,D\rangle}(\phi) \geq \alpha$ and $RM_{\langle C,D\rangle}(\phi) \not\geq \alpha + 1$.

By the similar way we can define local Morley degree of a formula. As usually we define Morley rank of a type. More generally, if r is some rank of a type from the stability theory, we may introduce a localization $r_{\langle C,D\rangle}$ of this rank by replacing the set of all types with the set off all types consistent with the cut $\langle C, D\rangle$.

2. Ordered o-stable groups: commutativity and general properties

Throughout this section G is an ordered group with unit e whose elementary theory is o-stable.

Let H be a convex subgroup of G (not necessary definable). Similar to stability for any formula $\varphi(x, \bar{y})$ there is a natural number n such that each chain $K_1 \cap H \subset K_2 \cap H \subset \cdots \subset K_m \cap H$ has length at most n, provided that K_i is definable by $\varphi(x, \bar{a}_i)$ for some $\bar{a}_i$ and that each K_i is not bounded in the convex subgroup H. We call this *trivial chain condition* for H. Here we use only the fact that K_i are subsets of G. For two subsets A and B of G denote

$$A \overrightarrow{\cap} B \triangleq \begin{cases} A \cap B & \text{if } A \text{ is not bounded in } B, \\ \{e\} & \text{otherwise.} \end{cases}$$

Note that here $A \overrightarrow{\cap} B$ is not necessary equal to $B \overrightarrow{\cap} A$. Using this notation we can rewrite the trivial chain condition as for any formula $\varphi(x, \bar{y})$ there is a natural number n such that each chain $K_1 \overrightarrow{\cap} H \subset K_2 \overrightarrow{\cap} H \subset \cdots \subset K_m \overrightarrow{\cap} H$ has length at most n, provided that $K_i = \varphi(G, \bar{a}_i)$.

Proofs of the following three lemmata are similar to proofs of the correspondent facts for stable groups.

Lemma 2.1 (Baldwin-Saxl condition [2])**.** *For any formula $\varphi(x, \bar{y})$ and any convex subgroup H there is a natural n such that the intersection of a family of subgroups of the form $K_i \overrightarrow{\cap} H$, where $K_i = \varphi(G, \bar{a}_i)$, is the intersection of just n of them. Consequently, subgroups which are finite or infinite intersections of K_i, form an almost uniform family in that sense that for any set of indices I there are $\bar{b}_0, \ldots, \bar{b}_{n-1}$ such that $\bigcap_{i\in I}(K_i \overrightarrow{\cap} H) = \bigcap_{j<n}(K_j \overrightarrow{\cap} H)$. So we may apply the trivial chain condition.*

Lemma 2.2 (O-superstable chain condition)**.** *In an o-superstable ordered group G for any convex subgroup H there is no infinite decreasing sequence $K_0 \overrightarrow{\cap} H \supset K_1 \overrightarrow{\cap} H \supset \cdots \supset K_n \overrightarrow{\cap} H \supset \ldots$, where K_i are definable subgroups of G, such that $|K_n \overrightarrow{\cap} H : K_{n+1} \overrightarrow{\cap} H| = \infty$ for each n.*

Lemma 2.3 (O-omega-stable chain condition)**.** *In an o-omega-stable ordered group G for any convex subgroup H there is no an infinite decreasing sequence $K_0 \overrightarrow{\cap} H \supset K_1 \overrightarrow{\cap} H \supset \cdots \supset K_n \overrightarrow{\cap} H \supset \ldots$, where K_i are definable subgroups of G.*

Lemma 2.4. *Let H and K be definable subgroups of G. Then $\overline{H \cap K} = \overline{H} \cap \overline{K}$, i.e. if $\overline{K} \leq \overline{H}$, then $H \cap K$ is not bounded in K.*

Proof. Let $\varphi(x,a) \triangleq \exists y \in K(e \leq y \leq a \wedge x \in H \cdot y)$. Obviously if $a < b$ then $\varphi(G,a) \cap (h, \sup H) \subseteq \varphi(G,b) \cap (h, \sup H)$ for any $h \in H$. Since G is o-stable, there is no strict order property inside any cut, and in particular in the cut defined by $\sup H$. Then there is $a_0 \in K$ such that for any $b \in K$, which is bigger than a_0 there is $h_b \in H$ such that $\varphi(G,a) \cap (h_b, \sup H) = \varphi(G,b) \cap (h_b, \sup H)$.

Let $k_1 > a_0$ be an element of K. Then $H \cdot k_1 \subseteq \varphi(G, a_0)$ holds eventually in the cut $\sup H$. Choose an arbitrary large element $h_1 \in H$. Since $h_1 k_1 \in \varphi(G, a_0)$ there are $h_2 \in H$ and $k_2 \in [e, a_0] \cap K$ such that $h_1 k_1 = h_2 k_2$. Rewrite this equality as $k_1 k_2^{-1} = h_1^{-1} h_2$. Obviously, $k_1 k_2^{-1} \in K \cap H$. Observe that k_2 is bounded by a_0 and k_1 may be chosen arbitrary large. This implies that $k_1 k_2^{-1}$ can be arbitrary large and consequently $K \cap H$ is not bounded in K. □

Lemma 2.5. *If H is a minimal definable unbounded in G subgroup, then H is the least definable unbounded in G subgroup.*

Proof. Assume that K is a definable unbounded subgroup. By Lemma 2.4 the intersection $K \cap H$ is not bounded in G. Since H is minimal, $K \cap H = H$. □

Lemma 2.6. *For any element $a \in G$ there is a convex subgroup H_a of G containing a, such that the center $Z(H_a)$ of H_a is not bounded in H_a.*

Proof. Let a be a positive element of G and $C(a)$ the centralizer of a. Consider a formula $\psi(x,a,b)$ which says that $a \leq b$ and $x \in \overline{C(c)}$ for all $c \in [a,b]$. Define a formula $\theta(y,a)$ as $\psi(y,a,y)$. Claim that $\theta(G,a)$ is not empty and $\sup \theta(G,a)$ defines a convex subgroup which we denote by H_a. Indeed, $\theta(G,a)$ contains a and b^2 whenever $b \in \theta(G,a)$.

By Lemma 2.4 for any $b_1 < b_2 < \cdots < b_n < \sup H_a$ with $b_1 > a$ the intersection of all $C(b_i)$ is not bounded in H_a. Since G is o-stable we may apply Baldwin-Saxl's condition to the intersection of uniformly definable family of subgroups inside the cut defined by $\sup H_a$. Then $\bigcap\{C(b) : b \in [a, \sup H_a)\}$ is an intersection of just n of them and is not bounded in H_a by Lemma 2.4. It is easy to see that this intersection is the center of H_a. Thus $Z(H_a)$ is not bounded in H_a. □

Lemma 2.7. *If the center of an arbitrary densely ordered group is dense in the group, then this group is abelian.*

Proof. First we claim that in an ordered group both functions $f_a(x) = ax$ and $g_a(x) = xa$ are continuous. Indeed, $ax_0\varepsilon^{-1} < ax < ax_0\varepsilon$ iff $x_0\varepsilon^{-1} < x < x_0\varepsilon$.

Let a and b be arbitrary elements. Since the center Z is dense, there is a sequence $\{c_\alpha\}$ of elements from the center, which converges to b. Then

$$ab = a \cdot \lim c_\alpha = \lim ac_\alpha = \lim c_\alpha a = (\lim c_\alpha) \cdot a = ba$$

Thus the group is abelian. □

Theorem 2.8. *An ordered o-stable group is abelian.*

Proof. Without loss of generality we may assume that $G = H_a$ for some element a. Indeed, if H_a is abelian for any element a, then G is abelian.

Let Z be the center of G and H the greatest subgroup of Z, which is convex in G. Obviously, H is a definable normal subgroup of G.

Assume the contrary, that G is not abelian, then H is a bounded subgroup of G. Define sets A and B as follows:

$$B \triangleq \{g \in Z : g > H\}, \quad A \triangleq \{g \in G : H < g < B\}$$

Claim, that the set A consists of cosets of H

Claim 2.8.1. *The index $|A : H|$ is finite.*

Proof. Consider the following formula:

$$\varphi(x, b) \triangleq \exists y(e \le y < b \wedge x \in yZ)$$

Let $a_1 < a_2 < \cdots < a_n$ be elements of A such that $a_i \cdot H < a_{i+1}$. Then $a_{i+1} \cdot Z \not\subseteq \varphi(G, a_i)$. By Lemma 2.6 the center Z is not bounded in G, which implies that any coset of Z is consistent with the cut $+\infty$. Then $\varphi(G, a_i)$ is eventually in the cut $+\infty$ a strict subset of $\varphi(G, a_{i+1})$. Since there is no the strict order property inside any cut, there is a bound on the length of $a_1 < a_2 < \cdots < a_n$. Thus $|A : H|$ is finite. $\square$

Claim 2.8.2. *The set A is empty.*

Proof. Assume the contrary, that the set A is not empty. Then in the virtue of Claim 2.8.1 the quotient group G/H is discretely ordered and there are an injective homomorphism $\tau : \mathbb{Z} \to G/H$ and a natural number n such that any representative of $\tau(n)$ in G is central.

Observe that $\tau(\mathbb{Z})$ is a subgroup of the center $Z(G/H)$ of the quotient group G/H. Indeed, if $\tau(1) \notin Z(G/H)$ then there is an element $g \in G/H$ such that $\tau(1)g \neq g\tau(1)$, say, $\tau(1)g < g\tau(1)$. Since $\tau(1)$ is positive, $g < \tau(1)g < g\tau(1)$. Eliminating g we obtain that $\tau(0) = g^{-1}g < g^{-1}\tau(1)g < \tau(1)$, which is contradictory, because $\tau(1)$ is the least positive element in G/H.

