The Finiteness Result à la Bogomolov

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Abstract

This paper reduces the Zilber-Pink conjecture to proving that weakly optimal, resp. optimal, points which are maximal with respect to inclusion among weakly optimal, resp. optimal, proper subvarieties are not Zariski dense.

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Introduction

Prerequisites and Setup

Let ƨ be any connected mixed Shimura variety over $\mathbb{C}$ and let V be some irreducible $\mathbb{C}$subvariety of ƨ. Endow ƨ with both the special pregeometry and the weakly special pregeometry, then define both the weakly and the special defect: [CassaniTDC] explains this in details.

Relevant History and Related Literature

Simpler cases of the THEOREM appeared originally in [HP16:OM] (for abelian varieties and powers of the level two modular curve), then in [DR18:HAS] (for pure Shimura varieties), where they reduce the finiteness of weakly optimal subvarieties (called “geodesic optimal” in [HP16:OM]) to the finiteness of the corresponding points.

I provide a twofold improvement over these results from the literature by using many of the same ingredients as in the literature, coupled with some extra “inductive” machinery:

(≎) the THEOREM is formulated for mixed Shimura varieties,

(≎) the THEOREM requires weaker conditions and provides stronger conclusions.

The THEOREM is somewhat similar to [BD21:DC], which was proven independently and simultaneously.

Preliminary Construction and statement of the LEMMA

Denote by $\langle\rangle_{ƨ}$ and by $\langle\rangle_{ƨa}$ the weakly special closure and the special closure, respectively. Denote

$$\delta_{ƨa}(V) \coloneqq \dim\langle V\rangle_{ƨa} - \dim(V),\ \delta_{ƨ}(V) \coloneqq \dim\langle V\rangle_{ƨ} - \dim(V).$$

Fix any connected mixed Shimura ƨ, any irreducible $\mathbb{C}$subvariety V of ƨ, and any two morphisms of connected mixed Shimura varieties ƨ”←ƨ’:r and m:ƨ’→ ƨ where m is a Shimura immersion.

$$ƨ'' \overset{r}{\leftarrow} ƨ' \overset{m}{\rightarrow} ƨ$$

Suppose the weakly special closure $\langle A\rangle_{ƨ}$ of some irreducible $\mathbb{C}$subvariety A of V arises as some irreducible $\mathbb{C}$component of the fibre $mr^{-1}(n_a)$ for some $n_a \in$ ƨ”.

Pick any irreducible component A’ of $m^{-1}(A)\cap r^{-1}(n_a)$ whose dimension equals $\dim(m^{-1}(A)\cap r^{-1}(n_a))$. Fix V’ any irreducible $\mathbb{C}$component of $m^{-1}(V)$ which contains A’. Denote by V” the Zariski closure of r(V’), as in the diagrams below:

$$V'' \overset{r}{\leftarrow} V' \overset{m}{\to} V.$$

LEMMA. Then V'' contains n_a and is such that the following happens: fix some nonempty open therefore dense $U'_\circ$ in V'' such that r restricted to $r^{-1}(U'_\circ)\cap V'$ is flat. Denote by q the restriction of r to V'. If A is not included in $W=\langle m(V'\backslash q^{-1}(U'_\circ))\rangle Zar$ and is weakly optimal, resp. optimal, in V then n_a is weakly optimal, resp. optimal, in V'' (which I defined above as being the Zariski closure of V').

COROLLARY C.

Fix any connected mixed Shimura 2, any irreducible ℂsubvariety V of 2, and any two morphisms of connected mixed Shimura varieties 2''←2':σ and [i]:2'→ 2 where i is a Shimura immersion.

Then there's some finite set of indexes J which correspond to all irreducible components of $[i]^{-1}(V)$, and for each j in J denote by q the restriction of σ to V'(j). Also denote some other finite collection of parameters i which correspond to all irreducible components of $\langle [i](V'\backslash q^{-1}(U'_\circ))\rangle Zar$ (same notations as in the LEMMA), which parametrise varieties $V_o''(j)$,$W_o(i(j))$ which correspond respectively to the Zariski closures of the image by σ, and to the irreducible components of $W_o(j)$ as defined in the LEMMA, such that for any n_o (in the image of σ) and for any irreducible subvariety A of V such that the weakly special closure of A arises as irreducible component of $[i]\sigma^{-1}(n_o)$, there's some j in J which verifies both the following:

(1*) $V_o''(j)$ contains n_o,

(2*) if A is not included in any of the $W_o(i(j))$ for this particular j, then n_o is weakly optimal, resp. optimal, in $V_o''(j)$ whenever A is weakly optimal, resp. optimal, in V.

