# DENUMERABLE COMPACT SPACES AND CANTOR DERIVATIVE

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Abstract. A continuous open finite-to-one surjection $f$ preserves Cantor rank. If the domain of $f$ is a Hausdorff compact space, Cantor degree variations are bounded by the maximal size of the finite fibres. If the fibres are infinite, we show inequalities involving the maximal and minimal rank of the fibres. A closed preorder on a denumerable Hausdorff compact space is the intersection of clopen preoders. We compute the Cantor rank of a cartesian product and build a semiring where the Cantor derivative is a derivation.

The Cantor derivative was introduced by Georg Cantor in 1872 to derivate sets of convergence of trigonometric series [1]. In Model Theory, one century later, Cantor-Bendixson rank gave birth to Morley rank in $\omega$-stable theories [6] and to Cantor rank in small ones. We shall specify some properties of this rank, well known by logicians when they refer to Morley rank. We first notice that the Cantor derivative of a topological sum and cartesian product is well-behaved. A continuous open finite-to-one surjection $f$ preserves Cantor rank. Moreover, if the domain of $f$ is a Hausdorff compact space, Cantor degree variations can be bounded by the maximal size of the finite fibres. If the fibres are infinite, there are still inequalities involving the maximal and minimal rank of the fibres. Thanks to Cantor rank, a closed preorder on a countable Hausdorff compact space can be shown to be the intersection of clopen preoders. This was almost noticed in [7, 4]. We build an ordered division semiring in which the Cantor derivative is a derivation. We consider when this semiring can be given a lattice structure. We finish by applying the results to first order theories having countably many types. This gives a new proof of a Theorem in [5] computing Cantor rank and degree over an algebraic tuple.

Most of section 1 and 2 must be well known. However, I could not find any reference except [3]. Proposition 5, Theorem 14 and Proposition 16 along with section 3, 4 and most of 5 seem to be new.

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## 1. Rank of a denumerable Polish space

Let $X$ be a Hausdorff topological space. We shall write $A^c$ for the complement of a subset $A$ of $X$. A point is *isolated* if it is an open set. Otherwise, it is a *limit point*. We shall write $X'$ for the *derivative* of $X$, that is the set of limit points with the induced topology. It is a closed subset. $X$ is *perfect* if all points are limit points. Define a descending chain of closed subsets $X^\alpha$ inductively by setting :

$$X^0 = X$$
$$X^{\alpha+1} = \big(X^\alpha\big)' \text{ for a successor ordinal}$$
$$X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha \text{ for a limit ordinal } \lambda$$

The *perfect kernel* of $X$ is the intersection of every $X^\alpha$ when $\alpha$ runs over ordinals, written $X^\infty$. A *Polish space* is a separable, completely metrisable topological space. We refer to [3] for more details about Polish spaces. Polish spaces can be partitioned into a perfect kernel and a countable rest. This is the well known Cantor-Bendixson Theorem :

**Theorem 1.** *(Cantor-Bendixson) Let $X$ be a Polish space. There is a unique decomposition of $X$ of the form $P \cup C$ where $P$ is perfect and $C$ countable open.*

Therefore, if $X$ is a denumerable Polish space, its perfect kernel is empty. We say that $X$ is *ranked* by Cantor-Bendixson rank, and call its *Cantor-Bendixson rank* the least ordinal $\beta$ such that $X^\beta$ is empty. If $T$ denotes the topology on $X$, we write $CB(X,T)$ or $CB(X)$ for the rank of $X$. Define the *Cantor-Bendixson rank* of a point $x$ in $X$, written $CB(x,X)$ or $CB(x)$, by the maximal ordinal $\alpha$ such that $X^\alpha$ contains $x$. We have

$$CB(X) = sup\{CB(x) : x \in X\}$$

Note that $x$ has rank at least $\alpha+1$ if and only if it is a limit point of points having rank at least $\alpha$. Moreover, $x$ is isolated from points of greater rank. Define the *Cantor rank* $CB(O)$ of every open set $O$ in $X$ as the Cantor rank of the induced topological space $O$. We have

$$CB(x) = min\{CB(O) : x \in O,\ O \text{ open set}\}$$

*Proof.* Let $\alpha$ be the rank of $x$. An open set $O$ isolates $x$ from points of greater ranks, so $O^{\alpha+1}$ is empty. Conversely, show inductively that if $x$ is in $X^\alpha$ and $O$ is an open set containing $x$, then $x$ is in $O^\alpha$ too. If $x$ is in $X^{\alpha+1}$, then $x$ is a limit point in $X^\alpha$. If $x$ is not a limit point in $O^\alpha$, then it is a limit point in $\big(O^c\big)^\alpha$, so $x$ belongs to $O^c$, a contradiction. □

