The Hart-Shelah example, in stronger logics.

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Abstract

We generalize the Hart-Shelah example [1] to higher infinitary logics. We build, for each natural number $k \geq 2$ and for each infinite cardinal λ, a sentence ψ_k^λ of the logic $L_{(2^\lambda)^+,\omega}$ that (modulo mild set theoretical hypotheses around λ and assuming $2^\lambda < \lambda^{+m}$) is categorical in $\lambda^+, \dots, \lambda^{+k-1}$ but not in $\beth_{k+1}(\lambda)^+$ (or beyond); we study the dimensional encoding of combinatorics involved in the construction of this sentence and study various model-theoretic properties of the resulting abstract elementary class $\mathcal{K}^*(\lambda, k) = (\mathrm{Mod}(\psi_k^\lambda), \prec_{(2^\lambda)^+,\omega})$ in the finite interval of cardinals $\lambda, \lambda^+, \dots, \lambda^{+k}$.

Keywords: Model theory, infinitary logic, categoricity, abstract elementary classes.

The study of categoricity transfer has been central to model theory since Morley's theorem; the question of finding extensions of this theorem to infinitary contexts and to abstract elementary classes has been a major source of results. Many central concepts of stability theory, both in first order and in its generalizations, are essential byproducts of the theory built in order to generalize the original Morley theorem.

One of the most important landmarks along this path was the Categoricity Transfer result for $L_{\omega_1,\omega}$ due to the first author: if a sentence ψ is categorical in $\aleph_n$ for all $n < \omega$ and the weak GCH holds for the $\aleph_n$'s ($2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$) then ψ is categorical in

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all cardinals (see [2] and [3]; although these are two references, they correspond to "Part A" and "Part B" of one big paper from 1983). Notice the unusually strong assumption!

An example from 1990 due to Bradd Hart and the first author of this paper [1] established the (surprising) necessity of that strong assumption: the existence of *few models* at all the $\aleph_n$'s is needed to get the eventual categoricity transfer for $L_{\omega_1,\omega}$: they provide, for each positive $k \in \omega$, an example of a sentence ψ_k in $L_{\omega_1,\omega}$ categorical in $\aleph_0, \aleph_1, \cdots, \aleph_k$ but not eventually categorical: there exists some cardinal greater than $\aleph_k$ where categoricity fails.

That important example has later been referred to as **the Hart-Shelah example.** In many ways, the existence of such sentences points to an interesting failure of the "categoricity transfer" (Morley's theorem for first order logic, for countable theories) at *small* cardinalities, in the absence of further set theoretical hypotheses. Our work extends those results.

The present paper shows that a similar example also exists for the stronger logic $L_{(2^\lambda)^+,\omega}$. A corollary of our result is that any extension of the results from [2] and [3] to stronger logics will require to assume categoricity at **all** cardinalities $\lambda, \lambda^+, \dots, \lambda^{+n}, \dots$ for all $n < \omega$.

Later, the first author has attempted an extension of the main result from [2] and [3] to Abstract Elementary Classes. These are more general than classes axiomatized by the logic $L_{(2^\lambda)^+,\omega}$.

Our construction provides, for each infinite cardinal λ and each $k \in (2, \omega)$, a sentence ψ_k of $L_{(2^\lambda)^+,\omega}$ that is categorical in $\lambda, \lambda^+, \dots, \lambda^{+k}$ but is not categorical in any cardinality $\mu \geq \beth_{k+1}(\lambda)^+$.

The shift of focus from infinitary logic to abstract elementary classes entails in many cases using Galois (orbital) types instead of syntactic types; although this shift is natural, compactness and locality properties in general do not transfer to Galois types. In particular, *tameness* and *type-shortness* do not hold in general for Galois types. Tameness was isolated by Grossberg and VanDieren [4]; later, Baldwin and Shelah [5] constructed an example of failure of tameness, based on an almost free non-Whitehead group. More recently, Boney and Unger have provided serious set theoretic reasons for the failure of tameness in AECs [6].

In [7], Baldwin and Kolesnikov study again the Hart-Shelah example: they prove that for the sentence ψ_k of $L_{\omega_1,\omega}$ of the example, the corresponding AEC (for $k \geq 3$)

$$\mathcal{K}^{HS}(\omega_1, k) = (\mathrm{Mod}(\psi_k), \prec_{\omega_1,\omega})$$

- has disjoint amalgamation,
- is Galois stable exactly in $\aleph_0, \aleph_1, \dots, \aleph_{k-1}$,
- is $(< \aleph_0, \leq \aleph_{k-1})$-tame.

Moreover, the AEC axiomatized by their sentence ψ_k fails $(\aleph_{k-1}, \aleph_k)$-tameness. This is an immediate consequence of the failure of categoricity transfer and the upward categoricity theorem for tame AECs due to Grossberg and VanDieren [8].

Baldwin and Kolesnikov really study a slight variant of the Hart-Shelah example, presented in the language of group actions and revealing the filiation to the early Baldwin-Lachlan example of an $\aleph_1$-categorical theory which is not almost strongly minimal.

More recently, Boney [9] has continued this study of the behavior of the Hart-Shelah example; he has proved that the class $\mathcal{K}^{HS}(\omega_1, k)$ has a "good $\aleph_m$-frame" for all $m \leq k-1$ but cannot have a good frame above by the failure of stability. Then, Boney and Vasey [10] continue this study and show first that the frame at $\aleph_{k-1}$ cannot be "successful". They study good frames in connection with the Hart-Shelah example: for frames around the $\aleph_n$'s ($n < \omega$) the Hart-Shelah example is a natural place to look for "boundary properties": being "successful up to some point" but failing to be successful above.

Our generalization of the Hart-Shelah example addresses the question of how necessary an assumption similar to "few models in all the $\aleph_n$'s" is for categoricity transfer in the case of stronger logics. Here of course the corresponding assumption would be of the form "few models in all the λ^{+n} ($n < \omega$)".

We build a sentence ψ_k^λ in $L_{(2^\lambda)^+,\omega}$, categorical in $\lambda, \lambda^+, \ldots, \lambda^{+k}$ but not categorical in $\beth_{k+1}(\lambda)^+$. Here are two important differences between our approach and earlier ones:

- The sentences are constructed in all cases by first building a "standard model" and then extracting the sequence from it. In the Hart-Shelah example, one predicate Q "ties together" various copies of groups in a way that ends up linking the "dimension" of the predicate to the length of induction in the proof of categoricity. In our example, we need a large family of predicates Q_s, $s \in S = [\lambda]^{<\aleph_0}$.

- The "failure of categoricity" argument at cardinals greater than or equal to $\beth_{k+1}(\lambda)^+$ here is done by using a regular filter $\mathfrak{D}$.

A natural question arises, on the "gap" between categoricity and failure of categoricity of ψ_k^λ. Here, we can guarantee categoricity in the interval $[\lambda, \lambda^{+k}]$ and failure of categoricity…at $\beth_{k+1}(\lambda)^+$. Admittedly, this is a very large gap, relatively much wider than what Baldwin and Kolesnikov have for their version of the Hart-Shelah sentence. The question remains open whether this gap may be reduced.

In our concluding remarks, we raise some questions connected with the tameness and frames, inspired by the paper [10]. In particular, we ask whether the methods from that paper (that worked for the Hart-Shelah sentence) may be generalized to our sentence ψ_k^λ.

A note on indexing: the previous papers dealing with constructing examples of sentences where categoricity "stops" are [1], [7] (which proved more model theoretic facts on a variant of the original example and studied the abstract elementary class determined by the example; in particular, Galois (=orbital) types, the amalgamation and tameness spectra associated with the class), [9] and [10], in which the connection to frames is worked out (analyzing the Hart-Shelah example enables Boney and Vasey to study limitations to the existence of good frames). Now, for [1], the "critical" cardinality (the last cardinality of categoricity) is $\aleph_k$. In [7], because of the way they analyze the construction, it is more natural to work with $k \geq 3$ and with $k-2$ as the critical cardinality. The two other papers follow this.

Since our paper is directly a generalization of [1], it is more natural for us to revert to the choice of critical cardinality from there, of course adapted to our context. So, the last cardinality where we will have categoricity is λ^{+k-1}.

Our notation is standard.

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1. Construction of the sentence ψ_k^λ, in $L_{(2^\lambda)^+,\omega}$

Context 1.1. *For the rest of the paper, we fix an infinite cardinal* λ *and a natural number* $k \geq 2$.

We build in this section a new sentence ψ_k^λ in the logic $L_{(2^\lambda)^+,\omega}$. Our construction of ψ_k^λ requires first building a model we will call "canonical", M_I, for an arbitrary index set I and later taking a conjunction of the first order theory of M_I along with several infinitary sentences describing the behavior of various components of M_I. The sentence ψ_k^λ has some similarity to the Hart-Shelah sentence and may be seen as a generalization, but important differences are also present and will be apparent later (the regular group G and the regular filter on λ, $\mathfrak{D}$). However, it is important to stress that prior knowledge of the Hart-Shelah is *not necessary* for an understanding of our construction, as we make it self-contained.

We will build the model M_I around a "spine" I, essentially by coding interactions between k-element subsets of $(k+1)$-element subsets of I, in some combinatorial ways. Namely, we will define various groups and encode in the model *their actions* on those k

and $(k+1)$-element subsets of I, focusing especially on the way different k-subsets of a given $(k+1)$-subset of I interact. Finally, a collection of predicates (called Q_s) will "tie" those combinatorial interactions.

Definition 1.2. Notation and general construction tools. *We fix the following basic objects to use in the construction later.*

- $S = S_\lambda := [\lambda]^{<\aleph_0} = \{u \subset \lambda | u \text{ is finite}\}$,
- $\mathfrak{D} = \mathfrak{D}_\lambda := \{A \subset S | \exists u_A \in S \quad \forall v \in S(u_A \subset v \to v \in A)\}$, *the* regular filter *on* S *generated by sets of the form* $\langle u \rangle = \{v \in S | u \subset v\}$,
- $G^+ = G^+_\lambda := {}^S(\mathbb{Z}_2)$, *as a group with the natural operation* $(f+g)(v) = f(v) +_{\mathbb{Z}_2} g(v)$,
- $G = G_\lambda := \{f \in {}^S(\mathbb{Z}_2) | \ker(f) = \{u \in S | f(u) = 0\} \in \mathfrak{D}\}$, *as a subgroup of* G^+ *(*$G \leq G^+$ *since, if* $f, g \in G$, *then* $\ker(f), \ker(g) \in \mathfrak{D}$, *so* $\ker(f+g) \supset \ker(f) \cap \ker(g) \in \mathfrak{D}$, *hence* $\ker(f+g) \in \mathfrak{D}$ *and* $f + g \in G$.

Note that $|G| = 2^\lambda$.

1.1. The model M_I

Fix a set I for the rest of this section.
We define the group H_I and then describe the model M_I.

