The Defect Condition

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Abstract

This paper proves the Defect Condition, which clarifies the relationship between the weakly special subvarieties and the special subvarieties of any ambient variety. The THEOREM suggests that properties which can be expressed by the weakly special structure automatically prove the corresponding for the special structure. This result explains why special subvarieties are often favoured when studying problems about atypical intersections, but also facilitates numerous applications to statements which are only proven for weakly special subvarieties.

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Introduction

Prerequisites and Setup

Any object $\mathcal{Z}$ in the category of connected mixed Shimura varieties over $\mathbb{C}$ comes equipped with two collections of irreducible sub$\mathbb{C}$varieties. The “special collection” on $\mathcal{Z}$ is essentially the natural structure on $\mathcal{Z}$ inherited from the category of connected mixed Shimura varieties over $\mathbb{C}$: precise definitions arise in the excellent [G17:ALW]. The “weakly special” collection represents all translates of special elements, and [HP16:OM] provides an explicit description in the particular case of abelian varieties and of powers of the modular curve.

Relevant History and Related Literature

Particular cases of the DEFECT CONDITION appeared originally in [HP16:OM, Proposition 4.3] (for abelian varieties and powers of the modular curve), then in [DR18:HAS, Proposition 4.4] (for pure Shimura varieties). The motivation in both cases was to reduce the Zilber-Pink conjecture to the corresponding problem of optimal pointcounting for then to be attacked through the Pila-Zannier strategy: the analogue of my DEFECT CONDITION kicks in for applying to optimal subvarieties some suitable version of the “finiteness result à la Bogomolov”, which is classically stated (and proven) only for weakly optimal subvarieties (called “geodesic optimal” in [HP16:OM]). The DEFECT CONDITION from this paper is stated in the context of mixed Shimura varieties, therefore generalises the formulations from both [HP16:OM] and [DR18:HAS] (thus proving [HP16:OM, Conjecture 4.4]) and paves the way to implementing the strategy described above in this setting, which I do in [CassaniFRB] and [CassaniPZS] (the latter with some modifications due to the stronger results which my [CassaniFRB] produces when compared to the analogous theorems from the literature).

The DEFECT CONDITION was proven independently and simultaneously by [BD21:DC].

Theorem and Applications

The existence ([P05:MLAO]) of the weakly special and of the special closures

$$\langle\rangle_{\mathcal{Z}} \text{ and } \langle\rangle_{\mathcal{Z}_a} \text{ respectively,}$$

paves the way to formulating “weakly atypical” and “atypical” properties: those which can be framed in terms of the weakly special collection and in terms of the special collection, respectively. Consider in fact the following two “defects”, which measure how well the special collection and the weakly special collection approximate some irreducible $\mathbb{C}$subvariety V of 2:

$$\delta_{2a}(V) \coloneqq \dim\langle V\rangle_{2a} - \dim(V), \ \delta_{2}(V) \coloneqq \dim\langle V\rangle_{2} - \dim(V).$$

Fix V any irreducible sub$\mathbb{C}$variety of 2: asserting that only finitely many irreducible $\mathbb{C}$subvarieties of V which are maximal with respect to the following order (we call them “optimal subvarieties”) exist, is an example “atypical property”:

$$X \leq Y \text{ exactly when } X \subseteq Y \text{ and } \delta_{2a}(Y) \leq \delta_{2a}(X).$$

[P05:MLAO, Lemma 3.2] allows for expressing the relationship between the weakly atypical and the atypical properties, which is captured by the DEFECT CONDITION below. This reveals that weakly atypical properties automatically prove the corresponding atypical property, therefore suggesting that the weakly special collection really is more powerful but the special collection is easier to deal with (for instance when arithmetic techniques intervene, because only the special subvarieties are defined over the field of algebraic numbers).