Therefore, the rank of a point does not depend on the neighbourhood in which it is calculated. We shall write $CB(x)$ for it. Note that the rank decreases when the topology gets finer :

**Proposition 2.** *Let $X$ be a set, $T_1 \subset T_2$ two topologies on $X$. Then $CB(X, T_1)$ is smaller than $CB(X, T_2)$.*

Given a family of topological spaces $A_i$ indexed by $I$, we write $\bigoplus_I A_i$ for their *direct sum*, that is their disjoint union with the topology defined as follows : an open set of $\bigoplus_I A_i$ is any set the intersection of which with every $A_i$ is open in $A_i$. Recall that any ordinal can be uniquely written as $\omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k$ where $\alpha_1, \ldots, \alpha_k$ is a strictly decreasing chain of ordinals and $n_1, \ldots, n_k$ are integers. This is known as *Cantor normal form.* If $\alpha$ and $\beta$ are two ordinals with normal form $\omega^{\alpha_1} \cdot m_1 + \ldots + \omega^{\alpha_k} \cdot m_k$ and $\omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k$ respectively, their *Cantor sum* $\alpha \oplus \beta$ is defined by :

$$\alpha \oplus \beta = \omega^{\alpha_1} \cdot (m_1 + n_1) + \ldots + \omega^{\alpha_k} \cdot (m_k + n_k)$$

See [2] for more explanations. Let us now compute the derivative and rank of a union, direct sum and cartesian product of topological spaces :

**Proposition 3.** *Let $A$ and $B$ be subsets of a topological space $X$.*

1) *If $A \subset B$, then $A' \subset B'$, and $CB(A) \leq CB(B)$.*
2) *$\big(A \cup B\big)^\alpha = A^\alpha \cup B^\alpha$ for all $\alpha$.*
3) *$CB(A \cup B) = max\{CB(A), CB(B)\}$*

*Proof.* 1) If $x$ is isolated in $B$, it is in $A$. 2) After 1), $A' \cup B' \subset (A \cup B)'$. Conversely, if $U$ and $V$ isolate $x$ in $A$ and $B$ respectively, then $U \cap V$ isolates $x$ in $A \cup B$, and $(A \cup B)' \subset A' \cup B'$. Finish inductively. □

**Proposition 4.** *Let $(A_i)_{i \in I}$ be topological spaces.*

1) *$\big(\bigoplus_{i \in I} A_i\big)^\alpha = \bigoplus_{i \in I} A_i^\alpha$ for every ordinal $\alpha$.*
2) *$CB\big(\bigoplus_{i \in I} A_i\big) = sup\{CB(A_i) : i \in I\}$*

*Proof.* 1) After the previous Proposition, $\bigoplus_{i \in I} A_i' \subset (\bigoplus A_i)'$. Conversely, if $x$ is a limit point in $\bigoplus_{i \in I} A_i$, it belongs to a unique $A_i$ in which it cannot be isolated. Finish inductively. 2) Follows from 1). □

**Proposition 5.** *Let $A$ and $B$ be non empty topological spaces.*

1) *$\big(A \times B\big)^\alpha = \bigcup_{\beta \oplus \gamma = \alpha} A^\beta \times B^\gamma$ for all ordinal $\alpha$.*
2) *$CB(A) \oplus CB(B) \leq CB(A \times B) + 1 \leq CB(A) \oplus CB(B) + 1$*

*Proof.* 1) If $a$ is isolated in $A$, and $b$ in $B$, so is $(a, b)$ in $A \times B$, and $(A \times B)'^c \supset A'^c \times B'^c$. Conversely, if $a, b$ is isolated in $A \times B$, there is pair $U, V$ of open sets in $A$ and $B$ respectively such that $U \times V$ isolates $a, b$. So $U$ isolates $a$, $V$ isolates $b$, and $(A \times B)'^c \subset A'^c \times B'^c$. Therefore,

$$\big(A \times B\big)' = A' \times B \cup A \times B'$$

Inductively, suppose that for an ordinal $\alpha$, we have

$$\big(A \times B\big)^\alpha = \bigcup_{\beta \oplus \gamma = \alpha} A^\beta \times B^\gamma$$

Note that the union runs over a finite set. Then

$$\big(A\times B\big)^{\alpha+1} = \Big(\bigcup_{\beta\oplus\gamma=\alpha} A^\beta\times B^\gamma\Big)' = \bigcup_{\beta\oplus\gamma=\alpha}\big(A^\beta\times B^\gamma\big)'$$

$$\bigcup_{\beta\oplus\gamma=\alpha}\Big(A^{\beta+1}\times B^\gamma\cup A^\beta\times B^{\gamma+1}\Big) = \bigcup_{\beta\oplus\gamma=\alpha+1} A^\beta\times B^\gamma$$