Theorem and Applications

We aim at applying pointcounting techniques and therefore chase statements that relate the distribution of weakly optimal, resp. optimal, subvarieties to the corresponding points.

Let 2 be any connected mixed Shimura variety and let V be any irreducible subvariety. Denote the following statements:

ZP(2,V,k,p): only finitely many weakly optimal, resp. optimal, subvarieties A of V whose difference in dimension between the weakly special closure and the special closure is bounded by p, and dim A=k, arise.

MZZP(2,V,0,p): the weakly optimal, resp. optimal, points A of V which verify $\delta_{2a}(P)\leq p$, which are maximal with respect to inclusion among weakly optimal, resp. optimal, proper subvarieties of V whose difference in dimension between the weakly special closure and the special closure is bounded by p, are not Zariski dense in V.

MP(Ƨ,V,0,p): the weakly optimal, resp. optimal, points A of V which verify $\delta_{Ƨa}(P)\leq p$, which are maximal with respect to inclusion among weakly optimal, resp. optimal, proper subvarieties of V whose difference in dimension between the weakly special closure and the special closure is bounded by p, are finitely many in V.

Apply to V both [CassaniTDC] and the "finiteness statement à la Bogomolov" and denote by σ the morphisms of connected mixed Shimura varieties which correspond to the finite set Σ from [G20:MAS, 8.2] (which only depend on V): it follows from [G17:AOMS, 5.6] that the weakly special closure $\langle A\rangle_2$ of any weakly optimal, resp. optimal, irreducible ℂsubvariety A of V, appears as the irreducible ℂcomponent of $[i]\sigma^{-1}(n_a)$ for some n_a and for some σ, where [i] is the morphism of connected mixed Shimura varieties which corresponds to the inclusion of connected mixed Shimura data, and σ is the morphism of connected mixed Shimura varieties which corresponds to some quotient of connected mixed Shimura data. All in all, [i] is a Shimura immersion and σ is a Shimura quotient. It follows from Pink's Thesis, 3.8, that [i] is finite and closed and preserves dimensions.

For each σ in Σ construct the i,j and Ƨ" as in COROLLARY C (they depend on σ!). Denote by "B" any positive dimensional proper weakly optimal, resp. optimal, subvariety of V which is maximal with respect to inclusion. Denote by Z the components of the Zariski closure of weakly optimal, resp. optimal, points A of V which are maximal with respect to inclusion among weakly optimal, resp. optimal, proper subvarieties of V.

THEOREM. Remark that there's finitely many σ,i,j by construction. Assume the following for all B,Z and for all σ such that dim Ƨ"<dim Ƨ, and their corresponding i,j:

(*1) ZP(Ƨ,W_o(i(j)),k,p),

(*2) ZP(Ƨ",V_o"(j),0,p),

(*3) ZP(Ƨ,dim B,0,p),

(*4) MP(Ƨ,Z,0,p).

Then ZP(Ƨ,V,k,p) holds.

The statement of the THEOREM suggests proving ZP(Ƨ,V,k,p) by recursion, where the hypothesis of the recursion is represented by the four stars above and the inductive step is

(*5) MZZP(Ƨ,V,0,p).

In fact:

(**1) $W_o(i(j))$ is always proper in V,

(**2) dim 2”<dim 2 by construction,

(**3) each B is proper in V,

(**4) each Z is proper in V by (*5).

COROLLARY. A. Suppose that MZZP holds recursively for all objects appearing in the statement of the THEOREM (remark further that 2” is obtained as a quotient of 2!): then ZP(2,V,k,p) holds.

Denote the following statement, which clearly alludes to Pink’s Conjecture:

PINK*(2,V,0,p): suppose that $\delta_{2a}(V)>k$. Then the points A of V which verify $\delta_{2a}(A)\leq p$, which are maximal with respect to inclusion among all weakly optimal, resp. optimal, proper subvarieties of V whose difference in dimension between the weakly special closure and the special closure is bounded by p, are not Zariski dense in V. Then:

COROLLARY. B. You can replace MZZP(2,V,0,p) by PINK*(2,V,0,p) in the statement of COROLLARY A.