Let b be a representative in G of $\tau(1)$. Since b is not central there is an element c such that $bc \neq cb$. From the other hand bH is central in G/H. Hence $[b, c] \in H$. By easy calculations

$$b^n c = b^{n-1}(bc) = b^{n-1}cb[b, c] = b^{n-2}cb[b, c]b[b, c] = b^{n-2}cb^2[b, c]^2 = \cdots = cb^n[b, c]^n$$

we obtain that $e = [b^n, c] = [b, c]^n$, because b^n is central as a representative in G of $\tau(n)$. This yields a contradiction, because any ordered group is torsion-free. $\square$

The consequence of Claim 2.8.2 is that $Z(G)/H$ is dense in G/H. By Lemma 2.7 G/H is abelian, because $Z(G)/H \le Z(G/H)$.

Assume that there are elements a and b of G such that $ab \neq ba$. Let $c = [a, b]$. Since G/H is abelian, the element c belongs to H. As in Claim 2.8.2 we obtain that $[a, b^n] = [a, b]^n = c^n$. Consider the following formula $\varphi(x, a, d) \triangleq d^{-1} < [a, x] < d$. Observe that $[a, f] = [a, b] = c$ for any element $f \in C(a) \cdot b$. Indeed, if $f = a_1 b$ then $aa_1b = a_1ab = a_1bac$. Thus $\varphi(G, a, d)$ consists of cosets of $C(a)$. It is easy to see that $C(a) \cdot b^n \not\subseteq \varphi(G, a, c^n)$ and $C(a) \cdot b^n \subseteq \varphi(G, a, c^{n+1})$. Thus we obtain the strict order property witnessed by $\varphi(x; a, y)$ in the cut defined by $\sup C(a)$. This contradicts to o-stability of G. $\square$

From now on we shall use the additional notation for the group operation.

As a corollary of Lemma 1.12 we obtain the following.

Lemma 2.9. *Let K be a definable unbounded subgroup. Then G/K with the full induced structure is stable.*

Let Σ be the family of all convex (not necessary definable) subgroups of G. Two convex subgroups A and $B \in \Sigma$ are called a *jump* if A is a subgroup of B and there is no subgroup $C \in \Sigma$ such that $A < C < B$.

Lemma 2.10. *For each natural $n \geq 2$ the number of jumps $A < B$, such that A/B is not n-divisible, is finite.*

Proof. Let $\varphi_n(x, a) \triangleq \exists y(0 \leq y \leq a \wedge x \in y + nG)$. If for some n the number of jumps $A < B$ such that A/B is not n-divisible is infinite, then $\varphi(x; y)$ has the strict order property inside the cut $+\infty$. Indeed, if $a \in A$ is such that $a + B$ is not divisible by n in the quotient group B/A, then for any $b \in B$ the coset $a + nG$ is not a subset of $\varphi(G, b)$. □

Lemma 2.11. *If the elementary theory of G is o-superstable, then $|G : nG| < \infty$ for any positive integer n.*

Proof. The lemma follows from the o-superstable chain condition. Indeed, nG is not bounded in G and if $|G : nG| = \infty$, then $|n^q G : n^{q+1} G| = \infty$. □

Lemma 2.12. *If the elementary theory of G is o-ω-stable, then G as a pure ordered group is elementary equivalent to the ordered group of rationals.*

Proof. Since an ordered o-stable group G is abelian, it is sufficient to prove that G is divisible. Because of the o-ω-stable chain condition any chain of the form $G \geq pG \geq p^2G \geq \cdots \geq p^kG \geq \ldots$ stabilizes in finite many steps. Since G is ordered, any chain $G > pG > p^2G > \cdots > p^kG > \ldots$ cannot be finite. Thus $G = pG$ and G is divisible. □

Question 1. *Characterize all pure ordered o-stable and o-superstable groups:*

(1) *is an ordered group G o-stable iff G is abelian and for each natural $n \geq 2$ the number of jumps $A < B$ such that B/A is not n-divisible is finite?*
(2) *is an ordered group G o-superstable iff G is abelian and $|G : nG| < \infty$ for each positive integer n?*

3. Definable subsets of ordered o-stable groups

Throughout this section G is a sufficiently saturated ordered group whose elementary theory is o-stable. We say that a formula φ is *eventually minimal inside a cut s* if φ is consistent with s and for any formula ψ exactly one of $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ is consistent with s. The aim of this section is to investigate the eventual stabilizer

$$K_\varphi(x) \triangleq \exists z \forall y \big[z < y \to (\varphi(y) \leftrightarrow \varphi(y + x))\big]$$

of a formula φ, which is eventually minimal in the cut $+\infty$. Obviously, $K_\varphi \triangleq K_\varphi(G)$ is a subgroup of G. Observe that for any $g \in G$ the formula $\varphi(x+g)$ is also eventually minimal.

Claim that for any definable convex set A each of $\inf A$ and $\sup A$ defines a convex subgroup H_A^- and H_A^+, respectively, in the following way:

$$\begin{aligned} H_A^- &\triangleq \{g \in G : a - |g| \in A \text{ for any } a \in A\} \\ H_A^+ &\triangleq \{g \in G : a + |g| \in A \text{ for any } a \in A\} \end{aligned}$$

Observe that the above restriction that we shall investigate the eventual stabilizer in the cut $+\infty$ is not essential. Indeed, let $\langle C, D\rangle$ be a cut, such that H_C^+ is not trivial. Since the expansion by a unary predicate naming a convex set preserve o-stability, we may assume that this convex subgroup is definable and that in fact $\sup C = \sup H_C^+$. So we can investigate the eventual stabilizer at the cut $+\infty$ in H_C^+, which is obviously equal to the eventual stabilizer at the cut $\langle C, D\rangle$ in the whole group G.

Now we prove some technical lemmata.

Lemma 3.1. *For any definable subset A the number of infinite convex components whose infimums belong to G is finite.*

Proof. Assume the contrary. Let $E(x, y)$ be an equivalence relation on A whose classes are convex components of A. Since G is supposed to be sufficiently saturated we may find countably many of these classes $\{[a_i] : i < \omega\}$ such that $[a_i] < [a_j]$ for all $i < j$ and that there is a positive element b such that nb is less than the length of convex components $[a]_E$ for any positive integer n (in the discrete case we may suppose that $\mathbb{Z} < G$ and that $b > \mathbb{Z}$).

Let a cut $\langle C, D\rangle$ in G be defined as $C = (-\infty, \sup \bigcup_i [a_i])$. Then

$$\inf[a]_E < \inf[a]_E + b < \sup[a]_E$$

The convex set $(\inf[a]_E, \inf[a]_E+b)$ is a proper subset of $(\inf[a]_E, \inf[a]_E+2b)$. Since there are infinitely many possibilities for b the following formula has the strict order property inside the cut $\langle C, D\rangle$:

$$\theta(x, y) \triangleq \exists z (z \in A \wedge x \in (\inf[z]_E, \inf[z]_E + y))$$

This contradicts to o-stability of G. □

Lemma 3.2. *Let G be densely ordered. Then for any infinite definable set there is an interval, where this set is dense.*

Proof. Assume that an infinite definable set A is not dense in any interval. Let

$$B = \{a \in A : (a, c) \cap A = \emptyset \text{ for some } c > a\}$$

By Lemma 3.1 the set B is finite. Thus without loss of generality we may assume that B is empty. Also we can suppose that for any $a \in A$ and $c < a$ there is an element $a_1 \in A$ with $c < a_1 < a$.

Let $a \in A$ and $b_1 > a$. Since A is not dense in (a, b_1) there is an interval (c_1, d_1) which has an empty intersection with A and which is a subinterval of (a, b_1). Since the set B is empty, $a < c_1$. Assume that we found out an element b_α and an interval (c_α, d_α) for $\alpha < \lambda < \omega_1$. Let $\lambda = \nu + 1$. We choose $b_{\nu+1} \in A \cap (a, c_\nu)$. Then there is an interval $(c_{\nu+1}, d_{\nu+1})$ which has an empty intersection with A and which is a subinterval of (a, b_ν). If λ is a limit ordinal then $b_\lambda \in \bigcap_{\alpha<\lambda}(a, c_\alpha)$. Since G is sufficiently saturated, this intersection is not empty. Claim that elements b_ν can be chosen so that the length of the convex component of the compliment of A which contains the interval (c_ν, d_ν) is bigger than those one which contains the interval $(c_{\nu+1}, d_{\nu+1})$.

Since the set A is infinite we can find a monotone (say, decreasing) sequence $\langle a_n : n < \omega\rangle$ of elements of A. Consider the cut $\langle C, D\rangle$ defined by the infimum of this sequence. Observe that for any a_n we can repeat the above construction. It means that in some neighborhood of a_n there are convex components of the compliment of A of an arbitrary small length.

Let a formula $\phi(x, y)$ says that x belongs to a convex component of the compliment of A, which has length less than y. Since there is no the strict order property, for any positive $g \in G$ there is a natural number n_g such that for any $n > n_g$ any convex component inside (a_{n+1}, a_n) of the compliment of A has length less than g. Let $k > n_g$. Consider an arbitrary convex component of the compliment of A in (a_{k+1}, a_k). Let h be smaller than the length of this convex component. Then $n_g < n_h$. Since G is assumed to be uncountable, we obtain a contradiction. □

As a corrolary of this lemma we obtain that each definable subset consists of dense components and of finitely many isolated point. The following lemma also is immediate from Lemma 3.2.