2) follows from 1). □

Finite-to-one continuous open surjections preserve Cantor rank :

**Proposition 6.** *Let $X$ and $Y$ be topological spaces, and $f$ a map from $X$ to $Y$. Then,*

1) *If $f$ is an open surjection, $CB(X)\geq CB(Y)$.*
2) *If $f$ continuous and finite-to-one, $CB(X)\leq CB(Y)$.*

*Proof.* 1) We Show $f^{-1}(Y')\subset f^{-1}(Y)'$. Let $y$ be a limit point in $Y$ and $x$ a preimage of $y$. For every neighbourhood $O$ of $x$, $f(O)$ is an infinite neighbourhood of $y$, so $O$ is infinite. Inductively, we have $f^{-1}(Y^\alpha)\subset f^{-1}(Y)^\alpha$. As $f$ is onto, $Y^\alpha\subset f(X^\alpha)$. 2) Let $x$ be a limit point in $X$ and $O$ a neighbourhood of $f(x)$. The neighbourhood $f^{-1}(O)$ of $x$ is infinite, and so are $f(f^{-1}(O))$ and $O$. Inductively, $f(X^\alpha)\subset f(X)^\alpha$. □

## 2. Degree of a denumerable compact space

If $X$ is a denumerable compact space, its rank is a successor ordinal $\alpha+1$. Write $CB^*(X)$ for $\alpha$. Moreover, $X^\alpha$ is a finite set ; we call *Cantor degree* of $X$ its cardinal, written $dCB(X)$. Among denumerable Polish spaces, Cantor degree characterises compact spaces :

**Proposition 7.** *Let $X$ be a Hausdorff space ranked by Cantor rank. If every closed subset of $X$ has a finite Cantor degree, then $X$ is a compact space.*

*Proof.* Let $(F_j)_{j\in J}$ be a family of closed sets with empty intersection. Among finite intersections of $F_j$, choose one, namely $I$, having minimal rank and degree. If $I$ is not empty, choose $x$ in $I$ with maximal rank. As the intersection of all $F_j$ is empty, there exists $F_j$ missing $x$, so either the rank or the degree of $I\cap F_j$ decreases. □

**Corollary 8.** *Let $X$ be a second countable Hausdorff space. $X$ is ranked by Cantor rank and every closed set has a finite degree if and only if $X$ is a denumerable compact.*

*Proof.* If every closed set of $X$ has a Cantor degree, then $X$ is compact. As $X$ is ranked by Cantor rank, $X$ embeds in its open basis. □

**Corollary 9.** *Let $A$ and $B$ be two compact spaces, then*

$$CB^*(A\times B) = CB^*(A)\oplus CB^*(B)$$

*Remark* 10. One can inductively build a set having Cantor rank $\alpha$. Let $X_1$ be the subset of fractions $1/n$ in $\mathbf{Q}$ together with zero. $X_1$ has only one limit point so has $CB^*$ rank one. If $\alpha$ is a successor's successor, take $X_\alpha \times X_1$ for $X_{\alpha+1}$. For a limit ordinal $\lambda$, set $X_\lambda$ the sum $\bigoplus_{\alpha<\lambda} X_\alpha$, which is locally compact, and $X_{\lambda+1}$ the one-point compactification of $X_\lambda$. Note that $X_\alpha$ has same cardinality as $\alpha$.

Let us compute the degree of a disjoint union and cartesian product :

**Proposition 11.** *Let $X$ and $Y$ be topological spaces, $A$ and $B$ two disjoint subsets in $X$ having same Cantor rank. Then,*

1) $dCB(A \cup B) = dCB(A) + dCB(B)$
2) $dCB(X \times Y) = dCB(X) \cdot dCB(Y)$

*Proof.* 1) Let $\alpha$ be the rank of $A$. The sets $A^\alpha$ and $B^\alpha$ are disjoint, so

$$dCB(A \cup B) = |A^\alpha \cup B^\alpha| = |A^\alpha| + |B^\alpha|$$

2) Let $\alpha$ be the rank of $X$ and $\beta$ the rank of $Y$.

$$dCB(X \times Y) = \Big| \bigcup_{\gamma\oplus\delta=\alpha\oplus\beta} X^\gamma \times X^\delta \Big| = |X^\alpha \times Y^\beta| = |X^\alpha| \cdot |Y^\beta|$$