The proof of THEOREM is achieved essentially via the “finiteness statement à la Bogomolov” as was proven by [G20:MAS], which says essentially that the weakly special closures of all weakly optimal subvarieties of V arise from finitely many families of weakly special subvarieties. On the other hand it must be noted that such “finiteness statement à la Bogomolov” does not provide any additional information when points are concerned: it follows that “Bogomolov” cannot be exploited further to obtain any refined version of the THEOREM above. As a consequence the THEOREM is essentially equivalent to the “finiteness statement à la Bogomolov” and this paper essentially represents a study of the latter.

Some attentive reader might wonder if anything is special about the order by which all intermediate results are shown in the proof before the THEOREM is achieved. Why is the order as follows?

(*) Reduction to points,

(*) reduction to maximality with respect to inclusion,

(*) reuction to Zariski nondensity,

Reducing to points first improves the exposition and does not affect the result. After reducing to points, the order between reduction to maximality and reduction to Zariski nondensity is indifferent: their application yields the same result.

Limitations and Areas of Further Investigation

The LEMMA (and COROLLARY C) which is part of the proof of the THEOREM can be probably made stronger, for example by adding to the properties "(∗)" in the statement, also the following:

(∗) weakly optimal proper subvariety of V and maximal with respect to inclusion among weakly optimal proper subvarieties of V.

However, given the strategy I decided to follow in the proof, this would be a useless complication.

I apply the Finiteness Statement to weakly optimal subvarieties A because this is how it was proven by [G20:MAS]. However it's apparent (at least from the statement) that, when operating the reduction to points, the morphisms which Bogomolov produces factor out the weakly special closure of A. This explains why only the difference between the special defect and the weakly special defect appears in the statement of my THEOREM. This behaviour is structural because only weakly special subvarieties arise in family. It follows that it would make much more sense to twist the concept of weak optimality so that it neglects whatever lies inside the weakly special closure: only then one would obtain the most natural Bogomolov statement. For instance this can be obtained as follows: replace in the definition of weak optimality $\dim\langle A\rangle_2$ by $\dim\langle A\rangle_{2a}$ and dim A by $\dim\langle A\rangle_2$. Incidentally this frees us up of the constant copresence of weak optimality and optimality. Then only the following will be needed:

(∗) prove Bogomolov for this new concept of (weak) optimality,

(∗) formulate all statements and conjectures as in this paper,

(∗) replay the proofs in this paper to get the corresponding results.
Notice that this new notion coincides with the old one for points, the same way that "whose weakly special closure is special" coincides with "special" for points.

Perhaps the theorems achieved this way bear less interest because the literature focuses on the classic notions of weak optimality and of optimality. However only by using this new notion then one can really express Bogomolov in its natural context.

Acknowledgments

Some technicalities in the proof of the LEMMA were overcome following a fruitful conversation with G. A. Dill.

Proof of the THEOREM

Proof of the THEOREM by using COROLLARY C

For the sake of clarity, in my proof I will ignore the hypothesis and the conclusion about the bound on the difference in dimension between the weakly special defect and the special defect: however the version stated in THEOREM follows imediately by replaying this proof and using both that the image of some special subvariety by any morphism of connected mixed Shimura varieties is special, together with the fact that all irreducible components of the preimage of a special subvariety by some Shimura morphism are special.

Reduction to weakly optimal, resp. optimal, points in V

Keep the same notations as in the statement. Apply COROLLARY C. to each $\sigma\in\Sigma$: it follows that for each weakly optimal, resp. optimal, irreducible subvariety A of V there is some $\sigma(A)$ in Σ and some n_o which verify the hypotheses of COROLLARY C: therefore there's some j in $J(\sigma)$ which verifies the conclusions of the COROLLARY C.

Now consider the collection of all weakly optimal, resp. optimal, irreducible subvarieties A of V such that there's no σ,j,n_o in $\Sigma,J(\sigma),\sigma(V)$ which verify (1*) from the conclusion of COROLLARY C and such that n_o is weakly optimal, resp. optimal, in $V_o''(j)$. Then it must be that each such A is included in $W_o(i(j))$ for some i,j,σ. This is where (*1) originates from.

Now focus on the remaining weakly optimal, resp. optimal, irreducible subvarieties A of V. Remark that for any such A, there's by definition some σ,j,n_o in $\Sigma,J(\sigma),\sigma(V)$ which verify (1*) from COROLLARY C and n_o is weakly optimal, resp. optimal, in $V_o''(j)$.

Remark now that each weakly optimal, resp. optimal, irreducible subvariety A of V is automatically an irreducible ℂcomponent of $\langle A\rangle_2 \cap V$, resp, $\langle A\rangle_{2a} \cap V$, therefore finiteness of some collection of $\langle A\rangle_2$, resp. $\langle A\rangle_{2a}$, is equivalent to finiteness of the corresponding collection of A's. This is where (*2) originates from.