Lemma 3.3. *Any definable subgroup of a dense o-stable ordered group G is dense in G.*

Lemma 3.4. *Let G be an ordered group with dense order whose elementary theory is o-stable. Then there is no definable function $f : A \to B$ such that the set A is definable and dense, $B \subseteq \overline{G}$, $\langle C, D\rangle$ is a cut in B and there are infinitely many sequences $\langle a_{n,i} : i \in I_n\rangle_{n<\omega}$ of elements of A converging to α_n with $\alpha_n \neq \alpha_k$ for all $n < k < \omega$ such that $\lim_{i\to\infty} f(a_{n,i}) = \sup C$ for any $n < \omega$, i.e. that $\{f(x) : |\alpha_n - x| < |\alpha_n - a_{n,i}|\}$ is consistent with $\langle C, D\rangle$ for any n and i.*

Proof. Without lost of generality we may assume that $\alpha_n < \alpha_k$ for $n < k$ and that for each n the sequence $\langle a_{n,i} : i \in I_n\rangle$ is strictly monotone, say, strictly increasing. Let $\langle \mathcal{C}, \mathcal{D}\rangle$ be a cut in G defined as $\sup \mathcal{C} = \sup\{\alpha_n : n < \omega\}$. By compactness we may assume that there are $c_1 \in C$ and $d_1 \in D$ such that for each n there is $i_{n,1}$ such that $f(a_{n,i_{n,1}}) \in (c_1, d_1)$. Assume that we have found $c_k \in C$ and $d_k \in D$. Then again by compactness we can find $c_{k+1} \in C$ and $d_{k+1} \in D$ such that for each n $f(a_{n,i_{n,k}}) \notin (c_{k+1}, d_{k+1})$ and there is $i_{n,k+1}$ such that $f(a_{n,i_{n,k+1}}) \in (c_{k+1}, d_{k+1})$. Then the following formula

$$\varphi(x; y, z) \triangleq y < f(x) < z$$

has the strict order property inside the cut $\langle \mathcal{C}, \mathcal{D}\rangle$, contradicting to o-stability of G. □

Lemma 3.5. *Let φ be such that for any $g \in G$ either $\varphi(G) \overset{\infty}{=} (\varphi(G) + g)$ or $\varphi(G) \cap (\varphi(G) + g) \overset{\infty}{=} \emptyset$. Then K_φ is unbounded.*

Proof. Assume the contrary, that K_φ is bounded. Let H_1 be the convex hull of K_φ. If G/H_1 is dense, then let $H \triangleq H_1$. If G/H_1 is discrete, then let g be a representative of the least positive coset of H_1. In this case let $H \triangleq \bigcup_{n\in\mathbb{Z}}(ng + H_1)$. Since G is sufficiently saturated, G/H is dense.

Then for any $g \notin H$ the intersection $\varphi(G) \cap (\varphi(G) + g)$ is eventually empty.

Define a formula

$$\psi(x,y,z) \triangleq \exists t(y < t < z \wedge \varphi(x-t))$$

Claim 3.5.1. *There are $a < b$ such that $b - a > \sup H$ and for any c and d with $a < c < d < b$ and $d - c > \sup H$ it holds that $\psi(x,a,b)$ and $\psi(x,c,d)$ are eventually equal at the cut $+\infty$.*

Proof. Since there is no the strict order property inside a cut. □

Fix some element $c \in (a,b)$ and some $e \in (\sup(c+H), b)$. Then

Claim 3.5.2. *Eventually $\varphi(G) + c \subseteq \psi(G,e,b)$*

Proof. Eventually $\psi(G,c,b) \subseteq \psi(G,e,b) \cup (\varphi(G)+c) \subseteq \psi(G,a,b) = \psi(G,e,b)$. □

Claim 3.5.3. *For any $d \in (e,b)$ the intersection $(\varphi(G)+c) \cap (\varphi(G)+d)$ is eventually empty at the cut $+\infty$.*

Now we define a function $f : (e,b) \to G$ as

$$f(x) = \sup[(\varphi(G)+c) \cap (\varphi(G)+x)]$$

By Claims 3.5.1–3.5.3 the function f is not bounded in any subinterval (b_1, b_2) of (e,b) with $b_1 + H < b_2$. So we may find infinitely many disjoint intervals $(b_{n,1}, b_{n,2})$ such that f is not bounded in each $(b_{n,1}, b_{n,2})$. In each interval we can find a sequence $\langle a_{n,i} : i \in I_n \rangle$ such that $\{f(a_{n,i}) : i \in I_n\}$ is unbounded. This contradicts to Lemma 3.4. Thus K_φ contains an element which does not belong to H. □

Claim that a formula φ such that for any $g \in G$ either $\varphi(G) \overset{\infty}{=} (\varphi(G)+g)$ or $\varphi(G) \cap (\varphi(G)+g) \overset{\infty}{=} \emptyset$ does exist in any o-stable group, because $\psi(x+y)$-rank of a formula $\psi(x)$ is finite.

Lemma 3.6. *Let $\varphi(x)$ be a formula such that the eventual stabilizer K_φ is not bounded. Then $\varphi(G) \cap (g + K_\varphi) \overset{\infty}{\neq} \emptyset$ for some g. If in addition $\varphi(x)$ is eventually minimal, then eventually $\varphi(G) \subseteq g + K_\varphi$.*

Proof. Take an arbitrary $a \in K_\varphi$. By the definition of K_φ there is some b such that for any c, if $b < c$ and $b < a + c$ then $\varphi(c) \leftrightarrow \varphi(c+a)$. Let $c \in \varphi(G)$ be bigger than b. Then $c + n \cdot a \in \varphi(G)$ for all natural n, and so $\varphi(G) \cap (c + K_\varphi)$ is infinite and is dense in some interval by Lemma 3.2.

We show that there is an element c, such that $\varphi(G) \cap (c + K_\varphi) \overset{\infty}{\neq} \emptyset$. Let

$$\begin{aligned} \psi_1(y,d) &\triangleq \exists u \exists v \big(d < u < v \wedge [\varphi(G) \cap (y + K_\varphi) \text{ is dense in } (u,v)]\big) \\ \psi(x,d) &\triangleq \exists y \big(\psi_1(y,d) \wedge x \in (y + K_\varphi)\big) \end{aligned}$$

Since there is no the strict order property inside the cut $+\infty$, there is d_1 such that $\psi(G,d_1) \overset{\infty}{=} \psi(G,d_2)$ for any $d_2 > d_1$. Also $\psi(G,d_1)$ is not eventually empty, because K_φ is unbounded and the above mentioned element c can be chosen arbitrary large. Let $c \in \varphi(G)$ be bigger than d_1. Then $\varphi(G) \cap (c + K_\varphi) \overset{\infty}{\neq} \emptyset$.

If $\varphi(x)$ is eventually minimal, then obviously eventually $\varphi(G) \subseteq g + K_\varphi$. □

Lemma 3.7. *Let K_φ be the eventual stabilizer of an arbitrary formula $\varphi(x)$. Assume that K_φ contains a positive element. Then there is a convex in K_φ subgroup H of K_φ such that eventually $\varphi(G)$ is equal to $\varphi(G) + H$.*

Proof. If K_φ is the zero-group, then the lemma is trivial. So, let $a \in K_\varphi$ be positive. Then by the definition of K_φ there is c such that $\varphi(G)\cap(c,\infty) = (\varphi(G)+a)\cap(c,\infty)$. We define a function f as

$$f(x) \triangleq \inf\{c \in G : \varphi(G) \cap (c,\infty) = (\varphi(G) + x) \cap (c,\infty)\}$$

By Lemma 3.4 in any interval (a_1, b_1) there is a subinterval $[a_2, b_2]$ such that the function f is bounded on $[a_2, b_2] \cap K_\varphi$ by some c. Then f is bounded on $[-b_2, -a_2] \cap K_\varphi$ with a bound $c + b_2$. Moreover, f is bounded on the intersection $[a_2 - b_2, b_2 - a_2] \cap K_\varphi$. Indeed, let $h \in [0, b_2 - a_2]$. Then

$$x \in \varphi(G) \iff x+(a_2+h) \in \varphi(G) \iff (x+a_2+h)-a_2 \in \varphi(G) \iff x+h \in \varphi(G)$$

Since $f(2x) \leq f(x)+x$ the function f is bounded on $[2(a_2-b_2), 2(b_2-a_2)]\cap K_\varphi$ as well as on $[n(a_2 - b_2), n(b_2 - a_2)] \cap K_\varphi$ for each natural n.

Let $H = \bigcup_{n\in\mathbb{N}}[n(a_2 - b_2), n(b_2 - a_2)] \cap K_\varphi$. By compactness we may assume that f is bounded on H. Then eventually $\varphi(G)$ is equal to $\varphi(G) + H$. $\square$

We say that G has *boundedly many definable convex subgroups* if there is a cardinal λ such that in any group which is elementary equivalent to G the number of convex definable subgroups does not exceed λ. Otherwise we say that G has *unboundedly many definable convex subgroups*. We say that a convex set A is *coset-infinite*, if it is not a finite union of cosets of definable subgroups.

Lemma 3.8. *If for some formula $\varphi(x)$ the number of coset-infinite convex components of $\varphi(G, \bar{a})$ is infinite, then G has unboundedly many definable convex subgroups.*

Proof. Assume the contrary, that such a formula $\varphi(x)$ does exist and G has boundedly many definable convex subgroups. Recall that for any convex A each of $\inf A$ and $\sup A$ defines a convex subgroup H^-_A and H^+_A, respectively.