□

In compact Hausdorff spaces, Cantor rank and degree have the following characterisation, well known by logicians :

**Proposition 12.** *Let $X$ be a denumerable Hausdorff compact space.*

1) *$CB(X) \geq \alpha + 1$ if and only if there are infinitely many open sets $(O_i)_{i\geq 1}$ of rank at least $\alpha$ with $CB(O_i \cap O_j) < \alpha$ for all $i \neq j$.*
2) *The degree of $X$ is the greatest number $d$ of open sets $O_1, \dots, O_d$ having rank $CB(X)$ with $CB(O_i \cap O_j) < CB(X)$ for all $i \neq j$.*

*Proof.* 1) Let $(O_i)_{i\geq 1}$ be a sequence of open sets having rank at least $\alpha$ with $CB(O_i \cap O_j) < \alpha$ for all $i \neq j$. The sets $O_i^\alpha$ are nonempty and disjoint, so $X^\alpha$ is infinite, and has a limit point. Conversely, if $X^{\alpha+1}$ is not empty, $X^\alpha$ has infinitely many isolated points $x_i$ isolated by $O_i$ respectively. So $CB(O_i) \geq \alpha$, and $CB(O_i \cap O_j) < \alpha$ for all $i \neq j$.
2) Let $O_1, \dots, O_d$ be open sets in $X$ having rank $\alpha$ with small ranked intersections. The sets $O_i^\alpha$ are disjoint and nonempty so $d \leq dCB(X)$. Conversely, let $x_1, \dots, x_d$ be an enumeration of $X^\alpha$. One can find open sets $O_i$ containing $x_i$ respectively, but no $x_j$ if $j \neq i$. Then, $CB(O_i \cap O_j) < \alpha$. The set $O_i^\alpha$ contains $x_i$, so $CB(O_i) \geq \alpha$. □

*Remark* 13. If $X$ is a denumerable compact, $X$ is a metric space and has a clopen basis. Replacing each $O_i$ in the previous Proposition by a basis clopen set included in $O_i$, which does not affect the rank of $O_i$, we can assume that every $O_i$ is clopen. Replacing inductively each $O_i$ by the clopen set $O_i \setminus (O_i \cap \bigcup_{j<i} O_j)$, which does not affect the rank of $O_i$ either, one can assume that the sets $O_i$ are disjoint.

**Theorem 14.** *Let $X$ and $Y$ be Hausdorff compact spaces. If $f$ is a continuous open $n$-to-one surjection from $X$ to $Y$, then $X$ and $Y$ have the same rank and*

$$dCB(Y) \leq dCB(X) \leq n \cdot dCB(Y)$$

*Proof.* If $O_1, \dots O_d$ are $d$ open sets in $Y$ having maximal rank with small ranked intersection, the open sets $f^{-1}(O_1), \cdots, f^{-1}(O_d)$ in $X$ have maximal rank and small rank intersection, so $dCB(X) \geq dCB(Y)$. Conversely, let $d$ be the degree of $Y$, and $O_0, \dots, O_{d \cdot n}$ a sequence of $d \cdot n + 1$ open sets in $X$ with maximal rank and small intersection. For every subset $I$ of $[0, d \cdot n]$ having at least $n+1$ points, the intersection $\bigcap_{i \in I} f(O_i^\alpha)$ is empty, so there exist $d+1$ disjoint subsets $I_0, \dots, I_d$ of $[0, d \cdot n]$ such that for all $j$, the set $\bigcap_{i \in I_j} f(O_i^\alpha)$ is nonempty, and $I_j$ is maximal with this property. Let us write $V_j$ for $\bigcap_{i \in I_j} f(O_i)$. Every $V_j$ is an open set in $Y$, with the same rank as $Y$, and $V_j \cap V_k$ has small rank for $k \neq j$ in $[0, d]$, a contradiction with $Y$ having degree $d$. □

Let $X$ be a topological space, $R$ an equivalence relation on $X$. For every subset $A$ of $X$, set $R^{-1}(A)$ the union of classes of $R$ intersecting $A$. $R$ is *continuous* if for every open set $O$ in $X$, $R^{-1}(O)$ is an open set in $X$. Recall that $R$ is continuous if and only if the quotient map from $X$ onto $X/R$ is open.