Definition 1.3 (The group H_I)**.** *For our (fixed)* I, *we let* $H_I := [[I]^k]^{<\aleph_0}$. *So,* H_I *is the set of finite subsets of* $[I]^k$, *with the group operation* $F_1 + F_2 := F_1 \Delta F_2$ *(symmetric difference). Equivalently,* H_I *may be seen as the set of functions* $[I]^k \to \mathbb{Z}_2$ *with finite support. In this case,* $F \in H_I$ *is coded by the function* $h_F : [I]^k \to \mathbb{Z}_2$ *with* $h_F(u) = 1$ *iff* $u \in F$ *and the group operation is given by* $(h_1 + h_2)(u) = h_1(u) +_{\mathbb{Z}_2} h_2(u)$.

We now start the (lengthy) description of M_I: the universe, basic predicates, projections between them, other partial functions coded by relations and, crucially, **the family of predicates** Q_s (for $s \in S$).

Definition 1.4 (Universe of M_I)**.** *The universe of* M_I *is the union of seven different sorts:*

$$|M_I| = I \cup [I]^k \cup [I]^{k+1} \cup ([I]^k \times S \times H_I) \cup ([I]^k \times S \times \mathbb{Z}_2) \cup H_I \cup ([I]^{k+1} \times G).$$

The following remarks on the universe of M_I are important:

- The natural way to think about the universe of M_I is as

$$|M_I|$$

consisting of two parts: the "support of the model" (I, $[I]^k$, $[I]^{k+1}$) and many copies of (the domains of) the three groups H_I, $\mathbb{Z}_2$ and G, *indexed* by elements of S and of the support part:

$$\overbrace{I \cup\ [I]^k \cup\ [I]^{k+1}}^{\text{`support part'}} \cup \overbrace{([I]^k \times S \times H_I) \cup ([I]^k \times S \times \mathbb{Z}_2) \cup H_I \cup ([I]^{k+1} \times G)}^{H_I \text{ and copies of the domains of } H_I,\ \mathbb{Z}_2 \text{ and } G}.$$

- Notice that the intersection between all the pieces of the model is empty.
- The universe of M_I depends directly on I and on G, as is clear from the various pieces. In particular, when the cardinality of I is $\geq \lambda$, the cardinality of M_I will be equal to $|I| + 2^\lambda$.
- The universe depends on k as well. Of course in our standard model this dependence is immediate, as seen from the superindices k and $k+1$. In general models later, we will need projection functions among the predicates in the model in order to axiomatize the connections between pieces corresponding to abstract versions of I, $[I]^k$, etc. *This dependence on* k *will be crucial in the "dimension" analysis later.*
- The universe also depends on λ, through the appearance of S and G among the pieces.

Definition 1.5 (Relations, functions of M_I - the predicates Q_s). *The structure of* M_I *consists of the following items:*

- λ-*many predicates* $P_0^M, P_{1,1}^M, P_{1,2}^M, P_2^M, (P_{2,s}^M)_{s\in S}, P_3^M, (P_{3,s}^M)_{s\in S}, P_4^M, P_5^M$,
- k-*many projections* $\pi_\ell^0 : P_{1,1}^M \to P_0^M$ *(*$\ell < k$*) and* $k+1$-*many projections* $\pi_\ell^1 : P_{1,2}^M \to P_0^M$,
- 2^λ-*many additional functions* $F_2^M, F_3^M, F_4^M, F_5^M, (F_{3,g^*}^M)_{g^*\in G}$,
- *a* $(3k+4)$-*ary predicate* Q_s*, for each* $s \in S$.

Each of these predicates and functions will be discussed in detail in the following paragraphs.

1.1.1. Descriptions of basic relations, functions, and the Q_s*-predicates*

Basic Relations: these consist of a family of λ-many predicates

$$P_0^M, P_{1,1}^M, P_{1,2}^M, P_2^M, (P_{2,s}^M)_{s\in S}, P_3^M, (P_{3,s}^M)_{s\in S}, P_4^M, P_5^M$$

defined by

- $P_0^M = I$,
- $P_{1,1}^M = [I]^k$,
- $P_{1,2}^M = [I]^{k+1}$,
- $P_2^M = [I]^k \times S \times H_I$,
- for $s \in S$, $P_{2,s}^M = \{(u, s, h) \in P_2^M | u \in [I]^k, h \in H_I\} = [I]^k \times \{s\} \times H_I$,
- $P_3^M = [I]^k \times S \times \mathbb{Z}_2$ (a copy of $\mathbb{Z}_2$ for each $b \in [I]^k$, $s \in S$),
- for $s \in S$, $P_{3,s}^M = \{(u, s, i) \in P_3^M | u \in [I]^k, i \in \mathbb{Z}_2\} = [I]^k \times \{s\} \times \mathbb{Z}_2$,
- $P_4^M = H_I$,
- $P_5^M = [I]^{k+1} \times G$

Remark 1.6. *The meaning of* $P_0^M, P_{1,1}^M, P_{1,2}^M, P_2^M$*,* P_3^M*,* P_4^M *is clear. In the case of* $P_{2,s}^M$*, the idea is that we stack "copies" of* H_I *for each* $b \in [I]^k$ *and each* $s \in S$*, and similarly for* P_3^M*,* $P_{3,s}^M$*. Another way of seeing this is thinking of the predicates as codifying families, as follows:*

- P_2^M *corresponds to* $(H_{\nu,s})_{\nu\in[I]^k, s\in S}$,
- P_3^M *corresponds to* $((\mathbb{Z}_2)_{\nu,s})_{\nu\in[I]^k, s\in S}$,
- P_5^M *corresponds to* $(G_u)_{u\in[I]^{k+1}}$.

Projections: We also include, for $\ell < k$, all the projections $\pi_\ell^0 : P_{1,1}^M \to P_0^M$:

$$\pi_\ell^0(\bar{a}) = a_\ell,$$

and for $\ell < k+1$, the projections $\pi_\ell^1 : P_{1,2}^M \to P_0^M$:

$$\pi_\ell^1(\bar{a}) = a_\ell.$$

The role of these projections is to tie the predicates $P_{1,1}^M$ and $P_{1,2}^M$ to P_0^M making them behave as the corresponding sets of ktuples or $k+1$-tuples.

Other Partial Functions: We also include 2^λ-many functions in M_I,

$$F_2^M, F_3^M, F_4^M, F_5^M, (F_{3,g^*}^M)_{g^*\in G} :$$

- A unary function F_2^M with domain P_2^M, given by

$$F_2^M(u, s, h) = u,$$

- A unary function F_3^M with domain P_3^M, given by

$$F_3^M(u, s, i) = u,$$

- for $g^* \in G$, a unary function F_{3,g^*}^M with domain P_5^M, given by

$$F_{3,g^*}^M(u, g) = (u, g^* + g),$$

- A binary function F_4^M with domain $P_2^M \times P_4^M$, given by

$$F_4^M\Big((v, s, h), h_1\Big) = (v, s, h +_H h_1)$$

- A unary function F_5^M with domain P_5^M, given by

$$F_5^M(u, g) = u,$$

A $(3k+4)$**-ary predicate** Q_s, for each $s \in S$. This is the crux of the construction of the model M_I. The predicate will encode interactions between the different parts of the model, in a way that will involve *dimensional* interactions between them. This predicate on the one hand *enables* later to move up in the proof of categoricity by induction $k-1$ times from λ to λ^k and on the other *blocks* the proof from moving up to λ^{k+1}. It is interpreted in M_I as the set of tuples

$$\langle a_0, \dots, a_k, u_0, \dots, u_k, x_0, \dots, x_{k-1}, y_k, z\rangle$$

satisfying (for fixed $s \in S$!!) for all $h_k \in H_I$, $i_\ell \in \mathbb{Z}_2 (\ell < k)$, $g \in G$:

(α) $a_\ell \in I$ with no repetitions ($\ell \leq k$),

(β) $u_\ell = \langle a_m | m \neq \ell\rangle \in P_{1,1}^M$ ($\ell \leq k$),

(γ) $y_k = (u_k, s, h_k) \in P_2^M$,

(δ) x_ℓ has the form $(u_\ell, s, i_\ell) \in P_3^M$ ($\ell < k$) so $i_\ell \in \mathbb{Z}_2$,

(ϵ) z is of the form $(u, g) \in P_5^M$, where $u = (a_0, \dots, a_k) \in [I]^{k+1}$ and

(ζ) **(main point)**

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s).$$

Some general remarks on this definition of the model M_I are in point, before giving a specific description for the case $k = 2$.

Remark 1.7. • (ζ) *is the crucial part of the definition of the predicates* Q_s*. It provides the connection between* k *copies of* $\mathbb{Z}_2$*, one copy of* H_I*, one copy of* G *and the* $(k+1)$*-many* k*-element subsets of a set of size* $k+1$ *in* I.

- *The role of* F_2^M *is to project* P_2^M *(essentially* $(H_{\nu,s})_{\nu\in[I]^k,s\in S}$*) onto its first coordinate; to trace the* k*-element subset of* I *it corresponds to. Similarly for* F_3^M *and* F_5^M*.*
- *The functions* F_{3,g^*}^M *and* F_4^M *encode the* actions *of the groups* G *and* H_I *on the corresponding "fibers" over* $u \in [I]^{k+1}$ *or* $(\nu, s) \in [I]^k \times S$*. The model* M_I *does not really include the group operations corresponding to* G *and* H_I*; it only has the effect of the group actions on the appropriate fibers.*
- *Notice that* $+^H$ *is definable - so in this case there is no need to add an analogue of* F_4 *for copies of* $\mathbb{Z}_2$*:*

$$F_4^M(F_4^M((u, s, h), h_1), h_2) = F_4^M((u, s, h), h_3) \Leftrightarrow H \models h_1 + h_2 = h_3.$$

1.1.2. Illustration of the definition of M_I*, when* $k = 2$

As an example to visualize the situation, we momentarily fix $k = 2$. We also fix $s \in S$ and choose some $u \in [I]^{k+1} = [I]^{2+1}$, $u = \langle a_0, a_1, a_2\rangle$. This determines automatically (using the projections) in the models we have described so far a_0, a_1, a_2 and $u_0 = \langle a_1, a_2\rangle$, $u_1 = \langle a_0, a_2\rangle$, $u_2 = \langle a_0, a_1\rangle$.

We then have

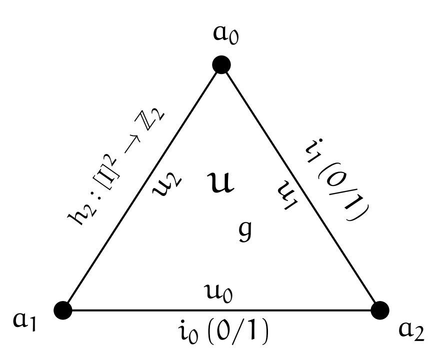
- copies of $\mathbb{Z}_2$ over both u_0 and u_1,
- a copy of the domain of H_I over u_2, together with the action of H_I on this copy,
- a copy of G over u, again with the action of G over this copy.