Since G has boundedly many definable convex subgroups, so convex components of $\varphi(G)$ define finitely many convex subgroups. Thus we can suppose that each convex component of $\varphi(G, \bar{a})$ is not a coset of a convex subgroup, is coset-infinite and $H^-_A = H^-_B$ and $H^+_A = H^+_B$ for any convex components A and B. Without loss of generality we may assume that $H^-_A \leq H^+_A$. Since G/H^-_A with the full induced structure is also o-stable, we can suppose that $H^-_A = \{0\}$. Then $\inf A \in G$. The lemma follows from Lemma 3.1. $\square$

Lemma 3.9. *Assume that G has boundedly many definable convex subgroups. Then for any formula $\varphi(x, \bar{y})$ there is a number k_φ such that for any $\bar{a}$ the number of infinite dense components of $\varphi(G, \bar{a})$ which are not cosets of subgroups is at most k_φ.*

Proof. Let A be the topological closure of $\varphi(G, \bar{a})$ in the topology induced by the ordering. Then the intersection of a convex component of A with $\varphi(G, \bar{a})$ gives a dense component of $\varphi(G, \bar{a})$. Thus the lemma follows from Lemma 3.1. $\square$

Recall that G is of non-valuational type if it contains no non-trivial definable convex subgroup.

Lemma 3.10. *Let in addition G be of non-valuational type. Then any equivalence relation in G has at most finitely many infinite convex classes.*

Proof. The lemma directly follows from Lemma 3.8, because there is no definable convex non-trivial subgroups. □

Lemma 3.11. *Let G be of non-valuational type with dense order. Let $\phi(x)$ be a formula and $\langle C, D\rangle$ a cut. Then $\phi(G)$ is either eventually dense in C or eventually empty in C. (The same holds for D with the inverse ordering)*

Proof. Let $E_\phi(x, y)$ be an equivalence relation with convex classes such that each class is a maximal convex set which either is infinite and has an empty intersection with $\phi(G)$, or is not necessary infinite and $\phi(G)$ is dense in this class. By Lemma 3.10 E_ϕ has at most finitely many convex classes, that is sufficient for us. □

We will prove that if G has boundedly many convex definable subgroups, then an eventually minimal formula is eventually equal to a coset of its eventual stabilizer, which is the least definable unbounded subgroup. Also we will construct an example of a group with unboundedly many definable convex subgroups, such that an eventually minimal formula is eventually a proper subset of its eventual stabilizer.

Theorem 3.12. *Let G have boundedly many concex definable subgroups, A a definable unbounded subset, and K_A its eventual stabilizer. Then eventually A is equal to $A + K_A$.*

Proof. Let H be a maximal convex in K_A definable subgroup such that eventually A is equal to $A+H$. By Lemma 3.7 H is not a zero group. If $H = K_A$ we are done. So assume that $H < K_A$. By the definition the eventual equality of A and of $A+h$ means that there is an element c such that $A\cap(c,\infty) = (A+H)\cap(c,\infty)$. Consider the quotient group $G_1 = G/H_1$ with the definable subset $A_1 = A\cap(\sup c+H, \infty)/H$. Then the eventuall stabilizer K_1 of A_1 in G_1 is equal to K_A/H. Let H_1 be the maximal convex in K_1 subgroup such that eventually in G_1 the equality of A_1 and $A_1 + H_1$ holds. By the choice of H the group H_1 must be a zero group and by Lemma 3.7 H_1 is not a zero group.

This implies that there is no maximal convex in K_A definable proper subgroup such that eventually A is equal to $A + H$. Let G' be the union of all definable convex proper subgroups of G. Since there is only boundedly many definable convex subgroups and G is considered to be sufficietly saturated, G' is a proper subgroup of G. We can expand our language adding a unary predicate naming G'. This preserves o-stability. Thus we obtained a maximal convex in K_A definable proper subgroup of G, that gives a contradaction. □

Theorem 3.13. *Let G have boundedly many convex definable subgroups, and A an eventually minimal definable subset. Then there is the least definable unbounded subgroup K, such that A is eventually equal to a coset of K. In particular, K is divisible.*

Proof. By Lemma 3.6 A is eventually a subset of $g + K_A$ for some g. For simplicity of notation we assume that $g = 0$. Then A is eventually a subset of K_A. By Theorem 3.12 the set A is eventually equal to $A + K_A$, which means that A is eventually equal to K_A. □

Theorem 3.14. *Let G be an o-stable ordered group with boundedly many definable convex subgroups. Assume that G is not weakly o-minimal, that is there is a definable subset A consisting of infinitely many convex components and a non-rational cut $\langle C, D\rangle$ such that both A and the compliment of A are consistent with this cut. Then there is an externally definable unbounded proper subgroup K of H_C^+ (where H_C^+ is the stabilizer of the set C). If in addition the cut $\langle C, D\rangle$ is definable then K is definable.*

Proof. Let A be a definable subset consisting of infinitely many convex components. Since G is sufficiently saturated there are convex components A_i of A such that $A_i < A_j$ for $i < j < \omega$. Let $C = \sup \bigcup_{i<\omega} A_i$. Claim that $\sup C$ defines a subgroup H_C^+, which is not zero-group, because G is sufficiently saturated. Thus $\sup C = \sup(H_C^+ + g)$ for some $g \in G$. For simplicity of notations we assume that $g = 0$. Obviously, H_C^+ is at least externally definable as a convex subset. By Theorem 1.8 the elementary theory of (G, H_C^+) is o-stable and by Lemma 1.10 the elementary theory of the full induced structure of H_C^+ is o-stable. So without loss of generality we may assume that A is an unbounded subset of G.

Let $\varphi(x; y) = x \in (y + A)$. Since G is o-stable $(2, \varphi)$-rank of A inside the cut $+\infty$ is finite. So there are $a_i \in G$ for $i < n$ and $\tau \in {}^n2$ such that the formula

$$\theta(x) \triangleq \bigwedge_{i<n} \varphi^{\tau(i)}(x; a_i)$$

satisfies the requirements of Lemma 3.5. Then K_θ is an externally definable unbounded subgroup of G.

Obviously, if C is definable, there is no need to expand the language and then K_θ is definable. □

Now I shall give an example of an o-ω-stable ordered group with a proper definable subset, whose eventual stabilizer is equal to the group.

Let G be the direct power $\mathbb{R}^{\mathbb{Q}}$ with lexicographical ordering, and a subgroup $H = \mathbb{Q}^{\mathbb{Q}}$. Let E be an equivalence relation with convex classes whose classes are archimedean classes of G. Then each positive element a define a convex subgroups $K_a = (-\inf E(G, a), \inf E(G, a))$. Let $B_a = E(G, a) \cap (H + K_a)$ and a unary predicate P names the union $\bigcup_{a\in G} B_a$. It will be shown that $(G, <, +, -, 0, P, E)$ is a group we are looking for. Obviously for any element $g \in G$ it holds that

$$P(G) \cap (\sup E(G, g), +\infty) = (P(G) + g) \cap (\sup E(G, g), +\infty)$$

Thus the eventual stabilizer K_P of P is equal to G.

In [14] it has been proved that $Th(G, <, +, -, 0, E)$ is weakly o-minimal and admits quantifier elimination. Below we prove that the elementary theory of the expansion of this structure by P also admits quantifier elimination and consequently is o-ω-stable.

Theorem 3.15. *The elementary theory T of $(G, <, +, -, 0, P, E)$ admits quantifier elimination.*

Proof. First, claim that any term in variables $x_1, \dots, x_n$ is equal to $k_1x_1 + \dots + k_nx_n$, where $k_i \in \mathbb{Z}$. Second we consider how we can reduce $E(t_1, t_2)$, where t_1, t_2 are terms, to a more simple form. Let x be a variable, and u, v terms. Note, that $0 < x < y$ implies $E(x + y, y)$, since $y < y + x < 2y$, $E(y, 2y)$ and E-classes are convex. If $0 < -x < y \wedge \neg E(x, y)$, then $y/2 < y + x < y$, and consequently $E(x + y, y)$ holds.

We shall use the following notations:

$$\begin{aligned}
E_1(x, y) &\triangleq E(x, y) \wedge \neg E(|x|, |x - y|)\\
\inf[y] < x &\triangleq y < x \vee E(x, y)\\
\sup[y] < x &\triangleq y < x \wedge \neg E(x, y)\\
\inf[y]_1 < x &\triangleq y < x \vee E_1(x, y)\\
\sup[y]_1 < x &\triangleq y < x \wedge \neg E_1(x, y)\\
E_g(x, y) &\triangleq (\inf[-|g|] < x, y < \sup[|g|]) \vee E(|x - y|, |g|)\\
E_g^1 &\triangleq E_g(x, y) \wedge \neg E_g(|x|, |x - y|)
\end{aligned}$$

Other notations $x < \inf[y]$, $x < \sup[y]$, $x < \inf[y]_1$, $x < \sup[y]_1$, $x < \inf[y]_g$, $x < \sup[y]_g$ (respectively E_g), $x < \inf[y]^1_g$, $x < \sup[y]^1_g$ (respectively E^1_g) are defined similarly. Obviously, $E(x, y)$ is equivalent to $\inf[y] < x < \sup[y]$ and similar equivalences hold for other equivalence relations.

In a natural way we can define addition on the set of cuts in G. Let $\langle C_i, D_i\rangle$ be cuts, for $i = 1, 2, 3$. Then $\langle C_3, D_3\rangle = \langle C_1, D_1\rangle + \langle C_2, D_2\rangle$ iff for any $c_3 \in C_3$ and $d_3 \in D_3$ there are $c_i \in C_i$ and $d_i \in D_i$ for $i = 1, 2$ such that $c_3 = c_1 + c_2$ and $d_3 = d_1 + d_2$. Then $\sup[a]_g = \sup[g] + a$, and $\sup[a]^1_g = \sup[g]_1 + a$. The similar hold for the infimum.