**Corollary 15.** *Let $X$ be a Hausdorff compact space and $R$ a continuous equivalence relation on $X$ every class of which is finite.*

1) $CB(X) = CB(X/R)$
2) *If every class of $R$ has cardinality at most $n$,*

$$dCB(X) \leq dCB(X/R) \leq n \cdot dCB(X)$$

*Proof.* The quotient map is $n$-to-one, continuous and open. □

**Proposition 16.** *Let $X$, $Y$ be compact spaces, $f$ a map from $X$ to $Y$.*

1) *If $f$ is open, surjective, with fibres having $CB^*$ rank at least $\alpha$, then $CB(X) \geq \alpha + CB(Y)$.*
2) *If $f$ is continuous with fibres having $CB^*$ rank at most $\alpha$, then $\alpha + CB(Y) \geq CB(X)$.*

*Proof.* 1) We show $f(X)^\beta \subset f(X^{\alpha+\beta})$ for all $\beta$. The space $X$ is the union of the fibres $F_i$, and $\bigcup F_i^\alpha \subset X^\alpha$, so $f(X)$ equals $f(X^\alpha)$. It was shown in Proposition 6 that $f(X)^\beta \subset f(X^\beta)$ for all $\beta$. So

$$f(X)^\beta = f(X^\alpha)^\beta \subset f((X^\alpha)^\beta) = f(X^{\alpha+\beta})$$

2) We show that $f(X^{\alpha+\beta}) \subset f(X)^\beta$ for all $\beta$. As $X^\alpha$ is in $\bigcup F_i^\alpha$, the restriction of $f$ to $X^\alpha$ is a continuous finite-to-one map, and

$$f(X^{\alpha+\beta}) = f((X^\alpha)^\beta) \subset f(X^\alpha)^\beta$$

□

## 3. Closed preorders in a countable compact space

Let $X$ be a topological space and $R$ a binary relation on $X$. We say that $R$ is *closed*, respectively *open* or *clopen* if the set of pairs being related by $R$ is a closed, respectively an open or clopen set in $X \times X$. Note that an open equivalence relation is also continuous. A *preorder* on $X$ is a reflexive transitive binary relation. The following Proposition is inspired by [7, 4].

**Proposition 17.** *Let $X$ be a countable Hausdorff compact space, and $R$ closed preorder on $X$. Then $R$ is the intersection of clopen preorders.*

*Proof.* $X$ has a clopen basis. Set $F$ to be the closed set of tuples in $X \times X$ related by $R$. If $x$ and $y$ are not related by $R$, there exists a basic open set $O_1 \times O_2$ outside $F$ containing the tuple $(x, y)$ ; the set $O_1 \cap O_2$ is empty as $R$ is reflexive. We choose $O_1$ and $O_2$ such that $(O_1 \cup O_2)^c$ have minimal Cantor rank and degree, and write $Y$ for $(O_1 \cup O_2)^c$. We show that $Y$ is empty ; otherwise, let $y$ be in $Y$ with maximal rank. If $(O_1 \times \{y\}) \cap F$ and $(\{y\} \times O_2) \cap F$ are both nonempty, as $R$ is transitive, $(O_1 \times O_2) \cap F$ is also nonempty, a contradiction. We may assume $(O_1 \times \{y\}) \cap F$ to be empty. The set $O_1 \times \{y\}$ is contained in the open set $F^c$. So we can choose a basic open set $Q_2$ containing $y$ with $O_1 \times Q_2 \subset F$. But $O_1 \times (Q_2 \cup O_2)$ is outside $F$. So $(O_1 \cup O_2 \cup Q_2)^c$ equals $Y^c \cap Q_2^c$, which misses $y$, a contradiction with the degree of $Y$ being minimal. So $Y$ is empty, $X$ equals $O_1 \cup O_2$, and $O_1 \times O_1^c \subset F^c$. Therefore, $F \subset (X \times O_1) \cup (O_1^c \times X)$, and $aRb$ implies $aR_{x,y}b$ where $R_{x,y}$ is the preorder defined by

$$aR_{x,y}b \iff \big(a \in O_1 \Rightarrow b \in O_1\big)$$

We have shown that $aRb$ is equivalent to $\bigwedge_{xRy \in F^c} aR_{x,y}b$. □

## 4. A word on Cantor's derivative

Let $X$ be a Hausdorff topological space, $2^X$ its power set. With the union law, $2^X$ is a semigroup. According to Proposition 3, the function mapping a subset of $X$ to its derivative is linear. Should $2^X$ be closed under cartesian product, Proposition 5 would be a Leibniz formula. Let us recall that limit points and derivated spaces have both been introduced by G. Cantor to derivate sets of convergence of trigonometric series [1]. Cantor named them *Grenzpunkt* and *abgeleitete Punktmenge* respectively, and already wrote $P'$ or $P^{(1)}$ for the first derivative of a set of points $P$. We do not know whether he had in mind a Leibniz formula or not, it is puzzling however that the class of topological spaces can naturally be turned into a semiring where Cantor's derivative is actually a derivation.