Furthermore, we have the predicate Q_s: it is in this case $3\cdot 2+4 = 10$-ary. The 10-uple associated with our u is then of the form

$$(a_0, a_1, a_2, u_0, u_1, u_2, x_0, x_1, y_2, z),$$

with $x_0 = (u_0, s, i_0)$, $x_1 = (u_1, s, i_1)$, $y = (u_2, s, h_2)$ and $z = (u, g)$ for some $i_0, i_1 \in \mathbb{Z}_2$, $h_2 \in H_I$ and $g \in G$.

We want to describe when this tuple belongs to Q_s. The following triangle summarizes the relevant information:



The tuple $(a_0, a_1, a_2, u_0, u_1, u_2, x_0, x_1, y_2, z)$ belongs to Q_s if and only if

$$\mathbb{Z}_2 \models i_0 + i_1 = h_2(u_0) + g(s).$$

Therefore, on top of the triangle u we have (when $k = 2$) four pieces of information playing: two elements (i_0, i_1) of $\mathbb{Z}_2$ associated to two sides of the triangle, one element h of H_I associated to the third side of the triangle (and the value of h at u_0) and finally one element g of G associated to the triangle u itself - and the value of g at… s.

1.2. The language, the sentence ψ_k^λ and the AEC $\mathcal{K}^*(\lambda, k)$

We now build the sentence ψ_k^λ.

Definition 1.8. *We deal with two vocabularies:*

- *Let τ^- be the vocabulary of all the construction above, except the predicates $\{Q_s | s \in S\}$ and*
- *let τ be the full vocabulary used in the construction of M_I.*

Specifically,

$$\text{let } \tau^- = \langle P_0, P_{1,1}, P_{1,2}, P_2, (P_{2,s})_{s\in S}, P_3, (P_{3,s})_{s\in S}, P_4, P_5,$$
$$\pi_0^0, \ldots, \pi_{k-1}^0, \pi_0^1, \ldots, \pi_k^1, F_2, F_3, F_4, F_5, (F_{3,g^*})_{g^*\in G}\rangle$$

and let $\tau = \tau^- \cup \{Q_s | s \in S\}$.

Notice that $|\tau| = |G_\lambda| + |S| + \aleph_0 = 2^\lambda$, since $|G_\lambda| = 2^\lambda$.

Definition 1.9 (The sentence ψ_k^λ). *The sentence $\psi_k^\lambda \in L_{(2^\lambda)^+,\omega}(\tau)$ is the conjunction*

$$\psi_k^\lambda \equiv \bigwedge T_0 \wedge \psi_G \wedge \psi_{\mathbb{Z}_2} \wedge \psi_H$$

of the first order theory T_0 of M_I (for infinite I) and the infinitary sentences

- $\psi_G \equiv \forall z_1 z_2([P_5(z_1) \wedge P_5(z_2) \wedge F_5(z_1) = F_5(z_2)] \to \bigvee_{g^*\in G} F_{3,g^*}(z_1) = z_2)$,
- $\psi_{\mathbb{Z}_2} \equiv \forall y(P_2(y) \leftrightarrow \bigvee_{s\in S} P_{2,s}(y))$,
- $\psi_H \equiv \forall y(P_3(y) \leftrightarrow \bigvee_{s\in S} P_{3,s}(y))$.

We describe in more detail some parts of the previous definition.

ψ_G says that G acts transitively (through the functions F_{3,g^*}) on copies of G (fibers of P_5).

ψ_H says that there are no "non-standard fibers" in P_2: every element of P_2 is in some $P_{2,s}$.

$\psi_{\mathbb{Z}_2}$ says that there are no "non-standard fibers" in P_3: every element of P_3 is in some $P_{3,s}$

Note that, although there are 2^{2^λ} sentences in the logic, we are only using 2^λ of them, as witnessed by $|G| = 2^\lambda$.

We will also use the following variant on the standard model: for a set I and a function

$$f : [I]^{k+1} \times S \to \mathbb{Z}_2,$$

we will now build models $M_{I,f}$ and $M^-_{I,f}$.

Definition 1.10. *[The models $M_{I,f}$ and $M^-_{I,f}$] Let $f : [I]^{k+1} \times S \to \mathbb{Z}_2$ and I a set. Then $M_{I,f}$ is the τ-model constructed just like M_I, with only one difference: the interpretation of Q_s, for $s \in S$, now is the set of tuples*

$$\langle a_0, \dots, a_k, u_0, \dots, u_k, x_0, \dots, x_{k-1}, y_k, z\rangle$$

(see page 8) with condition (ζ) replaced by

$$(\zeta)^*_f \qquad \mathbb{Z}_2 \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s) + f(u, s).$$

The τ^--model $M^-_{I,f}$ is then defined as $M_{I,f} \restriction \tau^-$.

We will use the models $M_{I,f}$ later as canonical ways of describing variants in the choices of elements of the groups when studying models of the sentence ψ^λ_k.

We call a τ-structure **strongly standard** if $M \restriction \tau^- = M_I \restriction \tau^-$ for $I = P^M_0$.

Definition 1.11. (Some abstract classes related to ψ^λ_k)

1. Let $K_1 := \{M | M \approx M_{I,f}$ for some infinite set I, for some f as in 1.10$\}$. Then K_1 is a class of τ-models.
2. Let $K^*(\lambda, k) :=$ Mod (ψ^λ_k) with the strong substructure relation

$$\prec_{K^*(\lambda,k)} := \prec_{L_{(2^\lambda)^+,\omega}} .$$

3. M from $K^*(\lambda, k)$ is **standard** if $P^M_{1,1} = [P^M_0]^k$ and $P^M_{1,2} = [P^M_0]^{k+1}$ and the π^t_ℓ's correspond to the actual projections sending $u \in [I]^k$ to its ℓ'th coordinate in I.

Claim 1.12. *For any $M_I \models \psi^\lambda_k$, $M_I \approx M_{I,\mathbf{0}}$, for the function $\mathbf{0} : [I]^{k+1} \times S \to \mathbb{Z}_2$ of constant value 0.*

The proof is immediate from the definition.

Claim 1.13. *Every* $N \models \psi^\lambda_k$ *is isomorphic to a strongly standard* M.

Proof Let $N \models \psi^\lambda_k$ and let $I := P^N_0$. Then $N \upharpoonright \tau^- \approx M_I \upharpoonright \tau^-$ (following the definition of the sorts of the vocabulary τ^-). Then *define* the interpretations of the relevant predicates Q_s on N by mapping directly from their definition on the strongly standard model M_I. □

Next, a straightforward observation.

Claim 1.14. $M_{I,f}$ *is strongly standard.*

Proposition 1.15. $(\mathcal{K}^*(\lambda, k), \prec_{K^*(\lambda,k)})$ *is an abstract elementary class with Löwenheim-Skolem number* 2^λ.

We do not investigate properties of this AEC in this paper; however, we propose some conjectures at the end of the paper on their properties and on their connection with good frames and the work of Boney and Vasey [10].

2. Categoricity of ψ^λ_k below λ^{+k}

In this section we study the categoricity spectrum of ψ^λ_k. The strategy consists of the following steps:

- Since the complexity of models of ψ^λ_k hinges on the predicates Q_s, and these ultimately depend on *choices* of elements of the copies of the groups above the "supports" (in the standard case, k-element subsets of $(k+1)$-sets of the index set), we will develop a language of **choice functions** to deal with these.
- Furthermore, comparing different models will amount to dealing with **correction functions** associated to the choice functions. We also set up a language for these.
- Later (Lemma 2.5) we establish that for every model N of ψ^λ_k and global choice for N with correction function f there are an index set I and an isomorphism $\mathbf{h}$ between N and $M_{I,f}$.
- Therefore, establishing categoricity in a cardinality κ amounts to showing that for every $N \models \psi^\lambda_k$ of cardinality κ there is a global choice for N with correction function 0 (for $\kappa < \lambda^{+k}$; see Theorem 2.14).
- The rest of the section is devoted to showing that if $M \models \psi^\lambda_k$ and $|M| < \lambda^{+k}$ then there is a global choice function for M with correction function 0 - with the cardinality restriction in place, we may conclude that ψ^λ_k is categorical in $\lambda^+, \lambda^{++}, \ldots, \lambda^{+k-1}$. This part requires several lemmas on extending choice functions while keeping the correction function 0; these lemmas depend crucially on the cardinality being of the form λ^{+m} for m a natural number *below* k. This is why the proof in this section only provides categoricity up to λ^{+k-1}.

2.1. Solutions, choices and correction functions

We will now define choice functions and correction functions. These will be used to study models of ψ_k^λ of cardinality $\lambda^+, \dots, \lambda^{+k-1}$.

Expanding choices from partial to global ones is the crucial issue.

Definition 2.1 (Partial M-(J_0, J_1, J_2)-choice). *For $M \models \psi_k^\lambda$, we say $(\bar{x}, \bar{y}, \bar{z})$ is a* **partial** M-(J_0, J_1, J_2)**-choice** *if*

(a) $J_0, J_1 \subset P_{1,1}^M$, $J_2 \subset P_{1,2}^M$,
(so, in the case of standard models, $J_0, J_1 \subset [I]^k$, $J_2 \subset [I]^{k+1}$)

(b) $\bar{x} = \langle x_{u,s} | s \in S, u \in J_0 \rangle$, *where*

$$x_{u,s} \in (P_{3,s}^M)^{-1}(u) \subset P_{3,s}^M.$$

(c) $\bar{y} = \langle y_{u,s} | s \in S, u \in J_1 \rangle$,

$$y_{u,s} \in H_{u,s}^M := (P_{2,s}^M)^{-1}(u) \subset P_{2,s}^M.$$

(d) $\bar{z} = \langle z_u | u \in J_2 \rangle$,

$$z_u \in G_u^M := (F_5^M)^{-1}(u) \subset P_5^M.$$

Therefore $\bar{x}$ essentially chooses an element i in the corresponding copy of $\mathbb{Z}_2$, $\bar{y}$ chooses a h in the corresponding copy of H_I, $\bar{z}$ chooses a g in the corresponding copy of G, for each relevant (u, s).

So, $x_{u,s}$ is *some* element in the 'fiber' of u via F_3^M, and analogously for $\bar{y}$ and $\bar{z}$.

Definition 2.2. *We call $(\bar{x}, \bar{y}, \bar{z})$ a* **partial** M-J-**choice** *if it is an M-(J, J, J_*^M)-choice, where*

$$J_*^M := \Big\{ a \in P_{1,2}^M \Big| \bigwedge_{m \le k} \exists b \in J [\bigwedge_{\ell < m} (\pi_\ell^1(a) = \pi_\ell^0(b) \wedge \bigwedge_{\ell \in [m,k[} \pi_\ell^0(b) = \pi_{\ell+1}^1(a)] \Big\}.$$

Similarly, we say that $(\bar{x}, \bar{y}, \bar{z})$ is a **global** M**-choice** *if it is a partial M-$P_{1,1}^M$-choice. We will sometimes just say "M-choice" (if clear from context).*

The previous is a way of describing, in our language of projections, that (in the standard case) J_*^M consists of the $k+1$-element sets such that *all* their ($k+1$-many) k-element subsets are in J).