Claim that

$$\begin{aligned}
E(x + y, z) \iff\ & [E(x + y, y) \wedge E(y, z)] \vee [E(x + y, x) \wedge E(x, z)] \vee\\
& [\neg E(x + y, y) \wedge \neg E(x + y, x) \wedge E(x, -y) \wedge E(x + y, z)]
\end{aligned}$$

and

$$\begin{aligned}
E(x + y, x) &\iff \inf[y] < x < \sup[y] \vee |x| > \sup[|y|] \vee\\
&\qquad (\inf[y] < -x < \sup[y] \wedge -x \notin (\inf[y]_1, \sup[y]_1))\\
E(x + y, y) &\iff \inf[y] < x < \sup[y] \vee |x| < \inf[|y|] \vee\\
&\qquad (\inf[y] < -x < \sup[y] \wedge -x \notin (\inf[y]_1, \sup[y]_1))\\
E(x + y, -x) &\iff \inf[y] < -x < \sup[y] \wedge -x \notin (\inf[y]_1, \sup[y]_1))
\end{aligned}$$

Now consider $E(x + y, x + z)$. Applying a reduction for $E(x + y, z)$ twice we obtain that it is equivalent to

$$\begin{aligned}
&[E(x + y, y) \wedge E(x + z, z) \wedge E(y, z)]\vee\\
&[E(x + y, y) \wedge E(x + z, x) \wedge E(y, x)]\vee\\
&[E(x + y, y) \wedge \neg E(x + z, z) \wedge \neg E(x + z, x) \wedge E(y, x + z)]\vee\\
&[E(x + y, x) \wedge E(x + z, x)]\vee\\
&[\neg E(x + y, x) \wedge \neg E(x + y, y) \wedge E(x + z, z) \wedge E(x + y, z)]\vee\\
&[\neg E(x + y, x) \wedge \neg E(x + y, y) \wedge E(x + z, x) \wedge E(x + y, x)]\vee\\
&[\neg E(x + y, x) \wedge \neg E(x + y, y) \wedge \neg E(x + z, z) \wedge \neg E(x + z, x)\wedge\\
&\qquad\qquad \wedge E(x + y, x + z)].
\end{aligned}$$

Claim that 3-d, 5-th and 6-th disjuncts are inconsistent. Consider 3-d disjunct. We have that $E(x, -z)$ and then $\forall t(E(|x|, t) \to |y| < t)$. Consequently $E(x +$

$y, x) \wedge \neg E(x+y, y)$, this is inconsistent with $E(x+y, y)$. Disjuncts 5-th and 6-th are considered analogously.

Consider the last disjunct: $\neg E(x+y, x) \wedge \neg E(x+y, y) \wedge \neg E(x+z, z) \wedge \neg E(x+z, x) \wedge E(x+y, x+z)]$. Let $E(x+y, g)$ hold. Then $E_g(y, -x)$, $E_g(-x, z)$, and $E_g(y, z)$. So the last conjunct is equivalent to

$$\begin{aligned}&-x = y = z \vee \\ &\big(\mathrm{sgn}(x+y) = \mathrm{sgn}(x+z) \wedge \big[(y > z \wedge \inf[y-z] < -x < \sup[y-z]) \vee \\ &\qquad\qquad (y \leq z \wedge \inf[z-y] < -x < \sup[z-y])\big]\big)\end{aligned}$$

In a similar way it is possible to show that $E(x+y, -x+z)$ is a boolean combination of convex sets of the same forms. Indeed, it is equivalent to

$$\begin{aligned}&[E(x+y, y) \wedge E(-x+z, z) \wedge E(y, z)] \vee \\ &[E(x+y, y) \wedge E(-x+z, -x) \wedge E(-x, y)] \vee \\ &[E(x+y, y) \wedge \neg E(-x+z, z) \wedge \neg E(-x+z, -x) \wedge E(x, z) \wedge E(y, -x+z)] \vee \\ &[E(x+y, x) \wedge E(-x+z, x)] \vee \\ &[\neg E(x+y, x) \wedge \neg E(x+y, y) \wedge \neg E(-x+z, z) \wedge \neg E(-x+z, -x) \wedge \\ &\qquad\qquad \wedge E(x+y, -x+z)].\end{aligned}$$

Consider the last disjunct. Let $E(x+y, g)$ hold. Then $E_g(y, -x)$, $E_g(x, z)$, $E_g(-x, -z)$ and $E_g(y, -z)$. So the last conjunct is equivalent to

$$\begin{aligned}&x = -y = z \vee \\ &\big(\mathrm{sgn}(x+y) = \mathrm{sgn}(-x+z) \wedge \big[(y > -z \wedge \inf[y+z] < -x < \sup[y+z]) \vee \\ &\qquad\qquad (y \leq -z \wedge \inf[z+y] < -x < \sup[z+y])\big]\big)\end{aligned}$$

Thus $E(x+y, \pm x+z)$ is a boolean combination of convex sets of the form (a_1, a_2), where either $a_i \in G \cup \{\pm\infty\}$ or $a_i = \sup[b_i]$, or $a_i = \inf[b_i]$, or $a_i = \sup[b_i]_1$, or $a_i = \inf[b_i]_1$.

Claim that $\sup[x] < y$ iff $x < \inf[y]$.

Thus, to prove the theorem it is enough to consider a formula $\exists x \bigwedge_i \varphi_i(x, \bar{y})$, where every φ_i (or $\neg\varphi_i$) is one of the following forms:

(1) $mx = u$,
(2) $mx < u$, or $mx > u$,
(3) $mx < \sup[v]$, or $mx > \sup[v]$,
(4) $mx < \inf[v]$, or $mx > \inf[v]$,
(5) $mx < \sup[v]_1$, or $mx > \sup[v]_1$,
(6) $mx < \inf[v]_1$, or $mx > \inf[v]_1$,
(7) $P(mx+u)$,
(8) $\neg P(mx+u)$.

Since $x \neq y \iff x > y \vee x < y$, $x \not< y \iff x > y \vee x = y$, and $mx \not< \sup[v] \iff mx > \sup[v]$ we may assume that each φ_i of the forms (1)–(6) occurs in the positive form.

Claim that $m(\sup[nx+u]+v) = \sup[nx+u]+mv$ and that $\sup[nx+u] = \sup[mnx+mu]$. Observe also that $P(nx+u) \iff P(mnx+mu)$. Thus by multiplying each equality and inequality by some number we can obtain that m's in all conjuncts are equal. Since the group is divisible for simplicity of notation we may assume that $m = 1$. If some φ_i is of the form $x = u$, then replacing each occurrence of x with u in each conjuncts we obtain an equivalent quantifier-free formula.

Now we suppose that there is no conjunct of the first form. Moreover, we may assume that that at most one of the φ_i is of the form (2)–(6). (where $<$ and $>$ count separately). Indeed, $x < u \wedge x < v$ is equivalent to $[x < u \wedge u \leq v] \vee [x < v \wedge v < u]$. The similar can be done for $x > u \wedge x > v$. Claim that $\mathrm{bound}_1[u_1] < \mathrm{bound}_2[u_2]$ is definable by a quantifier-free formula, where $\mathrm{bound}_i \in \{\inf, \sup\}$. Then a formula of the form $x < \mathrm{bound}_1[v_1] \wedge x < \mathrm{bound}_2[v_2]$ is equivalent to $(x < \mathrm{bound}_1[v_1] \wedge \mathrm{bound}_1[v_1] \leq \mathrm{bound}_2[v_2]) \vee (x < \mathrm{bound}_2[v_2] \wedge \mathrm{bound}_2[v_2] \leq \mathrm{bound}_1[v_1])$.

Then $\exists x \bigwedge_i \varphi_i(x, \bar{y})$ is equivalent to a formula saying that there is an element x, which belongs to a convex set and satisfies $\bigwedge_j P(x + u_i)$ and $\bigwedge_j \neg P(x + u_k)$.

Consider $P(x + u)$. It is equivalent to $(|x| < \inf[|u|] \wedge P(u)) \vee (|u| < \inf[|x|] \wedge P(x)) \vee (E(|x|, |u|) \wedge P(u + x))$. Taking into account that

$$E(x, u) \wedge P(u + x) \wedge E(x, v) \wedge P(v + x)$$

is equivalent to $E(x, u) \wedge P(x + u) \wedge E(u, v) \wedge P(u - v)$, we can suppose that at most one of φ_i is of the form $P(x + u)$.

Since $\neg P(x+u)$ is equivalent to $(|x| < \inf[|u|] \wedge \neg P(u)) \vee (|u| < \inf[|x|] \wedge \neg P(x)) \vee (E(|x|, |u|) \wedge \neg P(u + x))$ it is possible to assume that if at most one of φ_i is of the form $P(x + u)$ then there is no conjunct of the form $\neg P(x + u)$.

Hence $\exists x \bigwedge_i \varphi_i(x, \bar{y})$ is equivalent to a formula saying that there is x, which belongs to a convex set U, which has non-empty intersection either with $P(x + u)$ or has no intersection with finitely many $P(x + u)$'s. Let $a \in U$. If the length of U is bigger than the length of $[a]_1$, then U has a non-empty intersection with $P(G)+g$ for any g with $|g| < \sup[a]$ as well as with the compliment of a finite union of $(P(G)+g)$'s with similar g's. If the length of U is less than or equal to the length of $[a]_1$, then U has a non-empty intersection with $P(G) + g$ iff $a \in P(G) + g$. So, it is sufficient to show that all of these is expressible by a quantifier-free formula.

Case 1. Assume that one of φ_i is of the form $P(x + g)$. Then we may suppose that there is no conjunct of the form $\neg P(x + g)$. If the considered convex set U is unbounded then obviously there is an element which satisfy $P(x + g)$ from U. So, assume that both bounds in the formula do exist. Adding to bounds $-g$ we obtain an equivalent formula $\exists x(\inf U - g < x < \sup U - g \wedge P(x))$. So for simplicity of notation we assume that $g = 0$.