We call *$\omega$-embedding* an open continuous finite-to-one map between topological spaces. We write $\mathcal{T}op_\omega$ for the category of topological spaces the arrows of which are the $\omega$-embeddings. The relation

$$A \leq B \text{ if there exists an } \omega\text{-embedding of } A \text{ in } B$$

is a partial order on $\mathcal{T}op_\omega$. We shall write $A \equiv B$ to mean $A \leq B$ and $B \geq A$. It is an equivalence relation.

**Proposition 18.** *Let $A$ and $B$ be topological spaces. $A \oplus B$ is the unique least $C$ modulo $\equiv$ such that $A \leq C$ and $B \leq C$.*

*Proof.* If $A$ and $B$ embed in $C$ via $f$ and $g$ say, then $A \oplus B$ embeds in $C$ via the map mapping $a \in A$ to $f(a)$ and $b \in B$ to $g(b)$. Conversely, $A$ and $B$ both embed in $A \oplus B$. $\square$

We call *topological union* of $A$ and $B$ the equivalence class of $A \oplus B$ modulo $\equiv$, written $A \vee B$. So $A \vee B \equiv B \vee A$, and $A \times B \equiv B \times A$. The laws $\oplus$ and $\times$ survive modulo $\equiv$ :

**Proposition 19.** *Let $A$, $B$ and $C$ be topological spaces.*

1) *If $A \leq B$, then $A \vee C \leq B \vee C$.*
2) $(A \vee B) \vee C \equiv A \vee (B \vee C)$
3) *If $A \leq B$, then $A \times C \leq B \times C$.*
4) $A \times (B \vee C) \equiv (A \times B) \vee (A \times C)$

*Remark* 20. If $A, B \subset X$, then $A \cup B \equiv A \oplus B$ and $A \oplus A \equiv A$.

*Proof.* 1) $B$ and $C$ embed in $B \vee C$, so $A$ and $C$ too, and $A \vee C$ embeds in $B \vee C$. 2) Follows from the definition of $\vee$. 3) is clear. 4) After the previous assertions,
$A \times (B \vee C) \equiv A \times (B \oplus C) \equiv (A \times B) \oplus (A \times C) \equiv (A \times B) \vee (A \times C)$ $\square$

For every space $A$, set $D(A)$ the subspace of $A$ minus its finite open sets. As $A$ is Hausdorff, $D(A)$ equals $A'$. The map $D$ together with its properties survives modulo $\equiv$ :

**Proposition 21.** *Let $A$ and $B$ be topological spaces.*

1) *If $A \leq B$, then $D(A) \leq D(B)$*
2) $D(A \vee B) \equiv D(A) \vee D(B)$
3) $D(A \times B) \equiv (D(A) \times B) \vee (A \times D(B)))$

*Proof.* 1) If $A$ embeds in $B$ via $f$, $A'$ embeds in $B'$ via its restriction.
2) $(A \vee B)' \equiv (A \oplus B)' \equiv A' \oplus B' \equiv A' \vee B'$
3) $(A \times B)' \equiv (A' \times B) \cup (A \times B') \equiv (A' \times B) \vee (A \times B')$ $\square$

**Theorem 22.** *For every subclass $\mathcal{C}$ of $\mathcal{T}op_\omega$ closed under finite disjoint union and cartesian product, the class $(\mathcal{C}/{\equiv}, \vee, \times, D, \emptyset)$ is a commutative differential division semiring.*

If $\mathcal{C}$ contains a finite set, write 1 for the class of this set modulo $\equiv$. The semiring $\mathcal{C}$ is then unitary.

*Remark* 23. A *condensation point* is any point all the neighbourhoods of which are uncountable. Mapping a topological space to the subspace of its condensation points makes sense modulo $\equiv$, defining a derivation on $\mathcal{T}op_\omega/\equiv$.

Consider the class $\mathcal{P}ol_0$ of denumerable Polish spaces and the subclass $\mathcal{K}_0$ of denumerable compact spaces.

**Corollary 24.** *$(\mathcal{P}ol_0/\equiv, \vee, \times, D)$ is a commutative differential division semiring.*

**Corollary 25.** *The map $CB^*$ from $(\mathcal{K}_0/\equiv, \leq, \vee, \times)$ to $(\omega_1, \leq, sup, \oplus)$ is a morphism of ordered semirings.*

Let $\mathcal{C}$ be a category. We write $\leq_\mathcal{C}$ for the preorder associated to the notion of arrows in $\mathcal{C}$, and $\equiv_\mathcal{C}$ for the symmetrised equivalence relation. Let $(A_i, f_{ij})_{i\leq j\in I}$ be an inductive system. If it exists, the inductive limit $\varinjlim A_i$ of this system is, up to equivalence modulo $\equiv_\mathcal{C}$, the least upper bound of the $A_i$. We say that the category $\mathcal{C}$ is *closed under bounded inductive limits* if every inductive system $(A_i, f_{ij})$ in $\mathcal{C}$ bounded by some $B$ in $\mathcal{C}$ has an inductive limit in $\mathcal{C}$.