So, when M is standard, we have that

$$J_*^M = \Big\{ \langle a_\ell | \ell \le k \rangle \Big| \bigwedge_{m \le k} \langle a_\ell | \ell \ne m \rangle \in J \Big\}.$$

Definition 2.3. *Fix a standard* M *and a* M*-*(J_0, J_1, J_2)*-choice* $(\bar{x}, \bar{y}, \bar{z})$. *Then we let the* **correction function** f *for* M *and* $(\bar{x}, \bar{y}, \bar{z})$ *be the function such that*

1. Dom (f) *is the set of pairs* (u, s) *such that*
   (α) $u = \langle a_\ell | \ell \leq k \rangle \in J_2 \subset P^M_{1,2}$,
   (β) *if* $u_m := \langle a_\ell | \ell \leq k, \ell \neq m \rangle$, $u_\ell \in J_0$ *for* $\ell < k$, $u_k \in J_1 \subset P^M_{1,1}$,
2. rng $(f) \subset \mathbb{Z}_2$, *and*
3. *(recall* $x_{u_\ell, s}, y_{u_k, s}, z_{u_k}$ *are from the choice)*

$$f(u, s) = 0 \Leftrightarrow \langle a_0, \ldots, a_k, u_0, \ldots, u_k, x_{u_0,s}, \ldots, x_{u_{k-1},s}, y_{u_k,s}, z_{u_k} \rangle \in Q^M_s.$$

The next claim is a general observation on correction functions and choices.

Claim 2.4. *For every* $M \in$ Mod (ψ^λ_k), *there is an* M*-choice* $(\bar{x}, \bar{y}, \bar{z})$.

Proof Immediate: just construct the tuples. There the demands are on each choice separately. There are no demands connecting different choices. □

The next lemma is a crucial step. It shows how to build possible isomorphisms from an arbitrary model N of ψ^λ_k to standard models $M_{I,f}$.

Lemma 2.5. *Let* $N \in$ Mod (ψ^λ_k) *and let* $(\bar{x}, \bar{y}, \bar{z})$ *be a global* N*-choice with correction function* f. *Then, there exist a set* I *and an isomorphism*

$$\mathbf{h} : N \to M_{I,f}.$$

Furthermore, the isomorphism behaves as follows on the global N*-choice* $(\bar{x}, \bar{y}, \bar{z})$*:*

$$\mathbf{h}(x_{u,s}) = (\mathbf{h}(u), s, 0_{\mathbb{Z}_2}), \quad \mathbf{h}(y_{u,s}) = (\mathbf{h}(u), s, 0_{H_I}), \quad \mathbf{h}(z_u) = (\mathbf{h}(u), 0_G).$$

Proof Let $N \models \psi^\lambda_k$, and fix a global N-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function f. We build I and $\mathbf{h}$ as in the statement.

First, we extract the predicates for the model $M = M_{I,f}$: let $I := P^N_0$. Clearly, $P^M_0 = P^N_0$.

We now define $\mathbf{h}$, following the predicates of the domain of N (remember that the domain of N is the disjoint union

$$P^N_0 \cup P^N_{1,1} \cup P^N_{1,2} \cup P^N_2 \cup P^N_3 \cup P^N_4 \cup P^N_5$$

and the predicates P^N_2 and P^N_3 are each partitioned into classes $P^N_{2,s}$, $P^N_{3,s}$ $(s \in S)$).

- $\mathbf{h}$ is the identity on $P^N_0 = I = P^M_0$.
- if $x \in P^N_{1,1}$, $\ell < k$, $\pi^0_\ell(x) = x_\ell$ $(\in P^N_0)$, then $\mathbf{h}(x) := (\mathbf{h}(x_0), \ldots, \mathbf{h}(x_{k-1}))$.

- similarly, if $x \in P^N_{1,2}$, $\ell < k+1$, $\pi^1_\ell(x) = x_\ell$ ($\in P^N_0$), then $\mathbf{h}(x) := (\mathbf{h}(x_0), \dots, \mathbf{h}(x_k))$.
- if $x \in P^N_{2,s}$ then $\mathbf{h}(x) = (\mathbf{h}(F^N_2(x)), s, -) \in [I]^k \times S \times H_I$. For now we only know the third coordinate must be an element of H_I. Also, **as soon as we know** the third coordinate of the image of one element x_0 of a fiber inside the predicate $P^N_{2,s}$, we also know the third coordinate for all other elements x of that fiber: since the action given by F^N_4 is transitive (as encoded by T_0), there is some $h_0 \in P^N_4$ such that $F^N_4(x_0, h_0) = x$. Then (if we also have a definition of $\mathbf{h}$ on elements of P^N_4), we have that $\mathbf{h}(x) = \mathbf{h}(F^N_4(x_0, h_0)) = F^N_4(\mathbf{h}(x_0), \mathbf{h}(h_0))$.
- Similarly, if $x \in P^N_{3,s}$, then $\mathbf{h}(x) = (\mathbf{h}(F^N_3(x)), s, -) \in [I]^k \times S \times \mathbb{Z}_2$ and just as before the value of $\mathbf{h}$ on one element of the fiber will determine the rest.
- And similarly, if $x \in P^N_5$, then $\mathbf{h}(x) = (F^N_5(x), -) \in [I]^{k+1} \times G$. Again, since $N \models \psi_G$, the action ("of G") encoded by the family of functions F^N_{3,g^*} is transitive, and therefore knowing a second coordinate for *one* element of a fiber of P^N_5 implies knowing it for all elements of the corresponding fiber that predicate.

It therefore remains, in order to complete the definition, to make **choices** of images of elements of P^N_4 (images in H_I - but this is easy, as H_I is definable in our structure) and selecting, for each $s \in S$, one image in each one of the relevant fibers. We now use the correction function f and the predicates Q_s.

So fix $s \in S$. Checking the equivalence we are looking for, namely

$$Q^N_s(a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k z)$$

$$\Updownarrow$$

$$Q^{M_{I,f}}_s(\mathbf{h}(a_0) \dots \mathbf{h}(a_k)\mathbf{h}(u_0) \dots \mathbf{h}(u_k)\mathbf{h}(x_0) \dots \mathbf{h}(x_{k-1})\mathbf{h}(y_k)\mathbf{h}(z)),$$

amounts to answering the question

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s) + f(u, s)$$

$$\Updownarrow ?$$

$$\mathbb{Z}_2 \models \sum_{\ell<k} \mathbf{h}(i_\ell) = h_k(\mathbf{h}(u_0)) + g(\mathbf{h}(s)) + f(\mathbf{h}(u), \mathbf{h}(s))$$

Now letting

$$\begin{cases} \mathbf{h}(x_{u,s}) = (\mathbf{h}(u), s, 0_{\mathbb{Z}_2}), \\ \mathbf{h}(y_{u,s}) = (\mathbf{h}(u), s, 0_H), \\ \mathbf{h}(z_u) = (\mathbf{h}(u), 0_G) \end{cases}$$

works for these equations: we are assigning 0 on the missing coordinates (third or second) – exactly to those elements of the fibers ($x_{u,s}$, $y_{u,s}$, $z(u)$) that had already been picked by the choice function.

Why is this enough?

Well, our definition turns the equation (at the choices) into

$$\mathbb{Z}_2 \models 0 = \sum_{\ell<k} 0 = 0(\star) + 0(\star) + f(\star).$$

But, since f was a correction function *for the choice function* $(\bar{x}, \bar{y}, \bar{z})$,

$$f(u, s) = 0 \Leftrightarrow \langle a_0, \ldots, a_k, u_0, \ldots, u_k, x_{u_0,s}, \ldots, x_{u_{k-1},s}, y_{u_k,s}, z_{u_k}\rangle \in Q^N_s,$$

and therefore our definition of **h** works. □

Definition 2.6 (Canonical choice). *Fix $M = M_{I,f}$, and let $(\bar{x}, \bar{y}, \bar{z})$ be the M-choice given by*

$$x_{u,s} = (u, s, 0_{\mathbb{Z}_2}),$$
$$y_{u,s} = (u, s, 0_{H_I}),$$
$$z_u = (u, 0_G).$$

*This is by definition the **canonical** M-**choice**.*

Claim 2.7.

1. *If $(\bar{x}, \bar{y}, \bar{z})$ is a global M-choice, $M \models \psi^\lambda_k$, and f is the M-correction function for $(\bar{x}, \bar{y}, \bar{z})$, and f is identically zero, then $M \approx M_I$ for some I.*
2. *If f above is zero on $P^M_{1,1}$, $P^M_{1,2}$ and $f = f' \restriction J_2 \times S$, then $M \approx M_{P_1,f'}$.*

Proof Part (1) is a consequence of 2.5. Part (2) is clear. □

Corollary 2.8. *The correction function for $M_{I,f}$ with the canonical M-choice $(\bar{x}, \bar{y}, \bar{z})$ is f.*

Proof Similar to the previous: add zeroes to f as in 2.7. □

2.2. Models of cardinality below λ^{+k}

The rest of the section contains several *extension lemmas* for models of ψ^λ_k of cardinalities λ, λ^+, etc.: the crucial issue is to build a choice function with null correction function. This may be started first at cardinality λ, and then pushed up. But each step up exacts an "amalgam of choices" possible only up to cardinality λ^{+k-1}.

The next lemma is the first step in the categoricity proof. It provides a specific kind of extension of choice: from an M-J-choice with correction function zero to a global M-choice with correction function zero *when J consists of k-subsets of the "support part" of M that* omit *some fixed set W of **at most** k-many elements.* Also, it is worth stressing the lemma is about standard M.

For instance, when $m = 2 < k = 3$, the lemma would mean we start with a choice function $(\bar{x}, \bar{y}, \bar{z})$ for all "triangles" and "tetrahedra" *omitting* some fixed pair $\{a, b\}$ ... and then would extend the choice function (with correction function zero) to all triangles and tetrahedra.

Lemma 2.9. *[Extension property for W of size $m < k$, $|P_0^M| \le \lambda$]*
Assume $m < k$, $M \models \psi_k^\lambda$, M is strongly standard, $|P_0^M| \le \lambda$, $W \subset P_0^M$, $W = \{b_\ell | \ell < m\}$ with no repetition, $J = \{u \in P_{1,1}^M | W \not\subset u\}$ (note that $u \in [P_0^M]^k$, as M is standard), $(\bar{x}, \bar{y}, \bar{z})$ is an M-J-choice with correction function f_0, identically zero. Then, *we can extend $(\bar{x}, \bar{y}, \bar{z})$ to an M-choice with correction function identically zero.*

Proof

Part A: Without loss of generality, by 1.13, since M is strongly standard, $I = P_0^M$. Let $\langle \bar{a}^\alpha | \alpha < \beta^* \rangle$ list $P_{1,1}^M$ with $\langle \bar{a}^\alpha | \alpha < \alpha^* \rangle$ listing J (we have also used u for naming these $\bar{a}^\alpha$'s). Let $\langle \bar{b}^\gamma | \gamma < \gamma^* \rangle$ list $\{\bar{a} \in [I]^{k+1} | \bar{a}$ with no repetition and $W \subset \operatorname{rng}(\bar{a})\}$ and $\gamma^* < \lambda^+$.