Case 1.1. $\mathrm{bound}_1 U = \mathrm{bound}_2[b]$. Then any non-empty convex set U has length bigger than $[b]_1$ and $\exists x(\inf U < x < \sup U \wedge P(x))$ is equivalent to $\inf U < \sup U$.

Case 1.2. $\mathrm{bound}_1 U = \mathrm{bound}_2[b]_1$, say $\sup U = \sup[b]_1$. Then the length of U is bigger that $[b]_1$ iff $\inf U < \inf[b]_1$. Otherwise either $\inf U = \inf[b_1]$ or $\inf[b]_1 < \inf U = c < \sup[b_1]$. In this case $\exists x(\inf U < x < \sup U \wedge P(x))$ is equivalent to $P(b)$.

Case 1.3. $\mathrm{bound}_1 U = \mathrm{bound}_2[b] + g$ or $\mathrm{bound}_1 U = \mathrm{bound}_2[b]_1 + g$ for some g with $|g| > \sup[|b|]$, say $\sup U = \sup[b] + g$. Then $\exists x(\inf U < x < \sup U \wedge P(x))$ is equivalent to $\inf U < \sup U \wedge \exists x(\inf U < x < \sup[b]_1 \wedge P(x))$ (case 1.2).

Case 1.4. $U = (b, c)$. Then $\exists x(b < x < c \wedge P(x))$ is equivalent to $b < c \wedge (\neg E_1(b, c) \vee P(b))$.

Case 2. No conjunct is of the form $P(x+g)$ and there are finitely many conjuncts of the form $\neg P(x + g)$. Claim that $\neg P(x + g) \wedge E(|x|, |g|) \wedge \neg P(x + h) \wedge E(|x|, |h|)$ implies $E(|g|, |h|)$, thus we may assume that the considered convex set U is a subset of $[g]$ and the absolute values of all g's are in the same E-class. Consider U.

Assume that $\sup U = \sup[b]_1$. If $E_1(-g, b)$, then $U \cap \neg P(x+g)$ is equivalent to $(\inf U, \min\{\sup[b]_1, b - \inf[|b+g|]_1\}) \wedge \neg P(x+g)$. Similar facts hold in the cases $\inf U = \inf[b]_1$, $\sup U = b$, $\inf U = b$, $\operatorname{bound} U = b + \operatorname{bound}[g]$, $\operatorname{bound} U = b + \operatorname{bound}[g]_1$. So we may assume that if one of these conditions holds, then $\neg E_1(b, -g)$.

Case 2.1. $\operatorname{bound}_1 U = \operatorname{bound}_2[b]$. Then any non-empty convex set U has length bigger than $[b]_1$ and $\exists x(\inf U < x < \sup U \wedge \bigwedge_i \neg P(x+g_i))$ is equivalent to $\inf U < \sup U$.

Case 2.2. $\operatorname{bound}_1 U = \operatorname{bound}_2[b]_1$, say $\sup U = \sup[b]_1$. Then the length of U is bigger that $[b]_1$ iff $\inf U < \inf[b]_1$. Otherwise either $\inf U = \inf[b_1]$ or $\inf[b]_1 < \inf U = c < \sup[b_1]$. In this case $\exists x(\inf U < x < \sup U \wedge \bigwedge_i \neg P(x+g_i))$ is equivalent to $\bigwedge_i \neg P(b+g_i)$.

Case 2.3. $\operatorname{bound}_1 U = \operatorname{bound}_2[b]+g$ or $\operatorname{bound}_1 U = \operatorname{bound}_2[b]_1+g$ for some g with $|g| > \sup[|b|]$, say $\sup U = \sup[b]+g$. Then $\exists x(\inf U < x < \sup U \wedge \bigwedge_i \neg P(x+g_i))$ is equivalent to $\inf U < \sup U \wedge \exists x(\inf U < x < \sup[b]_1 \wedge \bigwedge_i \neg P(b+g_i))$ (case 2.2).

Case 2.4. $U = (b, c)$. Then $\exists x(b < x < c \wedge \bigwedge_i \neg P(x+g_i))$ is equivalent to $b < c \wedge (\neg E_1(b,c) \vee \bigwedge_i \neg P(b+g_i) \vee \bigwedge_i \neg P(c+g_i))$. □

Theorem 3.16. *The elementary theory T of $(G, <, +, -, 0, P, E)$ is o-ω-stable. More precisely, each cut over a model has finitely many extensions up to complete type over the model.*

Proof. By the previous theorem $(G, <, +, -, 0, E)$ is weakly o-minimal, thus each cut has at most 2 completions. Let $\langle C, D\rangle$ be a cut and $p(x)$ a type over G extending $\langle C, D\rangle$. For simplicity we may assume that $x > 0 \in p(x)$.

Let $\langle C, D\rangle$ be a cut and K_C be the stabilizer of C.

Claim that if $0 < |x| < \inf[|a|]$, then $P(x+a)$ is equivalent to $P(a)$. If $0 < |a| < \inf[|x|]$, then $P(x+a)$ is equivalent to $P(x)$. The formula $E(x,a) \wedge P(x+a) \wedge E(|x|, |b|) \wedge P(x+b) \wedge \neg E_1(x, -b)$ is equivalent to $E(x,a) \wedge P(x+a) \wedge E(a,b) \wedge P(b-a) \wedge \neg E(c, -b)$ for some c such that $E_1(x,c) \in p$. If there is no c such that $E_1(x,c) \in p$, then $\langle C/E_1, D_1\rangle$ is an irrational cut in G/E_1. Then $\neg E_1(x, -b)$ holds for any $b \in G$ and the above formula is equivalent in $Th(G) \cup \{c < x < d : c \in C, d \in D\}$ to $E(x,a) \wedge P(x+a) \wedge E(a,b) \wedge P(b-a)$.

Let $\langle C/E_1, D_1\rangle$ be an irrational cut in G/E_1. If $E(x,a) \in p$ then there are extensions of the cut of two kinds: $\{P(x+b), E(x,a)\}$ for some $b \in [a]$ defines a complete type over G and $\{\neg P(x+b), E(x,a) : b \in G\}$ defines a complete type. If $E(x,a) \notin p$ for any p then $\langle C/E, D\rangle$ is an irrational cut in G/E. Then there are two extension of the cut: with $P(x)$ or with $\neg P(x)$.

So, assume that $E_1(x,a) \in p$ for some $a \in G$.

Case 1. There is $b \in E_1(G,a)$ such that $b \in D$ and $\inf\{b - c : c \in C\}$ is a minimal or $b \in C$ and $\inf\{d - b : d \in D\}$ is a minimal. Then we can consider the following cut, which has the same number of extensions and been considered above: $\langle C - b, D - b\rangle$.

Case 2. There is no such $b \in E_1(G,a)$ as in Case 1. Then $K_C = \{0\}$. Assume that $P(x+c) \in p$ and $|x+c| < \inf[|c|]$. Then there is c' such that $|x+c'| < \inf[|x+c|]$. By the definition of P it holds that $P(x+c) \iff P(x+c-(x+c'))$. So we may replace $P(x+c)$ with $P(c-c')$. Thus this cut has at most two extensions: with $P(x+c)$ or with $\neg P(x+c)$. □

4. Ordered o-stable fields

Let $\mathcal{R} = (R, <, +, \cdot, 0, 1, \dots)$ be an ordered o-stable field.

Lemma 4.1. *Any definable subgroup of the additive group of $\mathcal{R}$ is convex.*

Proof. Assume the contrary, that H is a definable subgroup of $(R, +)$, which is not convex. Let K be the maximal convex subgroup of H, and for some $b > a > 0$ an interval (a, b) contains infinitely many cosets of K. Then infinitely many of these cosets are not subset of H. Let $h \in H$ and $c \in R$ be such that $0 < h < a$ and $ch \in (a, b) \setminus H$. Then $c > 1$ and $\sup H \leq \sup cH$. By Lemmata 2.4 and 3.3 the intersection $H \cap cH$ is not bounded in H. By Baldwin-Saxl condition (Lemma 2.1) there are $c_0, \dots, c_{n-1}$ such that

$$H \cap \bigcap \{cH : c \in (ah^{-1}, bh^{-1})\} = H \cap \bigcap_{i<n} c_i H$$

Hence $H_1 \triangleq H \cap \bigcap \{cH : c \in (ah^{-1}, bh^{-1})\}$ is not bounded in H. By the trivial chain condition inside the cut $\sup H$ we may assume that H_1 is minimal. Let $c \neq c_i$ for $i < n$. Then eventually in the cut defined by $\sup H$ the equality $H_1 \cap cH_1 = H_1$ holds.

Let $h \in H_1$ and $g < h$ be not in H_1. Let $c = hg^{-1}$. Then $c > 1$ and $\sup cH_1 \geq \sup H_1$. So, the intersection $H_1 \cap cH_1$ is unbounded in H_1 and since H_1 is minimal this intersection is equal to H_1. By the choice of c it holds that $cg = h \in H_1 \subseteq cH_1$. Then $cg = ch_1$ for some $h_1 \in H_1$ and $g \in H_1$, giving a contradiction. □

In [8] it has been shown that each weakly o-minimal ordered field is real closed.

Theorem 4.2. *Let $\mathcal{R}$ be an o-stable ordered field with boundedly many convex definable subgroups of the additive group. Then R is weakly o-minimal and real closed. In particular, if $\mathcal{R}$ is a strongly o-stable ordered field, then it is o-minimal.*

Proof. Assume the contrary, that R is not weakly o-minimal. Then there is a definable subset A with infinitely many convex components. So we may assume that there are convex components $[a_i]$ for $i < \omega_0$ such that $[a_i] < [a_j]$ iff $i < j$. Let $\langle C, D \rangle$ be a cut defined by $\sup \bigcup_{i<\omega_0} [a_i]$. Then $\sup C = \sup(H + a)$ for some convex definable subgroup H. Since the φ-rank of A inside $\langle C, D \rangle$ is finite, for $\varphi(x; y) \triangleq x \in (y + A)$ there is a definable subset B of A which at the cut $\langle C, D \rangle$ has the following property: if $B \cap (B + g)$ is not eventually an empty set, then these sets are eventually equal.