THEOREM (Defect Condition). Let A and B be two irreducible $\mathbb{C}$subvarieties of 2 and assume that A is included in B. Then:

$$\dim\langle A\rangle_{2a} - \dim\langle A\rangle_{2} \geq \dim\langle B\rangle_{2a} - \dim\langle B\rangle_{2}.$$

In particular,

$$\delta_{2a}(B) - \delta_{2a}(A) \leq \delta_{2}(B) - \delta_{2}(A).$$

The following LEMMA is central to the proof of the DEFECT CONDITION:

LEMMA. Fix any morphism [f] of connected mixed Shimura varieties over $\mathbb{C}$. The fibre dimension of [f] is constant.

The proof is easy and goes as follows. By [P06:MLAO, Proposition 2.8] then [f] factors in $[f]=[i']\circ[\psi]\circ[\pi]$, where

(A) $[\pi]$ is a quotient Shimura morphism, therefore the dimension is constant over the collection of irreducible $\mathbb{C}$components of all fibres,

(B) $[\psi]$ is a Shimura covering, therefore finite

(C) $[i']$ is a Shimura immersion, therefore finite.

By (B) + (C) then $[i']\circ[\psi]=[i'\circ\psi]$ is finite: it follows that every irreducible component of a fibre of [f] is also an irreducible $\mathbb{C}$component of a fibre of $[\pi]$, apply (A).

Limitations and Areas of Further Investigation

It would be nice to prove the analogous of LEMMA for morphisms between weakly special subvarieties. This requires extending [P06:MLAO, Proposition 2.8] to the weakly special case, I think it shouldn't be difficult.

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Proof of the DEFECT CONDITION

Keep the same notations as in the statement. Fix $[i]:T'\to\langle B\rangle_{2a}$ and $[\psi]:T'\to T''$ morphisms of connected mixed Shimura varieties such that $\langle B\rangle_2$ is an irreducible $\mathbb{C}$component of $\langle[i][\psi]^{-1}(t'')\rangle Zar$:

$$\langle B\rangle_{2a} \leftarrow T' \to T'' \ni t''.$$

Apply [P06:MLAO, 4.4] and [P89:ACMS, 3.8]: you can assume that

(A) $[i]$ is finite and closed and preserves dimensions,

(B) $[\psi]$ is some Shimura quotient, therefore the fibre dimension is constant over T''.

Fix some B' in T' as follows.

PRELIM. Consider $[\psi]^{-1}(t'')\cap[i]^{-1}(B)$ and remark that because $[i]$ restricted to $[\psi]^{-1}(t'')$ surjects onto B (because $\langle B\rangle_2$ is included in $[i][\psi]^{-1}(t'')$) then $[i]([i]^{-1}(B)\cap[\psi]^{-11}(t'')) = B$.

DEF. Pick some irreducible component B' of $[i]^{-1}(B)\cap[\psi]^{-1}(t'')$ whose dimension is maximal: B' is irreducible by definition.

PROP1. Because $[i]$ is finite and closed and therefore it preserves dimensions, then $\dim [i](B') = \dim B' = \dim [i]^{-1}(B)\cap[\psi]^{-1}(t'') = \dim [i][i]^{-1}(B)\cap[\psi]^{-1}(t'') = \dim B$: it follows from the fact that $[i](B')\subseteq B$ (by how we defined B') that $[i](B') = B$.

PROP2. Suppose that $\langle B'\rangle_2$ (which is irreducible contained in $[\psi]^{-1}((t'')$) were strictly contained in some component K: then $\dim \langle B\rangle_2 = \dim \langle[i]B'\rangle_2 \leq \dim [i]\langle B'\rangle_2 \leq \dim \langle B'\rangle_2 < \dim K = \dim [i](K)$ (because $[i]$ is finite closed and preserves dimensions) which is irreducible and closed because $[i]$ is finite closed and preserves dimensions, contained in $[i][\psi]^{-1}(t'')$ contradiction.

PROP3. Finally use again that $[i]$ is finite closed and preserves dimensions: $[i]\langle B'\rangle_2$ is closed, irreducible, contained in $[i][\psi]^{-1}(t'')$ and contains $\langle B\rangle_2$: it follows that $[i]\langle B'\rangle_2=\langle B\rangle_2$.