Our hypothesis is then that we have choice functions for all $u \in P_{1,1}^M$ such that $u \not\supset W$, with correction function zero.

We list these choice functions as follows: Let, for $\alpha < \alpha^*$,

$$x_{\bar{a}^\alpha,s} = (\bar{a}^\alpha, s, i_{\alpha,s}) \in (\mathbb{Z}_2)_{\bar{a}^\alpha,s},\ i_{\alpha,s} \in \mathbb{Z}_2,$$
$$y_{\bar{a}^\alpha,s} = (\bar{a}^\alpha, s, h_{\alpha,s}) \in H_{\bar{a}^\alpha,s},\ h_{\alpha,s} \in H_I,$$
$$z_{\bar{b}^\gamma} = (\bar{b}^\gamma, g^\gamma),\ g^\gamma \in G.$$

We now have to extend these choice functions to those u such that $u \supset W$.

We will now choose $x_{\bar{a}^\alpha,s} = (\bar{a}^\alpha, s, i_{\alpha,s})$, $y_{\bar{a}^\alpha,s} = (\bar{a}^\alpha, s, h_{\alpha,s})$, $z_{\bar{b}^\gamma} = (\bar{b}^\gamma, g^\gamma)$ for $\alpha^* \le \alpha < \beta^*$ and appropriate γ.

Without loss of generality, $\beta^* \le \alpha^* + \lambda$, $\gamma^* \le \lambda$. (Remember $S = [\lambda]^{<\aleph_0}$.)

Part B: **First,** we choose $i_{\alpha,s} = 0_{\mathbb{Z}_2}$ for $\alpha^* \le \alpha < \beta^*$, $s \in S$. This provides the choices $x_{\bar{a}^\alpha,s}$ for $\alpha^* \le \alpha < \beta^*$.

Second, we choose the relevant h functions. We try a value for $h_{\alpha,s}$ for $\alpha^* \le \alpha < \beta^*$ and $s \in S$ so that

(*) if $\gamma \in s \subset \lambda$, $\bar{b}^\gamma = \langle b_\ell^\gamma | \ell \le k \rangle$, $u_n^\gamma = \langle b_\ell^\gamma | \ell \le k, \ell \ne n \rangle$, let $\varepsilon(\gamma, n) < \beta^*$ be such that $u_n^\gamma = \bar{a}^{\varepsilon(\gamma,n)}$ then

$$h_{\varepsilon(\gamma,k),s}(\bar{a}^{\varepsilon(\gamma,0)}) = 0$$
$$\Updownarrow$$
$$\langle b_0^\gamma, \ldots, b_k^\gamma, u_0^\gamma, \ldots, u_k^\gamma, x_{\varepsilon(\gamma,0),s}, \ldots, x_{\varepsilon(\gamma,k-1),s}, (\bar{a}^{\varepsilon(\gamma,k)}, s, 0_H), (u^\gamma, 0_G) \rangle \in Q_s^M.$$

Note that all the elements in the bottom part of the previous have already been defined previously.

Let $t(\gamma, s)$ be 0 if the bottom statement is true, 1 otherwise. For our fixed $s \in S$, let A_s be the (finite) set $\{\varepsilon(\gamma, k) \mid \gamma \in s\}$; we now define $h_{\alpha,s}$ for our fixed s and at the relevant u. If $\alpha \notin A_s$, then let $h_{\alpha,s}(u) = 0$ for all u. If $\alpha \in A_s$, we proceed as follows. First we consider the set $s_\alpha := \{\gamma \in s \mid \varepsilon(\gamma, k) = \alpha\}$ and we then define

$$h_{\alpha,s}(u) = \begin{cases} t(\gamma, s), & \text{if } u = a^{\varepsilon(\gamma,0)} \text{ for some } \gamma \in s_\alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that these decisions are made **for each** s **separately,** and that as we fix s we really deal with one $\alpha \in [\alpha^*, \beta)$: when we choose $h_{\alpha,s}$ we only have to consider $\gamma < \gamma^*$ such that $\varepsilon(\gamma, \ell) \in s$. There are only finitely many such γ's. Moreover, if $\gamma_1 \neq \gamma_2 \in s$ and $\varepsilon(\gamma_1, k) = \alpha = \varepsilon(\gamma_2, k)$ then necessarily $\varepsilon(\gamma_1, 0) \neq \varepsilon(\gamma_2, 0)$, as $\bar{b}^\gamma$ is reconstructible from α and $\varepsilon(\gamma_1, 0)$.

So, our definition of the functions $h_{\varepsilon(\gamma,k),s}$ does not have contradictory demands; since the set s_α is finite, the function defined has finite support.

Part C: Having extended the choices x and y, it only remains to extend the z part. Let us now fix γ and find a $g \in G$ that will provide a choice (with correction function zero) for the corresponding $\bar{b}^\gamma$. [Recall that if $\bar{b} \in [I]^{k+1}$ is such that $\bar{b} \supset W$, then $\bar{b} = \bar{b}^\gamma$ for some $\gamma < \gamma^*$.]

But then the set

$$S^*_\gamma = \Big\{ s \in S \Big| M \models Q_s\big(b^\gamma_0, \dots, b^\gamma_k, \bar{a}^{\varepsilon(\gamma,0)}, \dots, \bar{a}^{\varepsilon(\gamma,k)}, \\ x_{u^\gamma_0,s}, \dots, x_{u^\gamma_{k-1},s}, y_{u^\gamma_k,s}, (u^\gamma, 0_G)\big) \Big\}$$

belongs to $\mathfrak{D}$. This last point holds by the regularity of $\mathfrak{D}$: if $s_0 \in S^*_\gamma$ then the tuple

$$\big(b^\gamma_0, \dots, b^\gamma_k, \bar{a}^{\varepsilon(\gamma,0)}, \dots, \bar{a}^{\varepsilon(\gamma,k)}, x_{u^\gamma_0,s_0}, \dots, x_{u^\gamma_{k-1},s_0}, y_{u^\gamma_k,s_0}, (u^\gamma, 0_G)\big)$$

belongs to Q_{s_0}; now, if $s \supset s_0$, the corresponding tuple

$$\big(b^\gamma_0, \dots, b^\gamma_k, \bar{a}^{\varepsilon(\gamma,0)}, \dots, \bar{a}^{\varepsilon(\gamma,k)}, x_{u^\gamma_0,s}, \dots, x_{u^\gamma_{k-1},s}, y_{u^\gamma_k,s}, (u^\gamma, 0_G)\big)$$

will belong to Q_s.

Next choose $z_{\bar{b}^\gamma} := (\bar{b}^\gamma, g)$ with g given by

$$g(s) = \begin{cases} 0 & \text{if } s \in S^*_\gamma \\ 1 & \text{if } s \notin S^*_\gamma \end{cases}$$

Now then, with these x, y and z, the equation holds.

$\square_{2.9}$

We now deal with **systems** of choices, trying to obtain extensions with correction function zero at cardinalities above λ. In what follows, as usual, $\mathcal{P}^-(m_2)$ denotes $\mathcal{P}(m_2) \setminus \{m_2\}$.

Definition 2.10 (Compatible system of choices). *Let $M \models \psi_k^\lambda$ be strongly standard, $A_\emptyset \subset P_0^M$, $m_1 + m_2 < k$ and $a_0, \dots, a_{m_2-1}$ different elements of $P_0^M \setminus A_\emptyset$. Then*

$$\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s | s \in \mathcal{P}^-(m_2) \rangle$$

*is a **compatible** λ^{+m_1}-$\mathcal{P}^-(m_2)$**-system of choices** iff*

1. $\bigcup_{s \in \mathcal{P}^-(m_2)} A_s = A_\emptyset \cup \{a_0, \dots, a_{m_2-1}\}$, $|A_\emptyset| \leq \lambda^{+m_1}$, $A_s = A_\emptyset \cup \{a_t | t \in s\}$.
2. $(\bar{x}, \bar{y}, \bar{z})_s$ *is a M-$[A_s]^k$-choice, for each $s \in \mathcal{P}^-(m_2)$.*
3. *For every $s, t \in \mathcal{P}^-(m_2)$, $s \subset t \Rightarrow (\bar{x}, \bar{y}, \bar{z})_s \subset (\bar{x}, \bar{y}, \bar{z})_t$*[4]*.*

Lemma 2.11. *If $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s | s \in \mathcal{P}^-(m_2) \rangle$ is a compatible λ-$\mathcal{P}^-(m_2)$-system with $m_2 < k$ (with correction function zero for each $s \in \mathcal{P}^-(m_2)$), <u>then</u> there is an M-$\bigcup_{s \in \mathcal{P}^-(m_2)} A_s$-choice $(\bar{x}, \bar{y}, \bar{z})$ extending all the $(\bar{x}, \bar{y}, \bar{z})_s$, for $s \in \mathcal{P}^-(m_2)$, with correction function zero.*

Proof Let $m_2 < k$ and let $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s | s \in \mathcal{P}^-(m_2) \rangle$ be a compatible λ-$\mathcal{P}^-(m_2)$-system, each choice in the system with correction function zero. Notice that

$$u \in [\bigcup_{s \in \mathcal{P}^-(m_2)} A_s]^k \setminus \bigcup_{s \in \mathcal{P}^-(m_2)} [A_s]^k,$$

if and only if $\{a_0 \dots a_{m_2-1}\} \subset u$.
We first notice that by compatibility, the union of the choices $(\bar{x}, \bar{y}, \bar{z})_s$ along $\mathcal{P}^-(m_2)$ is an M-choice for $\bigcup_{s \in \mathcal{P}^-(m_2)} [A_s]^k$. It remains to extend that choice to an M-$\bigcup_{s \in \mathcal{P}^-(m_2)} A_s$-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function zero.
We may apply Lemma 2.9 (here, the set W of cardinality $m_2 < k$ is $\{a_0, \dots, a_{m_2-1}\}$ and the lemma provides the extension from a M-$\bigcup_{s \in \mathcal{P}^-(m_2)} [A_s]^k$-choice with correction function zero to a M-$\bigcup_{s \in \mathcal{P}^-(m_2)} A_s$-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function zero - we extend the choice from those k-sets omitting W to all of them). $\square_{2.11}$

Lemma 2.12. *Let $m_1 + m_2 < k$. If $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s | s \in \mathcal{P}^-(m_2) \rangle$ is a compatible λ^{+m_1}-$\mathcal{P}^-(m_2)$-system of choices with correction function zero, <u>then</u> there is a $\bigcup_{s \in \mathcal{P}^-(m_2)} A_s$-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function zero such that $(\bar{x}, \bar{y}, \bar{z})_s \subset (\bar{x}, \bar{y}, \bar{z})$ for every $s \in \mathcal{P}^-(m_2)$.*

[4] here, of course, we are abusing notation - by $(\bar{x}, \bar{y}, \bar{z})_s \subset (\bar{x}, \bar{y}, \bar{z})_t$ we mean $\bar{x}_s \subset \bar{x}_t$, $\bar{y}_s \subset \bar{y}_t$ and $\bar{z}_s \subset \bar{z}_t$.