Let H be a non-zero subgroup. Then the eventual stabilizer K_B of $B - a$ is not bounded in H. By Lemma 4.1 K_B is convex. Then $K_B = H$ By Theorem 3.12 eventually B equals to $B + K_B = B + H$, which is eventually equal to H. That gives a contradiction.

Thus H is a zero subgroup. Let $b_1 < c_1 < b_2 < c_2 < \dots < b_n < c_n < \dots$ be an infinite sequence of elements such that $b_n \in B$ and $c_n \notin B$. Let α be an infinitesimal relatively $b_{n+1} - c_n$ and $c_n - b_n$ for all n. Let $C = \sup_{n<\omega} b_n$. Then $\sup C = \sup_{n<\omega} b_n = \sup_{n<\omega} b_n + \alpha = \sup C + \alpha$. Thus we find another cut whose eventual stabilizer now is not a zero group. So we may repeat the above consideration. □

Due to this theorem it is quite natural to ask the following

Question 2. *Is there an ordered o-stable field which is not weakly o-minimal? Is an ordered o-stable (o-superstable, or o-ω-stable) field real closed?*

5. Ordered non-valuational o-omega-stable groups

Throughout this section G is an ordered group whose elementary theory is non-valuational o-ω-stable.

Simple examples of non-valuational (or even strongly) o-ω-stable ordered groups which are not o-minimal are the following: $(\mathbb{R}, <, +, \alpha\cdot\mathbb{Q})$, where α is some real number. Other examples will be considered in Subsection 5.1. Note that by Lemma 2.4 the elementary theory of $(\mathbb{R}, <, +, \mathbb{Q}, \alpha \cdot \mathbb{Q})$ is not o-stable for any irrational α.

Lemma 5.1 (Descending chain condition)**.** *There is no infinite descending chain of definable subgroups in an ordered non-valuational o-ω-stable group.*

Proof. Since there is no bounded subgroups, all non-zero definable subgroups are consistent with the cut $+\infty$. Now we can apply the o-omega-stable chain condition. □

Claim that since an o-ω-stable group is divisible, the ordering of G is dense.

The following theorem immediately follows from Theorem 3.13:

Theorem 5.2. *Any formula with the least Morley rank and degree in the cut $+\infty$ eventually is a coset of the least definable non-trivial subgroup of G. In particular, for any two definable eventually minimal subsets A and B there is an element g such that eventuall $A + g$ equals B.*

Immediately we obtain the following

Corollary 5.3. *Any formula of Morley rank 1 in the cut $+\infty$ eventually is a finite union of cosets of the least definable non-trivial subgroup of G.*

Since any type consistent with the cut $+\infty$ is definable, we can speak of the eventual fixer of a type. Let $p \in S_1(G)$ be consistent with the cut $+\infty$, contain $\varphi(x)$ and have the same Morley rank and degree. Let

$$Fix_{+\infty}(p) = \{g \in G : g + \alpha \models p \text{ for some } (\sim \text{ any}) \ \alpha \models p\}$$

It is easy to check that if $\varphi(x)$ has the least positive Morley rank and degree then $Fix_{+\infty}(p) = K_\varphi$.

Lemma 5.4. *Morley rank of the eventual fixer of a type in the cut $+\infty$ is not bigger than Morley rank of the type in this cut.*

Proof. If Morley rank of a type in the cut $+\infty$ is 1, then obviously, Morley rank of its eventual fixer is 1. If Morley rank of a type in the cut $+\infty$ is bigger than 1, then by Corollary 5.3 we can consider the quotient group G' of G modulo the least definable non-trivial group. By Lemma 1.12 G' is a ω-stable. Claim that the preimage of the fixer of the quotient type in G' will be the eventual fixer of the type in G. Since for ω-stable group the similar fact holds, the lemma follows. □

5.1. Examples of ordered strongly o-ω-stable groups.

Consider a divisible subgroup $\mathcal{K} = (K, +, -, \dots) = (K, L_\mathcal{K})$ of the group of reals, possibly with an extra structure, which has zero intersection with the group of rationals $\mathbb{Q}$. Let $G = \mathbb{Q} + K$. We may assume that the elementary theory $T_\mathcal{K}$ of $\mathcal{K} = (K, +, -, \dots)$ admits quantifier elimination.

We construct the following structure: $\mathcal{G} = (G, <, +, -, Q, \dots)$, where Q stands for unary predicate naming the set $\mathbb{Q}$ of rational numbers. For each relation R^n of $\mathcal{K}$ we define a relation with the same name R on G by

$$\mathcal{G} \models R^n(a_1, \dots, a_n) \iff \mathcal{K} \models R(a_1 + Q, \dots, a_n + Q)$$

By other words we define the structure of $\mathcal{K}$ on the quotient group G/Q.

Theorem 5.5. *The elementary theory T of $\mathcal{G}$ admits quantifier elimination.*

Proof. Let $\varphi(x, \bar{y})$ be a conjunction of formulae of the following forms and its negation:

(1) $nx = t$;
(2) $nx < t$;
(3) $t < nx$;
(4) $Q(nx + t)$;
(5) $\psi(n_1x + t_1, \dots, n_kx + t_k)$, where $\psi(z_1, \dots, z_k)$ is a formula of $L_{\mathcal{K}}$;

here t and t_i are terms in $\bar{y}$.

We are going to eliminate existential quantifier in the formula: $\exists x \varphi(x, \bar{y})$.

Since formulae of the form $(1) - (4)$ are stable under multiplication by a positive natural number, we may assume that n's in subformulae of φ of these form are the same. Obviously, $nx = t_1 \wedge nx = t_2$ is equivalent to $nx = t_1 \wedge t_1 = t_2$, and $\neg nx = t$ is equivalent to $nx < t \vee nx > t$. Thus we may assume that the formula φ contains at most one positive occurrence of a formula of the form (1). Also we may assume that the formula φ contains at most one positive occurrence of a formula of the form (2) and of the form (3). Indeed, $nx < t_1 \wedge nx < t_2$ is equivalent to $(nx < t_1 \wedge t_1 < t_2) \vee (nx < t_2 \wedge t_2 \leq t_1)$.

Consider $Q(nx + t_1) \wedge Q(nx + t_2)$. It is equivalent to $Q(nx + t_1) \wedge Q(t_2 - t_1)$. Hence, we may assume that the formula φ contains at most one subformula of the form (4).

Assume that φ contains a subformula of the form (1). Then we can replace nx with t in subformulae of the form $(2) - (4)$. Consequently, $\exists x \varphi(x, \bar{y})$ is equivalent to $(\exists x(nx = t \wedge \psi(n_1x + t_1, \dots, n_kx + t_k))) \wedge \theta(\bar{y})$. For simplicity of notation we omit θ. By the construction of $\mathcal{G}$ the formula $\psi(n_1x + t_1, \dots, n_kx + t_k)$ is equivalent to $\psi(n_1x + t_1 + q_1, \dots, n_kx + t_k + q_n)$ for arbitrary rational q_i. Then $\exists x(nx = t \wedge \psi(n_1x + t_1, \dots, n_kx + t_k))$ is equivalent to $\exists x(nx = t + q \wedge \psi(n_1x + t_1, \dots, n_kx + t_k))$ for arbitrary q, which in turn is equivalent to $\exists x(Q(nx - t) \wedge \psi(n_1x + t_1, \dots, n_kx + t_k))$, which can be considered as a formula $\exists x(nx = t \wedge \psi(n_1x + t_1, \dots, n_kx + t_k))$ of the language $\mathcal{L}_{\mathcal{K}}$ in $T_{\mathcal{K}}$. Since by our supposition $T_{\mathcal{K}}$ admits quantifier elimination, we can omit the existential quantifier.

Thus we may assume that φ does not contain a subformula of the form (1). Then $\varphi(x, \bar{y})$ has the following general form:

$$v_1 < nx < v_2 \wedge Q(nx, v_3) \wedge \bigwedge_i \neg Q(nx, w_i) \wedge \psi(n_1x + t_1, \dots, n_kx + t_k)$$

By the construction of $\mathcal{G}$ we may replace x with $x + q$ for any rational q. So the subformula $v_1 < x < v_2$ can be replaced with $v_1 < v_2$. Again by the construction of $\mathcal{G}$ we may replace considering a formula

$$\exists x \left(Q(nx, v_3) \wedge \bigwedge_i \neg Q(nx, w_i) \wedge \psi(n_1x + t_1, \dots, n_kx + t_k) \right)$$

with considering the following formula of $\mathcal{K}$

$$\exists x \left(nx = v_3 \wedge \bigwedge_i nx \neq w_i \wedge \psi(n_1 x + t_1, \dots, n_k x + t_k) \right)$$

Since $T_{\mathcal{K}}$ admits quantifier elimination, we may eliminate the existential quantifier.

Clearly, other cases are similar. So, T admits quantifier elimination. $\square$

Now let $\mathcal{K}$ be an arbitrary abelian torsion-free omega-stable group with cardinality at most continuum. Claim that here by an omega-stable group we mean totally transcendental group, not o-omega-stable group. Then K as a pure group is a direct product of copies of the set $\mathbb{Q}$ of rationals and the number of copies is equal to the cardinality of K. Let $\{a_\alpha : \alpha < |K|\}$ be a linearly independent over $\mathbb{Q}$ subset of the set of reals $\mathbb{R}$. Then there is an isomorphism τ of K as a pure group onto a group of the form $\sum_{\alpha<|K|} a_\alpha \cdot \mathbb{Q}$, which can be ordered by the natural ordering of reals.