Reduction to maximality with respect to inclusion in V

Now concentrate on weakly optimal, resp. optimal, points in V (which certainly cover the remaining A's): pick any such point P. Fix some B (same notations as in the statement of the THEOREM) containing P. Then P is weakly optimal, resp. optimal, for B. Every such B we already came across in the section above. This is where (*3) originates from.

Further reduction from finiteness to Zariski nondensity

Now focus on weakly optimal, resp. optimal, points A of V which are maximal with respect to inclusion among weakly optimal, resp. optimal, proper subvarieties of V (which certainly cover the remaining A's) and denote by Z any irreducible component of their Zariski closure. Because the number of such Z is always finite, then this is where (*4) originates from.

The THEOREM is proven.

Proof of the LEMMA

I will use the symbol * as a place holder: replacing * by $_2$ yields the proof for the property "weakly optimal", whereas replacing it by $_{2a}$ yields the proof for the property "optimal".

We need to fix in the following specific order:

(⋄) some irreducible $\mathbb{C}$subvariety A' of $m^{-1}(V)$ which verifies the following properties which are needed for the proof:

(⋄⋄⋄⋄⋄) A' is such that m(A') dominates A,

(⋄⋄⋄⋄⋄⋄) $\langle A'\rangle_2$ is an irreducible component of $r^{-1}(n_a)$ and $m\langle A'\rangle_2$ is included in $\langle A\rangle_2$.

(⋄⋄⋄⋄) A' is contained in the fibre of r over n_a.

As suggested by the statement, choose the following A': consider $m^{-1}(A)\cap r^{-1}(n_a)$ and remark that because m restricted to $r^{-1}(n_a)$ surjects onto A (because $\langle A\rangle_2$ is included in $mr^{-1}(n_a)$) then $m(m^{-1}(A)\cap r^{-1}(n_a))$ = A.

Pick some irreducible component A' of $m^{-1}(A)\cap r^{-1}(n_a)$ whose dimension equals $\dim(m^{-1}(A)\cap r^{-1}(n_a))$: A' $\subseteq m^{-1}(V)$ and is irreducible by definition, let's verify that such A' is suitable.

Because m is finite and closed and therefore it preserves dimensions, then dim m(A') = dim A' = dim $m^{-1}(A)\cap r^{-1}(n_a)$ = dim $m(m^{-1}(A)\cap r^{-1}(n_a))$ = dim A: it follows from the fact that m(A')$\subseteq$ A (by how we defined A') that m(A') = A.

Suppose that $\langle A'\rangle_2$ (which is irreducible contained in $r^{-1}(n_a)$) were strictly contained in some component K: then dim $\langle A\rangle_2$ = dim $\langle mA'\rangle_2 \leq$ dim $m\langle A'\rangle_2 \leq$ dim $\langle A'\rangle_2 <$ dim K = dim m(K) (because m is finite closed and preserves dimensions) which is irreducible and closed because m is finite closed and preserves dimensions, contained in $mr^{-1}(n_a)$ contradiction.

Finally use again that m is finite closed and preserves dimensions: $m\langle A'\rangle_2$ is closed, irreducible, contained in $mr^{-1}(n_a)$ and contains $\langle A\rangle_2$: it follows that $m\langle A'\rangle_2=\langle A\rangle_2$.

(⋄) After A' is fixed, then we can fix the V' from the statement. We need for the proof that

(⋄⋄⋄) V' contains A'.

Therefore, as suggested by the statement:

fix V' any irreducible ℂcomponent of $m^{-1}(V)$ which contains A' (such V' is not guaranteed to dominate V but it doesn't matter).

Denote by q the restriction of r to V', that is the map $V'' \overset{q}{\leftarrow} V'$. Consider $\langle r(V')\rangle Zar$ and suppose that $q^{-1}(n_a)$ -which is nonempty because because n_a is contained in $r(V')$- is not included in $q^{-1}(U'_\diamond)$, which is open in V' and nonempty since q is dominant by construction: therefore $q^{-1}(n_a)$ is contained in the smaller dimensional $V'\backslash q^{-1}(U'_\diamond)$. It follows from ⋄⋄⋄,⋄⋄⋄⋄,⋄⋄⋄⋄⋄ that A is contained in $\langle m(V'\backslash q^{-1}(U'_\diamond))\rangle Zar$.