Proof By induction on m_1. For $m_1 = 0$, this is lemma 2.11. For $m_1 > 0$, suppose $A_s = A_\emptyset \cup \{b_j | j \in s\}$. Enumerate $A_\emptyset$ as $\langle a_\beta | \beta < \lambda^{+m_1} \rangle$. Let $A^\alpha_\emptyset = \{a_\beta | \beta < \alpha\}$ and $A^\alpha_s = A^\alpha_\emptyset \cup \{b_j | j \in s\}$ for every $s \in \mathcal{P}^-(m_2)$. Finally, let $(\bar{x}, \bar{y}, \bar{z})^\alpha_s$ be the restriction of the choice we have $(\bar{x}, \bar{y}, \bar{z})_s$ from the compatible system (with correction function zero) to an M-A^α_s-choice (also immediately with correction function zero).

The plan is to obtain an M-$\bigcup_{s \in m_2} A^\alpha_s$-choice $(\bar{x}, \bar{y}, \bar{z})_\alpha$ with correction function zero for each $\alpha < \lambda^{+m_1}$, such that $\alpha < \beta < \lambda^{+m_1}$ implies $(\bar{x}, \bar{y}, \bar{z})_\alpha \subset (\bar{x}, \bar{y}, \bar{z})_\beta$.

We build $(\bar{x}, \bar{y}, \bar{z})_\alpha$ by another induction, on $\alpha < \lambda^{+m_1}$. For $\alpha = 0$, the empty choice function is an M-$\emptyset$-choice ($A^0_s = \emptyset$ for each s). When α is a limit ordinal, the union of the chain of choices $((\bar{x}, \bar{y}, \bar{z})_\beta)_{\beta < \alpha}$ is a M-$\bigcup_{s \in m_2} A^\alpha_s$-choice with correction function zero. Finally, for $\alpha = \beta + 1$, we proceed as follows: we already have, by induction hypothesis, an M-$\bigcup_{s \in \mathcal{P}^-(m_2)} A^\beta_s$-choice with correction function zero, $(\bar{x}, \bar{y}, \bar{z})_\beta$; consider also the choices $(\bar{x}, \bar{y}, \bar{z})^\alpha_s$ for $s \in \mathcal{P}^-(m_2)$. Since the cardinalities of all their domains are $< \lambda^{+m_1}$, we may without loss of generality regard the previous choices as forming a compatible λ^{+m_1-1}-$\mathcal{P}^-(m_2+1)$-system of choices with correction function zero: the set $\{b_i \mid i \in s\} \cup \{\beta\}$ has cardinality $m_2 + 1$. Since $(m_1 - 1) + (m_2 + 1) = m_1 + m_2 < k$, we may apply the induction hypothesis; we obtain $(\bar{x}, \bar{y}, \bar{z})_\alpha$ an M-$\bigcup_{s \in \mathcal{P}^-(m_2)} A^\alpha_s$-choice with correction function zero.

Having constructed this chain $(\bar{x}, \bar{y}, \bar{z})_\alpha$ for $\alpha < \lambda^{+m_1}$, we just let

$$(\bar{x}, \bar{y}, \bar{z}) := \bigcup_{\alpha < \lambda^{+m_1}} (\bar{x}, \bar{y}, \bar{z})_\alpha.$$

This is a $\bigcup_{s \in \mathcal{P}^-(m_2)}$-choice with correction function zero, extending all the choices in the system.

$\square_{2.12}$

We may now obtain our general extension property.

Lemma 2.13. (Full extension)
Let $M \models \psi$ be strongly canonical, $J_1 \subset J_2 \subset P^M_0$, with $|J_2| < \lambda^{+k-1}$ and $(\bar{x}, \bar{y}, \bar{z})$ an M-J_1-choice with correction function identically zero. Then $(\bar{x}, \bar{y}, \bar{z})$ can be extended to an M-J_2-choice with correction function identically zero.

Proof Without loss of generality, $J_2 = J_1 \cup \{b\}$. If J_1 has size $\leq \lambda$, this is lemma 2.9. Now suppose $|J_1| = \lambda^{+m_1} < \lambda^{+k-1}$ (therefore $m_1 < k - 1$) and enumerate J_1 as $\langle a_\beta | \beta < \lambda^{+m_1} \rangle$. Let $J^\alpha_1 = \{a_\beta | \beta < \alpha\}$, and let $(\bar{x}, \bar{y}, \bar{z})_\alpha$ be the restriction of $(\bar{x}, \bar{y}, \bar{z})$ to an M-J^α_1-choice. We define by induction M-J^α_1 choices with correction function identically zero $(\bar{x}, \bar{y}, \bar{z})'_\alpha \supset (\bar{x}, \bar{y}, \bar{z})_\alpha$.

We may use here lemma 2.12 for $m_2 = 2$ to extend $(\bar{x}, \bar{y}, \bar{z})'_\alpha \cup (\bar{x}, \bar{y}, \bar{z})_{\alpha+1}$ to an M-$J^{\alpha+1}_1 \cup \{b\}$-choice with correction function identically zero: since $m_1 < k - 1$, along the induction the cardinality is $< \lambda^{+m_1}$, say $\lambda^{+m'_1}$ for some $m'_1 < m_1$. Since we also have that $m_1 < k - 1$, then $m'_1 + 2 < k$ and we can use $m_2 = 2$ when invoking lemma 2.12.

At limits take unions; finally,

$$\left(\bigcup_{\alpha<\lambda^{+m_1}} \bar{x}'_\alpha, \bigcup_{\alpha<\lambda^{+m_1}} \bar{y}'_\alpha, \bigcup_{\alpha<\lambda^{+m_1}} \bar{z}'_\alpha \right)$$

is an M-J_2-solution extending $(\bar{x}, \bar{y}, \bar{z})$. □

Theorem 2.14. *If $M \models \psi_k^\lambda$ is strongly standard and $|M| < \lambda^{+k}$ then there is an M-choice with correction function identically zero.*

PROOF We apply Lemma 2.13 (starting from the empty choice function, and taking unions at limits): the lemma gives an extension of a choice function with correction function zero from $J_1 \subset P_{1,1}^M$ to J_2 with $J_1 \subset J_2 \subset P_{1,1}^M$ provided $|J_2| < \lambda^{+k-1}$. Here $|M|$ may be *equal to* λ^{+k-1} (at "worst"); if (in that case) we enumerate $P_{1,1}^M$ as $\left\{a_\beta \mid \beta < \lambda^{+k-1}\right\}$ then given $\alpha < \lambda^{+k-1}$, $|\{a_\beta \mid \beta < \alpha\}| < \lambda^{+k-1}$ and we can apply Lemma 2.13 to get an extension of the choice with correction function zero to $\{a_\beta \mid \beta < \alpha\}$. □

Theorem 2.15. (Categoricity and amalgamation up to λ^{+k})

1. *For $m < k$, $\mathrm{Mod}\ (\psi_k^\lambda)$ has a unique strongly standard model M, $|P_0^M| = \lambda^{+m}$, modulo isomorphism.*
2. *For $m < k-1$, if $2^\lambda \leq \lambda^{+m}$, then $\mathcal{K}^*(\lambda, k)$ has amalgamation in λ^{+m}.*
3. *If $m < k$, $\lambda^{+m} \geq 2^\lambda$, then $\mathcal{K}^*(\lambda, k)$ is categorical in λ^{+m}.*

PROOF

1. Let $N \models \psi_k^\lambda$ be a strongly standard model with $\lambda \leq |P_0^N| < \lambda^{+k}$. By Lemma 2.5, once we have $(\bar{x}, \bar{y}, \bar{z})$ a global N-choice with correction function f, then $N \approx M_{I,f}$ for $I = P_0^M$. Now, since N is standard and $|P_0^N| \in [\lambda, \lambda^{+k-1}]$, Theorem 2.14 gives a global N-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function identically zero (as N is strongly standard). So, $N \approx M_I$.
2. In the proofs of the previous lemmas, amalgamation of choices (along systems) with correction function zero is carried out in detail. These give rise to the corresponding embeddings and amalgams of models, if the size of these is controlled by the size of their P_0^M parts. The only part of a standard model where this cardinality may increase is given by the coding of the action of G (remember $|G| = 2^\lambda$). If $2^\lambda \leq \lambda^{+m}$, the model M will have the same size as P_0^M.
3. Let $m < k$, $\lambda^{+m} \geq 2^\lambda$ and let M be a model in $\mathcal{K}^*(\lambda, k)$ of size λ^{+k}. Then by Lemma 1.13, M is isomorphic to a strongly standard model N; also, since $2^\lambda \leq \lambda^{+m}$, $|P_0^M| = \lambda^{+m}$. Thus by part (1) $M \approx N \approx M_I$ for some I of size λ^{+m}.

□

3. Failure of categoricity of ψ_k^λ at $\beth_{k+1}(\lambda)$

We have proved in 2.15 that ψ is categorical in λ^{+m} if $m < k$ and $2^\lambda < \lambda^{+m}$. We now prove that our sentence is not categorical in any cardinality $\kappa \geq \mu = \beth_{k+1}(\lambda)^+$. (It is also possible to show that ψ_k^λ has the maximal number of models possible in μ for each $\mu \geq \beth_{k+1}(\lambda)^+$. We do not do that in this paper.)

As before, we use our terminology of "solutions and corrections functions" to count the number of models.

3.1. *Combinatorial criteria for (failure of) isomorphism*

In this section we prove a combinatorial criterion for **non-isomorphism** between two models of the form $M_{I,f}$.

Before giving the purely combinatorial criterion, we prove the following lemma (a criterion for isomorphism in terms of **choices** and **correction functions**).