The former paragraphs gave us a notion of least upper bound compatible with the sum. The next ones will give a notion of greatest lower bound. Let $\mathcal{C}$ be a category every two objects $A$, $B$ of which have a least upper bound $A \vee B$. Suppose that $\mathcal{C}$ is closed under projective and bounded inductive limits.

**Proposition 26.** *Let $A$ and $B$ be in $\mathcal{C}$. There is a unique greatest lower bound of $A$ and $B$ up to equivalence modulo $\equiv_\mathcal{C}$.*

*Proof.* Set $D$ the set of lower bounds of both $A$ and $B$, and show that $(D, \leq_\mathcal{C})$ is inductive. Let $C_i$ an increasing chain of objects in $D$. The inductive limit $\varinjlim C_i$ is a greater bound of this chain, and still a lower bound of $A$ and $B$ after the universal property. The existence follows from Zorn's Lemma. As for uniqueness : Let $C$ and $C'$ two greater lower bounds of $A$ and $B$, then $C \vee C'$ is a lower bound of $A$ and $B$ as $C \vee C'$ is the least upper bound of $C$ and $C'$. But $C$ and $C'$ are lower bounds of $C \vee C'$. So $C \equiv_\mathcal{C} (C \vee C') \equiv_\mathcal{C} C'$ by maximality of $C$ and $C'$. □

We shall write $A \wedge B$ for this greastest lower bound.

**Proposition 27.** *Let $A$, $B$, $C$ and $D$ be in $\mathcal{C}$.*

1) *If $A \leq_\mathcal{C} C$ and $B \leq_\mathcal{C} D$, then $A \wedge B \leq_\mathcal{C} C \wedge D$.*
2) *$A \wedge (B \wedge C) \equiv_\mathcal{C} (A \wedge B) \wedge C$*
3) *$A \wedge (B \vee C) \geq_\mathcal{C} (A \wedge B) \vee (A \wedge C)$*

*Proof.* 1) $A \wedge B$ embeds in $A$ and $B$, so in $C$ and in $D$ too. As $C \wedge D$ is maximal with this property, $A \wedge B$ embeds in $C \wedge D$.

2) Follows from the very definition of $\wedge$.
3) $A \wedge B \leq A$ and $A \wedge C \leq A$, so $(A \wedge B) \vee (A \wedge C) \leq A$. Similarly, $(A \wedge B) \vee (A \wedge C) \leq B \vee C$. So $(A \wedge B) \vee (A \wedge C) \leq A \wedge (B \vee C)$. □

**Proposition 28.** *Suppose $(\mathcal{C}/\equiv_{\mathcal{C}}, \leq, \wedge, \vee)$ is a distributive lattice. Let $A \leq_{\mathcal{C}} B$. Modulo $\equiv_{\mathcal{C}}$, there is a unique least $C$ such that $A \vee C \equiv_{\mathcal{C}} B$.*

*Proof.* We show that the set $D$ of classes of $C \leq B$ mod $\equiv_{\mathcal{C}}$ so that $A \vee C \geq_{\mathcal{C}} B$ is inductive. Let $C_i$ be an increasing chain in $\mathcal{C}$ such that $B \leq_{\mathcal{C}} A \vee C_i$. The projective limit $\underleftarrow{lim}\, C_i$ is a lower bound belonging to $D$. By Zorn's Lemma, there is a smaller $C$ such that $A \vee C \geq_{\mathcal{C}} B$. As for uniqueness : if $C$ and $C'$ are two smaller elements with $A \vee C \geq_{\mathcal{C}} B$ and $A \vee C' \geq_{\mathcal{C}} B$, then $A \vee (C \wedge C') \equiv_{\mathcal{C}} (A \vee C) \wedge (A \vee C') \geq_{\mathcal{C}} B$, so $C \equiv_{\mathcal{C}} (C \wedge C') \equiv_{\mathcal{C}} C'$ by minimality of $C$ and $C'$. □