Now compute always keeping in mind that $[i]$ is finite closed therefore preserves dimensions:

$$(£)\ \delta(B)-\delta_2(B)=\dim\langle B\rangle_{2a}-\dim\langle B\rangle_2 \leq \dim\langle B'\rangle_{2a}-\dim\langle B'\rangle_2,$$

by PROP 1 and PROP 3, together with the fact that the image of any special subvariety is special (however applying [P05:MLAO, 3.2] yields an equality).

Now the first inequality below follows from the fibre dimension theorem and PROP2 (however the LEMMA yields an equality), whereas the equality follows from [P05:MLAO, 3.2]:

$$(££)\ \dim\langle B'\rangle_{2a}-\dim\langle B'\rangle_2 \leq \dim[\psi]\langle(B')\rangle_{2a}=\dim\langle[\psi](B')\rangle_{2a}.$$

PRELIM*. Consider $B'\cap[i]^{-1}(a)$ and remark that because [i] restricted to B' surjects onto A (by PROP1 above and because A is included in B) then $[i]([i]^{-1}(A)\cap B')=A$.

DEF*. Pick some irreducible component A' of $[i]^{-1}(A)\cap B'$ whose dimension is maximal: A' is irreducible by definition.

PROP1*. Because [i] is finite and closed and therefore it preserves dimensions, then $\dim[i](A')=\dim A'=\dim[i]^{-1}(A)\cap B=\dim[i][i]^{-1}(A)\cap B'=\dim A$: it follows from the fact that $[i](A')\subseteq A$ (by how we defined A') that $[i](A')=A$.

Now use [P05:MLAO, 3.2] together with PROP1* to conclude that $\dim\langle A\rangle_{2a}=\dim\langle A'\rangle_{2a}$. Also, the fact that the image of some weakly special is weakly special implies the following (however applying [P05:MLAO, 3.2] yields an equality):

$$(£££££)\ \delta(A)-\delta_2(A)=\dim\langle A\rangle_{2a}-\langle A\rangle_2 \geq \dim\langle A'\rangle_{2a}-\dim\langle A'\rangle_2.$$

Now the first inequality below follows from the LEMMA together with the fact that A' is included in B', and cannot be replaced by an equality (this is really where the fact that A is possibly strictly smaller than B comes into play).

$$(££££)\ \dim\langle A'\rangle_{2a}-\dim\langle A'\rangle_2 \geq \dim[\psi]\langle(A')\rangle_{2a}$$

and the next inequality follows form the fact that the image of a special subvariety is special (however [P05:MLAO, 3.2] yields an equality here):

$$(£££)\ \dim[\psi]\langle(A')\rangle_{2a} \geq \dim\langle[\psi](A')\rangle_{2a}.$$

Now use that $[\psi](A')=[\psi](B')$ and chain together (£) to (£££££) (in this order!).

References

[G17:ALW] Ziyang Gao (2017), Towards the André-Oort conjecture for mixed Shimura varieties: The Ax-Lindemann theorem and lower bounds for Galois orbits of special points, Journal für die reine und angewandte Mathematik (Crelles Journal).

[BD21:DC] F. Barroero and G. A. Dill, Distinguished categories and the Zilber-Pink Conjecture, preprint, 2021.

[P89:ACMS] Richard Pink (1989), Arithmetical Compactification of Mixed Shimura Varieties, Bonner Mathematische Schriften.

[HP16:OM] Philipp Habegger, Jonathan Pila (2016) O-minimality and certain atypical intersections, Annales scientifiques de l'École normale supérieure.

[DR18:HAS] Christopher Daw, Jinbo Ren (2018), Applications of the hyperbolic Ax-Schanuel conjecture, Compositio Mathematica.

[P06:MLAO] Richard Pink (2006), Combination of the Conjectures of Mordell-Lang and André-Oort, Progress in Mathematics - Geometric Methods in Algebra and Number Theory.

[CassaniFRB] Cassani, The Finiteness Result à la Bogomolov, eprint.

[CassaniPZS] Cassani, The Pila-Zannier Strategy, eprint.

[P05:MLAO] Richard Pink, A Common Generalization of the Conjectures of André-Oort, Manin-Mumford, and Mordell-Lang, eprint.