Lemma 3.1. *If M_1 and M_2 are strongly standard, and $(\bar{x}, \bar{y}, \bar{z})_\ell$ is an M_ℓ-choice for M_ℓ ($\ell = 1, 2$), $P_0^{M_1} = P_0^{M_2}$ with correction function f_ℓ for $\ell = 1, 2$ then the following are equivalent:*

(a) *there is an isomorphism from M_1 onto M_2 over the identity on $P_0^{M_1} \cup P_1^{M_1}$*

(b)$_1$ *there is an M_2-choice $(\bar{x}, \bar{y}, \bar{z})$ whose correction function is f_1,*

(b)$_2$ *there is an M_1-choice $(\bar{x}, \bar{y}, \bar{z})$ whose correction function is f_2,*

(c) *there are functions g_1, g_2, g_3 ("to correct the choice of zeros"), with*

1. $g_1 : [I]^k \times S \to \mathbb{Z}_2$ *(like the $x_{u,s}$'s above),*
2. $g_2 : [I]^k \times S \to H_I$ *(like the $y_{u,s}$'s above),*
3. $g_3 : [I]^{k+1} \to G$ *(like the z_u's above),*
4. *if $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z\rangle$ are like in Definition 1.1.1 for M_1, or M_2* *<u>then</u>*

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell - h_k(u_0) - g(s) = \sum_{\ell<k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s)$$

Proof

(a) $\to$ (b)$_1$ Recall that $M_1 \restriction \tau^- = M_2 \restriction \tau^-$, so M_1 and M_2 have the same universes. Fix $F : M_1 \xrightarrow{\approx}_{P_0^{M_1} \cup P_1^{M_1}} M_2$. We have, since f_1 is a correction function for M for the choice $(\bar{x}, \bar{y}, \bar{z})_1$, that

$$f_1(u, s) = 0 \Leftrightarrow \langle a_0 \dots a_k u_0 \dots u_k x^1_{u_0 s} \dots x^1_{u_{k-1} s} y^1_{u_k s} z^1_u \rangle \in Q_s^{M_1}.$$

But the right hand side holds iff

$$\langle a_0 \dots a_k u_0 \dots u_k F(x^1_{u_0 s}) \dots F(x^1_{u_{k-1} s}) F(y^1_{u_k s}) F(z^1 u) \rangle \in Q_s^{M_2},$$

since F is an isomorphism fixing $P_0^{M_1} \cup P_1^{M_1}$, and $a_0, \dots, a_k \in P_0^{M_1}$. This gives us the M_2-choice for which f_1 is a correction function: given $u_\ell \subset u$, $u_\ell \in [I]^k$, $u \in [I]^{k+1}$, let $x'_{u_\ell,s} = F(x^1_{u_\ell,s})$, $y'_{u,s} = F(y^1_{u_k,s})$, $z'_u = F(z^1_u)$.

(a) $\to$ **(b)**$_2$ Same.

(b)$_\ell$ $\to$ **(c)** $(\ell = 1, 2)$ The point of (c) is that we may find concrete representations g_1, g_2, g_3, that act *independently from* M_1 *or* M_2 as 'corrected choice functions' for the zeros for f_1 and f_2. So, suppose we have a M_2-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function f_1. Then for any $u \in P_0^{M_2}$ and any $s \in S$, if $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z \rangle$ are like in Definition 1.1.1

$$\langle a_0 \dots a_k u_0 \dots u_k x_{u_0 s} \dots x_{u_{k-1} s} y_{u_k s} z_u \rangle \in Q_s^{M_2}$$

$$\Updownarrow$$

$$f_1(u, s) = 0.$$

But since f_1 is also a correction function for the M_1-choice $(\bar{x}, \bar{y}, \bar{z})_1$,

$$f_1(u, s) = 0$$

$$\Updownarrow$$

$$\langle a_0 \dots a_k u_0 \dots u_k x^1_{u_0 s} \dots x^1_{u_{k-1} s} y^1_{u_k s} z^1_u \rangle \in Q_s^{M_1}.$$

So, we have both $\mathbb{Z} \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s)$ and $\mathbb{Z} \models \sum_{\ell<k} i^1_\ell = h^1_k(u_0) + g^1(s)$, so setting

$$g_1(u_\ell, s) = i^1_\ell, \quad g_2(u_k, s) = h^1_k, \quad g_3(u) = g^1$$

yields

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell - h_k(u_0) - g(s) = \sum_{\ell<k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s).$$

Since f_1 does this for all possible $(k+1)$-tuples, we have all the compability we need.

(c) $\to$ **(a)** If the predicates are the same modulo g_1, g_2 and g_3 then obtaining **(a)** becomes a matter of building $F : M_1 \xrightarrow{\approx}_{P_0^{M_1} \cup P_1^{M_1}} M_2$. Clearly we can start by $F \restriction P_0^{M_1} = id$, and then extend its definition to all the other portions of the model. The only strong restriction to the extension of this to the whole model is given by the relations $Q_s^{M_1}$ and $Q_s^{M_2}$. But part (4) of **(c)** provides this: the functions g_1, g_2, g_3 provide the definition of the isomorphism. Precisely, let $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z\rangle$ be a tuple from M_1; we use (4) to find simultaneously $F(x_\ell)$, $F(y_k)$ and $F(z)$. Compute (in $\mathbb{Z}_2$) the value $\sum_{\ell<k} i_\ell - h_k(u_0) - g(s)$ corresponding to the tuple. For every $s \in S$, this value is 0 iff the tuple belongs to Q_s. By (4), this value is equal to $\sum_{\ell<k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s)$. But also by (4), this value also corresponds to a corresponding tuple $\langle a_0 \dots a_k u_0 \dots u_k x'_0 \dots x'_{k-1} y'_k, z'\rangle$ in M_2. This provides the values $F(x_\ell) = x'_\ell$, $F(y_k) = y'_k$ and $F(z) = z'$: $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z\rangle \in Q_s^{M_1}$ if and only if $\langle a_0 \dots a_k u_0 \dots u_k x'_0 \dots x'_{k-1} y'_k, z'\rangle \in Q_s^{M_2}$.

□

Remark 3.2. *Counting the number of isomorphism types here is akin to the study of* $\mathrm{Ext}(G, \mathbb{Z})$ *in the work of the first author and Väisänen in [13]*[5]*.*

Lemma 3.3. *If* M_{I_1,f_1} *and* M_{I_2,f_2} *are models of* ψ*, and* $h : I_1 \to I_2$ *is one-to-one and onto,* then *the following are equivalent:*

(a) *there is an isomorphism from* M_{I_1,f_1} *onto* M_{I_2,f_2} *extending* h*.*

(b)$_1$ *there is an* M_{I_2,f_2}*-choice* $(\bar{x}, \bar{y}, \bar{z})$ *whose correction function is* f_1*,*

(b)$_2$ *there is an* M_{I_1,f_1}*-choice* $(\bar{x}, \bar{y}, \bar{z})$ *whose correction function is* f_2*,*

(c) *there are functions* g_1*,* g_2*,* g_3 *("to correct the choice of zeros"), with*

1. $g_1 : [I]^k \times S \to \mathbb{Z}_2$ *(like the* $x_{u,s}$*'s above),*
2. $g_2 : [I]^k \times S \to H_I$ *(like the* $y_{u,s}$*'s above),*
3. $g_3 : [I]^{k+1} \to G$ *(like the* z_u*'s above),*
4. *if* $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z\rangle$ *are like in Definition 1.1.1 for* M_{I_1,f_1}*, or* M_{I_2,f_2} *<u>then</u>*

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell - h_k(u_0) - g(s) = \sum_{\ell<k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s)$$

[5]Here $I(\lambda, \psi)$ is counted by the group of correction functions, derived from *some* g_1, g_2, g_3:

$$I(\lambda, \psi_k^\lambda) = \left\{ f \text{ a correction function} \middle| f(u, s) = \sum_{\ell<k} g_1(u_\ell, s) - g_2(u_0, s) - g_3(u) \right\}.$$

PROOF The proof is almost the same as that of the previous lemma (3.1). The main change is that now the identity on I is replaced by a bijection from I_1 onto I_2; the rest of the proof amounts to a renaming via the bijection $F \restriction I_1$. □

An important special case of the previous lemma happens when $I_1 = I = I_2$ *but the isomorphism is not the identity on* I. In this case, the restriction of the isomorphism between M_{I,f_1} and M_{I,f_2} is a *permutation* of I. Our combinatorial criterion for non-isomorphism will focus on this case.

Recall $\mathfrak{D}$ is the regular filter on S generated by sets of the form $\langle u \rangle = \{v \in S | u \subset v\}$, where $S = [\lambda]^{<\aleph_0}$:

$$\mathfrak{D} = \mathfrak{D}_\lambda := \{A \subset S | \exists u_A \in S \forall v \in S (u_A \subset v \to v \in A)\}$$

(see definition 1.2).

The notion of an I-function, which we define next, is central to our combinatorial criterion.

Definition 3.4. $f : [I]^{k+1} \times S \to \mathbb{Z}_2$ *is an* I*-function iff*

$$\{s \in S | f_u(s) \neq 0\} \in \mathfrak{D}, \text{ for all } u \in [I]^{k+1},$$

where $f_u : S \to \mathbb{Z}_2$ *is given by* $f_u(s) = f(u, s)$.

Lemma 3.5. *Let* $f : [I]^{k+1} \times S \to \mathbb{Z}_2$ *be an* I*-function.* Then, *the following is a sufficient condition for*

$$M_{I,f} \not\approx M_I :$$

(⋆) *for every* $F_1 : [I]^k \to [I]^{\leq \lambda}$, $F_2 : [I]^k \to {}^S(\mathbb{Z}_2)$ *and* π *a permutation of* I, *there exists* $u = \{t_0, \ldots, t_k\} \in [I]^{k+1}$ *(i.e., with no repetitions) such that*

- (α) $t_k \notin F_1(\{t_0 \ldots t_{k-1}\})$,
- (β) $f_{\pi\{t_0,\ldots,t_k\}} - \sum_{\ell<k} F_2(\{t_0, \ldots, t_k\} \setminus \{t_\ell\}) \notin G$.

Before proving Lemma 3.5, we note:

- First, (⋆) is a purely combinatorial statement; this will enable us to focus solely on combinatorial principles to prove the failure of categoricity.
- Also, by the definition of G, (β) says that for $\mathfrak{D}$-few elements $s \in S$ do we have

$$f_{\pi(t_0,\ldots,t_k)}(s) = \sum_{\ell<k} F_2(\{t_0, \ldots, t_k\} \setminus \{t_\ell\})(s).$$

Notice the role of the permutation π of I in the combinatorics that follows.

Proof of 3.5. Assume that $M_{I,f} \approx M_I$. Then, since $M_I \approx M_{I,o}$ (the null correction function) we may apply Lemma 3.3 and (by $(b)_2$ of that lemma) assume that $(\bar{x}, \bar{y}, \bar{z})$ witnesses $M_{I,f} \approx M_{I,0}$, with correction function identically zero.

We construct F_1, F_2 such that $(\star)$ of 3.5 does not hold (for the permutation π induced by the isomorphism between $M_{I,f}$ and M_I).

We first let $F_1 : [I]^k \to [I]^{\leq\lambda}$ be

$$F_1(v) = \bigcup \{w \in [I]^k | \text{ for some } s_1 \in S, y_{v,s_1}(w) \neq 0\}.$$

This is well defined, as $F_1(u)$ is a union of $|S|$-many finite sets. Also, let

$$F_2(v) = \langle x_{v,s} | s \in S \rangle.$$

We will show that **no** $u \in [I]^{k+1}$ satisfies both (α) and (β) of condition $(\star)$.