Let $\mathcal{G}$ be constructed from $\tau(\mathcal{K})$ as above. The previous theorem implies the following:

Theorem 5.6. *Let the elementary theory $T_{\mathcal{K}}$ of $\mathcal{K}$ is ω-stable. Then the theory T is strongly o-ω-stable. Moreover if in $\mathcal{K}$ a formula $\varphi(x)$ has Morley rank n, then for any cut $\langle C, D\rangle$ of G Morley rank of the corresponding formula $\varphi(x)$ inside the cut $\langle C, D\rangle$ equals $n+1$.*

6. Unary definable function and predicates

Throughout this subsection G is an ordered group whose elementary theory is non-valuational o-stable and is not weakly o-minimal. We assume also that there is the least non-trivial definable subgroup G_1 of G. Claim that if G is o-ω-stable and non-valuational, then G_1 does exists. One more supposition is that the full induced structure on G_1 is a non-valuational weakly o-minimal structure. Claim that it cannot be o-minimal because for any $g \notin G_1$ the intersection $(-\infty, g) \cap G_1$ is not an interval in G_1.

Weak o-minimality of G_1 implies that for any cut $s = \langle C, D\rangle$ and any definable subset A eventually in the cut s the set A consists of cosets of G_1. Indeed $A \cap (g + G_1)$ is a finite union of convex sets. Hence either eventually in s the intersection $A \cap (g + G_1)$ is equal to $g + G_1$ or is empty. Let

$$B \triangleq A \setminus (\cup\{g + G_1 : A \cap (g + G_1) \text{ eventually in } s \text{ equals } g + G_1\})$$

Obviously, the intersection $B \cap (g + G_1)$ is empty for any g. It is sufficient now to show that eventually in s the set B is empty. If not we define function

$$f(g, c) \triangleq \sup B \cap (g + G_1) \cap (-\infty, c)$$

where g runs over G and c over C. Since the formula

$$\varphi(x, a, c) \triangleq x \in \cup\{g + G_1 : f(g, c) < a\}$$

has not the strict order property inside any cut the image of f is finite. It means that B is bounded in C, that is eventually in s it is empty.

We shall use the following fact, proved by R. Wencel:

Fact 6.1. [17] *Let M be an ordered non-valuational weakly o-minimal group. Then any definable unary function is piecewise strictly monotone and continuous, where pieces are convex.*

Obviously that this theorem is applicable to any coset of G_1:

Corollary 6.2. *The restriction of a definable function to any coset of G_1 is piecewise strictly monotone and continuous.*

Theorem 6.3. *Let G be an ordered non-valuational o-stable group with a non-valuational weakly o-minimal minimal non-trivial definable subgroup Any definable unary function is piecewise monotone and continuous, i.e. there is a finite definable partition of the group G such that the restriction of the function to any element of the partition is strictly monotone and continuous. Observe, it is not necessary that we partition G into convex sets.*

Proof. Let f be a definable unary function in G. Since any coset of G_1 is non-valuational weakly o-minimal, the restriction of f onto any coset of G_1 is piecewise monotone and continuous. Let $E(x, y)$ be an equivalence relation which says that $x - y \in G_1$ and the function f is monotone and continuous on the set $((x, y) \cup (y, x)) \cap (G_1 + x)$. Since $Th(G_1)$ is non-valuational weakly o-minimal there is a natural number n such that there are at most n equivalence classes of E on any coset of G_1. Observe that E-classes are convex in cosets of G_1.

We say that two cosets $[a]$ and $[b]$ of G_1 are of the same type if they have the same number of E-classes, the restriction of f to the first E-class in $[a]$ is strictly increasing iff the restriction of F to the first class in $[b]$ is strictly increasing, the same holds if the restriction of f is strictly decreasing or constant and the same holds for the second E-class, the third one, $\ldots$, and the last one. Clearly there are finitely many types of cosets of G_1 and these types are definable. So, for simplicity we may assume that all cosets of G_1 are of the same types.

We say that a coset $[a]$ of G_1 is less than a coset $[b]$ of G_1 if the supremum in G of the first E-class in $[a]$ is less than the supremum in G of the first E-class in $[b]$.

Consider a formula $\varphi(x; y)$ which says that the supremum of the first E-class in $x + G_1$ is less than y. Since the cut $+\infty$ has no the strict order property, the set of supremums of the first E-classes of all coset of G_1 is finite. So, we may assume that this set consists of a unique elements, as well as the set of supremums of the second E-classes, of the third E-classes and so on. Moreover we may assume that there is only one E-class in each coset of G_1.

Since G_1 is dense in G and the restriction of f to any coset of G_1 is strictly monotone and continuous, we may construct an extension f_a of the restriction of f on a coset $[a]$ of G_1 to the whole group G as follows. Let $a_i \to g$, where $a_i \in [a]$. Then $f_a(g) = \lim f(a_i)$.

If the restrictions of f and f_a to a coset $[b]$ of G_1 are not equal, there is a first convex set in $[b]$ where for any element c either $f(c) < f_a(c)$, or $f(c) > f_a(c)$. If the first inequality holds we say that the coset $[a]$ is bigger than the coset $[b]$. Since there is no strict order property inside the cut $+\infty$, there is a finite definable partition of G into $A_0, \ldots A_{m-1}$ such that for any $i < m$ and for any a, $b \in A_i$ it holds that the restriction of f on the coset $[b]$ of G_1 is equal to the restriction of f_a on the coset $[b]$.

Again we may assume that $m = 0$. Then f is continuous on G. We prove that f is strictly increasing on G. It follows from the above that f is increasing. Assume that there are a and b such that $f(a) = f(b)$. Since G_1 is dense in G there is an element $c \in a + G_1$ such that $a < c < b$. Then $f(a) = f(c)$, that contradicts to our supposition that the restriction of f on any coset of G_1 is strictly increasing.

If f is strictly decreasing or constant in any coset of G_1 then the proof is similar. □

Theorem 6.4. *In an ordered non-valuational o-stable group with a non-valuational weakly o-minimal minimal non-trivial definable subgroup the algebraic closure satisfies the exchange principle: if* $c \in \mathrm{acl}(Ab) \setminus \mathrm{acl}(A)$ *then* $b \in \mathrm{acl}(Ac) \setminus \mathrm{acl}(A)$.

Proof. Let $c \in \mathrm{acl}(Ab) \setminus \mathrm{acl}(A)$. Since in any ordered structure the algebraic closure coincides with the definable closure, there is an A-definable function f such that $f(b) = c$. By Theorem 6.3 there is a definable subset B containing the element b such that the restriction of f on B is continuous and is either strictly increasing, or strictly decreasing. It cannot be constant because then $c \in \mathrm{acl}(A)$. Obviously, the restriction of f on B is a bijection between B and $C = f(B)$. Then f^{-1} can be defined and $b = f^{-1}(c)$. The last implies that $b \in \mathrm{acl}(Ac)$. □

In [5] it was proved the following. Let T be stable one-based and let G be a group interpreted in T. The induced structure on G is of the following kind: G is abelian-by-finite, there are subgroups H_i, definable over $\mathrm{acl}(\emptyset)$, such that in G every formula is equivalent to a boolean combination of cosets mod the H_i's. Here we can apply this result for o-stable groups.

Theorem 6.5. *Let G be an ordered non-valuational o-stable group with a non-valuational weakly o-minimal minimal non-trivial definable subgroup G_1. Let the stable quotient group G/G_1 with the full induced structure be one-based. Then any definable subset of G is a finite union of cosets of definable subgroups intersected with definable convex sets.*

Proof. Let A be a definable subset of G, and $\langle C, D\rangle$ a cut in G such that A is not eventually empty at this cut. First we consider the case when this cut is $+\infty$. Then eventually A consists of cosets of G_1 and we can define the eventual quotient A/G_1 of the set A by the following way: $g + G_1 \in A/G_1$ iff eventually $g + G_1$ is a subset of A. Since G/G_1 is one-based, A/G_1 is a finite union of cosets of definable subgroups of G/G_1. Let B be the lift of the eventual quotient A/G_1 to G. So for each $g \in G$ eventually $B \cap (g + G_1)$ is equal to $A \cap (g + G_1)$. Let

$$f(g) = \inf\{a \in G : B \cap (g + G_1) \cap (a, \infty) = A \cap (g + G_1) \cap (a, \infty)\}$$

Since there is no the strict order propery inside the cut $+\infty$ witnessed by the formula $\varphi(x, a) \triangleq x \in \bigcup_{f(g)<a} g + G_1$ the image of f must be finite. So there is a definable convex unbounded set U such that $A \cap U = B \cap U$.

Now assume that $\langle C, D\rangle$ is of the form $(a - 0, a)$. Since eventually in $(a - 0, a)$ the set A consists of cosets of G_1 we can repeat the above arguments.

Thus in some neighbourhood of a point in G set A equals the finite union of cosets of definable groups of G.

Let E be an equivalence relation which says that each equivalence class is a maximal convex set such that for any a and b from this set eventually in the cut $(a - 0, a)$ the set A is equal to $A + a - b$, that is the eventual in $(a - 0, a)$ quotient A/G_1 is equal to the eventual in $(b - 0, b)$ quotient A/G_1. Since the group G is non-valuational, the equivalance relation may contain at most finitely many infinite E-classes. Claim that since G is dense each finite E-class consists of one element. If there are infintely many one-element E-classes then there is an open interval consisting of one-element E-classes. Let a be in this interval. Then by the above arguments there is $b < a$ such that $A \cap (b,a) = B \cap (b,a)$, implying that (b,a) is a subset of an E-class. Thus E has finitey many equivalence classes and the theorem is proved. $\square$

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