To prove the statement assume that A is not contained in $\langle m(V'\backslash q^{-1}(U'_\diamond))\rangle Zar$: then $n_a \in U'_\diamond$.

Consider any irreducible ℂsubvariety C of V'' containing n_a and such that $\delta*(C)\leq\delta*(n_a)$, then fix B any irreducible ℂcomponent of $q^{-1}(C)$ which contains A': this is made possible by ⋄⋄⋄ and ⋄⋄⋄⋄.

Consider q restricted to $q^{-1}(U'_\diamond\cap C)$, which is the base change of a flat (by how we defined $U'_\diamond$) morphism, therefore is flat. From the two facts that it's the base change and is flat, it follows that

(£) every irreducible ℂcomponent of $q^{-1}(U'_\diamond\cap C)$ has dimension dim $U'_\diamond\cap C$+dim V'−dimV'' (see Corollary 9.6 and Proposition 10.1 Hartshorne).

Instead it follows just from flatness of such base change (or, alternatively, of the original r restricted to $q^{-1}(U'_\diamond)$) that

(££) any such irreducible ℂcomponent dominates $C\cap U'_\diamond$ (see EGA Ⅳ.2, Proposition 2.3.4 (ⅲ)).

Now $U'_\diamond\cap C$ is open nonempty in C beause contains n_a, therefore dense in C and dim $U'_\diamond\cap C$= dim C. Moreover, dim B = dim B ∩ $q^{-1}(U'_\diamond)$ and it's immediate that B ∩ $q^{-1}(U'_\diamond)$ is an irreducible ℂcomponent of $q^{-1}(U'_\diamond\cap C)$. It follows from (£) and (££) both that

(¬) B dominates C,

and

(∗∗∗∗) dim B = dim C + dim V' − dimV''.

Now use the hypothesis $\delta*(C)\leq\delta*(n_a)$ and (¬) to compute

(££££) $\delta*(B)=(\dim\langle B\rangle*-\dim\langle C\rangle*+\delta*C+(\dim C-\dim B)\leq(\dim\langle B\rangle*-\dim\langle q(B)\rangle*+\delta*(n_a)+(\dim\langle q(B)\rangle Zar-\dim B)$.

Now:

(+) $\langle q(B)\rangle*=q\langle B\rangle*$ by [CassaniTDC],

(+) $\delta*(n_a)\leq\dim\langle A\rangle*-\dim\langle A\rangle_2$ because for $*=2$ it's clear. Whereas for $*$ the special defect then $\dim\langle A'\rangle-\dim\langle n_a\rangle=\dim q^{-1}(n_a)\cap\langle A'\rangle$(by the LEMMA in [CassaniTDC]) $\geq\dim\langle A'\rangle_2$ (by ⋄⋄⋄⋄, ⋄⋄⋄) $\geq\dim\langle A\rangle_2$ (by ⋄⋄⋄⋄⋄),

(+) $\dim\langle q(B)\rangle Zar-\dim B\leq\dim V''-\dim V'$ follows from (****),

(+) $\dim\langle B\rangle*-\dim q\langle B\rangle*\leq\dim\langle A\rangle_2$ because $\dim\langle A\rangle_2\geq\dim\langle A'\rangle_2$ (by ⋄⋄⋄⋄⋄⋄) $\geq\dim\langle B\rangle*-\dim q\langle B\rangle*$ (by the fibre dimension theorem and because by ⋄⋄⋄⋄⋄⋄ $\langle A'\rangle_2$ is an irreducible component of $r^{-1}(n_a)$, therefore is an irreducible component of the fibre of the restriction to $\langle B\rangle*$ over n_a).

(+) we have by (⋄⋄⋄⋄⋄), (⋄⋄⋄⋄) and because $n_a\in U'_0$ that $\dim A\leq\dim V'-V''$.

Now plug all the bulletpoints above in (££££): it follows that $\delta*(B)\leq\delta*(A)$. Now use that m is finite closed therefore preserves dimensions, and that A is weakly optimal, resp. optimal, to conclude that m(B) = A. Because B contains A', then B=A' otherwise dim B > dim A' then dim A = dim m(B) (by what we just proved) =dim B (because m is finite closed therefore preserves dimensions) > dim A' = dim m(A') (because m is finite closed therefore preserves dimensions) ≥ dim A (by ⋄⋄⋄⋄⋄). It follows that $n_a=\langle q(A')\rangle Zar$ (by ⋄⋄⋄⋄) $=\langle q(B)\rangle Zar=C$ (by ¬).

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