Suppose otherwise; let then $u = \{t_0, \dots, t_k\} \in [I]^{k+1}$ satisfy $(\alpha) + (\beta)$. Let as usual $u_\ell = u \setminus \{t_\ell\}$. By (α), for each s,

$$y_{u_k,s}(u_0) = 0.$$

[Why? Just notice that by (α),

$$t_k \notin F_1(u_k) = \bigcup \{v \in [I]^k | \text{ for some } s_1 \in S, y_{u,s_1}(v) \neq 0\},$$

so for all $v \in [I]^k$, if $t_k \in v$, then for all $s_1 \in S$ we have $y_{u_k,s_1}(v) = 0$. In particular, as $t_k \in u_0$, $y_{u_k,s_1}(u_0) = 0$.]

Now, since $(\bar{x}, \bar{y}, \bar{z})$ is an $M_{I,f}$-choice with correction function identically zero, for each $s \in S$ we have that

$$\langle a_0, \dots, a_k, u_0, \dots, u_k, x_{u_0,s}, \dots, x_{u_{k-1},s}, y_{u_k,s}, z_u \rangle$$

belongs to $Q_s^{M_{I,f}}$ if and only if (by the definition of this predicate in the model $M_{I,f}$)

$$\mathbb{Z}_2 \models \sum_{\ell<k} x_{u_\ell,s} = y_{u_k,s}(u_0) + z_u(s) + f_{\pi(u)}(s).$$

But

But we also have that $z_u(s) = 0$ for the $\mathfrak{D}$-majority of $s \in S$ (by the definition of G); since we also have that $y_{u_k,s}(u_0) = 0$ for our particular u,

(*) For the $\mathfrak{D}$-majority of $s \in S$

$$\sum_{\ell<k} x_{u_\ell,s} = f_{\pi(u)}(s).$$

But this contradicts (β). □

Remark 3.6. *We can then regard* F_2 *as*

$$F_2 : [I]^k \to {}^S(\mathbb{Z}_2)/G.$$

Corollary 3.7. *If* f_1, f_2 *are* I*-functions, and* $f = f_1 - f_2$ *(coordinatewise) satisfies* $(\star)$*, then* $M_{I,f_1} \not\approx M_{I,f_2}$.

Proof We already know that since f satisfies $(\star)$, $M_{I,f} \not\approx M_I$. Now suppose we have an isomorphism $F : M_{I,f_1} \xrightarrow{\approx} M_{I,f_2}$. As before, $\pi = F \restriction I$ is a permutation of I, and the automorphism lifts in a natural way to all components of $M_{I,f}$ in the vocabulary τ^-. Now, the remaining part of τ: if $s \in S$, then a tuple $\langle a_0, \dots, a_k, u_0, \dots, u_k, x_{u_0,s}, \dots, x_{u_{k-1},s}, y_{u_k,s}, z_u\rangle$ belongs to Q_s in $M_{I,f}$ if and only if (for the corresponding indices)

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s) + f(u,s)$$

but this holds if and only if

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s) + f_1(u,s) - f_2(u,s).$$

Now, since F is an isomorphism, this is true if and only the tuple $\langle F(a_0), \dots, F(a_k), F(u_0), \dots, F(u_k), F(x_{u_0,s}), \dots, F(x_{u_{k-1},s}), F(y_{u_k,s}), F(z_u)\rangle$ belongs to Q_s in M_I, this is if and only if

$$\mathbb{Z}_2 \models \sum_{\ell<k} i'_\ell = h'_k(u'_0) + g'(s)$$

where the primes denote the values corresponding to the F-images of components of the long tuple. But this witnesses that F is also an isomorphism between M_{I,f_1} and M_{I,f_2}, which contradicts the hypothesis. □

We now have what we need for a proof of failure of categoricity at some μ above the categoricity cardinals. Notice we do *not* give an optimal (minimal) such μ; this is left for (possible) later work.

Theorem 3.8. *For some* $\mu > \lambda^{+k}$*, the sentence* ψ^λ_k *is not categorical in* μ.

Proof Let μ be a cardinal with the following properties:

$\otimes_1$ $\mu \to (\omega)^k_{2^\lambda}$,

$\otimes_2$ $\mu \not\to (\omega)^{k+1}_{2^\lambda}$,

$\otimes_3$ μ regular.

The existence of such a μ uses the Erdös-Rado theorem (the partition

$$\beth_k(\lambda)^+ \to \left((2^\lambda)^+\right)^k_{2^\lambda}$$

is an instance) for $\otimes_1$ and the negative partition relation $\beth_{k+1}(\lambda) \not\to (k+2)^{k+1}_{2^\lambda}$ (a consequence of [12, Lemma 24.1(e)]) for $\otimes_2$; we may therefore take μ as $\beth_k(\lambda)^+$.

Let then I have cardinality μ and let $f : [\mu]^{k+1} \to G^+/G$ be an I-function witnessing $\otimes_2$ (recall that G^+ denotes the group ${}^S\mathbb{Z}_2$). We use our criterion 3.5 to show that $M_{I,f}$ and M_I can not be isomorphic from which we conclude that the sentence ψ^λ_k is not categorical in μ.

Let $F_1 : [I]^k \to [I]^{\leq\lambda}$, $F_2 : [I]^k \to G^+$ and π a permutation of I.

Now find $E \subset \mu$ club such that

$$\alpha_0 < \cdots < \alpha_k \in E \Longrightarrow \begin{cases} F_1(\alpha_0, \dots, \alpha_{k-1}) \subset \alpha_k, \\ \pi(\alpha_0), \dots, \pi(\alpha_{k-1}) < \alpha_k. \end{cases}$$

This is possible by the regularity of μ.

Now apply $\otimes_1$ to $F_2 \restriction E$: since $\mu \to (\omega)^k_{2^\lambda}$, there must be an infinite ω-sequence $T = \{\alpha_0 < \alpha_1 < \dots \alpha_n < \dots\}$ such that $F_2 \restriction [T]^k$ is constant. Therefore we have, for $u = \{\alpha_0, \dots, \alpha_k\}$ and $u_\ell = u \setminus \{\alpha_\ell\}$:

- $\alpha_k \notin F_1(\{\alpha_0, \dots, \alpha_{k-1}\})$ (since these are elements from the club E) and
- the equation $f_{\pi\{\alpha_0,\dots,\alpha_k\}}(s) = \sum_{\ell<k} F_2(u_\ell)(s)$ holds for $\mathfrak{D}$-few elements s: as F_2 is constant on u_ℓ from the monochromatic sequence, the sum on the right hand side will be 0 when k is even (and 1 when k is odd) whereas the value on the left hand side will not be constant, by $\otimes_2$ applied to f.

The previous two properties correspond to (α) and (β) of the criterion from Lemma 3.5. Therefore, $M_{I,f} \not\approx M_I$ and the sentence ψ^λ_k is not categorical in μ. □

The result also holds for all $\kappa \geq \beth_{k+1}(\lambda)^+$ (we will show monotonicity of the crucial criterion).

Conclusion 3.9. *The sentence ψ^λ_k is not categorical in any $\kappa \geq \beth_{k+1}(\lambda)^+$.*

Proof Let $\kappa \geq \mu = \beth_{k+1}(\lambda)^+$. If $\kappa = \mu$, Theorem 3.8 shows how to get **two** non-isomorphic models. If $\kappa > \mu$ then let J be a set of cardinality κ.

We show that as $\kappa > \mu$ we may pick a J-function $f : [J]^{k+1} \times S \to \mathbb{Z}_2$ satisfying the criterion $(\star)$ of Lemma 3.5 (which will enable us to conclude that $M_{J,f} \not\approx M_J$, and thus conclude failure of categoricity at κ).

Let first $F_1 : [J]^k \to [J]^{\leq\lambda}$, $F_2 : [J]^k \to {}^S\mathbb{Z}_2$ and π a permutation of J. Let $I \subset J$ with $|I| = \mu$, I closed under π and such that $F_1 \restriction [I]^k : [I]^k \to [I]^{\leq\lambda}$. [Such an I exists by closing first under iterating taking the unions of F_1-images of k-tuples from I and taking the union of μ many sets of cardinality $\leq \lambda < \mu$ - after an ω—iteration the result is closed under F_1-images. Similarly, we close under images and preimages under the permutation π and alternate these closure operations ω many times.]

Let now $f : [J]^{k+1} \times S \to \mathbb{Z}_2$ be a J-function that witnesses $\otimes_2$ on the set I; as $|I| = \mu$, this is possible.

Furthermore, for the set I, the functions $F_1 \restriction I$, F_2 and $\pi \restriction I$ are in the situation of Lemma 3.5. The proof of Theorem 3.8 applies then, as $|I| = \mu$. We then obtain $u = \{t_0, \dots, t_k\} \in [I]^{k+1}$ such that (α) and (β) of the criterion hold. But these properties are also true of the original F_1, F_2, π. Therefore, $M_{J,f} \not\approx M_J$. □

Remark 3.10. *Here are some important differences between the structure of this proof and that of [1]:*

1. *The use of the filter $\mathfrak{D}$ is central here - it is not needed there.*
2. *The way the group itself is used is slightly different at the end of the proof.*

We conjecture that the class has the maximal number of models at all $\mu > \beth_{k+1}(\lambda)^+$.

4. Further directions

After our generalization of the original Hart-Shelah example to the stronger logic $L_{(2^\lambda)^+,\omega}$, we have the following situation:

- Any generalization of the early results from 1983 for the logic $L_{(2^\lambda)^+,\omega}$ must necessarily use as hypothesis few models in all cardinalities $\lambda, \lambda^+, \dots, \lambda^{+k}, \dots$ for all $k < \omega$. The first author has written several papers in this direction (see [14]), in the (wider) context of AECs.
- On the other hand, the necessity of an interval of $\aleph_0$-many cardinals with few models to start the machinery for categoricity transfer seems interesting per se; and even more so the fact that this would happen along all strengthenings of $L_{\omega_1,\omega}$ (inside $L_{\infty,\omega}$).
- Finally, we conjecture that our sentence may be analyzed in terms of building frames, in the spirit of the work of Boney and Vasey [10]. Specifically, we conjecture that our abstract elementary class

$$\mathcal{K}^*(\lambda, k) = (Mod(\psi_k^\lambda), \prec_{(2^\lambda)^+,\omega})$$

is $(<\lambda, \lambda^{+k-1})$-tame, $(<\lambda, \lambda^{+k-1})$-typeshort over models of size λ^{+k-2}, and that

1. for each $m \leq k-1$ there is a frame $\mathfrak{s}^*(\lambda, k)_m$ that is type-full and λ^{+m}-good on $\mathrm{Mod}(\psi_k^\lambda)$,
2. The (type-full and λ^{+k-1}-good) frame $\mathfrak{s}^*(\lambda, k)_{k-1}$ is not weakly successful.

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