## 5. Rank of a denumerable space of types

Eventually, we apply the results to the space of types of a first order small theory. Let $L$ be a countable language, $T$ a $L$-theory, $a$ a finite tuple. Write $S_n(a)$ the set of $n$-types with parameters in $a$, with the topology associated to the clopen sets $[\phi(x_1, ..., x_n)]$ where $\phi(x_1, ..., x_n)$ runs over formulae in the language $L \cup \{x_1, ..., x_n\} \cup a$. It is a compact space. Suppose $T$ is *small*, that is $S_n(\emptyset)$ is countable for all $n$. As $a$ is finite, $S_n(a)$ is countable : every type $p$ has an ordinal rank $CB_a(p)$. We call *Cantor rank over $a$ of a formula $\phi$* with parameters in $a$, the Cantor-Bendixson rank of the open set $[\phi]$ in $S_n(a)$. The *Cantor degree* of $\phi$ is the Cantor-Bendixson degree of $[\phi]$. Write $dCB_a(\phi)$. For a formula $\phi$, we have

$$CB(\phi) = max\{CB(p) : p \in [\phi]\}$$
$$CB(p) = min\{CB(\psi) : \psi \in p\}$$

According to the previous pages, we have the following well known statements :

**Corollary 29.** *Let $C$ and $D$ be two $a$-definable sets.*

1) *$CB_a(D) < \aleph_1$*
2) *If $C \subset D$, then $CB_a(C) \leq CB_a(D)$*
3) *$CB_a(C \cup D) = max\{CB_a(C), CB_a(D)\}$*
4) *If $C$ and $D$ are disjoint with the same Cantor rank,*
$$dCB_a(C \cup D) = dCB_a(C) + dCB_a(D)$$
5) *$CB_a(D) \geq \alpha + 1$ if and only if there are infinitely many $a$-definable disjoint sets $D$ with Cantor rank at least $\alpha$.*
6) *The degree of $D$ is the greatest number $d$ of $a$-definable disjoint sets in $D$ having same Cantor rank as $D$.*

**Proposition 30.** *Let $A$ and $B$ be definable sets, and $f$ a definable map from $A$ to $B$. Then $f$ induces a map $\tilde{f}$ from $[A]$ to $[B]$ and*

1) *$\tilde{f}$ is continuous and open.*
2) *If $f$ is finite-to-one, so is $\tilde{f}$.*
3) *If $f$ is onto, so is $\tilde{f}$.*
4) *If $g$ is a definable map from $C$ to $A$, then $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$.*
5) *If $h = id_A$, then $\tilde{h} = id_{[A]}$.*

*Proof.* For a type $p$, let $a$ be a realisation, and $\tilde{f}(p)$ be $tp(f(a))$. Note that $\tilde{f}(p)$ does not depend on $a$. 1) Let $[D]$ be a basic open set in $[B]$,

$$\tilde{f}^{-1}([D]) = \{tp(a) \in [A] : f(a) \in D\} = [f^{-1}(D)]$$

So $\tilde{f}$ is continuous, and open as the space is compact with clopen basis. 2) If $f(a) \models tp(f(b))$, then there is an automorphism $\sigma$ such that $f(\sigma(a))$ equals $f(b)$, so $\sigma(a)$ is in the fibre of $f(b)$. □

This shows that there is a *covariant functor* $\sim$ from the category of definable sets in a structure $M$, with definable maps as arrows, to the category of topological spaces together with continuous open maps. The functor $\sim$ preserves finite-to-one maps, inducing a a functor from the subcategory of definable sets with finite-to-one maps to $\mathcal{T}op_\omega$.

**Corollary 31.** *Let $C$ and $D$ be $a$-definable sets, $f$ a $a$-definable map from $C$ to $D$.*

1) *If $f$ is onto, $CB_a(C) \geq CB_a(D)$.*
2) *If $f$ is finite-to-one, $CB_a(D) \geq CB_a(C)$.*
3) *If $f$ is $n$-to-one, onto, then $C$ and $D$ have same Cantor rank over $a$, and*

$$dCB_a(D) \leq dCB_a(C) \leq n \cdot dCB_a(D)$$

**Corollary 32.** *Let $X$ be definable without parameters, $a$ an algebraic tuple having degree $n$, then*

1) $CB_a(X) = CB_\emptyset(X)$
2) $dCB_\emptyset(X) \leq dCB_a(X) \leq n \cdot dCB_\emptyset(X)$

*Proof.* Write $R_a$ the relation on $S(X,a)$ "being conjugated under the action of the group of automorphisms fixing $a$ pointwise". It is a continuous equivalence relation every class of which has at most $n$ elements. On the other hand, $S(X,\emptyset)$ is homeomorphic to $S(X/R_a, a)$, so $CB_\emptyset(X)$ equals $CB_a(X/R_a)$. Apply Corollary 15. □

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